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On local integrability of fundamental solutions

Lars Hörmander

1. Introduction

Let $P(D)$, $D=-i\partial/\partial x$, be a partial differential operator in $\mathbf{R}^n$ with constant coefficients. In my thesis [1] I proved that $P(D)$ is hypoelliptic if and only if one of the following equivalent conditions is fulfilled:

(i) $\operatorname{Im}\zeta\to\infty$ if $\mathbf{C}^n\ni\zeta\to\infty$ and $P(\zeta)=0$;

(ii) $P(\xi)\neq 0$ for large $\xi\in\mathbf{R}^n$, and $P^{(\alpha)}(\xi)/P(\xi)\to 0$ when $\mathbf{R}^n\ni\xi\to\infty$, if $\alpha\neq 0$.

The sufficiency was proved by constructing a fundamental solution, that is, a *distribution* E with $P(D)E=\delta$, and verifying that (ii) implies that $E\in C^\infty$ in $\mathbf{R}^n\setminus\{0\}$. In a conversation with Marcel Riesz, who had been my mentor but was then retired, he reproached me for relying on the notion of distribution and told me that I ought to prove that E is in fact a locally integrable function. This reaction was quite typical of the reluctance of the mathematical community to accept the notion of distribution. It was not unexpected, and I had in fact avoided using distributions as far as I could.

Although it is quite irrelevant for the purposes of [1], I have never quite been able to dismiss the question whether the fundamental solutions of a hypoelliptic operator in $\mathbf{R}^n$ are always locally integrable. In Section 2 we shall prove that the answer is positive when $n=2$, but in Section 4 we shall give an example proving that the answer is negative for every $n\geq 14$. At last this settles the question except for dimensions $3,\dots,13$, and proves that distributions are essential and not only convenient in this context.

If $P(D)$ is an elliptic differential operator then $P^{(\alpha)}(D)E$ is essentially the inverse Fourier transform of $P^{(\alpha)}(\xi)/P(\xi)$, which behaves at infinity as a function which is homogeneous of degree $-|\alpha|$. When $|\alpha|=1$ it follows that $P^{(\alpha)}(D)E$ is singular at the origin as a homogeneous function of degree $1-n$, which gives that $P^{(\alpha)}(D)\in L^p_{\mathrm{loc}}$ if and only if $p<n/(n-1)$. For arbitrary $\alpha\neq 0$ we have $P^{(\alpha)}(D)E\in L^p_{\mathrm{loc}}$ if $1/p>1-|\alpha|/n$. More generally, if $P(D)$ is semielliptic in the sense of [2, Chapter XI, p. 67] of orders $m_1\geq m_2\geq\dots\geq m_n$, then it is easy to see that $P^{(\alpha)}(D)E\in L^p_{\mathrm{loc}}$

if $1\le p<\left(\sum_{j=1}^n 1/m_j\right)/\left(\sum_{j=2}^n 1/m_j\right)$ and $\alpha\neq 0$. By the following simple result only hypoelliptic operators can have such a strong regularity property.

Proposition 1.1. *Let $F\in\mathcal{D}'(\mathbf{R}^n)$ be a parametrix of $P(D)$, that is, $P(D)F-\delta_0\in C^\infty$. If $P^{(\alpha)}(D)F\in L^1$ in a neighborhood of 0 for all $\alpha\neq 0$, then $P(D)$ is hypoelliptic, and*

$$P^{(\alpha)}(D)F\in L^1_{\mathrm{loc}}(\mathbf{R}^n),\quad \alpha\neq 0, \tag{1.1}$$

for every parametrix F of $P(D)$.

Proof. Let Ω be an open neighborhood of 0 such that $P^{(\alpha)}(D)F\in L^1(\Omega)$ when $\alpha\neq 0$, and choose $\varphi\in C_0^\infty(\Omega)$ with $\varphi(0)=1$. If $G=\varphi F$ then

$$P(D)G-\delta_0=\sum_{\alpha\neq 0}\frac{(D^\alpha\varphi)P^{(\alpha)}(D)F}{\alpha!}+\varphi(P(D)F-\delta_0)\in L^1(\mathbf{R}^n),$$

$$P^{(\alpha)}(D)G=\sum_{\beta}\frac{(D^\beta\varphi)P^{(\alpha+\beta)}(D)F}{\beta!}\in L^1(\mathbf{R}^n),\quad \text{if } \alpha\neq 0.$$

Taking Fourier transforms we obtain when $\mathbf{R}^n\ni\xi\to\infty$,

$$P(\xi)\widehat{G}(\xi)-1\to 0,\quad P^{(\alpha)}(\xi)\widehat{G}(\xi)\to 0,\quad \text{if } \alpha\neq 0.$$

Hence $P(\xi)\neq 0$ when $|\xi|$ is large, and

$$\frac{P^{(\alpha)}(\xi)}{P(\xi)}=\frac{P^{(\alpha)}(\xi)\widehat{G}(\xi)}{P(\xi)\widehat{G}(\xi)}\to 0,\quad \text{as } \mathbf{R}^n\ni\xi\to\infty,$$

which proves hypoellipticity. Since parametrices of a hypoelliptic operator are smooth except at the origin and differ by functions which are smooth everywhere, we obtain (1.1) for every parametrix F.

The converse of Proposition 1.1 is not even true when $n=2$. In fact, in Section 3 we shall give for the two dimensional case a necessary and sufficient condition for the parametrices to have the property (1.1), and it is not fulfilled by all hypoelliptic operators. In Section 5 we shall discuss some consequences of the existence of parametrices with this property. They indicate that there may be some interest in characterizing this class of hypoelliptic operators also in the case of several variables.

2. The integrability of the fundamental solution itself

Let $P(D)$ be a hypoelliptic differential operator in $\mathbf{R}^2$, of order m. By condition (ii) in the introduction we can choose $M>0$ so large that $P(\xi)\neq 0$ when $|\xi|\geq M$. Set $B_t=\{\xi\in\mathbf{R}^2; |\xi|<t\}$. If $\chi\in C^\infty(\mathbf{R}^2)$ and $\chi=0$ in B_M, $\chi=1$ in $\complement B_{2M}$, then the inverse Fourier transform F of χ/P is a parametrix which is rapidly decreasing at infinity.

Theorem 2.1. *The function χ/P belongs to L^q when $q>(m+1)/m$, and $F\in L^p$ when $1\leq p<m+1$.*

Proof. If $m=1$ then P is elliptic (essentially the Cauchy–Riemann operator), and the statement follows then from the observations preceding Proposition 1.1. From now on we assume that $m>1$. If we can prove that $\chi/P\in L^q$ when $q>(m+1)/m$, then it follows from the Hausdorff–Young theorem that $F\in L^p$ when $2\leq p<m+1$. Since F is rapidly decreasing this implies $F\in L^p$ when $1\leq p<m+1$.

Thus it only remains to prove that $\chi/P\in L^q$ when $q>(m+1)/m$. If A is a closed angle in $\mathbf{R}^2$ containing no characteristic of P, then $|\xi|^m\leq C|P(\xi)|$ if $\xi\in A$ and $|\xi|\geq M$. Since $\int_M^\infty r^{1-mq}\,dr=M^{2-mq}/(mq-2)$ if $mq>2$ and since $mq-2>m(q-(m+1)/m)$, it follows that $\chi/P\in L^q(A)$ when $q>(m+1)/m$. It remains to prove that this is also true when A is a small angular neighborhood of a characteristic ray. We can choose the coordinates so that the ray is defined by $\xi_2=0$ while the ξ_2 axis is not a characteristic. Then we can write for large ξ_1

$$P(\xi)=a\prod_{j=1}^{m}(\xi_2-\tau_j(\xi_1)), \tag{2.1}$$

where a is a constant and each τ_j has a Puiseux series expansion with $\tau_j(\xi_1)=O(\xi_1)$, as $\xi_1\to+\infty$. (See e.g. [2, Appendix A].) When $\tau_j(\xi_1)=a_j\xi_1+o(\xi_1)$ with $a_j\neq 0$ then $|\xi|\leq C|\xi_2-\tau_j(\xi_1)|$ for large ξ_1 if $\xi\in A$ and A is a sufficiently small angular neighborhood of the positive ξ_1 axis. Denote by μ the number of such factors. When $a_j=0$ we have for large positive ξ_1,

$$\tau_j(\xi_1)=\sum_{k\leq s}c_{jk}(\xi_1^{1/r})^k, \tag{2.2}$$

where r is an integer >1, $0<s<r$, $c_{js}\neq 0$, and $\operatorname{Im} c_{jk}\neq 0$ for some $k>0$ (by condition (i)). Replacing $\xi_1^{1/r}$ by $\xi_1^{1/r}e^{2\pi i\varrho/r}$ gives different zeros for $\varrho=0, 1, \dots, r-1$. The absolute values of their imaginary parts are bounded below by a positive constant times $\xi_1^{1/r}$. Hence $|\xi_2-\tau_j(\xi_1)|(\xi_1^{1/r})^{r-1}$ can be bounded by a constant times the product of the corresponding factors in (2.1), when $\xi\in A$ is large, and the factor

$|\xi_2-\tau_j(\xi_1)|$ can be replaced by another factor $\xi_1^{1/r}$. Let ν be the number of such groups of zeros. Then we have for large $\xi\in A$,

$$|\xi_1|^\mu \xi_1^{\nu-1/r}|\xi_2-\tau(\xi_1)|\le C|P(\xi)|,$$

where τ denotes one of the zeros (2.2) and r is the number of zeros in the corresponding group. If $q>1$ we have

$$\int_{\mathbf{R}}|\xi_2-\tau(\xi_1)|^{-q}\,d\xi_2=|\operatorname{Im}\tau(\xi_1)|^{1-q}\int_{\mathbf{R}}|t+i|^{-q}\,dt\le C_q|\xi_1|^{(1-q)/r}.$$

Hence it follows that

$$\int_A\left|\frac{\chi(\xi)}{P(\xi)}\right|^q d\xi\le C_q\int_{M/2}^\infty \xi_1^{1/r-\nu q-\mu q}\,d\xi_1.$$

The integral converges if $(\mu+\nu)q>1+1/r$. Since $\tau_j(\xi_1)=o(\xi_1)$ we have $r\ge 2$, so this is true for all $q\ge 1$ if $\mu+\nu\ge 2$. Otherwise we must have $\mu=0$ and $\nu=1$, for $\nu\ne 0$, and then $r=m$ so the theorem is proved.

Since parametrices of a hypoelliptic operator only differ by smooth functions, the following is an immediate consequence of Theorem 2.1.

Theorem 2.2. *For every hypoelliptic operator $P(D)$ in $\mathbf{R}^2$ there is a parametrix F with $F\in L^p$ for $1\le p<m+1$, and every parametrix is in L^p_{loc} when $1\le p<m+1$.*

Example 2.1. For the heat operator $P(D)=\partial/\partial x_1-\partial^2/\partial x_2^2$ in $\mathbf{R}^2$ we have the fundamental solution

$$E(x)=\begin{cases}(4\pi x_1)^{-1/2}\exp(-x_2^2/4x_1), & \text{when } x_1>0,\\ 0, & \text{when } x_1\le 0,\end{cases}$$

$$\iint_{|x_j|<1}E(x)^p\,dx=\int_0^1(4\pi x_1)^{(1-p)/2}\int_{|t|<1/\sqrt{4\pi x_1}}e^{-\pi p t^2}\,dt.$$

The integral converges if and only if $\frac12(1-p)>-1$, that is, $p<3$. This proves that the L^p class in Theorems 2.1 and 2.2 cannot be improved in general when $m=2$. More generally, for the semielliptic operator $P(D)=iD_1+D_2^m$ of order $m\ge 2$ it is also true that the L^p class in these theorems is optimal. To prove this we choose $\varphi\in C_0^\infty(\mathbf{R}^2\setminus\{0\})$ with $\int(\varphi(\xi)/P(\xi))\,d\xi=1$ and set $\varphi_s(\xi)=\varphi(\xi_1/s^m,\xi_2/s)/s$. Then

$$\int\frac{\varphi_s(\xi)}{P(\xi)}\,d\xi=1,\quad \widehat{\varphi}_s(x)=s^m\widehat{\varphi}(s^m x_1,sx_2).$$

With $\widehat{F}=\chi/P$ as in Theorem 1 it follows that for large s,

$$1=\widehat{F}(\varphi_s)=F(\widehat{\varphi}_s).$$

Assume that $F\in L^p$ in a neighborhood Ω of 0. Then $F\widehat{\varphi}_s$ converges rapidly to 0 in $\complement\Omega$ as $s\to\infty$, and if $1/p+1/q=1$ it follows that for large s,

$$\begin{aligned}1=F(\widehat{\varphi}_s)&\le\frac{1}{2}+\Big(\int_\Omega|F|^p\,dx\Big)^{1/p}\Big(\int|\widehat{\varphi}_s|^q\,dx\Big)^{1/q}\\&\le\frac{1}{2}+C\Big(\int_\Omega|F|^p\,dx\Big)^{1/p}s^{m-(m+1)/q}.\end{aligned}$$

Since we can choose Ω with $C\big(\int_\Omega|F|^p\,dx\big)^{1/p}<\frac{1}{2}$, this gives a contradiction when $s\to\infty$ unless $m-(m+1)/q>0$, that is, $p<m+1$ as claimed.

3. The strong local integrability property

With P of the form (2.1) we have

$$\frac{1}{P(\xi)}\frac{\partial P(\xi)}{\partial\xi_2}=\sum_{j=1}^{m}\frac{1}{\xi_2-\tau_j(\xi_1)}\tag{3.1}$$

and similarly for derivatives of higher order. This suggests that the study of the inverse Fourier transform of $P^{(\alpha)}(\xi)/P(\xi)$ can be reduced to the study of the inverse Fourier transform of one of the terms in the sum in (3.1). It suffices to examine those with $\tau_j(\xi_1)=o(\xi_1)$, corresponding to a branch of the zeros asymptotic to the characteristic ξ_1 axis. To simplify notation we drop the subscript j temporarily and note that the properties of the Puiseux series expansion prove that $\tau\in C^\infty([c,\infty))$ for some $c>0$, and that there exist exponents γ_0, γ_1 with $0<\gamma_1\le\gamma_0<1$ and constants $c_0\in\mathbf{C}\setminus\{0\}$, $c_1\in\mathbf{R}\setminus\{0\}$ such that for every integer $j\ge0$,

$$\begin{aligned}\tau^{(j)}(\xi_1)\xi_1^{j-\gamma_0}&\to c_0\gamma_0(\gamma_0-1)\dots(\gamma_0-j+1),\quad \xi_1\to+\infty,\\ \operatorname{Im}\tau^{(j)}(\xi_1)\xi_1^{j-\gamma_1}&\to c_1\gamma_1(\gamma_1-1)\dots(\gamma_1-j+1),\quad \xi_1\to+\infty.\end{aligned}\tag{3.2}$$

We may assume that $c_1>0$, and replacing c by a larger number we may also assume that

$$\operatorname{Im}\tau(\xi_1)>\tfrac{1}{2}c_1\xi_1^{\gamma_1},\quad \xi_1\ge c.\tag{3.3}$$

If $\gamma_0=\gamma_1$ then $\operatorname{Im} c_0=c_1$, but otherwise $c_0\in\mathbf{R}\setminus\{0\}$.

With $a\in C^\infty(\mathbf{R})$ vanishing in $(-\infty,c]$ and $a(\xi_1)=1$ for large ξ_1 we wish to estimate the inverse Fourier transform u of $a(\xi_1)/(\xi_2-\tau(\xi_1))$ which is a C^∞ function of $\xi\in\mathbf{R}^2$ bounded by $C\xi_1^{-\gamma_1}$. Thus

$$u(x)=(2\pi)^{-2}\iint\frac{e^{i\langle x,\xi\rangle}a(\xi_1)}{\xi_2-\tau(\xi_1)}\,d\xi_1\,d\xi_2 \tag{3.4}$$

in the sense of distribution theory. The inverse Fourier transform with respect to ξ_2 vanishes when $x_2<0$, and when $x_2>0$ we have

$$u(x)=\frac{i}{2\pi}\int e^{i(x_1\xi_1+x_2\tau(\xi_1))}a(\xi_1)\,d\xi_1. \tag{3.5}$$

The following lemma proves that $u\in L^p$ for every p, outside an arbitrary neighborhood of the origin.

Lemma 3.1. *If j is an integer with $j(1+\gamma_1-\gamma_0)>1$ then $x_1^ju\in L^\infty(\mathbf{R}^2)$, and if $j\gamma_1>1$ then $x_2^ju\in L^\infty(\mathbf{R}^2)$.*

Proof. The inverse Fourier transform of $a(\xi_1)(\xi_2-\tau(\xi_1))^{-j-1}$ is $(ix_2)^ju/j!$. Since $|\xi_2-\tau|^2=|\xi_2-\operatorname{Re}\tau|^2+|\operatorname{Im}\tau|^2$, it follows when $j>0$ that

$$\int\frac{1}{|\xi_2-\tau|^{j+1}}\,d\xi_2\le C_j|\operatorname{Im}\tau|^{-j}.$$

We have $\int|a(\xi_1)|\,|\operatorname{Im}\tau(\xi_1)|^{-j}\,d\xi_1<\infty$ if $j\gamma_1>1$.

The inverse Fourier transform of $\partial^j(a(\xi_1)(\xi_2-\tau(\xi_1))^{-1})/\partial\xi_1^j$ is $(-ix_1)^ju$. Expanding by Leibniz' rule we obtain a number of terms vanishing for large ξ_1 where a has been differentiated, and by (3.5) their inverse Fourier transform is obviously bounded. The terms where a is not differentiated are of the form

$$\frac{a(\xi_1)\tau^{(k_1)}(\xi_1)\dots\tau^{(k_\mu)}(\xi_1)}{(\xi_2-\tau(\xi_1))^{1+\mu}},\quad \mu>0,\ k_1>0,\dots,k_\mu>0,\ k_1+\dots+k_\mu=j.$$

Such a term is bounded by a constant times

$$\frac{\xi_1^{\mu\gamma_0-k_1-\dots-k_\mu}}{|\xi_2-\tau(\xi_1)|^{1+\mu}}=\frac{\xi_1^{\mu\gamma_0-j}}{|\xi_2-\tau(\xi_1)|^{1+\mu}}.$$

This is integrable if $\mu\gamma_0-j-\mu\gamma_1<-1$, that is, $j>1+\mu(\gamma_0-\gamma_1)$. Here $\mu\le j$, so this is true if $j(1+\gamma_1-\gamma_0)>1$. The proof is complete.

From Lemma 3.1 it follows that $u \in L^p(\mathbf{R}^2)$ if $u \in L^p_{\mathrm{loc}}(\mathbf{R}^2)$. If $\chi \in C_0^\infty((0,\infty))$ and R is large, then

$$\chi(D_1/R)u(x) = u_R(x) = \frac{i}{2\pi}\int e^{i(x_1\xi_1 + x_2\tau(\xi_1))}\chi(\xi_1/R)\,d\xi_1 \tag{3.6}$$
$$= \frac{iR}{2\pi}\int e^{i(Rx_1\xi_1 + x_2\tau(R\xi_1))}\chi(\xi_1)\,d\xi_1,$$

is also in L^p, and $\|u_R\|_{L^p} \to 0$, as $R \to \infty$, provided that $1 \le p < \infty$. In fact, $\chi(D_1/R)$ is equivalent to convolution with respect to x_1 with $R\check{\chi}(Rx_1)$, where $\check{\chi} \in \mathcal{S}(\mathbf{R})$, so the operator norm in L^p is bounded by $\|\check{\chi}\|_{L^1} < \infty$ and $\chi(D_1/R)v \to 0$ in L^p, as $R \to \infty$, if $v \in \mathcal{S}(\mathbf{R}^2)$. We can choose $\chi \in C_0^\infty\big((\frac{1}{2}, 2)\big)$ so that $\sum_{\nu=-\infty}^{\infty}\chi(\xi_1/2^\nu) = 1$ if $\xi_1 > 0$. Then $u = \sum_{\nu \ge \nu_0} u_{2^\nu}$ if $2^{\nu_0} \le c$ so we shall be able to decide if $u \in L^p$ by examining $\|u_R\|_{L^p}$, as $R \to \infty$. This will be done in the following two lemmas.

Lemma 3.2. *If $\gamma_0 = \gamma_1$ then $u_R(y_1/R, y_2/R^{\gamma_0})/R$ converges in $\mathcal{S}$ to*

$$\Psi(y) = \frac{i}{2\pi}\int e^{i(y_1\xi_1 + y_2 c_0 \xi_1^{\gamma_0})}\chi(\xi_1)\,d\xi_1$$

when $R \to \infty$ and $y_2 \ge 0$. Hence

$$\|u_R\|_{L^p}^p R^{1+\gamma_0-p} \to \iint_{y_2>0} |\Psi(y)|^p\,dy. \tag{3.7}$$

Proof. Since

$$\frac{u_R(y_1/R, y_2/R^{\gamma_0})}{R} = \frac{i}{2\pi}\int e^{i(y_1\xi_1 + y_2\tau_R(\xi_1))}\chi(\xi_1)\,d\xi_1,$$

where $\tau_R(\xi_1) = R^{-\gamma_0}\tau(R\xi_1) \to c_0\xi_1^{\gamma_0}$ in C^∞ near $\operatorname{supp}\chi$, the stated convergence is obvious when y_2 is bounded. We can write

$$\frac{u_R(y_1/R, y_2/R^{\gamma_0})}{R} = \frac{i}{2\pi}e^{-y_2c_1/8}\int e^{iy_1\xi_1}a(y_2,\xi_1)\,d\xi_1,$$
$$a(y_2,\xi_1) = e^{y_2(i\tau_R(\xi_1)+c_1/8)}\chi(\xi_1),$$

and since $\operatorname{Re}\big(i\tau_R(\xi_1) + \frac{1}{8}c_1\big) \le c_1\big(-\frac{1}{2}\xi_1^{\gamma_1} + \frac{1}{8}\big) \le -\frac{1}{8}c_1$ in $\operatorname{supp}\chi$ by (3.3), it follows that $\xi_1 \mapsto a(y_2,\xi_1)$ is bounded in C_0^∞ when $y_2 > 0$. Hence we have uniform bounds

$$\left|\frac{u_R(y_1/R, y_2/R^{\gamma_0})}{R}\right| \le C_N(1+|y_1|)^{-N}e^{-c_1y_2/8}, \quad y_2 > 0,$$

for all N. This proves the lemma, for similar estimates are obtained in the same way for the derivatives with respect to y.

Lemma 3.3. *If $\gamma_0>\gamma_1$ and $1\le p<4$ then as $R\to\infty$,*

$$\frac{\|u_R\|^p_{L^p}}{R^\Gamma}\to(2\pi)^{-p/2}\iint_{y_2>0}e^{-py_2c_1\xi_1^{\gamma_1}}|\psi''(\xi_1)y_2|^{1-p/2}|\chi(\xi_1)|^p\,d\xi_1\,dy_2, \tag{3.8}$$

where

$$\Gamma=\gamma_0-1-2\gamma_1+p\big(1-\tfrac{1}{2}(\gamma_0-\gamma_1)\big),\quad \psi(\xi_1)=c_0\xi_1^{\gamma_0}. \tag{3.9}$$

Proof. With $y_1=R^{1+\gamma_1-\gamma_0}x_1$ and $y_2=R^{\gamma_1}x_2$ we have when R is large

$$\begin{aligned}\frac{u_R(x)}{R}=\frac{u_R(R^{\gamma_0-1-\gamma_1}y_1,R^{-\gamma_1}y_2)}{R}&=\frac{i}{2\pi}\int e^{i\varphi_R(y,\xi_1)}\chi(\xi_1)\,d\xi_1,\\ \operatorname{Im}\varphi_R(y,\xi_1)=y_2R^{-\gamma_1}\operatorname{Im}\tau(R\xi_1)\to y_2c_1\xi_1^{\gamma_1}&\qquad\text{when } R\to\infty,\\ R^{\gamma_1-\gamma_0}\operatorname{Re}\varphi_R(y,\xi_1)=y_1\xi_1+y_2R^{-\gamma_0}\operatorname{Re}\tau(R\xi_1)\to y_1\xi_1+y_2\psi(\xi_1)&\quad\text{when } R\to\infty.\end{aligned}$$

Hence the stationary phase method proves that

$$|u_R(R^{\gamma_0-1-\gamma_1}y_1,R^{-\gamma_1}y_2)R^{-1+(\gamma_0-\gamma_1)/2}|\to\frac{e^{-y_2c_1\xi_1^{\gamma_1}}\chi(\xi_1)}{\sqrt{|2\pi y_2\psi''(\xi_1)|}}, \tag{3.10}$$

if $y_1+y_2\psi'(\xi_1)=0$. Let $I\subset\mathbf{R}\setminus\{0\}$ be a compact interval which is a neighborhood of $\{-\psi'(\xi_1);\xi_1\in\operatorname{supp}\chi\}$. The stationary phase method also gives the bounds

$$\begin{aligned}|u_R(R^{\gamma_0-1-\gamma_1}y_1,R^{-\gamma_1}y_2)|R^{-1+(\gamma_0-\gamma_1)/2}&\le C|y|^{-1/2}e^{-by_2},\quad\text{if } y_1/y_2\in I,\ y_2>0,\\ \frac{|u_R(R^{\gamma_0-1-\gamma_1}y_1,R^{-\gamma_1}y_2)|}{R}&\le C_N(1+R^{\gamma_0-\gamma_1}|y|)^{-N},\quad\text{if } y_1/y_2\notin I.\end{aligned}$$

Here $b>0$, and N is arbitrary. Hence

$$\int_{\substack{y_2>0\\ y_1/y_2\notin I}}|u_R(R^{\gamma_0-1-\gamma_1}y_1,R^{-\gamma_1}y_2)|^p\,dy\le CR^{p+2(\gamma_1-\gamma_0)}.$$

Since $p+2(\gamma_1-\gamma_0)+\gamma_0-1-2\gamma_1<\Gamma$ when $p<4$ and

$$\iint_{\substack{y_2>0\\ y_1+y_2\psi'(\xi_1)=0}}\left|\frac{e^{-y_2c_1\xi_1^{\gamma_1}}\chi(\xi_1)}{\sqrt{2\pi y_2|\psi''(\xi_1)|}}\right|^p dy_1\,dy_2=\iint_{y_2>0}\left|\frac{e^{-y_2c_1\xi_1^{\gamma_1}}\chi(\xi_1)}{\sqrt{2\pi y_2|\psi''(\xi_1)|}}\right|^p y_2|\psi''(\xi_1)|\,dy_2\,d\xi_1,$$

(3.8) follows from the dominated convergence theorem.

Note that although the proofs of Lemmas 3.2 and 3.3 were rather different, the exponent Γ in (3.9) reduces to the exponent $p-1-\gamma_0$ in (3.7) when $\gamma_1=\gamma_0$.

Proposition 3.4. *The inverse Fourier transform u defined by (3.5) is in $L^p_{\mathrm{loc}}(\mathbf{R}^2)$ if and only if*

$$1\le p<1+\frac{3\gamma_1-\gamma_0}{2+\gamma_1-\gamma_0}=2-\frac{2-2\gamma_1}{2+\gamma_1-\gamma_0}, \tag{3.11}$$

and $u\in L^p(\mathbf{R}^2)$ then.

Proof. This is an immediate consequence of Lemmas 3.1–3.3, for (3.11) means precisely that $\Gamma<0$.

Remark. Note that (3.11) implies that $\gamma_0<3\gamma_1$. If τ has the Puiseux series expansion (2.2) and $\gamma_0<3\gamma_1$, then $3\gamma_1\ge\gamma_0+1/r$ and $\gamma_1\ge 1/r$. Hence

$$\frac{3\gamma_1-\gamma_0}{2+\gamma_1-\gamma_0}\ge\frac{1/r}{2+\gamma_1-3\gamma_1+1/r}=\frac{1}{1+2r(1-\gamma_1)}\ge\frac{1}{2r-1}\ge\frac{1}{2m-1}$$

if τ is an algebraic function of degree m. Thus $u\in L^p$ if $p<2m/(2m-1)$.

We can now return to studying the parametrices of a hypoelliptic operator $P(D)$ in $\mathbf{R}^2$. Let P_0 be the principal part of P, set

$$Z=\{\zeta\in\mathbf{C}^2\,;P(\zeta)=0\},\quad Z_0=\{\zeta\in\mathbf{C}^2\,;P_0(\zeta)=0\}, \tag{3.12}$$

and denote by $d_{Z_0}(\zeta)$ the distance from ζ to Z_0,

$$d_{Z_0}(\zeta)=\inf_{\zeta_0\in Z_0}|\zeta-\zeta_0|. \tag{3.13}$$

It is clear that $d_{Z_0}(\zeta)$ is homogeneous of degree 1, $d_{Z_0}(\zeta)\le|\zeta|$, and if V is a conic neighborhood of Z_0 then $|\zeta|\le C_V d_{Z_0}(\zeta)$ when $\zeta\notin V$. If $(1,0)\in Z_0$ then $d_{Z_0}(\zeta)=|\zeta_2|$ when $\zeta=(\zeta_1,\zeta_2)$ is in a sufficiently small conic neighborhood of $(1,0)$. When $\zeta=(\zeta_1,\tau(\zeta_1))$ with τ as in (3.2), then

$$d_{Z_0}(\zeta)=|c_0|\,|\zeta_1|^{\gamma_0}(1+o(1))\le 2|c_0|(\operatorname{Re}\zeta_1)^{\gamma_0}(1+o(1)),$$
$$|\operatorname{Im}\zeta|^2=|\operatorname{Im}\zeta_1|^2+(c_1(\operatorname{Re}\zeta_1)^{\gamma_1}+o(\operatorname{Re}\zeta_1)^{\gamma_1}+O(|\zeta_1|^{\gamma_0-1}\operatorname{Im}\zeta_1))^2>\tfrac12 c_1^2(\operatorname{Re}\zeta_1)^{2\gamma_1},$$

when $\zeta_1\to\infty$ and $|\operatorname{Im}\zeta_1|<\operatorname{Re}\zeta_1$. The condition $\gamma_0<3\gamma_1$ in Proposition 3.4 is therefore equivalent to

$$\frac{d_{Z_0}(\zeta)}{|\operatorname{Im}\zeta|^3}\to 0\quad\text{when } Z\ni\zeta\to\infty. \tag{3.14}$$

Theorem 3.5. *If $P(D)$ is a hypoelliptic operator in $\mathbf{R}^2$ of order m and (3.14) is fulfilled then $P^{(\alpha)}(D)F\in L^p_{\mathrm{loc}}$ if F is a parametrix of $P(D)$ and*

$$1\le p<\frac{2m}{2m-|\alpha|}. \tag{3.15}$$

When $|\alpha|=m$ we conclude that $F\in L^p_{\mathrm{loc}}$ when $1\le p<2$, which is a much weaker result than Theorem 2.1. We have not aimed for the best exponent here since local integrability is the main issue.

Proof. As in Section 2 we choose $M>0$ so that $P\neq 0$ in $\complement B_M$, $B_t=\{\xi\in\mathbf{R}^2; |\xi|<t\}$. If $\chi\in C^\infty(\mathbf{R}^2)$ vanishes in B_M and $\chi(t\xi)=\chi(\xi)$ when $t\ge 1$ and $\xi\notin B_{2M}$, then $\chi(\xi)P^{(\alpha)}(\xi)/P(\xi)$ is a classical symbol of order $-|\alpha|$ if $P_0\neq 0$ in $\operatorname{supp}\chi$, so the inverse Fourier transform is in $L^q(\mathbf{R}^2)$ if $1\le q<2$ when $|\alpha|=1$ and for all $q\in[1,\infty)$ when $|\alpha|\ge 2$. Hence it is in L^q when $1\le q<2m/(2m-|\alpha|)$.

Assuming that the ξ_1 axis is in Z_0 and writing P in the form (2.1) we note that

$$\frac{1}{P(\xi)}\frac{\partial^k P(\xi)}{\partial\xi_2^k}=k!\sum_{|J|=k}\prod_{j\in J}\frac{1}{\xi_2-\tau_j(\xi_1)}, \tag{3.16}$$

where J runs over subsets of $\{1,\dots,m\}$ with k elements. If χ is as above with M replaced by a sufficiently large number and $\operatorname{supp}\chi$ is sufficiently close to the positive ξ_1 axis, then the inverse Fourier transform u_j of $\chi(\xi)(\xi_2-\tau_j(\xi_1))^{-1}$ is in L^p if $1\le p<2m/(2m-1)$. When $\tau_j(\xi_1)$ is not $o(\xi_1)$ as $\xi_1\to+\infty$ this follows at once from the beginning of the proof, for then we have a symbol of order -1. When $\tau_j(\xi_1)=o(\xi_1)$ it follows from Proposition 3.4 and the remark after its proof that the inverse Fourier transform v_j of $a(\xi_1)(\xi_2-\tau_j(\xi_1))^{-1}$ is in L^p when $1\le p<2m/(2m-1)$ for some a which equals 1 in $[M,\infty)$, and it is rapidly decreasing at infinity by Lemma 3.1. The inverse Fourier transform of χ is rapidly decreasing and smooth outside the origin, so it is the sum of a function in $\mathcal{S}(\mathbf{R}^2)$ and a function of compact support which as a convolution operator is continuous on L^p for every $p\in(1,\infty)$. Hence it maps v_j to a function which is rapidly decreasing and belongs to L^p when $1<p<2m/(2m-1)$ and therefore also when $p=1$. Since $u_j=\chi(D)v_j$ we have proved the claim about u_j. From the classical inequality

$$\|u_1*\dots*u_k\|_p\le\|u_1\|_{p_1}\dots\|u_k\|_{p_k},\quad 1-\frac{1}{p}=\sum_{j=1}^{k}\Big(1-\frac{1}{p_j}\Big),$$

it follows that the inverse Fourier transform of (3.16) multiplied by $\chi(\xi)^k$ is in L^p when $p<2m/(2m-k)$. By choosing other ξ_2 axes we get the same conclusion for the inverse Fourier transform of $\chi(\xi)^k P^{(\alpha)}(\xi)/P(\xi)$ for all α with $|\alpha|=k$.

Summing up, we can find functions $\chi_j \in C^\infty$, $j=0,\dots,\mu$, where μ is the number of real characteristics of P, such that the inverse Fourier transform of the function $\chi_j(\xi)P^{(\alpha)}(\xi)/P(\xi)$ is in L^p for $1\le p<2m/(2m-|\alpha|)$ and $\chi=\sum_{j=0}^{\mu}\chi_j$ is equal to 1 outside a compact set. Then the inverse Fourier transform F of $\chi(\xi)/P(\xi)$ is a parametrix with the required L^p class also globally; it is of course rapidly decreasing. The proof is complete.

Our final goal in this section is to prove that (3.14) is necessary in order that $P^{(\alpha)}F\in L^1_{\mathrm{loc}}$ when $|\alpha|=1$. When $\alpha=(0,1)$, this is essentially the inverse Fourier transform of $\sum_{j=1}^{m}(\xi_2-\tau_j(\xi_1))^{-1}$. For the individual terms we know the necessity from Proposition 3.4, but we must prove that there cannot be cancellations which make the sum locally integrable although the individual terms are not. The proof is fairly long so we shall first give an example where this problem does not occur.

Example. For $P(\xi)=(\xi_2^2-2i\xi_1)^2-\xi_1(\xi_1-1)^2$ the zeros are given by

$$\xi_2=\xi_1^{3/4}+i\xi_1^{1/4}$$

with the four possible determinations of $\xi_1^{1/4}$. For one of them we have $\gamma_0=\frac{3}{4}=3\gamma_1$ and for the other three we have $\gamma_0=\gamma_1=\frac{3}{4}$. Hence the inverse Fourier transform of $a(\xi_1)\sum_{j=1}^{4}(\xi_2-\tau_j(\xi_1))$ is not integrable, for three of the terms are but the fourth is not, by Proposition 3.4. The assumption (3.14) in Theorem 3.5 is therefore not superfluous.

To rule out the possibility of cancellations in general will require a more precise version of (3.10) which also takes into account the phase factor given by the method of stationary phase. If $\gamma_0>\gamma_1$ then $c_0\in\mathbf{R}$ and the phase factor is equal to

$$\exp\bigl(i\operatorname{Re}\varphi_R(y,\xi_1)-\tfrac{1}{4}\pi i\operatorname{sgn}c_0\bigr),\quad \text{where } y_1+y_2R^{1-\gamma_0}\operatorname{Re}\tau'(R\xi_1)=0.$$

Assume now that τ has a Puiseux series expansion of the form (2.2). Then $\tau(\xi_1)\xi_1^{-\gamma_0}$ has a convergent expansion in powers of $\xi_1^{-1/r}$, equal to c_0 at infinity, and the equation for ξ_1 can be written

$$y_1+y_2\sum_{k=0}^{\infty}C_k(\xi_1)\varrho^k=0,\quad \varrho=R^{-1/r},$$

where $C_0(\xi_1)=c_0\gamma_0\xi_1^{\gamma_0-1}$, all the functions C_k are analytic when $\xi_1>0$, and the series converges in a neighborhood of $\{(\xi_1,\varrho);\xi_1>0,\varrho=0\}$. By the implicit function theorem this defines ξ_1 as an analytic function of $y_1/y_2\in I$ and ϱ which is equal to $(-y_1/y_2c_0\gamma_0)^{1/(\gamma_0-1)}$ when $\varrho=0$. Hence

$$\operatorname{Re}\varphi_R(y,\xi_1)=\Phi(y,R)=R^{\gamma_0-\gamma_1}\sum_{k=0}^{\infty}\Phi_k(y)R^{-k/r},$$

where Φ_k are analytic when $y_1/y_2 \in I$ and homogeneous of degree 1. With this notation (3.10) can be refined to

(3.17)
$$u_R(R^{\gamma_0-1-\gamma_1}y_1, R^{-\gamma_1}y_2)R^{-1+(\gamma_0-\gamma_1)/2}e^{-i\Phi(y,R)+\pi i/4\,\mathrm{sgn}\,c_0} \to \frac{e^{-y_2c_1\xi_1^{\gamma_1}}\chi(\xi_1)}{\sqrt{|2\pi y_2\psi''(\xi_1)|}}.$$

Let τ_ν, $\nu=1,\dots,\mu$, be Puiseux series satisfying the hypotheses of Lemma 3.3 with the same values of γ_0 and γ_1. Let u_R^ν, Φ^ν, c_j^ν, $I^\nu,\dots$ also be defined as u_R, Φ, c_j, $I,\dots$ with τ replaced by τ_ν, and assume that the coefficients c_1^ν are positive. Then we claim that for suitably chosen $\chi \in C_0^\infty((0,\infty))$

$$\varliminf_{R\to\infty}\Big\|\sum_{\nu=1}^{\mu}u_R^\nu\Big\|_{L^1}R^{(3\gamma_1-\gamma_0)/2}>0. \tag{3.18}$$

For the proof we observe that

$$\begin{aligned}\Big\|\sum_{\nu=1}^{\mu}u_R^\nu\Big\|_{L^1}R^{(3\gamma_1-\gamma_0)/2} &= R^{-1+(\gamma_0-\gamma_1)/2}\int\Big|\sum_{\nu=1}^{\mu}u_R^\nu(R^{\gamma_0-1-\gamma_1}y_1,R^{-\gamma_1}y_2)\Big|\,dy\\ &\ge \mathrm{Re}\int_K\sum_{\nu=1}^{\mu}u_R^\nu(R^{\gamma_0-1-\gamma_1}y_1,R^{-\gamma_1}y_2)R^{-1+(\gamma_0-\gamma_1)/2}e^{-i\Phi^1(y,R)}\,dy\end{aligned}$$

for every compact set $K\subset\{(y_1,y_2);y_1/y_2\in I^1,\ y_2>0\}$. We choose K so that if $\Phi^\nu(y,R)-\Phi^1(y,R)$ is unbounded as $R\to\infty$, then it is asymptotic to a positive power of R times a nonvanishing function of y in K. Then it follows from the homogeneity that there is no stationary point in K, so these terms converge to 0 by the Riemann–Lebesgue lemma. For the other terms $\Phi^\nu(y,R)-\Phi^1(y,R)$ converges to a function which is homogeneous of degree 0. If K is chosen close to the origin then this limit takes its values in $\left(-\frac{1}{8}\pi,\frac{1}{8}\pi\right)$, so the argument of the integrand belongs to the interval $\left(-\frac{3}{8}\pi,\frac{3}{8}\pi\right)$, if $\chi\ge 0$. If χ is chosen so that the limit of the term with $\nu=1$ is positive, this implies (3.18) since there cannot be any cancellation.

We have now made the preparations required for the proof that (3.14) is a necessary hypothesis in Theorem 3.5.

Theorem 3.6. *If a hypoelliptic operator $P(D)$ in $\mathbf{R}^2$ has a parametrix F such that $P^{(\alpha)}(D)F\in L^1_{\mathrm{loc}}$ when $|\alpha|=1$, then* (3.14) *is fulfilled.*

Proof. Since parametrices differ by smooth functions the hypothesis is fulfilled for every parametrix F, so we can assume that $\widehat{F}\in C^\infty$ and that $\widehat{F}(\xi)=1/P(\xi)$ for large $|\xi|$. Then $P^{(\alpha)}(D)F$ is rapidly decreasing so $P^{(\alpha)}(D)F\in L^1$, when $|\alpha|=1$. We may also assume the coordinates chosen so that $P(\xi)$ is of the form (2.1) for large $\xi_1>0$. When $\alpha=(0,1)$ the Fourier transform of $P^{(\alpha)}(D)F$ is then equal to $\sum_{\nu=1}^m 1/(\xi_2-\tau_\nu(\xi_1))$ for large positive ξ_1. With $\chi\in C_0^\infty((0,\infty))$ to be chosen later it follows that the inverse Fourier transform $\chi(D_1/R)F$ of

$$\chi(\xi_1/R)\frac{P^{(\alpha)}(\xi)}{P(\xi)}=\sum_{\nu=1}^m \frac{\chi(\xi_1/R)}{\xi_2-\tau_\nu(\xi_1)}$$

converges to 0 in L^1 as $R\to\infty$.

As in the proof of Theorem 3.5 we choose $M>0$ so that $P\neq 0$ in $\complement B_M$, where $B_t=\{\xi\in\mathbf{R}^2;|\xi|<t\}$. Let h_0 be the set of functions $\psi\in C^\infty(\mathbf{R}^2)$ with $\psi=0$ in B_M and $\psi(t\xi)=\psi(\xi)$ when $t\geq 1$ and $\xi\notin B_{2M}$. If $\psi_0\in h_0$ and $P_0\neq 0$ in $\operatorname{supp}\psi_0$ then $\psi_0(\xi)P^{(\alpha)}(\xi)/P(\xi)$ is a classical symbol of order $-|\alpha|=-1$, so the inverse Fourier transform is in $L^q(\mathbf{R}^2)$ if $1\leq q<2$. If $\psi_1\in h_0$ and $\psi_0+\psi_1=1$ in $\complement B_{2M}$, then it follows that the inverse Fourier transform of $\psi_1(\xi)P^{(\alpha)}(\xi)/P(\xi)$ is also in L^1, since that of $(\psi_0(\xi)+\psi_1(\xi))/P(\xi)$ is a parametrix. Hence

$$\chi(\xi_1/R)\psi_1(\xi)\frac{P^{(\alpha)}(\xi)}{P(\xi)}\to 0 \quad \text{in } \mathcal{F}L^1, \text{ as } R\to\infty. \tag{3.19}$$

We can choose ψ_0 so that ψ_1 is a finite sum of functions $\psi_{1,j}\in h_0$, $j=1,\dots,J$, with disjoint supports, each of which is equal to 1 in the intersection of B_{2M} and a conic neighborhood of one of the real characteristics and vanishes in a neighborhood of the others. Then it follows that (3.19) is valid with ψ_1 replaced by $\psi_{1,j}$, for $j=1,\dots,J$. In fact, if B_j, $j=1,\dots,J$, are disjoint compact sets in $\mathbf{R}^2$ then

$$\sum_{j=1}^J \|u_j\|_{L^1}\leq C\Big\|\sum_{j=1}^J u_j\Big\|_{L^1} \tag{3.20}$$

for all $u_j\in L^1(\mathbf{R}^2)$ with $\operatorname{supp}\hat{u}_j\subset B_j$. This is clear since we can choose $d_j\in\mathcal{S}(\mathbf{R}^2)$ with $\hat{d}_j=1$ in B_j and $\hat{d}_j=0$ in B_k for $k\neq j$, which gives $u_j=d_j*\sum_{k=1}^J u_k$ and proves (3.20). A dilation shows that (3.20) remains valid if $\operatorname{supp}\hat{u}_j\subset RB_j$, $j=1,\dots,J$, for some $R>0$, and this proves that ψ_1 may be replaced by $\psi_{1,j}$ in (3.19), for $j=1,\dots,J$.

If the ξ_1 axis is a characteristic we conclude in particular that

$$\chi(\xi_1/R)\psi_2(\xi)\sum_{\nu=1}^m \frac{1}{\xi_2-\tau_\nu(\xi_1)}\to 0 \quad \text{in } \mathcal{F}L^1, \text{ as } R\to\infty, \tag{3.21}$$

for some $\psi_2\in h_0$ which outside B_{2M} is equal to 1 in a conic neighborhood of the positive ξ_1 axis and vanishes in a conic neighborhood of the other characteristics.

When $\tau_\nu(\xi)$ is not $o(\xi_1)$ then $\psi_2(\xi)/(\xi_2-\tau_\nu(\xi_1))\in\mathcal{F}L^1$ so the corresponding term converges to 0. The summation in (3.21) may therefore be restricted to the zeros with $\tau_\nu(\xi_1)=o(\xi_1)$. We may assume that the zeros are labelled so that they are the zeros with $1\le\nu\le\mu$, and we shall assume that $1\le\nu\le\mu$ in what follows.

To be able to use Lemmas 3.2 and 3.3, with the refinements preceding the statement of Theorem 3.6, we must remove the factor ψ_2. To do so we choose $\psi_3\in h_0$ equal to 0 in a conic neighborhood of the ξ_1 axis so that $\psi_2+\psi_3=1$ in $\complement B_{2M}$. Then $\psi_3(\xi)/\xi_2$ is a classical symbol of order -1, so the inverse Fourier transform is in L^q when $1\le q<2$. Thus $\chi(\xi_1/R)\psi_3(\xi)/\xi_2\to 0$ in $\mathcal{F}L^1$, as $R\to\infty$. To prove that $\chi(\xi_1/R)\psi_3(\xi)(\xi_2-\tau_\nu(\xi_1))^{-1}\to 0$ in $\mathcal{F}L^1$ it is therefore sufficient to prove that

$$\frac{\chi(\xi_1/R)\psi_3(\xi)\tau_\nu(\xi_1)}{\xi_2(\xi_2-\tau_\nu(\xi_1))}\to 0 \quad \text{in } \mathcal{F}L^1, \text{ as } R\to\infty,\ 1\le\nu\le\mu. \tag{3.22}$$

The L^2 norm is $O(R^{\gamma_0^\nu-1})\to 0$, as $R\to\infty$, so (3.22) follows from Parseval's formula if we prove that the inverse Fourier transform is rapidly decreasing. The L^1 norm of any derivative is $O(R^{\gamma_0^\nu-1})\to 0$, as $R\to\infty$. This is obvious for the derivatives with respect to ξ_2. A differentiation with respect to ξ_1 improves the L^1 norm by a factor R^{-1} if it falls on $\tau_\nu(\xi_1)$, $\chi(\xi_1/R)$ or $\psi_3(\xi)$, and by a factor $R^{\gamma_0^\nu-2}=O(R^{-1})$ if it falls on the factor $(\xi_2-\tau_\nu(\xi_1))^{-1}$, which proves the claim. Hence the inverse Fourier transform of the function in (3.22) can be estimated by $C|x|^{-3}R^{\gamma_0^\nu-1}$ outside the origin, which completes the proof of (3.22).

From (3.21), with the summation restricted to $\nu\le\mu$, and (3.22) it follows that

$$\chi(\xi_1/R)\sum_{\nu=1}^{\mu}\frac{1}{\xi_2-\tau_\nu(\xi_1)}\to 0 \quad \text{in } \mathcal{F}L^1, \text{ as } R\to\infty. \tag{3.23}$$

For $1\le\nu\le\mu$ we know by Lemmas 3.2 and 3.3 that

$$\left\|\frac{\chi(\xi_1/R)}{\xi_2-\tau_\nu(\xi_1)}\right\|_{\mathcal{F}L^1} R^{(3\gamma_1^\nu-\gamma_0^\nu)/2}$$

has a positive limit as $R\to\infty$. Let $\Gamma=\max_{1\le\nu\le\mu}\frac{1}{2}(\gamma_0^\nu-3\gamma_1^\nu)$. If we prove that

$$\varliminf_{R\to\infty}\left\|\chi(\xi_1/R)\sum_{\nu=1}^{\mu}\frac{1}{\xi_2-\tau_\nu(\xi_1)}\right\|_{\mathcal{F}L^1}/R^\Gamma>0, \tag{3.24}$$

it will follow from (3.23) that $\Gamma<0$, that is, that $\gamma_0^\nu<3\gamma_1^\nu$, $\nu=1,\dots,\mu$, and this will prove the theorem.

In (3.24) the terms with $\frac{1}{2}(\gamma_0^\nu-3\gamma_1^\nu)<\Gamma$ play no role so we can drop them and assume from now on that $\frac{1}{2}(\gamma_0^\nu-3\gamma_1^\nu)=\Gamma$ when $1\le\nu\le\mu$. Assume for example that $c_1^1>0$. From (3.18) we know then that

$$\varliminf_{R\to\infty}\frac{\|u_R\|_{L^1}}{R^\Gamma}>0,\quad \text{if } \hat{u}_R(\xi)=\chi(\xi_1/R)\sum{}'\frac{1}{\xi_2-\tau_\nu(\xi_1)}$$

with the summation taken for the indices ν with $1\le\nu\le\mu$, $\gamma_0^\nu=\gamma_0^1$, $\gamma_1^\nu=\gamma_1^1$, and $c_1^\nu>0$. The last condition can be dropped at once, for the terms in the inverse Fourier transform with $c_1^\nu<0$ have their support in the half plane where $x_2\le 0$. The proof of (3.18) gives more; it proves that for some compact set $K\subset\{(y_1,y_2);y_1/y_2\in I^1,\ y_2>0\}=Q^1$,

$$\text{(3.25)}\quad \varliminf_{R\to\infty}\int_{K_R}|u_R(x)|\,dx/R^\Gamma>0,\quad K_R=\{(x_1,x_2);(R^{1+\gamma_1^1-\gamma_0^1}x_1,R^{\gamma_1^1}x_2)\in K\}.$$

If $\gamma_1^\nu\ne\gamma_1^1$ or $\gamma_0^\nu\ne\gamma_0^1$ then both inequalities are valid, and if $x\in K_R$ then

$$\begin{aligned}Y&=(R^{1+\gamma_1^\nu-\gamma_0^\nu}x_1,R^{\gamma_1^\nu}x_2)=(R^{\gamma_1^\nu-\gamma_1^1-\gamma_0^\nu+\gamma_0^1}y_1,R^{\gamma_1^\nu-\gamma_1^1}y_2),\\ y&=(R^{1+\gamma_1^1-\gamma_0^1}x_1,R^{\gamma_1^1}x_2)\in K,\end{aligned}$$

so Y stays outside any given compact subset of Q^ν for large R. Hence the proof of (3.8) using an integrable majorant proves that the integral of the inverse Fourier transform of $\chi(\xi_1/R)(\xi_2-\tau_\nu(\xi_1))^{-1}/R^\Gamma$ over K_R converges to 0, as $R\to\infty$. Now (3.24) follows from (3.25) and the proof is complete.

4. A counterexample in high dimensions

If $P(D)$ is a hypoelliptic operator in $\mathbf{R}^n$, of order m, with principal part $P_0(D)$, then there are no simple characteristics, that is, $P_0(\xi)=0$ implies $P_0'(\xi)=0$ if $0\ne\xi\in\mathbf{R}^n$. This follows from condition (ii) since $P(t\xi)=O(t^{m-1})$, as $t\to\infty$, and $P'(t\xi)/t^{m-1}\to P_0'(\xi)$. On the other hand, Theorem 11.1.12 in [2] shows that the characteristic set may be quite arbitrary if the multiplicity is high. The following result is closely related.

Proposition 4.1. *Let $P(\xi)=P_0(\xi)+iQ(\xi)$ where $P_0\ge 0$ is homogeneous of order m, Q is real valued and homogeneous of order $m-1$, and $Q(\xi)\ne 0$ when $0\ne\xi\in\mathbf{R}^n$ and $P_0(\xi)=0$. Then $P(D)$ is hypoelliptic.*

Proof. Since P_0 is homogeneous and $P_0\ge 0$ we have (see e.g. [2, Lemma 7.7.2])

$$|P_0'(\xi)|^2\le CP_0(\xi)|\xi|^{m-2},\quad \xi\in\mathbf{R}^n.$$

From the estimate

$$|P'(\xi)|^2=|P_0'(\xi)|^2+|Q'(\xi)|^2\le CP_0(\xi)|\xi|^{m-2}+|Q'(\xi)|^2$$
$$\le CP_0(\xi)^2|\xi|^{-1}+C|\xi|^{2m-3}+|Q'(\xi)|^2$$

and the fact that $|\xi|^{m-1}\le C|Q(\xi)|\le C|P(\xi)|$ in a conic neighborhood V of the characteristic set $\{\xi\in\mathbf{R}^n;P_0(\xi)=0\}$, it follows that $P'(\xi)/P(\xi)=O(|\xi|^{-1/2})$ in V, and this is obviously true also in $\complement V$. When $|\alpha|\ge 2$ we have

$$\left|\frac{P^{(\alpha)}(\xi)}{P(\xi)}\right|=O(|\xi|^{m-|\alpha|-(m-1)})=O(|\xi|^{1-|\alpha|})=O(|\xi|^{-|\alpha|/2}),$$

which completes the proof of condition (ii).

Remark. The preceding estimates show that solutions of the differential equation $P(D)u=0$ are in fact of Gevrey class 2.

The special case we shall study in this section is the square of the wave operator

$$\square=(\partial/\partial x_0)^2-(\partial/\partial x_1)^2-\ldots-(\partial/\partial x_n)^2$$

in $\mathbf{R}^{1+n}$, modified as in Proposition 4.1 by adding $-(\partial/\partial x_0)^3$. Thus we set

$$P(\xi)=(\xi_0^2-\xi_1^2-\ldots-\xi_n^2)^2+i\xi_0^3,$$

and we shall examine if the parametrices of the hypoelliptic operator $P(D)$ are locally integrable. If $F\in\mathcal{S}'(\mathbf{R}^{1+n})$ and $\widehat{F}\in C^\infty$, $\widehat{F}=1/P$ outside a compact set, then F is a parametrix which is rapidly decreasing and smooth except at the origin, so $F\in L^1$ if $F\in L^1_{\mathrm{loc}}$.

We change notation now and write

$$x_0=t,\quad x=(x_1,\ldots,x_n),\quad \xi_0=\tau,\quad \xi=(\xi_1,\ldots,\xi_n).$$

If $\chi\in C_0^\infty(\mathbf{R}^n\setminus\{0\})$ and $\chi(D/R)$ is the convolution operator multiplying the Fourier transform by $\chi(\xi/R)$, then we see as in Section 3 that $\chi(D/R)F\to 0$ in $L^1(\mathbf{R}^n)$, as $R\to\infty$, if $F\in L^1(\mathbf{R}^n)$. We have for large R,

$$\begin{aligned}F_R(t,x)=\chi(D/R)F(t,x)&=(2\pi)^{-1-n}\iint\frac{e^{i(t\tau+\langle x,\xi\rangle)}}{P(\tau,\xi)}\chi(\xi/R)\,d\tau\,d\xi\\&=(2\pi)^{-1-n}R^{1+n}\iint\frac{e^{iR(t\tau+\langle x,\xi\rangle)}}{P(R\tau,R\xi)}\chi(\xi)\,d\tau\,d\xi.\end{aligned}\tag{4.1}$$

To evaluate the integral with respect to τ we must examine the zeros of $P(\tau,\xi)$.

Lemma 4.2. *The roots $\tau_j(r)$ of the equation*

$$(\tau^2-r^2)^2+i\tau^3=0 \tag{4.2}$$

have Puiseux series expansions in powers of $r^{-1/2}$ at $+\infty$,

$$\begin{aligned}\tau_1(r)&=r-\tfrac{1}{2}e^{-\pi i/4}\sqrt{r}-\tfrac{1}{4}i+O(1/\sqrt{r}),\\ \tau_2(r)&=-r+\tfrac{1}{2}e^{\pi i/4}\sqrt{r}-\tfrac{1}{4}i+O(1/\sqrt{r}),\\ \tau_3(r)&=r+\tfrac{1}{2}e^{-\pi i/4}\sqrt{r}-\tfrac{1}{4}i+O(1/\sqrt{r}),\\ \tau_4(r)&=-r-\tfrac{1}{2}e^{\pi i/4}\sqrt{r}-\tfrac{1}{4}i+O(1/\sqrt{r}).\end{aligned} \tag{4.3}$$

At these roots the derivative of the polynomial with respect to τ is

$$\begin{gathered}4\tau_j(r)(\tau_j(r)^2-r^2)+3i\tau_j(r)^2=r^{5/2}A_j(r),\\ A_1(\infty)=-4e^{-\pi i/4}=\overline{A_4(\infty)},\quad A_2(\infty)=4e^{\pi i/4}=\overline{A_3(\infty)},\end{gathered} \tag{4.4}$$

where A_j is a convergent power series in $r^{-1/2}$.

Proof. With $\tau=\pm r+sr^{1/2}$ the equation can be written

$$(2s\pm s^2\varrho)^2+i(\pm1+s\varrho)^3=0,\quad \varrho=r^{-1/2}.$$

When $\varrho=0$ the equation reduces to $4s^2\pm i=0$ which has two simple zeros, so the implicit function theorem gives that the equation is satisfied by an analytic function of ϱ equal to a square root of $\mp\frac{1}{4}i$ at 0. Differentiation of the equation at $\varrho=0$ gives

$$8s\frac{ds}{d\varrho}\pm4s^3+3is=0,\quad \text{that is,}\quad 8\frac{ds}{d\varrho}=\mp4s^2-3i=-2i$$

which proves (4.3), and (4.4) is an immediate consequence.

The imaginary parts of τ_1 and τ_2 are positive and those of τ_3 and τ_4 are negative. When $t>0$ it follows that

$$\begin{aligned}F_R(t,x)=\chi(D/R)F(t,x)&=i(2\pi)^{-n}\sum_{j=1}^{2}\int\frac{e^{i(t\tau_j(|\xi|)+\langle x,\xi\rangle)}}{P'_\tau(\tau_j(|\xi|),\xi)}\chi(\xi/R)\,d\xi\\ &=i(2\pi)^{-n}R^n\sum_{j=1}^{2}\int\frac{e^{i(t\tau_j(R|\xi|)+R\langle x,\xi\rangle)}}{P'_\tau(\tau_j(R|\xi|),R\xi)}\chi(\xi)\,d\xi.\end{aligned} \tag{4.1$'$}$$

The fundamental solution of $\Box^2$ is singular on the Lorentz cone $\{(t,x);t^2-|x|^2=0\}$, and it is not locally integrable if $n\geq5$. It is natural to expect that as a smoothing of this fundamental solution, F is concentrated near the wave cone. This suggests that we study

$$(4.5)\qquad G_R(t,x)=F_R\big(|x|/\sqrt{R}+t/R, x/\sqrt{R}\big).$$

When $|t|<|x|\sqrt{R}$ it follows from $(4.1)'$ and Lemma 4.2 that

$$(4.6)\qquad G_R(t,x)=i(2\pi)^{-n}R^{n-5/2}\sum_{j=1}^{2}\int e^{i\sqrt{R}\,\varphi_j(x,\xi)}a_j^R(t,x,\xi)\chi(\xi)\,d\xi,$$

$$(4.7)\qquad \varphi_1(x,\xi)=|x||\xi|+\langle x,\xi\rangle,\quad \varphi_2(x,\xi)=-|x|\,|\xi|+\langle x,\xi\rangle.$$

Moreover, a_j^R converges as $R\to\infty$ in the C^∞ topology in a neighborhood of $\mathbf{R}^{1+n}\times$ supp χ to

$$(4.8)\qquad \begin{aligned} a_1^\infty(t,x,\xi)&=\frac{e^{i(t|\xi|-e^{-\pi i/4}|x|\sqrt{|\xi|}\,/2)}}{|\xi|^{5/2}A_1(\infty)},\\ a_2^\infty(t,x,\xi)&=\frac{e^{i(-t|\xi|+e^{\pi i/4}|x|\sqrt{|\xi|}\,/2)}}{|\xi|^{5/2}A_2(\infty)}.\end{aligned}$$

The phase function φ_1 is critical with respect to ξ precisely when $\xi/|\xi|+x/|x|=0$, that is, ξ and x have opposite directions, and φ_2 is critical when ξ and x have the same direction. The critical values are zero by the homogeneity in ξ. Since the critical points are degenerate we introduce polar coordinates $\xi=\varrho\omega$ where $\varrho>0$ and $|\omega|=1$. The phase functions $\omega\mapsto\pm|x|\varrho+\varrho\langle x,\omega\rangle$ have non-degenerate critical points when $\omega=\pm x/|x|$ and $\omega=\mp x/|x|$, but only the latter are relevant. Hence the method of stationary phase gives when $R\to\infty$, with some constants $C_j\neq 0$ which we do not have to specify (see e.g. [2, Theorem 7.7.14]),

$$R^{5/2-n}R^{(n-1)/4}G_R(t,x)\to\sum_{j=1}^{2}C_j|x|^{(1-n)/2}\int a_j^\infty(t,x,\varrho\omega_j)\chi(\varrho\omega_j)\varrho^{(n-1)/2}\,d\varrho$$

for $x\neq 0$, where $-\omega_1=\omega_2=x/|x|$. If we take χ with support in a half space the two terms will have disjoint supports. For a suitable choice of $\chi\in C_0^\infty(\mathbf{R}^n\setminus\{0\})$ and of a compact set $K\subset\mathbf{R}\times(\mathbf{R}^n\setminus\{0\})$, it follows that

$$\varliminf_{R\to\infty} R^{(9-3n)/4}\int_K|G_R(t,x)|\,dt\,dx>0.$$

Since

$$\int_K|G_R(t,x)|\,dt\,dx\le R^{1+n/2}\int|F_R(t,x)|\,dt\,dx$$

for large R, this implies that

$$\varliminf_{R\to\infty} R^{(13-n)/4}\int |F_R(t,x)|\,dt\,dx>0,$$

and we conclude that $n<13$ if $F\in L^1$. Thus we have proved the following proposition.

Proposition 4.3. *The differential operator $\Box^2-\partial_0^3$ in $\mathbf{R}^{1+n}$ is hypoelliptic, but the parametrices are not locally integrable if $1+n\geq 14$.*

5. A remark on local spaces

Let $B\subset\mathcal{D}'(\mathbf{R}^n)$ be a Banach space which is semilocal in the sense of [2, Definition 10.1.18], that is, $C_0^\infty(\mathbf{R}^n)B\subset B$. If $X\subset\mathbf{R}^n$ is open, we denote by $B^{\mathrm{loc}}(X)$ the corresponding local subspace of $\mathcal{D}'(X)$,

$$B^{\mathrm{loc}}(X)=\{u\in\mathcal{D}'(X)\,;\varphi u\in B \text{ if } \varphi\in C_0^\infty(X)\}.$$

Theorem 5.1. *Let $P(D)$ be a hypoelliptic operator in $\mathbf{R}^n$ which has a parametrix F such that $P^{(\alpha)}(D)F\in L^1_{\mathrm{loc}}$ when $\alpha\neq 0$. Then*

$$\{u\in\mathcal{D}'(X)\,;P(D)u\in B^{\mathrm{loc}}(X)\}\tag{5.1}$$

is a local space if B is a semilocal space containing $C_0^\infty(\mathbf{R}^n)$ which is invariant under convolution with functions in $L^1(\mathbf{R}^n)$ with compact support.

Proof. We must prove that $P(D)(\varphi u)=\sum D^\alpha\varphi P^{(\alpha)}(D)u/\alpha!\in B$ if $\varphi\in C_0^\infty(X)$ and u is in the space (5.1). To do so we choose $\psi\in C_0^\infty(\mathbf{R}^n)$ equal to 1 in a neighborhood of the origin with support so close to the origin that $\operatorname{supp}\varphi-\operatorname{supp}\psi\Subset X$. Then $F_1=\psi F$ is also a parametrix, that is, $P(D)F_1=\delta_0+\omega$ where $\omega\in C_0^\infty$. Thus

$$u=F_1*P(D)u-\omega*u,\quad P^{(\alpha)}(D)u=(P^{(\alpha)}(D)F_1)*P(D)u-(P^{(\alpha)}(D)\omega)*u$$

in a neighborhood of $\operatorname{supp}\varphi$. If $\Psi\in C_0^\infty(X)$ is equal to 1 in a neighborhood of $\operatorname{supp}\varphi-\operatorname{supp}\psi$ then

$$D^\alpha\varphi P^{(\alpha)}(D)u=D^\alpha\varphi((P^{(\alpha)}(D)F_1)*(\Psi P(D)u)-(P^{(\alpha)}(D)\omega)*(\Psi u)).$$

Both terms are in B which proves that (5.1) is a semilocal space. To prove that it is local assume that $u\in\mathcal{D}'(X)$ and that $P(D)(\varphi u)\in B^{\mathrm{loc}}(X)$ for every $\varphi\in C_0^\infty(X)$. If $\psi\in C_0^\infty(X)$ we can choose $\varphi\in C_0^\infty(X)$ equal to 1 in a neighborhood of $\operatorname{supp}\psi$

and obtain $\psi P(D)u = \psi P(D)(\varphi u) \in B$, hence $P(D)u \in B^{\mathrm{loc}}(X)$ which completes the proof. (The second part of the proof is of course valid for every differential operator.)

When $B = L^2$ or more generally one of the spaces $B_{p,k}$ in [2, Section 10.1] then Theorem 5.1 holds for every hypoelliptic operator. The point of Theorem 5.1 and of the strong integrability property of fundamental solutions is that this is also true for L^p spaces, Hölder spaces and so on which are not defined in terms of Fourier transforms.

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Lars Hörmander
Department of Mathematics
University of Lund
Box 118
S-221 00 Lund
Sweden
email: lvh@maths.lth.se