Ideal theory on non-orientable Klein surfaces

NORMAN L. ALLING*

University of Rochester, Rochester, N.Y., U.S.A.

and

BALMOHAN V. LIMAYE

Tata Institute of Fundamental Research, Bombay, India

§ 0. Introduction

That non-empty, connected, non-orientable compact surfaces Y, with non-empty boundary ∂Y, can bear a structure, a dianalytic structure, which allows one to define the notion of analytic "functions" on them has been known essentially since Klein's 1882 monograph [6]. However, only recently have the standard algebra $A(Y)$ of all continuous "functions" on Y that are analytic on $Y \setminus \partial Y \equiv Y^{\circ}$, and the algebra $H^{\infty}(Y^{\circ})$ of all bounded analytic "functions" on Y°, been studied by Alling, Campbell, and Greenleaf [2, 3, 4, 5]. In these papers it has been shown that $A(Y)$ is a real Banach algebra which does not admit a complex scalar multiplication, whose maximal ideal space is Y, and whose Silov boundary is ∂Y. Further it has been shown that $A(Y)$ is a hypo-Dirichlet algebra whose deficiency is $c - 1$, c being the first Betti number of Y, and that $A^{-1}(Y)/\exp A(Y)$, the factor group of units modulo exponentials, is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}^{c-1}$.

The purpose of this paper is the study of the closed ideals of $A(Y)$. Y admits an unramified double covering morphism p of a compact bordered Riemann surface X such that $p^{-1}(\partial Y) = \partial X$, and X admits an antianalytic involution τ that commutes with p. This *orienting double* (X, p, τ) of Y is unique up to an analytic isomorphism. Let $A(X)$ be the standard algebra on X and for f in $A(X)$, let $\sigma(f) \equiv \varkappa \circ f \circ \tau$, $\varkappa$ being complex conjugation. Then σ is an $\mathbf{R}$-automorphism of $A(X)$ of period 2 that is an isometry. $A(Y)$ is naturally $\mathbf{R}$-isomorphic and

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isometric to $\{f$ in $A(X) : \sigma(f) = f\}$: i.e., to the set of all *symmetric elements* of $A(X)$. Let $A(Y)$ then be identified with this set. If X is any compact bordered Riemann surface, the ideal theory of $A(X)$ has been extensively studied. If b is the first Betti number of X, the classical case $b = 0$ where X is the closed unit disk, was done by Beurling [unpublished] and by Rudin [10]. The case $b > 0$ was handled in [7, 11, 12] by Limaye, Stanton, and Voichick. While in [12], Voichick used so-called multiple-valued "functions", the results of Stanton [11] are not as strong as those of the classical case. We have, therefore, included in § 2 the intrinsic characterization given in the second author's thesis [7], written under the direction of the first author, which proves to be particularly useful in pushing the results down to Y from its orienting double X.

Since the above characterization is in terms of inner functions, we felt it necessary to give in § 1 the foundations of the theory of generalized inner functions, functions which are independent of the choice of a basis of $A^{-1}(X)$ modulo $\exp A(X)$. In order to get sharp technical results, though, we do use inner functions relative to a fixed basis. A particularly suitable basis had to be found to be able to define symmetric inner functions in a satisfactory fashion. This we achieve in § 3 and § 4. It also enables us to carry the factorization of functions in $H^\infty(X^\circ)$, Limaye [8, 9], to those in $H^\infty(Y^\circ)$, and to show that each "function" in $H^\infty(Y^\circ)$ is in some sense a σ-square root of a function in $H^\infty(X^\circ)$. This we do in § 5. In § 6 the notion of symmetric closed ideals of $A(X)$ is introduced and is shown to reflect exactly the theory of the closed ideals pulled up from $A(Y)$. We illustrate this by considering the standard algebra on the Möbius strip. While Theorem 4.2 is the crucial technical result, the main theorem of the paper is Theorem 6.2.

§ 1. Inner and outer functions

Let z_0 be a point in the interior X° of a compact bordered Riemann surface X, and consider the harmonic measure $m_{z_0} \equiv m$ on ∂X with respect to z_0. It is well known that the space $L^\infty(\partial X, dm)$ does not depend on the choice of the point z_0 in X°. Let $H^\infty(dm)$ denote the weak-star closure of $\{f|_{\partial X} : f$ in $A(X)\}$ in $L^\infty(dm)$. It is isometrically isomorphic to the space $H^\infty(X^\circ)$ of all bounded analytic functions on X°, where a function f in $H^\infty(dm)$ corresponds to the function $\hat{f}$ in $H^\infty(X^\circ)$ whose non-tangential boundary values are equal to f.

Let $\langle \log |A^{-1}(X)| \rangle$ denote the real span of $\{\log |f| : f$ in $A^{-1}(X)\}$, considered as imbedded in $L^1(dm)$. If f is a non-zero element of $H^\infty(dm)$, then $\log |f|$ belongs to $L^1(dm)$, by the Jensen inequality.

Definition 1.1. A function f in $H^\infty(dm)$ is called *a generalized inner function* (g.i.f.) if $\log |f|$ belongs to $\langle \log |A^{-1}(X)| \rangle$. It is called *an outer function* if $\log |\hat{f}(z_0)| = \int_{\partial X} \log |f| dm$.

Remark 1.2. The definition of the generalized inner functions is intrinsic for the algebra $A(X)$. It is also easy to see that the definition of the outer functions does not depend on the choice of the point z_0 in X°. It is natural to investigate the overlap between the g.i.f.s and the outer functions. For this we need to know the structure of $A^{-1}(X)$. The basic result, proved by Wermer [14, Lemma 1] states that if $\{\gamma_1, \ldots, \gamma_b\}$ is a basis of the first homology group of X with integral coefficients then there exist $Z_1, \ldots, Z_b$ in $A^{-1}(X)$, which are analytic across ∂X, such that

$$(1/2\pi) \int_{\gamma_j}^{*} d(\log |Z_k|) = \delta_{j,k}, \quad 1 \leq j, k \leq b .$$

It shows that $\{\log |Z_1|, \ldots, \log |Z_b|\}$ is a basis of the real vector space V of all real-valued harmonic functions on X° modulo the subspace of the real parts of all analytic functions on X°, and that $\{Z_1, \ldots, Z_b\}$ is a basis of the multiplicative group $A^{-1}(X)$ modulo the exponentials in $A(X)$. This, in turn, implies that if $\{W_1, \ldots, W_b\}$ is any other basis of $A^{-1}(X)$ modulo $\exp A(X)$, then $\{\log |W_1|, \ldots, \log |W_b|\}$ is a basis of V.

PROPOSITION 1.3. *A function f in $H^\infty(dm)$ is a g.i.f. as well as an outer function if and only if f belongs to $A^{-1}(X)$.*

Proof. A function f in $A^{-1}(X)$ is a g.i.f. by definition. That it is an outer function follows by using the Jensen inequality both ways. Conversely, let f be an outer function and let $\log |f| = \sum_{j=1}^{n} \alpha_j \log |f_j|$, where α_j are real numbers and f_j are in $A^{-1}(X)$. Then

$$\log |\hat{f}(z)| = \sum_{j=1}^{n} \int_{\partial X} \alpha_j \log |f_j| dm_z = \sum_{j=1}^{n} \alpha_j \log |f_j(z)| ,$$

for each z in X°, since the definition of the outer functions does not depend on the point z_0 in X°. Let now $\{Z_1, \ldots, Z_b\}$ be a basis of $A^{-1}(X)$ modulo $\exp A(X)$, as in Remark 1.2. Then, $f_j = Z_1^{n_{1,j}} \cdot \ldots \cdot Z_b^{n_{b,j}} \cdot \exp g_j$, where $n_{k,j}$ is an integer and g_j is in $A(X)$, for $1 \leq k \leq b$. Hence there exist real numbers a_j and a g in $A(X)$ such that

$$\log |\widehat{f}(z)| = \sum_{k=1}^{b} a_k \log |Z_k(z)| + \operatorname{Re} g(z) .$$

Now if we integrate the conjugate differentials over the homology cycle γ_j, $1 \leq j \leq b$, it becomes clear that each a_k is an integer, and that $f = Z_1^{a_1} \ldots Z_b^{a_b} \cdot \exp g$.

The above proposition indicates that if we use the g.i.f.s in the description of the closed ideals of $A(X)$, the characterization could be unique only up to a factor of a function in $A^{-1}(X)$. Since we know the structure of $A^{-1}(X)$ modulo $\exp A(X)$, we shall get more precise information if we define and use inner functions relative to a fixed basis of $A^{-1}(X)$ modulo $\exp A(X)$.

Definition 1.4. Let $\{Z_1, \ldots, Z_b\}$ be a basis of $A^{-1}(X)$ modulo $\exp A(X)$. Then a function f in $H^\infty(dm)$ is called *an inner function relative to* $\{Z_1, \ldots, Z_b\}$ if $\log |f|$ belongs to $\langle \log |Z_1|, \ldots, \log |Z_b| \rangle$. If $\log |f| = \sum_{j=1}^{b} \alpha_j \log |Z_j|$, then the real numbers α_j are uniquely determined by f.

A function f in $H^\infty(dm)$ would be an inner function relative to $\{Z_1, \ldots, Z_b\}$ as well as an outer function if and only if $f = cZ_1^{m_1} \cdot \ldots \cdot Z_b^{m_b}$, where c is a complex constant of absolute value 1, and $m_1, \ldots, m_b$ are integers. Such a function will be called a *trivial inner function relative to* $\{Z_1, \ldots, Z_b\}$.

Remark 1.5. It is clear that an inner function relative to $\{Z_1, \ldots, Z_b\}$ is a g.i.f., and conversely if f is a g.i.f., then there exists a g in $A(X)$ such that $f \cdot \exp g$ is an inner function relative to $\{Z_1, \ldots, Z_b\}$.

Remark 1.6. A factorization of a function in $H^\infty(dm)$ into an inner function relative to $\{Z_1, \ldots, Z_b\}$ and an outer function, unique up to a factor of a trivial inner function, was given by Limaye in [9]. In developing the ideal theory of $A(X)$, we shall use the following two facts from this function theory.

(i) Given a point z in X°, there exists an inner function f in $A(X)$ such that the only zero of f on X is a simple zero at z.

(ii) Given a closed subset E of ∂X, of m-measure zero, there exists an outer function g in $A(X)$ such that E is precisely the zero set of g on X.

§ 2. Closed ideals of $A(X)$

The characterization of the non-zero closed ideals of $A(X)$ which we give here is based on two celebrated function algebraic results, we now quote.

THE F. AND M. RIESZ THEOREM. *Let μ be a finite complex Baire measure on ∂X and let $\mu_a + \mu_s$ be its Lebesgue decomposition w.r.t. the harmonic measure m. Then μ is orthogonal to an ideal I of $A(X)$ if and only if μ_a and μ_s are orthogonal to I. If, in particular, μ is orthogonal to $A(X)$, then $\mu_s = 0$.* (Cf., Theorem 3.1 of Ahern's and Sarason's [1], and Lemma 3 of Wermer's [14].)

THE INVARIANT SUBSPACE THEOREM. *Let M be a simply invariant subspace of $H^\infty(dm)$: that is, let M be a weak-star closed subspace of $H^\infty(dm)$ such that $A(X)M \subset M$, and the closure of A_0M is strictly contained in M, where $A_0 = \{f \text{ in } A(X) : f(z_0) = 0\}$; then there exists an inner function w (relative to*

$\{Z_1, \ldots, Z_b\}$) *such that* $M = wH^\infty(dm)$. (Cf., Lemma 14.3 of Ahern's and Sarason's [1].)

We now prove a lemma which shows that a closed ideal I of $A(X)$ is determined by its weak-star closure in $L^\infty(dm)$ and its hull on ∂X.

LEMMA 2.1. *Let* I *be a closed ideal of* $A(X)$, *and let* $[I]_*$ *denote its weak-star closure in* $L^\infty(dm)$. *Let* $E = \{x \text{ in } X : f(x) = 0, \text{ for every } f \text{ in } I\}$, *and* $I(E) = \{g \text{ in } A(X) : g = 0 \text{ on } E\}$. *Then*

$$I = [I]_* \cap I(E) .$$

Proof. Let $C(\partial X)^*$ denote the continuous dual of $C(\partial X)$: that is, the space of all finite complex Baire measures on ∂X. We shall write $f \perp \mu$, for f in $C(\partial X)$ and μ in $C(\partial X)^*$, if f is orthogonal to μ; $\mu_1 \ll \mu_2$, if μ_1 is absolutely continuous w.r.t. μ_2; and $\mu_1 \perp \mu_2$, if μ_1 and μ_2 are mutually singular. Define

$$I^\perp = \{\mu \text{ in } C(\partial X)^* : \mu \perp I\} ,$$

$$I_a^\perp = \{\mu \text{ in } I^\perp : \mu \ll m\} \text{ and } I_s^\perp = \{\mu \text{ in } I^\perp : \mu \perp m\} .$$

By the F. and M. Riesz theorem, $I^\perp = I_a^\perp + I_s^\perp$. Thus,

$$\begin{aligned} I &= \{f \text{ in } C(\partial X) : f \perp I^\perp\} \\ &= \{f \text{ in } C(\partial X) : f \perp I_a^\perp\} \cap \{f \text{ in } C(\partial X) : f \perp I_s^\perp\} . \end{aligned}$$

But $I \subset [I]_* \cap C(\partial X) \subset \{f \text{ in } C(\partial X) : f \perp I_a^\perp\}$, and the second part of the F. and M. Riesz theorem shows that $I \subset I(E) \subset \{f \text{ in } C(\partial X) : f \perp I_s^\perp\}$. Thus, $I = [I]_* \cap I(E)$.

THEOREM 2.2. *Let* I *be a non-zero closed ideal of* $A(X)$. *Then there exists an inner function* w *(relative to* $\{Z_1, \ldots, Z_b\}$*) and a unique closed subset* E *of* ∂X, *of harmonic measure zero, such that* $I = wI(E)$. *The inner function* w *is determined up to a trivial inner function relative to* $\{Z_1, \ldots, Z_b\}$.

Proof. Lemma 2.1 gives $I = [I]_* \cap I(E)$. Now suppose that there exists a function f in I such that $f(z_0) \neq 0$. Then $[I]_*$ is a simply invariant subspace of $H^\infty(dm)$ and by the Invariant Subspace Theorem, there exists an inner function w relative to $\{Z_1, \ldots, Z_b\}$ such that $[I]_* = wH^\infty$. If, on the contrary, every function in I vanishes at z_0, then let k be the smallest positive integer such that each f in I has a zero of order at least k at z_0. If f is an inner function in $A(X)$ relative to $\{Z_1, \ldots, Z_b\}$ having its only zero on X a simple zero at z_0, $[f^{-k}I]_*$ is again a simply invariant subspace of $H^\infty(dm)$. Thus, in any case, there exists an inner function w relative to $\{Z_1, \ldots, Z_b\}$ such that $[I]_* = wH^\infty$ and hence $I = wH^\infty \cap I(E)$.

We now note that if w is an inner function relative to $\{Z_1, \ldots, Z_b\}$, and x a point in ∂X, then $\hat{w}$ can be extended continuously to a neighbourhood of x in ∂X if and only if $\hat{w}$ is bounded away from zero in a neighbourhood of x in X (cf. § 3 of Limaye's [8]). This, in particular, shows that if w is invertible in $L^\infty(dm)$, then w is invertible in $A(X)$, and hence is a trivial inner function. Also, it shows that if $f = wg$, where f is in $A(X)$ and g is in H^∞, then g is, in fact, in $A(X)$ and g vanishes on ∂X wherever f does. These considerations enable us to conclude that $wH^\infty \cap I(E) = wI(E)$.

Finally, let us consider the question of uniqueness. Assume that I is also given by $w_1 I(E_1)$. Let g be an outer function in $A(X)$ that vanishes precisely on E. Then $wg = w_1 g_1$, where g_1 is in $I(E_1)$. Since g is outer, the uniqueness of the inner-outer factorization shows that w_1 divides w in $L^\infty(dm)$, and since g vanishes precisely on E, $E_1 \subset E$. Similarly, w divides w_1 in $L^\infty(dm)$ and $E \subset E_1$. Thus, w/w_1, being invertible in $L^\infty(dm)$, is a trivial inner function, and $E = E_1$.

COROLLARY 2.3. *Every closed ideal of $A(X)$ is the principal closed ideal generated by a function in $A(X)$.*

Proof. Let I be a non-zero closed ideal of $A(X)$ and $I = wI(E)$ as in Theorem 2.2. Let g be an outer function in $A(X)$ which vanishes precisely on E. Then I is the closure in $A(X)$ of the ideal generated by wg.

§ 3. Symmetric and antisymmetric harmonic functions

In this section we shall give an account of symmetric and antisymmetric real-valued harmonic functions and solve the obstruction problem occasioned by asking when such functions are real parts of global analytic functions. These results will then be applied in § 4 to questions involving the groups $A^{-1}(X)/\exp A(X)$ and $A^{-1}(Y)/\exp A(Y)$.

Let c be the first Betti number of Y; then $2c - 1 = b$, the first Betti number of the orienting double X (Cf., Alling [4], (2.b) and (2.c)).

Let $L_{\mathbf{R}}(X^\circ)$, $\Delta(X^\circ)$, and $A(X^\circ)$ denote the spaces of all real-valued harmonic functions on X°, of all analytic differentials on X°, and of all analytic functions on X°, respectively. For h in $L_{\mathbf{R}}(X^\circ)$, let $\tau^*(h) \equiv h \circ \tau$; for ω in $\Delta(X^\circ)$, represented by $(\omega_j)_{j \in J}$, where $(U_j, z_j)_{j \in J}$ is the maximal analytic structure on X°, let $\sigma(\omega)$ be represented by $(\sigma(\omega_j))_{j \in J}$; and for f in $A(X^\circ)$ let $\sigma(f) \equiv \varkappa \circ f \circ \tau$, where $\varkappa$ is the complex conjugation. τ^* and σ are $\mathbf{R}$-linear involutions of $L_{\mathbf{R}}(X^\circ)$, $\Delta(X^\circ)$, and $A(X^\circ)$.

Definition 3.1. A function h in $L_{\mathbf{R}}(X^\circ)$ will be called *symmetric* (resp. *antisymmetric*) if $\tau^*(h) = h$ (resp. $\tau^*(h) = -h$).

Let $L_{\mathbf{R}}(X^\circ)^s$ (resp. $L_{\mathbf{R}}(X^\circ)^a$) be the subspace of all symmetric (resp. antisymmetric) elements of $L_{\mathbf{R}}(X^\circ)$. Since $h = (h + \tau^*(h))/2 + (h - \tau^*(h))/2$, $L_{\mathbf{R}}(X^\circ) = L_{\mathbf{R}}(X^\circ)^s \oplus L_{\mathbf{R}}(X^\circ)^a$. Of course, each of the direct summands is invariant, as a set, under τ^*: i.e., each is a submodule over the group algebra $\mathbf{R}(1, \tau^*) \simeq \mathbf{R}(\mathbf{Z}_2)$. $L_{\mathbf{R}}(X^\circ)^s$ and $L_{\mathbf{R}}(Y^\circ)$ are naturally isomorphic, since $Y^\circ = X^\circ/\{1, \tau\}$. We can similarly define $\varDelta(X^\circ)^s$, $\varDelta(X^\circ)^a$; $A(X^\circ)^s$ and $A(X^\circ)^a$. For h in $L_{\mathbf{R}}(X^\circ)$, let $\delta(h) = dh + i^*dh$; thus δ is an $\mathbf{R}$-linear map of $L_{\mathbf{R}}(X^\circ)$ into $\varDelta(X^\circ)$. The kernel of δ is easily seen to be $\mathbf{R}$. For $f = u + iv$, u and v being real-valued, let $\mathrm{Re}\,(f) = u$.

LEMMA 3.2. *The following diagram is commutative:*

$$\begin{array}{ccccc} A(X^\circ) & \xrightarrow{\mathrm{Re}} & L_{\mathbf{R}}(X^\circ) & \xrightarrow{\delta} & \varDelta(X^\circ) \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\tau^*} & & \downarrow{\scriptstyle\sigma} \\ A(X^\circ) & \xrightarrow[\mathrm{Re}]{} & L_{\mathbf{R}}(X^\circ) & \xrightarrow[\delta]{} & \varDelta(X^\circ)\,. \end{array}$$

Thus Re *and* δ *are* $\mathbf{R}(\mathbf{Z}_2)$*-linear; and* $\mathrm{Re}\,(A(X^\circ)^j) \subset L_{\mathbf{R}}(X^\circ)^j$ *and* $\delta(L_{\mathbf{R}}(X^\circ)^j) \subset \varDelta(X^\circ)^j$, *for* $j = s$ *or* a.

Proof. That the first half of the diagram commutes is clear. As for the second half, let h be in $L_{\mathbf{R}}(X^\circ)$ and $\omega = \delta(h)$. If z is a point in X°, then ω is locally of the form dg for some locally defined analytic function g at z, of the form $h + ik$. If $\tau(z) = z'$, $\sigma(g)$ is an analytic function locally defined at z', of the form $\tau^*(h) - i\tau^*(k)$. Now, at z', the real part of $\sigma(\omega) = \sigma(dg) = d(\sigma(g))$ is $d(\tau^*(h))$, which is also the real part at z' of $\delta(\tau^*(h))$. Since this is true for all z in X°, the real parts of $\sigma(\delta(h))$ and $\delta(\tau^*(h))$ agree globally. Since an analytic differential is uniquely determined by its real part, $\sigma(\delta(h)) = \delta(\tau^*(h))$.

Consider now the spaces $L_{\mathbf{R}}(X^\circ)^j/\mathrm{Re}\,A(X^\circ)^j$, for $j = \emptyset$, s and a.

THEOREM 3.3. $L_{\mathbf{R}}(X^\circ)/\mathrm{Re}\,A(X^\circ) \simeq (L_{\mathbf{R}}(X^\circ)^s/\mathrm{Re}\,A(X^\circ)^s) \oplus (L_{\mathbf{R}}(X)^a/\mathrm{Re}\,A(X^\circ)^a)$.

$$\dim_{\mathbf{R}} L_{\mathbf{R}}(X^\circ)^j/\mathrm{Re}\,A(X^\circ)^j = \begin{cases} b, & \text{if } j = \emptyset \\ c - 1, & \text{if } j = s \\ c, & \text{if } j = a\,. \end{cases}$$

Proof. Case $j = \emptyset$ is classical. (cf. Remark 1.2.) In case $j = s$, the question was settled by Alling in [4, (4.7)]. Since $b = 2c - 1$, case $j = a$ follows from the above two.

Let $\Gamma_1, \ldots, \Gamma_c$ be oriented Jordan curves in Y° that form a basis for the first homology group of Y° with integral coefficients such that $\Gamma_1, \ldots, \Gamma_{c-1}$ lie in tubular annular neighbourhoods and Γ_c lies in a Möbius strip neighbourhood. For $1 \leq j < c$, the preimage of Γ_j, under the canonical map p of X° onto $X^\circ/\{1, \tau\} = Y^\circ$, has two components, Γ_j' and Γ_j'', which inherit an orientation from Γ_j. Γ_c has a connected oriented preimage Γ_c' under p. $\{\Gamma_1', \Gamma_1'', \ldots, \Gamma_{c-1}', \Gamma_{c-1}'', \Gamma_c'\}$ then forms a basis for the first homology group $H_1(X^\circ, \mathbf{Z})$, of X° with integral coefficients.

LEMMA 3.4. *Let Γ be an oriented Jordan arc or curve in X°; then so is $\tau(\Gamma)$. If ω belongs to $\Delta(X^\circ)$, then $\int_{\tau(\Gamma)} \omega = \varkappa \circ \int_\Gamma \sigma(\omega)$.*

Proof. This follows by a simple modification of Lemma 3.1. of Alling [4].

Let $\Lambda_1 \equiv (\Gamma_1' - \Gamma_1'')/2, \ldots, \Lambda_{c-1} \equiv (\Gamma_{c-1}' - \Gamma_{c-1}'')/2$, $\Lambda_c \equiv (\Gamma_1' + \Gamma_1'')/2, \ldots$, $\Lambda_{2c-2} \equiv (\Gamma_{c-1}' + \Gamma_{c-1}'')/2$ and $\Lambda_{2c-1} \equiv \Gamma_c'$; and for h in $L_{\mathbf{R}}(X^\circ)$ let

$$\lambda_k(h) = \operatorname{Im} \int_{\Lambda_k} \delta(h), \quad 1 \leq k \leq 2c-1\,.$$

Each λ_k is an $\mathbf{R}$-linear functional on $L_{\mathbf{R}}(X^\circ)$. Since $\delta(h)$ has an exact real part, namely dh, it is only the imaginary periods of $\delta(h)$ that concern us.

LEMMA 3.5. *For h in $L_{\mathbf{R}}(X^\circ)^s$, $\lambda_c(h) = \ldots = \lambda_{2c-1}(h) = 0$. For h in $L_{\mathbf{R}}(X^\circ)^a$, $\lambda_1(h) = \ldots = \lambda_{c-1}(h) = 0$.*

Proof. If h is symmetric (resp. antisymmetric), so is $\delta(h)$, by Lemma 3.2. Hence for h in $L_{\mathbf{R}}(X^\circ)^j$, by Lemma 3.4,

$$\int_{\tau(\Gamma)} \delta(h) = \begin{cases} -\displaystyle\int_\Gamma \delta(h), & \text{if } j = s \\ \displaystyle\int_\Gamma \delta(h), & \text{if } j = a\,. \end{cases}$$

Since each λ_k is trivial on $\operatorname{Re} A(X^\circ)$, each induces an $\mathbf{R}$-linear functional $\tilde{\lambda}_k$ on $L_{\mathbf{R}}(X^\circ)/\operatorname{Re} A(X^\circ)$. Given h in $L_{\mathbf{R}}(X^\circ)$ such that $\lambda_1(h) = \ldots = \lambda_{2c-1}(h) = 0$, we find that $*dh$ is exact and h, in fact, belongs to $\operatorname{Re} A(X^\circ)$; thus we have the following main result of this section.

THEOREM 3.6. *$\{\tilde{\lambda}_1, \ldots, \tilde{\lambda}_b\}$ is a basis of $\operatorname{Hom}_{\mathbf{R}} (L_{\mathbf{R}}(X^\circ)/\operatorname{Re} A(X^\circ), \mathbf{R})$. $L_{\mathbf{R}}(X^\circ)^s/\operatorname{Re} A(X^\circ)^s \simeq \{\tilde{\lambda}_c, \ldots, \tilde{\lambda}_b\}^\perp$ and $L_{\mathbf{R}}(X^\circ)^a/\operatorname{Re} A(X^\circ)^a \simeq \{\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{c-1}\}^\perp$.*

Proof. Since $\tilde{\lambda}_1(\tilde{h}) = \ldots = \tilde{\lambda}_b(\tilde{h}) = 0$ implies $\tilde{h} = 0$, as noted above, $\{\tilde{\lambda}_1, \ldots, \tilde{\lambda}_b\}$ spans. By Theorem 3.3, $\dim_{\mathbf{R}} L_{\mathbf{R}}(X^\circ)/\operatorname{Re} A(X^\circ) = b$; thus it is a basis. Using Lemma 3.5 and Theorem 3.3, the rest follows.

§ 4. Relations between $A^{-1}(X)/\exp A(X)$ and $A^{-1}(Y)/\exp A(Y)$

Since there exists a natural $\mathbf{R}$-isomorphism of $A(Y)$ into $A(X)$, and since under this isomorphism units go to units and exponentials go to exponentials, there exists a homomorphism α of $A^{-1}(Y)/\exp A(Y) \equiv G(Y)$ into $A^{-1}(X)/\exp A(X) \equiv G(X)$. Since -1 belongs to $\exp A(X)$ but not to $\exp A(Y)$, its image in $G(Y)$ is a non-trivial element of the kernel of α. It is well known (cf. Remark 1.2) that $G(X) \simeq \mathbf{Z}^b$, and (Alling [4, 5]) that $G(Y) \simeq \mathbf{Z}_2 \oplus \mathbf{Z}^{c-1}$. In this section we shall analyse the kernel of α, the image of α, and the action of σ on $G(X)$. While doing so we shall find a basis of $G(X)$ that will serve us well in § 5.

Given f in $A^{-1}(X)$, let $dl(f) \equiv df/2\pi f$ on X°; then dl is a homomorphism of $A^{-1}(X)$ into $\Delta(X^\circ)$. It is easy to see that $dl(f)$ is an exact differential if and only if f is an exponential. Thus dl induces a monomorphism of $G(X)$ into $\Delta(X^\circ)/dA(X^\circ)$. For each f in $A^{-1}(X)$, $dl(f)$ has only imaginary periods.

LEMMA 4.1. *Given f in $A^{-1}(X)$, then $dl(\sigma(f)) = \sigma(dl(f))$.*

Proof. If ω in $\Delta(X^\circ)$ has a representation gdh locally, then $\sigma(\omega)$ has a representation $\sigma(g)d\sigma(h)$ locally. Now, $\sigma(dl(f)) = \sigma(df/2\pi f) = d\sigma(f)/2\pi\sigma(f) = dl(\sigma(f))$.

In order to investigate the action of σ on $A^{-1}(X)$, let us proceed, following Wermer [14], as was done in Alling [4, § 5], by imbedding X in a slightly larger non-compact Riemann surface X' in such a way that $\Gamma'_1, \Gamma''_1, \ldots, \Gamma'_{c-1}, \Gamma''_{c-1}, \Gamma'_c$ (see § 3) is a basis of $H_1(X', \mathbf{Z})$. Choose $u_1, \ldots, u_{c-1}$ in $L_{\mathbf{R}}(X')^s$ and $u_c, \ldots, u_b$ in $L_{\mathbf{R}}(X')^a$ such that their images $\{\tilde{u}_1, \ldots, \tilde{u}_b\}$ in $L_{\mathbf{R}}(X')/\mathrm{Re}\, A(X')$ form a basis dual to $\{\tilde{\lambda}_1, \ldots, \tilde{\lambda}_b\}$. (See Theorem 3.6 for details.)

THEOREM 4.2. *There exists a basis $\{Z_1, \ldots, Z_{2c-1}\}$ of $A^{-1}(X)$ modulo* $\exp A(X)$ *such that*

$$\sigma(Z_j) = \begin{cases} Z_j, & 1 \le j \le c-1\,, \\ Z_{j-(c-1)}Z_j^{-1}, & c \le j \le 2c-2\,, \\ -Z_j^{-1}, & j = 2c-1\,. \end{cases}$$

Moreover, each Z_j is analytic across ∂X.

Proof. Let $v_1 \equiv u_1, \ldots, v_{c-1} \equiv u_{c-1}$,

$$v_c \equiv (u_1 - u_c)/2, \ldots, v_{2c-2} \equiv (u_{c-1} - u_{2c-2})/2, \text{ and}$$

$$v_{2c-1} \equiv u_{2c-1}\,.$$

It can be verified that $\{\tilde{v}_1, \ldots, \tilde{v}_{2c-1}\}$ is a basis of $L_{\mathbf{R}}(X')/\mathrm{Re}\, A(X')$ dual to $\{\tilde{\mu}_1, \ldots, \tilde{\mu}_{2c-1}\}$, where, for h in $L_{\mathbf{R}}(X')$,

$$\left.\begin{aligned}\mu_j(h) &\equiv \operatorname{Im}\int_{\Gamma_j'} \delta(h)\,,\\ \mu_{j+(c-1)}(h) &\equiv \operatorname{Im}\int_{-(\Gamma_j'+\Gamma_j'')} \delta(h)\,,\end{aligned}\right\}\quad 1 \leq j \leq c-1, \text{ and}$$

$$\mu_{2c-1}(h) \equiv \operatorname{Im}\int_{\Gamma_c'} \delta(h)\,.$$

Moreover, we have

$$\tau^*(v_j) = \begin{cases} v_j, & 1 \leq j \leq c-1\,,\\ v_{j-(c-1)} - v_j, & c \leq j \leq 2c-2\,,\\ -v_j, & j = 2c-1\,.\end{cases}$$

Let now $\omega_j = \delta(v_j)$ for $1 \leq j \leq 2c-1$. Then, by Lemma 3.2, $\sigma(\omega_j) = \delta(\tau^*(v_j))$, which gives

$$\sigma(\omega_j) = \begin{cases} \omega_j, & 1 \leq j \leq c-1\,,\\ \omega_{j-(c-1)} - \omega_j, & c \leq j \leq 2c-2\,,\\ -\omega_j, & j = 2c-1\,.\end{cases}$$

Let x_0 be a point in Γ_c' and let Γ be a positively oriented Jordan arc in Γ_c' from x_0 to $\tau(x_0)$; thus $\Gamma + \tau(\Gamma) = \Gamma_c'$.

Let $\int_{x_0}^{\tau(x_0)} \omega \equiv \int_\Gamma \omega$ and $\int_{\tau(x_0)}^{x_0} \omega \equiv \int_{\tau(\Gamma)} \omega$, for ω in $\Delta(X')$. For $1 \leq j \leq 2c-1$, and for x in X, let

$$Z_j(x) \equiv \exp\left(2\pi\int_{x_0}^{x} \omega_j - \pi\int_{x_0}^{\tau(x_0)} \omega_j\right).$$

(Cf. Alling [4], § 5.) Since all the real periods of ω_j are zero and since the imaginary periods of it, via $\{\Gamma_1', \Gamma_1'', \ldots, \Gamma_{c-1}', \Gamma_{c-1}'', \Gamma_c'\}$ — a basis for $H_1(X', \mathbf{Z})$ — are all integral, Z_j is a well defined analytic function on X'. Now, for all x in X,

$$\begin{aligned}\sigma(Z_j)(x) &= \varkappa \circ \exp\left(2\pi\int_{x_0}^{\tau(x)} \omega_j - \pi\int_{x_0}^{\tau(x_0)} \omega_j\right)\\ &= \exp\left(2\pi\varkappa \circ \int_{\tau(x_0)}^{\tau(x)} \omega_j + \pi\varkappa \circ \int_{x_0}^{\tau(x_0)} \omega_j\right).\end{aligned}$$

By Lemma 3.4 and the relations between $\sigma(\omega_j)$ and ω_j,

$$\varkappa \circ \int_{\tau(x_0)}^{\tau(x)} \omega_j = \begin{cases} \displaystyle\int_{x_0}^{x} \omega_j\,, \text{ if } 1 \leq j \leq c-1\,, \\ \displaystyle\int_{x_0}^{x} (\omega_{j-(c-1)} - \omega_j)\,, \text{ if } c \leq j \leq 2c-2, \\ \displaystyle -\int_{x_0}^{x} \omega_j, \text{ if } j = 2c-1. \end{cases}$$

Also, for $1 \leq j \leq 2c-2$, $0 = \int_{\Gamma_c'} \omega_j = \int_\Gamma \omega_j + \int_{\tau(\Gamma)} \omega_j = \int_\Gamma \omega_j + \varkappa \circ \int_\Gamma \sigma(\omega_j)$. Hence, for $1 \leq j \leq c-1$, $\operatorname{Re} \int_\Gamma \omega_j = 0$, and $\varkappa \circ \int_{x_0}^{\tau(x_0)} \omega_j = -\int_{x_0}^{\tau(x_0)} \omega_j$; while for $c \leq j \leq 2c-2$, $\varkappa \circ \int_\Gamma \omega_j = = -\int_\Gamma \sigma(\omega_j) = -\int_{x_0}^{\tau(x_0)} \omega_{j-(c-1)} + \int_{x_0}^{\tau(x_0)} \omega_j$. From these calculations it follows that $\sigma(Z_j) = Z_j$, if $1 \leq j \leq c-1$; and $= Z_{j-(c-1)} Z_j^{-1}$, if $c \leq j \leq 2c-2$. Finally, consider the case $j = 2c-1$. Now, $1 = \operatorname{Im} \int_{\Gamma_c'} \omega_{2c-1}$ $= 2 \operatorname{Im} \int_\Gamma \omega_{2c-1}$. Let $\int_{x_0}^{\tau(x_0)} \omega_{2c-1} = s + it$, s and t reals; then $t = 1/2$. Hence, $\sigma(Z_{2c-1}) = \exp(-2\pi \int_{x_0}^{x} \omega_{2c-1} + \pi s - \pi i/2) = \exp(-2\pi \int_{x_0}^{x} \omega_{2c-1} + \pi(s + i/2) - i\pi)$ $= -Z_{2c-1}^{-1}$.

Since each Z_j is analytic on $X' \supset X$, each is analytic on ∂X; thus it only remains to show that $\{Z_1, \ldots, Z_b\}$ is a basis of $A^{-1}(X)$ modulo $\exp A(X)$. Let the following homomorphisms be defined for f in $A^{-1}(X)$: $a_1(f) \equiv \operatorname{Im} \int_{\Gamma_1'} dl(f)$, $\ldots, a_{c-1}(f) \equiv \operatorname{Im} \int_{\Gamma_{c-1}'} dl(f)$, $a_c(f) \equiv \operatorname{Im} \int_{-(\Gamma_1'+\Gamma_1'')} dl(f), \ldots, a_{2c-2}(f) \equiv$ $\equiv \operatorname{Im} \int_{-(\Gamma_{c-1}'+\Gamma_{c-1}'')} dl(f)$, and $a_{2c-1}(f) \equiv \operatorname{Im} \int_{\Gamma_c'} dl(f)$. Note that each a_k is a homomorphism of $A^{-1}(X)$ into $\mathbf{Z}$, having $\exp A(X)$ in its kernel. Let $a(f) \equiv (a_1(f), \ldots, a_b(f))$; then a is a homomorphism of $A^{-1}(X)$ into $\mathbf{Z}^b$, having $\exp A(X)$ as its kernel. Thus a induces a homomorphism $\tilde{a}$ of $A^{-1}(X)/\exp A(X)$ $(= G(X))$ into $\mathbf{Z}^b$. Since $dl\,(Z_j) = \omega_j$, and $\omega_j = \delta(v_j)$, $\{\tilde{v}_1, \ldots, \tilde{v}_b\}$ being a basis dual to the basis $\{\tilde{\mu}_1, \ldots, \tilde{\mu}_b\}$, it follows that $a_k(Z_j) = \delta_{j,k}$, $1 \leq j, k \leq b$. Thus the map $\tilde{a}$ of $G(X)$ into $\mathbf{Z}^b$ is surjective. Since $G(X)$ is known to be a free abelian group of rank b, $\tilde{a}$ is an isomorphism, and the images of $Z_1, \ldots, Z_b$ generate $G(X)$.

Remark 4.3. It should be noticed that the above theorem has a rather easy proof as follows. Let $\{W_1', W_1'', \ldots, W_{c-1}', W_{c-1}'', W_b'\}$ be a basis of $A^{-1}(X)$ modulo $\exp A(X)$ corresponding to the basis $\{\Gamma_1', -\Gamma_1'', \ldots, \Gamma_{c-1}', -\Gamma_{c-1}'', \Gamma_b'\}$ of $H_1(X', \mathbf{Z})$. (Cf., Remark 1.2.) Then we can assume without loss of generality that $W_1'' = \sigma(W_1'), \ldots,$ $W_{c-1}'' = \sigma(W_{c-1}')$. Let now $Z_1 = W_1' \cdot \sigma(W_1'), \ldots,$ $Z_{c-1} = W_{c-1}' \cdot \sigma(W_{c-1}')$, $Z_c = W_1', \ldots, Z_{2c-2} = W_{c-1}'$, and $Z_b' = W_b'$. Clearly $Z_1, \ldots, Z_{c-1}$ are symmetric. Also, $\sigma(Z_j) = \sigma(W_{j-(c-1)}') = Z_{j-(c-1)} Z_j^{-1}$, for

$c \leq j \leq 2c-2$. Finally, since $\{W_1', W_1'', \ldots, W_{c-1}', W_{c-1}'', W_b'\}$ is a basis of $A^{-1}(X)$ modulo $\exp A(X)$, $\sigma(Z_b') = \sigma(W_b') = W_1'^{m_1} W_1''^{n_1} \ldots W_{c-1}'^{m_{c-1}} W_{c-1}''^{n_{c-1}} W_b'^{m_b} \exp g$, where m's and n's are integers, and g belongs to $A(X)$. If we consider the differential $dl\ (\sigma(W_b'))$ and its periods on the above basis of $H_1(X', \mathbf{Z})$, it is apparent that $m_1 = n_1 = \ldots = m_{c-1} = n_{c-1} = 0$, and $m_b = -1$. Thus, $\sigma(Z_b') = Z_b'^{-1} \exp g$. Since $\exp g = Z_b' \sigma(Z_b')$, $\exp g$ is symmetric, and $\sigma(g) = g + 2k\pi i$, where k is an integer. Let $Z_b = Z_b' \exp(-g/2)$, then $\sigma(Z_b) = \pm Z_b^{-1}$. It only needs to consider the winding number of Z_b around Γ_c' to discard the positive sign and obtain $\sigma(Z_b) = -Z_b^{-1}$.

Even though the proof of Theorem 4.2 which we have given earlier is much longer, it is more instructive in the sense that it actually gives the construction of the required basis starting with the symmetric and antisymmetric harmonic functions.

THEOREM 4.4. *Let α be the natural homomorphism of $G(Y)$ into $G(X)$. Then the kernel of α is isomorphic to $\mathbf{Z}_2$ and the image of α is a free abelian group which is a direct summand of $G(X)$ of rank $c-1$. σ induces an action $\tilde{\sigma}$ on $\mathbf{Z}^b$ which takes $(m_1, \ldots, m_{c-1}, m_c, \ldots, m_{2c-2}, m_b)$ to $(m_1, \ldots, m_{c-1}, m_1 - m_c, \ldots, m_{c-1} - m_{2c-2}, -m_b)$.*

Proof. Let $\tilde{f}$ be in the kernel of α; then $f = \sigma(f)$ and $f = \exp g$ for some g in $A(X)$. Hence $\exp g = \exp \sigma(g)$ which implies that $g - \sigma(g) = 2k\pi i$, for some integer k. Thus $f = \exp(g - \sigma(g))/2 \cdot \exp(g + \sigma(g))/2 = \pm \exp(g + \sigma(g))/2$, depending on k being even or odd. Since $(g + \sigma(g))/2$ is symmetric, $\tilde{f} = \tilde{1}$ or $-\tilde{1}$; and $\ker \alpha \simeq \mathbf{Z}_2$. Let now $\{\tilde{Z}_1, \ldots, \tilde{Z}_b\}$ be a basis of $G(X)$, as in Theorem 4.2. Since $Z_1, \ldots, Z_{c-1}$ are symmetric, $\tilde{Z}_1, \ldots, \tilde{Z}_{c-1}$ belong to $\operatorname{im} \alpha$. Now let $\tilde{f}$ be in $\operatorname{im} \alpha$; then $f = \sigma(f)$, and $\tilde{f} = \tilde{Z}_1^{m_1} \cdot \ldots \cdot \tilde{Z}_b^{m_b}$ for some integers $m_1, \ldots, m_b$. But $\tilde{\sigma}(f) = \tilde{Z}_1^{m_1+m_c} \cdot \ldots \cdot \tilde{Z}_{c-1}^{m_{c-1}+m_{2c-2}} \tilde{Z}_c^{-m_c} \cdot \ldots \cdot \tilde{Z}_{2c-2}^{-m_{2c-2}} \cdot \tilde{Z}_b^{-m_b}$. Thus, $m_c = \ldots = m_b = 0$. This shows that $\operatorname{im} \alpha$ is generated by $\tilde{Z}_1, \ldots, \tilde{Z}_{c-1}$. Since we know the action of σ on Z_j, we know the action of $\tilde{\sigma}$ on $\mathbf{Z}^b$, which is written out above.

§ 5. Symmetric inner functions

The algebra $H^\infty(Y^\circ)$ of all bounded analytic functions on Y° is isometrically $\mathbf{R}$-isomorphic to $\{f \text{ in } H^\infty(X^\circ) : \sigma(f) = f\}$, and we will identify it with this algebra.

Definition 5.1. A function f in $H^\infty(dm)$ such that $\sigma(f) = f$ is called a *symmetric generalized inner function* (s.g.i.f.) if $\log|f|$ belongs to $\langle \log |A^{-1}(Y)| \rangle$. If $\{W_1, \ldots, W_{c-1}\}$ is a basis of the free part of $A^{-1}(Y)$ modulo $\exp A(Y)$, a function f in $H^\infty(dm)$ such that $\sigma(f) = f$ is called *a symmetric inner function relative to* $\{W_1, \ldots, W_{c-1}\}$ if $\log|f|$ belongs to $\langle \log|W_1|, \ldots, \log|W_{c-1}| \rangle$.

It should be noticed that if $\{Z_1, \ldots, Z_{c-1}\}$ is any basis of the free part of $A^{-1}(Y)$ modulo $\exp A(Y)$, there exist $Z_c, \ldots, Z_b$ in $A^{-1}(X)$ such that $\{Z_1, \ldots, Z_b\}$ is a basis of $A^{-1}(X)$ modulo $\exp A(X)$ satisfying the conditions of Theorem 4.2. If f is an inner function relative to such a basis, then so is $\sigma(f)$. We choose such a basis and in this section by an inner function we shall mean an inner function relative to this basis.

PROPOSITION 5.2. *Let f be in $H^\infty(dm)$ such that $\sigma(f) = f$. If f is a g.i.f., then, in fact, f is a s.g.i.f.; and if f is an inner function then f is a symmetric inner function.*

Proof. Let $\log|f| = \sum_{j=1}^{b} \alpha_j \log|Z_j|$, where $\alpha_1, \ldots, \alpha_b$ are real numbers; then $\log|\sigma(f)| = \sum_{j=1}^{c-1} (\alpha_j + \alpha_{j+c-1}) \log|Z_j| - \sum_{j=c}^{b} \alpha_j \log|Z_j|$. Since $f = \sigma(f)$, and since α_j is uniquely determined by f, $\alpha_c = \ldots = \alpha_b = 0$; thus f is a symmetric inner function. If f were a g.i.f., it similarly follows that, in fact, f is a s.g.i.f.

We now prove a lemma which would prove to be crucial in the description of the closed ideals of $A(Y)$.

LEMMA 5.3. *If w is an inner function such that $\sigma(w)$ differs from w by a factor of a trivial inner function, then there exists a symmetric inner function u which differs from w by a factor of a trivial inner function.*

Proof. Let $w = cZ_1^{m_1} \cdot \ldots \cdot Z_b^{m_b} \cdot \sigma(w)$; then

$$\sigma(w) = (-1)^{m_b} \cdot \bar{c} Z_1^{m_1+m_c} \cdot \ldots \cdot Z_{c-1}^{m_{c-1}+m_{2c-2}} \cdot Z_c^{-m_c} \cdot \ldots \cdot Z_{2c-2}^{-m_{2c-2}} \cdot Z_b^{-m_b} \cdot w\,.$$

Putting this value in the expression for w, we obtain $m_c = -2m_1, \ldots, m_{2c-2} = -2m_{c-1}$ and $(-1)^{m_b} = 1$, so that $m_b = 2n_b$, for some integer n_b. If we now let $u \equiv dZ_c^{m_1} \cdot \ldots \cdot Z_{2c-2}^{m_{c-1}} \cdot Z_b^{-n_b} \cdot w$, where d is the complex number of absolute value 1 satisfying $d = (-1)^{n_b} \cdot \bar{c} \cdot \bar{d}$; then it is easily seen that $\sigma(u) = u$.

Here we leave the main line of attack on the ideal theory of $A(Y)$ for a while and indulge ourselves in examining some of the striking consequences of the above lemma.

An essentially unique factorization of a function in $H^\infty(X^\circ)$ into a Blaschke product, a singular function and an outer function was given in Limaye [9, § 2]. We now give a factorization of a "function" in $H^\infty(Y^\circ)$. Note first that if f is a Blaschke product, a singular function, or an outer function, then so is $\sigma(f)$, respectively.

THEOREM 5.4. *Let f be a function in $H^\infty(dm)$ such that $\sigma(f) = f$. Then there exists a symmetric inner function u and an outer function g satisfying $\sigma(g) = g$*

such that $f = ug$. *Further, the symmetric inner function* u *factors into a symmetric Blaschke product and a symmetric singular function.*

Proof. Let $f = wF$ where w is inner and F is outer, then $\sigma(f) = \sigma(w)\sigma(F)$. Since $\sigma(f) = f$, $\sigma(w)$ is inner and $\sigma(F)$ is outer, and since the factorization is unique up to a factor of a trivial inner function, w and $\sigma(w)$ differ by such a factor. Lemma 5.3 then gives a symmetric inner function u such that $w = uw'$, where w' is a trivial inner function. It then also follows that $w'F$ is a symmetric outer function. If $u = BS$, where B is a Blaschke product and S is a singular function, then $u = \sigma(u) = \sigma(B)\sigma(S)$. Now, since the zeros of $\hat{B}$ and $\widehat{\sigma(B)}$ on X° are the same, B and $\sigma(B)$ differ by a factor of a trivial inner function (cf. Limaye [8], Proposition 3.1). The required result now follows.

There are two important maps from $H^\infty(X^\circ)$ to $H^\infty(Y^\circ)$, namely the trace map $T(f) \equiv f + \sigma(f)$, and the norm map $N(f) \equiv f \cdot \sigma(f)$. Since, for f in $H^\infty(Y^\circ)$, $T(f/2) = f$, the trace map is surjective. We now prove that the norm map is also surjective, which shows that each f in $H^\infty(Y^\circ)$ has a sort of σ-square root in $H^\infty(X^\circ)$.

THEOREM 5.5. *The norm map* N *from* $H^\infty(X^\circ)$ *to* $H^\infty(Y^\circ)$ *is surjective.*

Proof. Let first f be a trivial inner function in $H^\infty(Y^\circ)$; then by Proposition 5.2, $f = rZ_1^{m_1} \cdot \ldots \cdot Z_{c-1}^{m_{c-1}}$, where $r = \pm 1$, and $m_1, \ldots, m_{c-1}$ are integers. Since $-1 = Z_b\sigma(Z_b)$, and $Z_j = Z_{j+(c-1)} \cdot \sigma(Z_{j+(c-1)})$, for $1 \leq j \leq c-1$, f is clearly in the range of the norm operator. Now, let B be a symmetric Blaschke product; then a point a in X° is a zero of $\hat{B}$ of order m if and only if $\tau(a)$ is a zero of $\hat{B}$ of order m. Thus, the sequence $(a_n)_n$ of the zeros of $\hat{B}$ on X° can be divided as follows: $(a_n)_n = (a_{n,1})_n \cup (a_{n,2})_n$, where $\tau(a_{n,1}) = a_{n,2}$, and $(a_{n,1})_n \cap (a_{n,2})_n = \emptyset$. Let B_1 be a Blaschke product with respect to $(a_{n,1})_n$. Then $\hat{B}$ and $\hat{B}_1 \cdot \sigma(\hat{B}_1)$ are both symmetric and have the same zeros on X°, and hence differ by a factor of a trivial inner function in $H^\infty(Y^\circ)$. Thus, B is in the range of N.

Now, if f is in $H^\infty(Y^\circ)$, $f = B \exp g$, where B is a symmetric Blaschke product and g is an analytic function on X° such that $\sigma(\exp g) = \exp \sigma(g)$, by Theorem 5.4. Since, $g - \sigma(g) = 2k\pi i$, for some integer k,

$$\exp g = (-1)^k \exp (g + \sigma(g))/2 = Z_b^k \exp [(g + \sigma(g))/4] \cdot \sigma(Z_b^k \cdot \exp (g + \sigma(g))/4)\,.$$

This proves the theorem.

§ 6. Closed ideals of $A(Y)$

Consider the $\mathbf{R}$-linear trace map $T(f) = f + \sigma(f)$, for each f in $A(X)$. Since $T(if) = if - i\sigma(f) = i(f - \sigma(f))$, $f = 1/2\, T(f) - i/2\, T(if) \equiv u + iv$, where u and

v are in $A(Y)$. Clearly they are unique. Thus $A(X) = A(Y)[i]$, i being quadratic over $A(Y)$.

Let $\mathcal{I}$ and $\mathcal{J}$ be the sets of all closed ideals of $A(X)$ and $A(Y)$ respectively. There is a map that sends I in $\mathcal{I}$ to $I \cap A(Y) \equiv I^c$ in $\mathcal{J}$ called *contraction*. There is also a natural map that sends J in $\mathcal{J}$ to $J \cdot A(X) \equiv J^e$, which is an ideal in $A(X)$, called *extension*. Clearly $J^e = J[i] = J \oplus iJ$, the sum being direct as a vector space sum over $\mathbf{R}$. Thus, J^e is closed and belongs to $\mathcal{I}$.

An ideal I in $\mathcal{I}$ will be called *symmetric* if $I = \sigma(I)$.

PROPOSITION 6.1. *The set $\mathcal{J}^e$ of extended ideals is exactly the set $\mathcal{I}_\sigma$ of symmetric ideals of $\mathcal{I}$. Given I in $\mathcal{I}_\sigma$, $I^{ce} = I$, and given J in $\mathcal{J}$, $J^{ec} = J$.*

Proof. Let J be in $\mathcal{J}$ and note that $J^e = J \oplus iJ$. Thus $\sigma(J^e) = J - iJ = J + iJ = J^e$; showing that $\mathcal{J}^e \subset \mathcal{I}_\sigma$. Now let I be in $\mathcal{I}_\sigma$, and let $J \equiv I^c$; then $J^e \subset I$. On the other hand, for f in I, $f = 1/2\, T(f) - i/2\, T(if)$, and both $T(f)$ and $T(if)$ are in J, thus $I \subset J^e$. This proves that $\mathcal{J}^e = \mathcal{I}_\sigma$ and that $I = I^{ce}$, for I in $\mathcal{I}_\sigma$. Given J in $\mathcal{J}$, $J^e = J \oplus iJ$, as a real vector space. The set of symmetric elements of J^e is then J, hence $J^{ec} = J$; proving the proposition.

THEOREM 6.2. *Let J be a non-zero closed ideal of $A(Y)$. Then there exists a symmetric inner function u relative to a basis $\{W_1, \ldots, W_{c-1}\}$ of the free part of $A^{-1}(Y)$ modulo* $\exp A(Y)$ *and a unique closed subset F of ∂Y such that $J = uJ(F)$, where $J(F) = \{f \text{ in } A(Y) : f = 0 \text{ on } F\}$. The symmetric inner function u is determined up to a factor of a trivial symmetric inner function.*

Proof. Let $\{W_1, \ldots, W_{c-1}\}$ be extended to a basis $\{W_1, \ldots, W_b\}$ of $A^{-1}(X)$ modulo $\exp A(X)$ such that

$$\sigma(W_j) = \begin{cases} W_j, & 1 \leq j \leq c-1 \\ W_{j-(c-1)} W_j^{-1}, & c \leq j \leq 2c-2 \\ -W_j^{-1}, & j = 2c-1 = b\,. \end{cases}$$

Since J is in $\mathcal{J}$, $J = J^{ec}$, where the ideal $J^e \equiv I$ belongs to $\mathcal{I}_\sigma$, by Proposition 6.1. Theorem 2.2 gives the following decomposition: $I = wI(E)$, where w is an inner function relative to $\{W_1, \ldots, W_b\}$ and E is a closed subset of ∂X of harmonic measure zero. Since $I = \sigma(I) = \sigma(w)I(\tau(E))$, the uniqueness part of Theorem 2.2 shows that $E = \tau(E)$ and $w = cW_1^{m_1} \cdot \ldots \cdot W_b^{m_b} \cdot \sigma(w)$, where c is a complex constant of absolute value 1 and $m_1, \ldots, m_b$ are integers. Hence by Lemma 5.3, there exists a symmetric inner function u relative to $\{W_1, \ldots, W_{c-1}\}$ such that $w = dW_1^{n_1} \cdot \ldots \cdot W_b^{n_b} \cdot u$, where d is a complex constant of absolute value 1 and $n_1, \ldots, n_b$ are integers. Thus $I = J^e = uI(E)$, where $E = \tau(E)$. Since, now, $u = \sigma(u)$ and $J = J^e \cap A(Y)$, we obtain $J = uJ(F)$, where $F =$

$p(E)$ and $J(F) = \{f \text{ in } A(Y) : f = 0 \text{ on } F\}$, p being the map from X to Y. The uniqueness part is similar to that of Theorem 2.2.

Example 6.3. Let Y be a Möbius strip and let its orienting double be the annulus $X = \{z : 1/r \leq |z| \leq r\}$, together with the antianalytic involution $\tau(z) = -1/\bar{z}$. Since in this case $c = 1, A^{-1}(Y) \simeq \mathbf{Z}_2$ and the free part of $A^{-1}(Y)$ modulo $\exp A(Y)$ is trivial; and since $b = 1$, the function $W_1(z) = z$ constitutes a required basis of $A^{-1}(X)$ modulo $\exp A(X)$. Here, a function u in $H^\infty(Y^\circ)$ is a symmetric inner function if and only if $|u| = 1$ a.e. on ∂Y. Thus we see that the non-zero closed ideals of the standard algebra on the Möbius strip Y are of the form $uJ(F)$ where u is in $H^\infty(Y^\circ)$ with $|u| = 1$ on ∂Y and F a closed subset of ∂Y. This gives a precise analog of the classical ideal theory of the standard algebra on the unit disk.

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Norman L. Alling
University of Rochester,
Rochester, N.Y., U.S.A.

Balmohan V. Limaye
Tata Institute of Fundamental Research,
Bombay, India