Interpolation between weighted L^p-spaces

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1. Introduction

One of the main problems in any application of interpolation space theory is the identification of the interpolation space $(\mathcal{X}_0, \mathcal{X}_1)_\Phi$ generated from a pair of Banach spaces $\mathcal{X}_0, \mathcal{X}_1$ by a given interpolation method Φ. The importance of interpolation space theory stems in part from the frequency with which $(\mathcal{X}_0, \mathcal{X}_1)_\Phi$ is identified with classical Banach spaces and the deeper understanding the theory then brings to these spaces. Applications to Lorentz L^{pq}-spaces and to Lipschitz spaces are good examples (cf. [4], for example).

In this paper the interpolation spaces $(L^p_{\omega_0}, L^p_{\omega_1})_{\theta, q}$ generated between weighted L^p-spaces by the (real) J-, K-methods of Peetre will be characterized. Without real loss of generality we shall assume $\omega_0 \equiv 1$. The »diagonal» spaces $(L^p_{\omega_0}, L^p_{\omega_1})_{\theta, p}$, it is known already, coincide with another weighted L^p-space:

$$(L^p_{\omega_0}, L^p_{\omega_1})_{\theta, p} = L^p_\nu, \quad \nu = \omega_0^{1-\theta}\omega_1^\theta, \quad 0 < \theta < 1$$

([7], [15]) and the associated interpolation theorem reduces to the Stein-Weiss extension of the classical M. Riesz theorem to spaces with changes of measure. Peetre began the characterization of the »off-diagonal» cases by identifying $(L^p_{\omega_0}, L^p_{\omega_1})_{\theta, 1}$, $1 < p < \infty$, with one of a family of spaces introduced by Beurling ([3]) in connection with problems of spectral synthesis. Herz later generalized Beurling's definition and considerably clarified Beurling's paper though without systematic recourse to interpolation space theory ([11]). In sections 4 and 5 we complete the characterizations of $(L^p_{\omega_0}, L^p_{\omega_1})_{\theta, q}$, $1 \leq q \leq \infty$, and identifications with the Beurling spaces. With these characterizations as well as the theory of (homogeneous) Besov spaces the various results of Herz can be obtained very easily using simple interpolation space techniques (see [6]).

The Beurling spaces have been considered in many contexts other than spectral synthesis (cf. [9], [10], [12], [13], [18]) with various characterizations being obtained or used. In section 3 we obtain one simple characterization (theorem (3.1)) from which all characterizations follow as special cases. This theorem, in addition, completes other results of Peetre ([17] theorem (2.1)). Since the only property of L^p-spaces needed is the fact that L^p is an L^∞-module via pointwise multiplication, the results in § 3 are presented from the point of view of Banach Function spaces $\mathcal{X}$; however, this section can be read equally well substituting L^p for $\mathcal{X}$.

Some applications of these results to harmonic analysis appear in [1] and [2].

2. Elements of interpolation space theory

Wherever possible the current notation and terminology of abstract interpolation space theory will be used (cf. [4], [7], [14]). Let $\mathcal{X}_0$, $\mathcal{X}_1$ be Banach spaces both continuously embedded in some Hausdorff topological vector space so that then $(\mathcal{X}_0, \mathcal{X}_1)$ forms a *compatible couple* in the sense of [7] chap. 1. Function norms $(K(t, (.))$ and $J(t, (.))$, $0 < t < \infty$, are defined on the sum of $\mathcal{X}_0 + \mathcal{X}_1$ and intersection $\mathcal{X}_0 \cap \mathcal{X}_1$ by

$$K(t,f) = K(t,f; \mathcal{X}_0, \mathcal{X}_1) = \inf_{f=f_0+f_1} (\|f_0\|_{\mathcal{X}_0} + t\|f_1\|_{\mathcal{X}_1}),$$
$$J(t, g) = J(t, g; \mathcal{X}_0, \mathcal{X}_1) = \max (\|g\|_{\mathcal{X}_0}, t\|g\|_{\mathcal{X}_1})$$

where $f \in \mathcal{X}_0 + \mathcal{X}_1$ and $g \in \mathcal{X}_0 \cap \mathcal{X}_1$ respectively. These function norms are both continuous and monotonic increasing in t ([4] p. 167). The interpolation space $(\mathcal{X}_0, \mathcal{X}_1)_{\theta,q;K}$ is the subspace of $\mathcal{X}_0 + \mathcal{X}_1$ of all f for which

$$\|f\|_{\theta,q;K} = \left(\int_0^\infty (t^{-\theta}K(t,f))^q \frac{dt}{t}\right)^{1/q} \quad \begin{matrix} 0 < \theta < 1, & 1 \le q \le \infty, \\ \theta = 0, 1, & q = \infty, \end{matrix} \tag{1}$$

is finite (obvious modifications when $q = \infty$); under the norm $\|(.)\|_{\theta,q;K}$ $(\mathcal{X}_0, \mathcal{X}_1)_{\theta,q;K}$ is a Banach space, non-trivial whenever θ, q are restricted as in (1). The interpolation space $(\mathcal{X}_0, \mathcal{X}_1)_{\theta,q;J}$ consists of all f in $\mathcal{X}_0 + \mathcal{X}_1$ for which there is a strongly measurable function $u: (0, \infty) \to \mathcal{X}_0 \cap \mathcal{X}_1$ satisfying

$$\text{(i)} \quad \int_0^\infty K(1, u(t)) \frac{dt}{t} < \infty, \quad \text{(ii)} \quad f = \int_0^\infty u(t) \frac{dt}{t}, \tag{2}$$

$$\text{(iii)} \quad \left(\int_0^\infty (t^{-\theta}J(t, u(t)))^q \frac{dt}{t}\right)^{1/q} < \infty, \quad \begin{matrix} 0 < \theta < 1, & 1 \le q \le \infty, \\ \theta = 0, 1, & q = 1. \end{matrix} \tag{3}$$

If we set

$$\|f\|_{\theta,q;J} = \inf \left(\int_0^\infty (t^{-\theta} J(t, u(t)))^q \frac{dt}{t} \right)^{1/q},$$

the infimum being taken over all u satisfying (2) and (3), then $(\mathcal{X}_0, \mathcal{X}_1)_{\theta,q;J}$ becomes a non-trivial Banach space. It is known that

$$(\mathcal{X}_0, \mathcal{X}_1)_{\theta,q;J} = (\mathcal{X}_0, \mathcal{X}_1)_{\theta,q;K}, \quad 0 < \theta < 1, \quad 1 \le q \le \infty,$$

at least up to equivalence of norms.

These spaces $(\mathcal{X}_0, \mathcal{X}_1)_{\theta,q;K}$, $(\mathcal{X}_0, \mathcal{X}_1)_{\theta,q;J}$ can be constructed in exactly the same way to yield the same spaces (with equivalent norms) by replacing the multiplicative group $R_* = (0, \infty)$ with the multiplicative discrete group $\{r^n : n \in Z\}$, r being fixed, $r > 1$, the Haar measure dt/t on $(0, \infty)$ being replaced by Haar measure on $\{r^n : n \in Z\}$. We shall refer to this method as the *discrete* method of construction (see [6] or [14] for full details). The L^p-spaces on R_* defined with respect to Haar measure will be denoted by L^p_*.

The previous methods of construction are all »real» methods in contrast to the »complex» method introduced by Calderón *et al.* (cf., for instance, [5]). In this method, to each compatible couple $(\mathcal{X}_0, \mathcal{X}_1)$ is associated a Banach space $[\mathcal{X}_0, \mathcal{X}_1]_\theta$, $0 \le \theta \le 1$. Since we are concerned exclusively with characterizations of interpolation spaces obtained by the J-, K-methods further details are omitted. Let us recall only that if (X, μ) is a totally σ-finite measure space and L^p $(= L^p(X, \mu))$ the usual Lebesgue spaces, then

$$(L^{p_0}, L^{p_1})_{\theta,q;K} = L^{pq}, \quad [L^{p_0}, L^{p_1}]_\theta = L^p$$

where $1/p = (1 - \theta)/p_0 + \theta/p_1$ and $L^{pq} (= L^{pq}(X, \mu))$ is the usual Lorentz space associated with (X, μ) (see [4] § 3.3.1, [5]).

3. Interpolation between weighted function spaces

For a totally σ-finite measure space (X, μ), $\mathcal{M}(X, \mu)$ denotes the space of μ-measurable complex-valued functions on X. A Banach space $\mathcal{X}$ of (equivalence classes of) functions in $\mathcal{M}$ is called a *Banach function space* provided

(a) if $g \in \mathcal{M}(X, \mu)$ and $|g| \le |f|$ μ-a.e. for some $f \in \mathcal{X}$ then g belongs to $\mathcal{X}$ and $\|g\|_{\mathcal{X}} \le \|f\|_{\mathcal{X}}$,

(b) if $\{f_n\}$ is a sequence of non-negative functions in $\mathcal{X}$ and $f_n \uparrow f$ μ-a.e. then $\sup_n \|f_n\|_{\mathcal{X}} = \|f\|_{\mathcal{X}}$.

Obviously the Lebesgue L^p-spaces:

$$\|f\|_p = \left(\int_X |f(x)|^p d\mu \right)^{1/p}, \quad 1 \le p \le \infty, \quad \|f\|_\infty = \text{ess. sup. } |f(x)| \tag{4}$$

are Banach Function spaces as are the Lorentz $L^{pq}(X, \mu)$ spaces, condition (b) being the classical Fatou property. Notice that (a) ensures $\mathcal{X}$ is an $L^\infty(X, \mu)$-module via pointwise multiplication and

$$\|fg\|_{\mathcal{X}} \le \|f\|_{\mathcal{X}} \|g\|_\infty, \quad f \in \mathcal{X}, \quad g \in L^\infty .$$

When ω is a function in $\mathcal{M}(X, \mu)$ with $\omega > 0$ μ-a.e., $\mathcal{X}_\omega$ denotes the *weighted* function space of all (equivalence classes of) functions f in $\mathcal{M}(X, \mu)$ for which $f\omega \in \mathcal{X}$; we set

$$\|f\|_{\mathcal{X}_\omega} = \|f\omega\|_{\mathcal{X}}, \quad f \in \mathcal{X}_\omega .$$

For weighted $L^p(X, \omega)$ spaces we shall write

$$\|f\|_{p,\omega} = \|f\omega\|_p, \quad 1 \le p \le \infty ,$$

with $\|(.)\|_p$ defined by (4). The pair $(\mathcal{X}, \mathcal{X}_\omega)$ forms a compatible couple: indeed, up to equivalence of norms,

$$\mathcal{X} + \mathcal{X}_\omega = \mathcal{X}_m , \quad \mathcal{X} \cap \mathcal{X}_\omega = \mathcal{X}_M$$

where

$$m(x) = \min(1, \omega(x)), \quad M(x) = \max(1, \omega(x))$$

(cf. [15]). In this and the succeeding section we shall characterize the interpolation spaces

$$(\mathcal{X}, \mathcal{X}_\omega)_{\theta,q;K}, \quad (L^p, L^p_\omega)_{\theta,q;K}, \qquad \begin{array}{l} 0 < \theta < 1, \\ 1 \le q \le \infty . \end{array}$$

For the complex method the interpolation space $[L^p, L^p_\omega]_\theta$ is known:

$$[L^p, L^p_\omega]_\theta = L^p_\nu, \quad \nu(x) = \omega(x)^\theta ,$$

(see [5] pp. 123—4 for even more general results). As frequently happens, however, the real methods yield directly a richer family of spaces.

Throughout, σ will denote a non-negative, piecewise continuous function in $L^\infty(0, \infty)$ such that $t\sigma(t)$ belongs to $L^\infty(0, \infty)$; thus, with σ_t defined for each fixed t by

$$\sigma_t(\lambda) = t\lambda\sigma(t\lambda), \quad t, \lambda \in (0, \infty) ,$$

our assumptions on σ ensure that

$$\max(\|\sigma\|_\infty, \sup_t \|\sigma_t\|_\infty) < \infty . \tag{5}$$

Furthermore, the composition $\sigma_t \circ \omega$ belongs to $L^\infty(X, \mu)$ and $f \cdot (\sigma_t \circ \omega)$ to $\mathcal{X}$ whenever $f \in \mathcal{X}$. All of our characterizations follow (more or less easily) from the following result.

THEOREM (3.1). *There exist constants depending only on* σ, θ *and* q *such that*

$$\text{const.}\, \|f\|_{\theta,q;J} \leq \left(\int_0^\infty (t^{-\theta}\|f \cdot (\sigma_t \circ \omega)\|_{\mathcal{X}})^q \frac{dt}{t} \right)^{1/q} \leq \text{const.}\, \|f\|_{\theta,q;K}$$

for all f *in* $(\mathcal{X}, \mathcal{X}_\omega)_{\theta,q;K}$, $0 < \theta < 1$, $1 \leq q \leq \infty$.

Proof. When $f = f_0 + f_1$ is a decomposition of f in $\mathcal{X} + \mathcal{X}_\omega$,

$$\|f \cdot (\sigma_t \circ \omega)\|_{\mathcal{X}} \leq \|f_0 \cdot (\sigma_t \circ \omega)\|_{\mathcal{X}} + \|f_1 \cdot (\sigma_t \circ \omega)\|_{\mathcal{X}} \leq (\sup_t \|\sigma_t\|_\infty)\|f_0\|_{\mathcal{X}} + t(\|\sigma\|_\infty)\|f_1\omega\|_{\mathcal{X}},$$

and so, in view of (5),

$$\|f \cdot (\sigma_t \circ \omega)\|_{\mathcal{X}} \leq \max (\sup_t \|\sigma_t\|_\infty, \|\sigma\|_\infty) K(t, f) \tag{6}$$

for all $t \in (0, \infty)$. The right hand inequality follows.

Now let ψ be a non-negative continuous function with compact support in $(0, \infty)$. Then the function $u = u(t)$:

$$u(t) = f(x) \cdot \sigma_t(\omega(x)) \cdot \psi(t\omega(x)), \quad t \in (0, \infty), \quad x \in X$$

is a measurable $\mathcal{X} \cap \mathcal{X}_\omega$-valued function such that

$$\|u(t)\|_{\mathcal{X}} \leq \|\psi\|_\infty \|f \cdot (\sigma_t \circ \omega)\|_{\mathcal{X}}$$

while

$$t\|u(t)\omega\|_{\mathcal{X}} = \|f \cdot (\sigma_t \circ \omega) \,.\, (\psi_t \circ \omega)\|_{\mathcal{X}} \leq (\sup_t \|\psi_t\|_\infty) \|f \cdot (\sigma_t \circ \omega)\|_{\mathcal{X}}\,.$$

Thus u: $(0, \infty) \to \mathcal{X} \cap \mathcal{X}_\omega$ and

$$J(t, u(t)) \leq \max (\|\psi\|_\infty, \sup_t \|\psi_t\|_\infty) \|f \cdot (\sigma_t \circ \omega)\|_{\mathcal{X}}\,.$$

But with a suitable normalization,

$$f = \int_0^\infty u(t) \frac{dt}{t}\,.$$

The left hand inequality of the theorem now follows.

If the discrete construction is used in the proof of theorem (3.1) the following »discrete» characterization of $(\mathcal{X}, \mathcal{X}_\omega)_{\theta,q;K}$ is obtained.

THEOREM (3.2). *For each fixed* $r > 1$ *there exist constants depending only on* θ, q *and* σ *such that*

$$\text{const.}\ \|f\|_{\theta,q;J} \leq \Big(\sum_{-\infty}^{\infty} (r^{-k\theta}\|f\cdot(\sigma_{r^k}\circ\omega)\|_{\mathcal{X}})^q\Big)^{1/q} \leq \text{const.}\ \|f\|_{\theta,q;K}$$

for all f *in* $(\mathcal{X}, \mathcal{X}_\omega)_{\theta,q;K}$, *where* $0<\theta<1$ *and* $1\leq q\leq\infty$.

The first application of theorems (3.1) and (3.2) gives what in practice is the most useful characterization of $(\mathcal{X}, \mathcal{X}_\omega)_{\theta,q;K}$. For fixed $r>1$ denote by τ the function

$$\tau(t) = \begin{cases} \dfrac{1}{t}, & 1<t\leq r \\ 0, & \text{elsewhere}. \end{cases} \tag{7}$$

When $X_t = \{x\in X\colon \omega(x)\leq 1/t\}$ and χ_t is the characteristic function of X_t, then $\tau_t\circ\omega = \chi_{t/r} - \chi_t$ and

$$\Delta(\chi_{r^k}) = \tau_{r^k}\circ\omega = \chi_{r^{k-1}} - \chi_{r^k}.$$

THEOREM (3.3). *For each* $r>1$ *there exist constants depending only on* θ *and* q, $0<\theta<1$, $1\leq q\leq\infty$, *such that*

$$\text{const.}\ \|f\|_{\theta,q;J} \leq \Big(\sum_{-\infty}^{\infty} (r^{-k\theta}\|f\Delta(\chi_{r^k})\|_{\mathcal{X}})^q\Big)^{1/q} \leq \text{const.}\ \|f\|_{\theta,q;K}$$

for all f *in* $(\mathcal{X}, \mathcal{X}_\omega)_{\theta,q;K}$.

Proof. The particular choice (7) of τ satisfies (5) and theorem (3.2) then applies.

Now define τ, τ' by

$$\tau(t) = \begin{cases} 1, & 0<t<1, \\ 0, & \text{elsewhere}, \end{cases} \qquad \tau'(t) = \begin{cases} \dfrac{1}{t}, & t\geq 1, \\ 0, & \text{elsewhere}. \end{cases}$$

Obviously τ and τ' satisfy (5) while

$$\tau_t\circ\omega = t\omega\chi_t, \quad \tau'_t\circ\omega = 1-\chi_t,$$

and

$$t\|f\omega\chi_t\|_{\mathcal{X}} \leq K(t,f), \quad \|f\cdot(1-\chi_t)\|_{\mathcal{X}} \leq K(t,f) \tag{8}$$

(cf. (6)). Theorem (3.1) thus gives:

THEOREM (3.4). *When* χ_t *is the characteristic function of the set* $X_t = \{x\in X\colon \omega(x)\leq 1/t\}$ *then*

$$\left(\int_0^\infty (t^{1-\theta}\|f\omega\chi_t\|_{\mathcal{X}})^q \frac{dt}{t}\right)^{1/q}, \quad \left(\int_0^\infty (t^{-\theta}\|f\cdot(1-\chi_t)\|_{\mathcal{X}})^q \frac{dt}{t}\right)^{1/q}$$

define equivalent norms on $(\mathcal{X}, \mathcal{X}_\omega)_{\theta,q;K}$, $0 < \theta < 1$, $1 \leq q \leq \infty$.

The Lipschitz character of $(\mathcal{X}, \mathcal{X}_\omega)_{\theta,q;K}$ can be described by setting

$$\tau(t) = e^{-t}, \quad \tau'(t) = \frac{1}{t}(e^{-t} - 1), \tag{9}$$

so that τ, τ' satisfy (8) and

$$\tau_t \circ \omega = t\omega e^{-t\omega}, \quad \tau_t' \circ \omega = e^{-t\omega} - 1 .$$

THEOREM (3.5). *For* $0 < \theta < 1$ *and* $1 \leq q \leq \infty$,

$$\left(\int_0^\infty (t^{1-\theta}\|f\omega e^{-t\omega}\|_{\mathcal{X}})^q \frac{dt}{t}\right)^{1/q}, \quad \left(\int_0^\infty (t^{-\theta}\|f\cdot(e^{-t\omega} - 1)\|_{\mathcal{X}})^q \frac{dt}{t}\right)^{1/q}$$

define equivalent norms on $(\mathcal{X}, \mathcal{X}_\omega)_{\theta,q;K}$.

Proof. Apply theorem (3.1) with τ, τ' defined by (9).

It is of interest to formulate the characterizations contained in theorems (3.4) and (3.5) for the special case $\mathcal{X} = L^p(X)$, $1 \leq p \leq \infty$. When $\sigma = (1/t)\min(1, t)$ in theorem (3.1) and $1 \leq p < \infty$,

$$\int_0^\infty (t^{-\theta}\|f\cdot(\sigma_t\circ\omega)\|_p)^p \frac{dt}{t} = \int_0^\infty \int_X t^{-p\theta}|f(x)|^p \min(1, t\omega(x))^p d\mu \frac{dt}{t} =$$

$$\int_X |f(x)|^p \omega(x)^{\theta p}\left(\int_0^\infty t^{-p\theta}\min(1, t)^p \frac{dt}{t}\right) d\mu .$$

In case $p = \infty$ the analogous result holds:

$$\sup_t t^{-\theta}\|f\cdot(\sigma_t\circ\omega)\|_\infty = \|f\omega^\theta\|_\infty \cdot \sup_t (t^{-\theta}\min(1, t)) .$$

Consequently, for all p, $1 \leq p \leq \infty$,

$$\left(\int_0^\infty (t^{-\theta}\|f\cdot(\sigma_t\circ\omega)\|_p)^p \frac{dt}{t}\right)^{1/p} = \|f\omega^\theta\|_p \cdot \|t^{-\theta}\min(1, t)\|_{L^p_*}$$

which establishes (cf. [7] p. 25, 31; [15]):

THEOREM (3.6). *For* $0 < \theta < 1$ *and* $1 \leq p \leq \infty$

$$(L^p(X), L^p_\omega(X))_{\theta,q;K} = \left\{ f: \left(\int_X |f(x)|^p \omega(x)^{\theta p} d\mu \right)^{1/p} < \infty \right\},$$

and

$$\text{const.}\ \|f\|_{\theta,p;J} \leq \|f\omega^\theta\|_p \leq \text{const.}\ \|f\|_{\theta,q;K}\,.$$

For non-diagonal values of q, i.e., $q \neq p$, theorems (3.3) and (3.4) yield:

THEOREM (3.7). *The expressions*

(i) $$\left(\sum_{-\infty}^{\infty} \left(r^{-k\theta} \left[\int_{r^{k-1} < \omega(x) \leq r^k} |f(x)|^p d\mu \right]^{1/p} \right)^q \right)^{1/q}, \quad r > 1,$$

(ii) $$\left(\int_0^\infty \left(t^{1-\theta} \left[\int_{t\omega(x) \leq 1} |f(x)^p \omega(x)|^p d\mu \right]^{1/p} \right)^q \frac{dt}{t} \right)^{1/q},$$

(iii) $$\left(\int_0^\infty \left(t^{-\theta} \left[\int_{t\omega(x) > 1} |f(x)|^p d\mu \right]^{1/p} \right)^q \frac{dt}{t} \right)^{1/q}$$

define equivalent norms on $(L^p(X), L^p_\omega(X))_{\theta,q;K}$ *for all* $0 < \theta < 1$, $1 \leq q \leq \infty$, *and* $1 \leq p \leq \infty$.

Remark. Partial results along the lines of theorem (3.7) were obtained by Peetre in [17] (cf. p. 63).

4. Beurling spaces

We come now to our main characterization of the interpolation spaces $(L^p(X), L^p_\omega(X))_{\theta,q;K}$, identifying these spaces with important Banach spaces introduced by Beurling ([3]).

With $\gamma = (1/p - 1/q)$ and $0 < \theta < 1$ denote by $\Omega(\theta, \gamma)$ the class of functions φ in L^1_*, $\varphi \geq 0$, such that

(a) $\|\varphi\|_{L^1_*} = 1$, (b) $t^\theta \varphi(t)^\gamma$ is monotonic increasing.

Such functions φ certainly exist; for instance, suitable normalizations of

$$\varphi(t) = t^{-\theta/\gamma} \min(1, t^{1/\gamma}), \quad \gamma \neq 0, \quad \varphi(t) = t^{-\theta} \min(1, t), \quad \gamma = 0$$

satisfy (a), (b). For fixed ω we then define $B^p_{\theta,q}(X, \omega)$ by

$$B^p_{\theta,q}(X, \omega) = \bigcup_{\varphi \in \Omega(\theta,\gamma)} \{L^p_w(X) \colon w = \omega^\theta(\varphi \circ \omega)^\gamma\}, \quad \gamma \leq 0 ,$$

$$B^p_{\theta,q}(X, \omega) = \bigcap_{\varphi \in \Omega(\theta,\gamma)} \{L^p_w(X) \colon w = \omega^\theta(\varphi \circ \omega)^\gamma\}, \quad \gamma \geq 0 .$$

In [3] additional convolution properties are imposed on φ to ensure that certain of the $B^p_{\theta,q}(X, \omega)$ spaces are Banach algebras. Discussion of this particular facet will be omitted from this paper.

THEOREM (4.1). *Let* $\gamma = 1/p - 1/q$ *where* $1 \leq p < \infty$ *and* $1 \leq q \leq \infty$. *Then for each* θ, $0 < \theta < 1$, *there exist constants such that for all* f *in* $(L^p(X), L^p_\omega(X))_{\theta,q;K}$

$$\text{const.}\ \|f\|_{\theta,q;J} \leq \inf_{\varphi \in \Omega(\theta,\gamma)} \|f\omega^\theta(\varphi \circ \omega)^\gamma\|_p \leq \text{const.}\ \|f\|_{\theta;q;K} \tag{10}$$

when $\gamma \leq 0$, *and*

$$\text{const.}\ \|f\|_{\theta,q;J} \leq \sup_{\varphi \in \Omega(\theta,\gamma)} \|f\omega^\theta(\varphi \circ \omega)^\gamma\|_p \leq \text{const.}\ \|f\|_{\theta,q;K} \tag{11}$$

when $\gamma \geq 0$. *In particular,*

$$(L^p(X), L^p_\omega(X))_{\theta,q;K} = B^p_{\theta,q}(X, \omega) .$$

Proof. (i) We establish first the left hand inequality in (10). Since $\gamma \leq 0$, automatically $q \leq p < \infty$.

Choose any non-negative continuous function ψ with compact support in $[1, \infty)$, $\psi \not\equiv 0$, and define σ on $(0, \infty)$ by $\sigma(t) = (1/t)\psi(t)$. In view of theorem (3.1) the left hand inequality in (10) follows provided

$$\left(\int_0^\infty (t^{-\theta}\|f \cdot (\sigma_t \circ \omega)\|_p)^q \frac{dt}{t}\right)^{1/q} \leq \text{const.} \inf_\varphi \|f \cdot \omega^\theta(\varphi \circ \omega)^\gamma\|_p . \tag{12}$$

But, since $t^\theta\varphi^\gamma$ is monotonic increasing,

$$t^{-\theta}(\sigma_t \circ \omega)(x) = t^{-\theta}\psi(t\omega(x)) \leq \omega(x)^\theta\varphi(1/t)^{-\gamma}\varphi(\omega(x))^\gamma\psi(t\omega(x))$$

because $\psi(t\omega(x))$ is non-zero only when $t\omega(x) \geq 1$ and then $\psi(t\omega(x)) \geq 0$. Thus

$$t^{-\theta}\|f \cdot (\sigma_t \circ \omega)\|_p \leq \varphi(1/t)^{-\gamma}\|f\omega^\theta(\varphi \circ \omega)^\gamma\psi(t\omega)\|_p$$

and so

$$\int_0^\infty (t^{-\theta}\|f \cdot (\sigma_t \circ \omega)\|_p)^q \frac{dt}{t} \leq \int_0^\infty \left(\int_X |f|^p\omega^{\theta p}(\varphi \circ \omega)^{\gamma p}\psi(t\omega)^p d\mu\right)^{q/p} \varphi\left(\frac{1}{t}\right)^{-\gamma q} \frac{dt}{t} . \tag{13}$$

Using crucially the fact that $q \leq p$, we deduce from Hölder's inequality applied to (13) that the left hand side of (12) is dominated by

$$\left(\int_0^\infty\left(\int_X |f|^p\omega^{\theta p}(\varphi\circ\omega)^{\gamma p}\psi(t\omega)^p d\mu\right)\frac{dt}{t}\right)^{1/p}(\|\varphi\|_{L^1_*})^{-\gamma} = \|f\omega^\theta(\varphi\circ\omega)^\gamma\|_p\|\psi\|_{L^p_*} \tag{14}$$

inverting the order of integration in (14). This establishes (12).

(ii) In proving the right hand inequality in (10) we may assume $q < p(< \infty)$, i.e., $\gamma < 0$, since the case $q = p$, i.e., $\gamma = 0$, has been taken care of already in theorem (3.6). For fixed f in $(L^p, L^p_\omega)_{\theta,q;K}$ define ψ on $(0, \infty)$ by

$$\psi(t)^\gamma = t^{-\theta}\left(\int_{1/t}^\infty \lambda^{-\theta q}K(\lambda, f)^{q-p}\frac{d\lambda}{\lambda}\right)^{1/p}.$$

Since $K(\lambda, f) \le \max(1, \lambda t)K(1/t, f)$,

$$\psi(t)^\gamma \ge t^{-\theta}K\left(\frac{1}{t}, f\right)^{\gamma q}\left(\int_{1/t}^\infty \lambda^{-\theta q}(\lambda t)^{q-p}\frac{d\lambda}{\lambda}\right)^{1/p} = \left[t^\theta K\left(\frac{1}{t}, f\right)\right]^{\gamma q}(p + \theta q - q)^{-1/p}.$$

Hence,

$$\psi(t) \le \text{const.}\left[t^\theta K\left(\frac{1}{t}, f\right)\right]^q.$$

For this function ψ,

$$(\|f\omega^\theta(\psi\circ\omega)^\gamma\|_p)^p = \int_X |f|^p\left(\int_{1/\omega(x)} \lambda^{-\theta q}K(\lambda, f)^{q-p}\frac{d\lambda}{\lambda}\right)d\mu =$$

$$= \int^\infty \lambda^{-\theta q}K(\lambda, f)^{q-p}\left[\int_X |f|^p(1-\chi_\lambda)d\mu\right]\frac{d\lambda}{\lambda} \le \int_0^\infty \lambda^{-\theta q}K(\lambda, f)^q\frac{d\lambda}{\lambda} = (\|f\|_{\theta,q;K})^q$$

(cf. (8)). Now set $\varphi = \psi/\|\psi\|_{L^1_*}$. Then φ belongs to $\Omega(\theta, \gamma)$ and

$$\|f\omega^\theta(\varphi\circ\omega)^\gamma\|_p \le (\|f\|_{\theta,q;K})^{q/p}(\|\psi\|_{L^1_*})^{-\gamma} \le \text{const.}\ \|f\|_{\theta,q;K}$$

completing the proof of (10).

(iii) We turn next to the left hand inequality in (11). First we assume $p \le q < \infty$. It is then enough to show that

$$\left(\int_0^\infty (t^{-\theta}\|f\cdot(1-\chi_t)\|_p)^q\frac{dt}{t}\right)^{p/q} \le \text{const.}\sup_{\varphi\in\Omega(\theta,\gamma)}(\|f\omega^\theta(\varphi\circ\omega)^\gamma\|_p)^p. \tag{15}$$

Now the left hand side of (15) is

$$\sup_g\left(\int_0^\infty g(t)t^{-\theta p}\left(\int_X |f|^p(1-\chi_t)d\mu\right)\frac{dt}{t}\right)$$

the supremum being taken over all $g \geq 0$ such that

$$\int_0^\infty g(t)^{1/p\gamma} \frac{dt}{t} = 1 \tag{16}$$

(remember $p\gamma + 1/(q/p) = 1$. But, with a change in the order of integration,

$$\int_0^\infty g(t)t^{-\theta p} \left(\int_X |f|^p (1 - \chi_t) d\mu \right) \frac{dt}{t} = \int_X |f|^p \omega^{\theta p} (\psi \circ \omega)^{\gamma p} d\mu$$

where

$$\psi(t)^\gamma = t^{-\theta} \left(\int_{1/t}^\infty g(\lambda) \lambda^{-\theta p} \frac{d\lambda}{\lambda} \right)^{1/p} = \left(\int_1^\infty g\left(\frac{\lambda}{t}\right) \lambda^{-\theta p} \frac{d\lambda}{\lambda} \right)^{1/p}.$$

Clearly $t^\theta \psi^\gamma$ is monotonic increasing. Also, by (16),

$$(\|\psi\|_{L^1_*})^{p\gamma} \leq \left(\int_0^\infty \left(\int_1^\infty g\left(\frac{\lambda}{t}\right) \lambda^{-\theta p} \frac{d\lambda}{\lambda} \right)^{1/p\gamma} \frac{dt}{t} \right)^{p\gamma} \leq \left(\int_0^\infty g(t)^{1/p\gamma} \frac{dt}{t} \right)^{p\gamma} \left(\int_1^\infty \lambda^{-\theta p} \frac{d\lambda}{\lambda} \right) = \frac{1}{\theta p}$$

regarding $\int_1^\infty g(\lambda/t)\lambda^{-\theta p} d\lambda/\lambda$ as a convolution on R_*. Thus with $\varphi = \psi/\|\psi\|_{L^1_*}$,

$$\int_0^\infty g(t)t^{-\theta p} \left(\int_X |f|^p (1 - \chi_t) d\mu \right) \frac{dt}{t} \leq (\|\psi\|_{L^1_*})^{\gamma p} \left(\int_X |f|^p \omega^{\theta p} (\varphi \circ \omega)^{\gamma p} d\mu \right)$$

$$\leq \text{const.} \left[\sup_{\varphi \in \Omega(\theta, \gamma)} (\|f\omega^\theta (\varphi \circ \omega)^\gamma\|_p)^p \right]$$

for all g satisfying (16). This establishes (15).

In case $q = \infty$ we must replace (15) by

$$\sup_t (t^{-\theta} \|f \cdot (1 - \chi_t)\|_p) \leq \text{const.} \sup_{\varphi \in \Omega(\theta, \gamma)} \|f\omega^\theta (\varphi \circ \omega)^\gamma\|_p. \tag{17}$$

The proof of (17) is entirely analogous to that of (8) if we remember that $p\gamma = 1$ when $q = \infty$.

(iv) Finally we establish the right hand side of (11). Choose any non-negative continuous function σ, $\sigma \not\equiv 0$, with support in $(0, 1]$. Then, because $t^\theta \varphi^\gamma$ is monotonic increasing,

$$\omega^{p\theta} (\varphi \circ \omega)^{p\gamma} \leq \text{const.} \int_0^\infty (\sigma_t \circ \omega)^p t^{-p\theta} \varphi \left(\frac{1}{t}\right)^{p\gamma} \frac{dt}{t},$$

the constant being independent of φ. In this case, with a change in the order of integration,

$$\int_X |f|^p \omega^{p\theta}(\varphi \circ \omega)^{p\gamma} d\mu \leq \text{const.} \int_0^\infty t^{-p\theta}\left(\int_X |f|^p(\sigma_t \circ \omega)^p d\mu\right)\varphi\left(\frac{1}{t}\right)^{p\gamma}\frac{dt}{t}$$

$$\leq \text{const.}\ (\|\varphi\|_{L^1_*})^{p\gamma}\left(\int_0^\infty (t^{-\theta}\|f\cdot(\sigma_t \circ \omega)\|_p)^q \frac{dt}{t}\right)^{p/q}$$

by Hölder's inequality using crucially the fact that $q \geq p$. This establishes the right hand side of (11) completing the proof of theorem (4.1).

On $L^p(X)$, $1 \leq p < \infty$, the semi-group $\{M(t): 0 < t < \infty\}$ of multiplication operators:

$$M(t): f(x) \to e^{-t\omega(x)}f(x), \quad f \in L^p(X)\,,$$

is a semi-group of contraction operators of class $(\mathscr{C}_0)$ whose infinitesimal generator A_ω is the multiplication operator $A_\omega: f \to -\omega f$ (cf. [4] p. 3 for definitions). The domain of definition $D(A_\omega)$ of A_ω becomes a Banach space under the graph norm

$$\|f\|_{D(A)_\omega} = \|f\|_p + \|\omega f\|_p, \quad f \in L^p(X)\,.$$

Thus, if $\omega(x) \geq \delta > 0$ on X as frequently is the case in applications, $L^p_\omega(X)$ is norm equivalent to $D(A_\omega)$. The general theory of the commutativity of interpolation functors as developed by Grisvard ([8] pp. 169, 171) enables us to identify the respective complex and real interpolation spaces

$$[B^{p_0}_{\theta_0,q_0}(X,\omega), B^{p_1}_{\theta_1,q_1}(X,\omega)]_\varphi, \quad (B^{p_0}_{\varphi,q}(X,\omega), B^{p_1}_{\varphi,q}(X,\omega))_{\varphi,q;K}$$

with

$$1 \leq p_0, p_1 < \infty, \quad 0 < \theta_0, \theta_1, \theta, \varphi < 1, \quad 1 \leq q_0, q_1, q \leq \infty\,. \tag{18}$$

THEOREM (4.2). *Under the restrictions* (18),

$$[B^{p_0}_{\theta_0,q_0}(X,\omega), B^{p_1}_{\theta_1,q_1}(X,\omega)]_\varphi = B^p_{\theta,q}(X,\omega)$$

and

$$(B^{p_0}_{\varphi,q}(X,\omega), B^{p_1}_{\varphi,q}(X,\omega))_{\varphi,q;K} = B^p_{\varphi,q}(X,\omega)$$

where in both cases

$$\frac{1}{p} = \frac{1-\varphi}{p_0} + \frac{\varphi}{p_1}, \quad \frac{1}{q} = \frac{1-\varphi}{q_0} + \frac{\varphi}{q_1}, \quad \theta = (1-\varphi)\theta_0 + \varphi\theta_1\,.$$

Remark (4.3). If ω is not bounded away from 0 on X the (non-homogeneous) Lipschitz spaces used by Grisvard have to be replaced with Lipschitz spaces of homogeneous character (cf. [16] p. 286). Though important in applications we shall omit any such considerations here; the theory of such spaces is developed systematically in [6], for instance, to which the reader is referred.

5. Examples

We shall consider briefly the case when $X = R^n$, μ is Lebesgue measure on R^n and $\omega(x) = (x_2^1 + \ldots + x_n^2)^{n/2} = |x|^n$; the case of $X = Z^n$, $\omega(m) = |m|^n$ is entirely analogous. Generalizing a definition of Beurling ([3] §§ 1, 2), Herz introduced a family of spaces ${}^pL^q(R^n)$ as follows (cf. [11] pp. 298—300): denote by Φ the set of functions φ in $L^1(0, \infty)$, $\varphi \geq 0$ satisfying

(i) $\int_0^\infty \varphi(t)dt = \|\varphi\| = 1$, (ii) φ is monotonic decreasing,

and, with $\gamma = 1/p - 1/q$, set

$$
{}^pL^q(R^n) = \bigcup_{\varphi \in \Phi} \{L^p_\sigma(R^n) \colon \sigma = (\varphi \circ \omega)^\gamma\}, \quad \gamma \leq 0 ,
$$

$$
{}^pL^q(R^n) = \bigcap_{\varphi \in \Phi} \{L^p_\sigma(R^n) \colon \sigma = (\varphi \circ \omega)^\gamma\}, \quad \gamma \geq 0 .
$$

These spaces are Banach spaces under the respective norms

$$
\inf_{\varphi \in \Phi} \|f(\varphi \circ \omega)^\gamma\|_p, \quad \gamma \leq 0 ,
$$

$$
\sup_{\varphi \in \Phi} \|f(\varphi \circ \omega)^\gamma\|_p, \quad \gamma \geq 0 .
$$

The Beurling spaces of great interest in harmonic analysis are the spaces $A^p = {}^pL^1$, $B^p = {}^pL^\infty$ ([3] p. 10). Further spaces $K_{pq}(R^n)$, $K^\alpha_{pq}(R^n)$ were defined by Herz to facilitate the discussion of the ${}^pL^q$ spaces ([11] pp. 301, 302):

$$
K_{pq}(R^n) = \{f \colon f\omega^\gamma \in {}^pL^q(R^n)\} ,
$$

$$
K^\alpha_{pq}(R^n) = \{f \colon f\omega^{\gamma+\alpha/n} \in {}^pL^q(R^n)\}, \quad -\infty < \alpha < \infty . \tag{19}
$$

The following relations between the above spaces and the interpolation spaces $(L^p(R^n), L^p_\omega(R^n))_{\theta,q;K}$, $(L^p(R^n), L^p_{1/\omega}(R^n))_{\theta,q;K}$ remove much of the mystery surrounding the Beurling-Herz spaces besides allowing use of the powerful abstract interpolation space theory in their study. It is interesting to note that Herz connected L^q and ${}^\infty L^q$ by complete interpolation (*loc.cit.*, p. 299).

THEOREM (5.1). *When* $\omega(x) = |x|^n$ *and* $1 \leq p < \infty$, $1 \leq q \leq \infty$,

$$
{}^pL^q(R^n) = (L^p(R^n), L^p_\omega(R^n))_{1/q-1/p,q;K}, \quad q < p ,
$$

$$
{}^pL^q(R^n) = (L^p(R^n), L^p_{1/\omega}(R^n))_{1/p-1/q,q;K}, \quad q > p ;
$$

in particular, the Beurling spaces satisfy

$$A^p(R^n) = (L^p(R^n), L^p_\omega(R^n))_{1-1/p,1;K}\,, \quad p \neq 1\,,$$

$$B^p(R^n) = (L^p(R^n), L^p_{1/\omega}(R^n))_{1/p,\infty;K}\,, \quad p \neq \infty\,.$$

Furthermore, when $K^\alpha_{pq}(R^n)$ *is defined by* (19),

$$K^{n\theta}_{pq}(R^n) = (L^p(R^n), L^p_\omega(R^n))_{\theta,q;K}\,, \quad 0 < \theta < 1\,.$$

Proof. Straightforward application of theorem (4.1).

Special cases of the characterizations in theorem (3.7) (ii), (iii) are worth noting:

$$\int_0^\infty \left(\frac{1}{t^{p+1}} \int_{|x|^n \le t} |f(x)|^p |x|^{np} dx\right)^{1/p} dt, \quad \int_0^\infty \left(\frac{1}{t} \int_{|x|^n \ge t} |f(x)|^p dx\right)^{1/p} dt \tag{20}$$

define equivalent norms on $A^p(R^n)$, $1 < p < \infty$, while

$$\sup \left(\frac{1}{t} \int_{|x|^n \le t} |f(x)|^p dx\right)^{1/p} \tag{21}$$

defines an equivalent norm on $B^p(R^n)$, $1 < p < \infty$. Beurling in fact took (21) as the defining property of $B^p(R^n)$. Properties (20) and (21) have been proved or used by several authors (cf. [9], [10], [12], [13], [18] for instance); the more general spaces considered by Sunouchi are nothing more than $(L^2(Z), L^2_\omega(Z))_{1/\beta-1/2,\beta;K}$, $1 \le \beta < 2$, and his characterizations are contained in theorem (3.7). The Banach algebra properties of $(L^p(Z^n), L^p_\omega(Z^n))_{\theta,q;K}$ (for appropriate values of θ, q) are studied in [1] and [2] making important use there of the interpolation space properties.

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