On the average order of the function $E(x) = \sum_{n \leq x} \phi(n) - \frac{3x^2}{\pi^2}$

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1. Introduction

Let n denote a positive integral variable and x denote a real variable, usually ≥ 3. Let $[x]$ denote the greatest integer $\leq x$ and $\{x\} = x - [x]$, called the fractional part of x. Let $\phi(n)$ denote the Euler totient function, which is defined to be the number of positive integers $\leq n$ and relatively prime to n. In 1874, F. Mertens [3] proved that

$$\sum_{n \leq x} \phi(n) = \frac{3x^2}{\pi^2} + O(x \log x) , \tag{1.1}$$

a proof of which may be found in many books on number theory (cf. [2], Theorem 330). It can be easily shown that

$$\sum_{n \leq x} \phi(n)n = \frac{2x^3}{\pi^2} + O(x^2 \log x) . \tag{1.2}$$

Let $\Phi(x)$, $\Phi'(x)$, $E(x)$ and $E'(x)$ be defined by

$$\Phi(x) = \sum_{n \leq x} \phi(n), \quad \Phi'(x) = \sum_{n \leq x} \phi(n)n , \tag{1.3}$$

$$E(x) = \sum_{n \leq x} \phi(n) - \frac{3x^2}{\pi^2} = \Phi(x) - \frac{3x^2}{\pi^2} , \tag{1.4}$$

and

$$E'(x) = \sum_{n \leq x} \phi(n)n - \frac{2x^3}{\pi^2} = \Phi'(x) - \frac{2x^3}{\pi^2} . \tag{1.5}$$

The best known O-estimate of $E(x)$ is due to Arnold Walfisz (cf. [11], Satz 1, p. 144), who proved that

$$E(x) = O(x \log^{2/3} x (\log\log x)^{4/3}) . \tag{1.6}$$

In 1930, S. S. Pillai and S. D. Chowla [4] proved that

$$\sum_{n \leq x} E(n) = \frac{3x^2}{2\pi^2} + o(x^2) \tag{1.7}$$

and also that

$$E(x) \neq o(x \log\log\log x) . \tag{1.8}$$

In 1931, M. L. N. Sarma [6] disproved the conjecture of J. J. Sylvester ([8], [9]), namely $E(n) > 0$, by showing that $E(820) < 0$.

In 1951, P. Erdös and H. N. Shapiro [1] proved that

$$E(x) = \Omega_{\pm}(x \log\log\log\log x) . \tag{1.9}$$

In this paper we prove that

$$\int_1^x E(t)dt = O(x^2\delta(x)) \tag{1.10}$$

and improve the result (1.7) to

$$\sum_{n \leq x} E(n) = \frac{3x^2}{2\pi^2} + O(x^2\delta(x)) , \tag{1.11}$$

where $\delta(x) = \exp(-A \log^{3/5} x (\log\log x)^{-1/5})$, A being a positive constant.

Further, we prove on the assumption of the Riemann hypothesis that

$$\int_1^x E(t)dt = O(x^{9/5}\omega(x)) \tag{1.12}$$

and

$$\sum_{n \leq x} E(n) = \frac{3x^2}{2\pi^2} + O(x^{9/5}\omega(x)) , \tag{1.13}$$

where $\omega(x) = \exp(A \log x (\log\log x)^{-1})$, A being a positive constant.

2. Preliminaries

In this section we state some known results which are needed in our discussion. We need the following best known estimate concerning the Möbius function $\mu(n)$ obtained by Arnold Walfisz [11]:

LEMMA 2.1 (cf. [11]; Satz 3, p. 191).

$$M(x) = \sum_{n \leq x} \mu(n) = O(x\delta(x)) , \tag{2.1}$$

where

$$\delta(x) = \exp(-A \log^{3/5} x(\log\log x)^{-1/5}) , \tag{2.2}$$

A being a positive constant.

LEMMA 2.2 (cf. [7], Lemma 2.2, $s = 2$).

$$\sum_{n \leq x} \frac{\mu(n)}{n^2} = \frac{6}{\pi^2} + O\left(\frac{\delta(x)}{x}\right). \tag{2.3}$$

LEMMA 2.3 (cf. [10], Theorem 14—26 (A), p. 316). *If the Riemann hypothesis is true, then*

$$M(x) = \sum_{n \leq x} \mu(n) = O(x^{1/2}\omega(x)) , \tag{2.4}$$

where

$$\omega(x) = \exp(A \log x(\log\log x)^{-1}) , \tag{2.5}$$

A being a positive constant.

LEMMA 2.4 (cf. [7], Lemma 2.6, $s = 2$). *If the Riemann hypothesis is true, then*

$$\sum_{n \leq x} \frac{\mu(n)}{n^2} = \frac{6}{\pi^2} + O(x^{-3/2}\omega(x)) . \tag{2.5}$$

LEMMA 2.5 (cf. [5], Lemma, p. 279, p. 765). *Let $f(n)$ be a function of a positive integral variable, and suppose*

$$\sum_{n \leq x} f(n) = g(x) + E(x) , \tag{2.6}$$

where $g(x)$ is twice continuously differentiable, and $g''(x)$ is of constant sign, for $x \geq 1$. Then

$$\sum_{n \leq x} E(n) = \tfrac{1}{2} g(x) + (1 - \{x\})E(x) + \int_1^x E(t)dt + O(|g'(x)|) + O(1) . \tag{2.7}$$

3. Proofs of (1.10) and (1.11)

It is well-known (cf. [2], (16.3.1) that $\phi(n) = \sum_{d\delta = n} \mu(d)\delta$. Hence

$$\sum_{n \leq x} \phi(n) = \sum_{d\delta \leq x} \mu(d)\delta , \tag{3.1}$$

where the summation on the right being taken over all ordered pairs (d, δ) such that $d\delta \leq x$.

Let $0 < \varrho = \varrho(x) < 1$, where the function $\varrho(x)$ will be suitably chosen later. If $d\delta \leq x$, then both $d > \varrho x$ and $\delta > \varrho^{-1}$ cannot simultaneously hold good. Hence from (3.1), we have

$$\sum_{n \leq x} \phi(n) = \sum_{\substack{d \leq \varrho x \\ d\delta \leq x}} \mu(d)\delta + \sum_{\substack{\delta \leq \varrho^{-1} \\ d\delta \leq x}} \mu(d)\delta - \sum_{\substack{d \leq \varrho x \\ \delta \leq \varrho^{-1}}} \mu(d)\delta$$
$$= S_1 + S_2 - S_3, \text{ say}. \tag{3.2}$$

Now,

$$S_1 = \sum_{\substack{d \leq \varrho x \\ d\delta \leq x}} \mu(d)\delta = \sum_{d \leq \varrho x} \mu(d) \sum_{\delta \leq \frac{x}{d}} \delta = \frac{1}{2} \sum_{d \leq \varrho x} \mu(d)\left(\left[\frac{x}{d}\right]^2 + \left[\frac{x}{d}\right]\right) =$$
$$= \frac{1}{2} \sum_{d \leq \varrho x} \mu(d)\left(\left(\frac{x}{d} - \left\{\frac{x}{d}\right\}\right)^2 + \left(\frac{x}{d} - \left\{\frac{x}{d}\right\}\right)\right) =$$
$$= \frac{x^2}{2} \sum_{d \leq \varrho x} \frac{\mu(d)}{d^2} - x \sum_{d \leq \varrho x} \frac{\mu(d)}{d}\left\{\frac{x}{d}\right\} + \frac{1}{2} \sum_{d \leq \varrho x} \mu(d)\left\{\frac{x}{d}\right\}^2 +$$
$$+ \frac{x}{2} \sum_{d \leq \varrho x} \frac{\mu(d)}{d} - \frac{1}{2} \sum_{d \leq \varrho x} \mu(d)\left\{\frac{x}{d}\right\}.$$

Hence by (2.3),

$$S_1 = \frac{x^2}{2}\left(\frac{6}{\pi^2} + 0\left(\frac{\delta(\varrho x)}{\varrho x}\right)\right) - x \sum_{n \leq \varrho x} \frac{\mu(n)}{n}\left\{\frac{x}{n}\right\} + O(\varrho x) + \frac{x}{2} \sum_{n \leq \varrho x} \frac{\mu(n)}{n} + O(\varrho x)$$
$$= \frac{3x^2}{\pi^2} - x \sum_{n \leq \varrho x} \frac{\mu(n)}{n}\left\{\frac{x}{n}\right\} + \frac{x}{2} \sum_{n \leq \varrho x} \frac{\mu(n)}{n} + O(\varrho x) + O(x\varrho^{-1}\delta(\varrho x)). \tag{3.3}$$

We have

$$S_2 = \sum_{\substack{\delta \leq \varrho^{-1} \\ d\delta \leq x}} \mu(d)\delta = \sum_{n \leq \varrho^{-1}} n \sum_{d \leq \frac{x}{n}} \mu(d) = \sum_{n \leq \varrho^{-1}} nM\left(\frac{x}{n}\right) = O\left(\sum_{n \leq \varrho^{-1}} x\delta\left(\frac{x}{n}\right)\right),$$

by (2.1). Since $\delta(x)$ is monotonic decreasing and $x/n \geq \varrho x$, we have $\delta(x/n) \leq \delta(\varrho x)$. Hence

$$S_2 = O(x\delta(\varrho x) \sum_{n \leq \varrho^{-1}} 1) = O(x\varrho^{-1}\delta(\varrho x)). \tag{3.4}$$

Also, we have

$$S_3 = \sum_{\substack{d \leq \varrho x \\ \delta \leq \varrho^{-1}}} \mu(d)\delta = (\sum_{\delta \leq \varrho^{-1}} \delta)(\sum_{d \leq \varrho x} \mu(d)) = (\sum_{\delta \leq \varrho^{-1}} \delta) M(\varrho x) = \tag{3.5}$$
$$= O(\varrho^{-2} \cdot \varrho x \delta(\varrho x)) = O(x\varrho^{-1}\delta(\varrho x)) .$$

Hence by (3.,2) (3.3), (3.4) and (3.5), we get that

$$\sum_{n \leq x} \phi(n) = \frac{3x^2}{\pi^2} - x \sum_{n \leq \varrho x} \frac{\mu(n)}{n} \left\{\frac{x}{n}\right\} + \frac{x}{2} \sum_{n \leq \varrho x} \frac{\mu(n)}{n} + O(\varrho x) + O(x\varrho^{-1}\delta(\varrho x)) . \tag{3.6}$$

Following the same procedure as above, starting from

$$\sum_{n \leqq x} \phi(n)n = \sum_{d\delta \leq x} \mu(d)d\delta^2 = \sum_{\substack{d \leq \varrho x \\ d\delta \leq x}} \mu(d)d\delta^2 + \sum_{\substack{\delta \leq \varrho^{-1} \\ d\delta \leq x}} \mu(d)d\delta^2 - \sum_{\substack{d \leq \varrho x \\ \delta \leq \varrho^{-1}}} \mu(d)d\delta^2 ,$$

we can show that

$$\sum_{n \leq x} \phi(n)n = \frac{2x^3}{\pi^2} - x^2 \sum_{n \leq \varrho x} \mu(n) \left\{\frac{x}{n}\right\} + \frac{x^2}{2} \sum_{n \leq \varrho x} \frac{\mu(n)}{n} + O(\varrho x^2) + O(x^2\varrho^{-1}\delta(\varrho x)) . \tag{3.7}$$

We get from (3.6), (3.7) and (1.3), that

$$\Phi'(x) - x\Phi(x) = - \frac{x^3}{\pi^2} + O(\varrho x^2) + O(x^2\varrho^{-1}\delta(\varrho x)) . \tag{3.8}$$

Now, we choose

$$\varrho = \varrho(x) = \{\delta(x^{1/2})\}^{1/2} , \tag{3.9}$$

and write

$$f(x) = \log^{3/5} (x^{1/2})(\log \log (x^{1/2}))^{-1/5} = (\tfrac{1}{2})^{3/5} u^{3/5} (v\text{-}\log 2)^{-1/5} , \tag{3.10}$$

where $u = \log x$ and $v = \log \log x$.

$$\text{For } v \geq 2 \log 2, \text{ that is, } u \geq 4, \ x \geq e^4 > 54 , \tag{3.11}$$

we have

$$v^{-1/5} \leq (v\text{-}\log 2)^{-1/5} \leq \left(\frac{v}{2}\right)^{-1/5}$$

and therefore

$$\left(\frac{1}{2}\right)^{3/5} u^{3/5} v^{-1/5} \leq f(x) \leq \left(\frac{1}{2}\right)^{3/5} u^{3/5} \left(\frac{v}{2}\right)^{-1/5} . \tag{3.12}$$

We assume without loss of generality that in (2.2)

$$A < 1 . \tag{3.13}$$

By (3.9), (2.2) and (3.10), we have

$$\varrho = \exp\left(- \frac{A}{2} f(x)\right) . \tag{3.14}$$

By (3.11), we have

$$\frac{1}{2}\left(\frac{1}{2}\right)^{2/5} u^{3/5}v^{-1/5} \le \frac{u}{2} .$$

Hence by (3.12), (3.13), (3.14) and the above,

$$\varrho \ge \exp\left(-\frac{A}{2}\left(\frac{1}{2}\right)^{2/5} u^{3/5}v^{-1/5}\right) \ge \exp\left(-\frac{1}{2}\left(\frac{1}{2}\right)^{2/5} u^{3/5}v^{-1/5}\right) \ge$$
$$\ge \exp\left(-\frac{u}{2}\right) = \exp\left(-\frac{\log x}{2}\right),$$

so that $\varrho \ge x^{-1/2}$.

Hence $\varrho x \ge x^{1/2}$. Since $\delta(x)$ is monotonic decreasing, $\delta(\varrho x) \le \delta(x^{1/2}) = \varrho^2$, by (3.9).

Hence the second O-term in (3.8) is also $O(\varrho x^2)$.

By (3.12) and (3.14), we have

$$\varrho = \exp\left(-\frac{A}{2} f(x)\right) \le \exp\left(-\frac{A}{2}\left(\frac{1}{2}\right)^{3/5} u^{3/5}v^{-1/5}\right) = \exp\left(-Bu^{3/5}v^{-1/5}\right), \tag{3.15}$$

where
$$B = A\left(\frac{1}{2}\right)^{8/5} .$$

Hence by (3.8) and (3.15),

$$\Phi'(x) - x\Phi(x) = -\frac{x^3}{\pi^2} + O(x^2 \exp(-Bu^{3/5}v^{-1/5})) = -\frac{x^3}{\pi^2} + O(x^2\delta(x)) , \tag{3.16}$$

with the constant B in place of the constant A in $\delta(x)$. From (1.4), (1.5) and (3.16), we get that

$$\frac{2x^3}{\pi^2} + E'(x) - \frac{3x^3}{\pi^2} - xE(x) = -\frac{x^3}{\pi^2} + O(x^2\delta(x)) ,$$

so that

$$E'(x) - xE(x) = O(x^2\delta(x)) . \tag{3.17}$$

By partial summation formula (cf. [2], Theorem 421), we have

$$\Phi'(x) = \sum_{n \le x} \phi(n)n = \Phi(x)x - \int_1^x \Phi(t)dt ,$$

so that

$$\frac{2x^3}{\pi^2} + E'(x) = \left(\frac{3x^2}{\pi^2} + E(x)\right) x - \int_1^x \left(\frac{3t^2}{\pi^2} + E(t)\right) dt =$$

$$= \frac{3x^3}{\pi^2} + xE(x) - \frac{x^3}{\pi^2} + O(1) - \int_1^x E(t)dt \,.$$

Hence by (3.17), we get that

$$\int_1^x E(t)dt = -\,(E'(x) - xE(x)) = O(x^2\delta(x)) \,. \tag{3.18}$$

Thus (1.10) is proved.

Now, to prove (1.11), we use Lemma 2.5. Taking $f(n) = \phi(n)$ in Lemma 2.5, we see by (1.1) that $g(x) = 3x^2/\pi^2$ and $E(x) = O(x \log x)$. Hence by Lemma 2.5, we get that

$$\sum_{n \leq x} E(n) = \frac{3x^2}{2\pi^2} + O(x \log x) + \int_1^x E(t)dt + O(x) + O(1) = \frac{3x^2}{2\pi^2} + O(x^2\delta(x)) \,. \tag{3.19}$$

Thus (1.11) is proved.

4. Proofs of (1.12) and (1.13)

Following the same procedure as in the proof of (1.10) and making use of (2.4) and (2.6) instead of (2.1) and (2.3), we get the following in place of (3.8):

$$\Phi'(x) - x\Phi(x) = -\,\frac{x^3}{\pi^2} + O(\varrho x^2) + O(x^{3/2}\varrho^{-3/2}\omega(\varrho x)) \,. \tag{4.1}$$

Now, choosing $\varrho = x^{-1/5}$, we see that

$$\varrho x^2 = x^{3/2}\varrho^{-3/2} = x^{9/5} \,.$$

Since $\omega(x)$ is monotonic increasing, we have $\omega(\varrho x) < \omega(x)$. Hence we get from (4.1) that

$$\Phi'(x) - x\Phi(x) = -\,\frac{x^3}{\pi^2} + O(x^{9/5}\omega(x)) \,. \tag{4.2}$$

Now, proceeding as before in the proofs (1.10) and (1.11), we get the following, one after another:

$$E'(x) - xE(x) = O(x^{9/5}\omega(x)) \,, \tag{4.3}$$

$$\int_1^x E(t)dt = O(x^{9/5}\omega(x)) \tag{4.4}$$

and

$$\sum_{n \leq x} E(n) = \frac{3x^2}{2\pi^2} + O(x^{9/5}\omega(x)) \,. \tag{4.5}$$

Thus the proofs of (1.12) and (1.13) are complete.

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