Distributions with bounded potentials and absolutely convergent Fourier series

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1. Introduction

In the first part of this paper we show that the capacity of a compact subset of the n-dimensional Euclidean space can be characterized by means of distributions (in the sense of L. Schwartz) which are carried by the set and which have bounded potentials.

In the second part we consider compact subsets of the real line with the property that no non-trivial function can locally be in the class of Fourier transforms of L^1-functions and yet be constant on the intervals of the complement of the set. Using the result from the first part we shall show that a sufficient condition for a set to have this property is that it has logarithmic capacity zero. This improves a result by Kahane and Katznelson [4, p. 21] concerning Cantor sets.

It will also be shown that a necessary condition is that the set has capacity zero with respect to all kernels $(\log^+ 1/|x|)^{2+\delta}$, $\delta > 0$.

The author is very grateful to prof. L. Carleson for his many valuable suggestions.

2. Notations and definitions

We denote the n-dimensional Euclidean space by $\mathbf{R}^n$, its points by $x = (x_1, \ldots, x_n)$ and we write $|x| = (x_1^2 + \ldots + x_n^2)^{1/2}$.

By A^{loc} we mean the space of all functions f on $\mathbf{R}^n$ with the property that for each $x \in \mathbf{R}^n$ there exists a neighbourhood V of x and a function g, whose Fourier transform is an integrable function, so that $f = g$ in V.

The Fourier transform of a function, measure or tempered distribution S is denoted by $\hat{S}$.

By a kernel we mean, in the case when $n \geq 2$, an integrable function on $\mathbf{R}^n$, with compact support, of the form $K(x) = H(\varphi(x))$, where H is a non-negative, continuous, increasing and convex function on $\mathbf{R}$ and where φ is a fundamental solution of Laplace's equation, i.e.

$$\varphi(x) = \begin{cases} \log \dfrac{1}{|x|}, & n = 2 \\ |x|^{2-n}, & n \geq 3 \end{cases}$$

When $n = 1$ we mean by a kernel an even, integrable and positive function with compact support on $\mathbf{R}$ which is convex on $(0, \infty)$.

We shall throughout this paper let K_α be a kernel which in a neighbourhood of the origin equals $|x|^{-\alpha}$ when $\alpha > 0$ or $\log 1/|x|$ when $\alpha = 0$ and which moreover is infinitely differentiable for $x \neq 0$.

We define as usual the capacity $C_K(E)$ of a compact set E with respect to a kernel K by

$$(C_K(E))^{-1} = \inf_{\mu} (\sup_{\mathbf{R}^n} U^\mu(x))$$

where the infimum is taken over all positive measures on E of mass 1. For the other basic concepts of classical potential theory we refer the reader to [2].

3. Distributions with bounded potentials

Let S be a distribution (in the sense of L. Schwartz) on $\mathbf{R}^n$. We define the potential of S with respect to the kernel K as the convolution of S and K and denote it by $U^S = S * K$.

Deny [3] has in great detail studied potentials of distributions with finite energy and has e.g. shown that a set of capacity zero with respect to some kernel cannot carry a non-zero distribution with finite energy with respect to the same kernel. In the special case when $n = 1$ and for kernels K_α this was also proved for $\alpha = 0$ in an Uppsala lecture by Beurling in 1940 and later by Broman [1] for $0 < \alpha < 1$.

Our aim is to study distributions whose potentials are bounded functions and we shall obtain some analogous results.

THEOREM 1. *Let E be a compact subset of $\mathbf{R}^n$ and let S be a distribution with support on E. If the potential of S with respect to a kernel K is a function U^S and if furthermore $S(1) = 1$ then*

$$\operatorname*{ess\,sup}_{x \in \mathbf{R}^n} |U^S(x)| \geq (C_K(E))^{-1}$$

Proof. Let $\varepsilon > 0$ be given and let σ_ε be the equilibrium measure of the set $E_\varepsilon = \{x \in \mathbf{R}^n;\ \text{dist}\,(x, E) \leq \varepsilon\}$. Let furthermore $k \in C_0^\infty$ be a positive function with support in the n-dimensional unit ball and assume $\int k dx = 1$. Write $k_\varepsilon(x) = \varepsilon^{-n} k(x_1/\varepsilon, \ldots, x_n/\varepsilon)$ and let $S_\varepsilon = S * k_\varepsilon$.

We now easily obtain the following inequalities by using wellknown properties of the equilibrium measure.

$$\text{ess sup}\,|U^S| \geq \sup\,(|U^S| * k_\varepsilon) \geq \sup\,|U^S * k_\varepsilon| = \sup\,|U^{S_\varepsilon}| \geq \int |U^{S_\varepsilon}| d\sigma_\varepsilon \geq$$

$$\geq \left| \int U^{S_\varepsilon} d\sigma_\varepsilon \right| = (S_\varepsilon * K * \sigma_\varepsilon)(0) = (S_\varepsilon * (K * \sigma_\varepsilon))(0) = (C_K(E_\varepsilon))^{-1} \int S_\varepsilon dx\ .$$

But $C_K(E_\varepsilon) \to C_K(E)$ as $\varepsilon \to 0$ and from $S(1) = 1$ it follows that $\int S_\varepsilon dx = 1$ which proves the theorem.

We are later on going to use this theorem in the classical cases of logarithmic capacity and α-capacity, i.e. capacity with respect to the kernels K_α. For these kernels we can prove the following corollary.

COROLLARY. *A compact set $E \subset \mathbf{R}^n$ has positive α-capacity (if $\alpha = 0$: logarithmic capacity) for* $\max\,(0, n-2) \leq \alpha < n$ *if and only if it carries a non-zero distribution whose potential with respect to K_α is a bounded function.*

Proof. Assume that $S \neq 0$ is carried by the set E and that $S * K_\alpha$ is a bounded function.

It is by the theorem sufficient to find a distribution S_0 which satisfies the same assumptions as S and which in addition has the property that $S_0(1) \neq 0$.

If S is non-vanishing there must exist $y_0 \in \mathbf{R}^n$ such that $S(e^{i(\cdot, y_0)}) \neq 0$. We shall prove that the distribution $S_0 = e^{i(\cdot, y_0)} S$ has the required properties. This follows immediately if we can show that $\hat{S}(y)(\hat{K}_\alpha(y - y_0) - \hat{K}_\alpha(y))$ is the Fourier transform of a bounded function.

We claim that we can write

$$\hat{K}_\alpha(y - y_0) - \hat{K}_\alpha(y) = \hat{K}_\alpha(y)(\hat{P}(y) + \hat{Q}(y) + \hat{R}(y)) \tag{1}$$

where $P, \hat{Q} \in L^1(\mathbf{R}^n), \hat{R} \in L^2(\mathbf{R}^n)$.

Since $\hat{K}_\alpha(y) = C|y|^{\alpha - n} + O(|y|^{-N})$ for any N as $|y| \to \infty$ we have that

$$\hat{K}_\alpha(y - y_0) - \hat{K}_\alpha(y) = \hat{K}_\alpha(y)\left(\frac{|y|^{n-\alpha}}{|y - y_0|^{n-\alpha}} - 1 + O(|y|^{-N})\right)$$

$$= \hat{K}_\alpha(y)\Big(\sum_1^{[\frac{n}{2}]} p_k(y)|y|^{-2k} + O(|y|^{-[\frac{n}{2}]-1})\Big)$$

where p_k are certain polynomials of degree k.

We can without loss of generality assume that $y_0 = (1, 0, \dots, 0)$ and write

$$\hat{K}_\alpha(y - y_0) - \hat{K}_\alpha(y) = \hat{K}_\alpha(y)\Big(\sum_{k=1}^{[\frac{n}{2}]} \sum_{l=0}^{k} \alpha_{lk} y_1^l |y|^{-2k} + O(|y|^{-[\frac{n}{2}]-1})\Big)$$

We claim that we can find an L^1-function such that the behaviour of its Fourier transform at infinity is close to the behaviour of the sum in the last member above.

Consider therefore one term, $y_1^l|y|^{-2k}$ and let $P_{lk} \in C^\infty(\mathbf{R}^n \setminus \{0\})$ be some function with compact support which in a neighbourhood of the origin equals $(-i\, \partial/\partial x_1)^l |x|^{2k-n}$ when $n \neq 2k$ or $(-i\, \partial/\partial x_1)^l \log 1/|x|$ when $n = 2k$. Then $\hat{P}_{lk}(y) = C|y|^{-2k} y_1^l + O(|y|^{-N})$ for all N as $|y| \to \infty$ and $P_{lk} \in L^1(\mathbf{R}^n)$.

By adding constant multiples of such functions P_{lk} we finally obtain a function $P \in L^1(\mathbf{R}^n)$ which satisfies $\hat{P}(y) - \sum p_k(y)|y|^{-2k} = O(|y|^{-N})$ for all N as $|y| \to \infty$ and it is easily seen that Q and R can be chosen so that (1) holds.

Since $\hat{S}\hat{K}_\alpha$ and $S * K_\alpha$ are bounded functions and since $S * K_\alpha \in L^2(\mathbf{R}^n)$ it now follows that $S * K_\alpha * (P + Q + R)$ is a bounded function which proves the sufficiency part of the corollary. The necessity follows directly from the definition of capacity.

It seems probable that the corollary also holds for general kernels K although we have not been able to prove this.

Let us now end this section by showing how the corollary could be used to prove the classical result (see e.g. [1, ch. VII]) that a set is »removable» for bounded harmonic functions if it has capacity zero with respect to K_{n-2}.

Let therefore $D \subset \mathbf{R}^n$, $n \geq 2$, be a bounded region whose boundary Γ is a smooth surface and let $E \subset D$ be a closed set strictly contained inside Γ. Assume that E has capacity zero with respect to K_{n-2} and let u be a bounded and harmonic function on $D \setminus E$. We claim that u can be extended to the whole of D.

Choose a function v with compact support which coincides with u on some neighbourhood of E and which is infinitely differentiable outside E.

Then $\Delta v = S + \varphi$, where S is a distribution carried by E and where $\varphi \in C_0^\infty$.

But the potential of S with respect to K_{n-2} is

$$S * K_{n-2} = \Delta v * K_{n-2} - \varphi * K_{n-2} = v * \Delta K_{n-2} - \varphi * K_{n-2} = v + v * \psi - \varphi * K_{n-2}$$

where $\psi \in C_0^\infty$.

We thus have that $U^S = S * K_{n-2}$ is a bounded function which by the corollary implies that $S = 0$ and hence that u can be extended as claimed.

4. A class of thin sets

Let E be a compact subset of the real line and assume that $f \in A^{\mathrm{loc}}$ is constant on each interval of the complement of E.

Our aim is to try to characterize the sets E for which no nontrivial function f can have the properties just stated.

Kahane and Katznelson [4, p. 21] have given an example of a Cantor set with the above property. The following theorem contains their result.

THEOREM 2. *Let* $E \subset \mathbf{R}$ *be a compact set of logarithmic capacity zero. If* $f \in A^{\mathrm{loc}}$ *is constant on each interval of the complement of* E *then* f *equals a constant.*

Proof. Let f be an arbitrary function that fulfills the hypothesis. Then there exists a function g which satisfies:

(i) $\hat{g} \in L^1(\mathbf{R})$.
(ii) $g = f$ on some interval $I \supset E$.
(iii) $g = 0$ outside some neighbourhood of $\bar{I}$.
(iv) g is infinitely differentiable outside I.

Let S be the derivative of f in the sense of distributions and let U^S be the logarithmic potential of S.

Then $S = dg/dx + \varphi$ where $\varphi \in C_0^\infty$ and $U^S = U^{g'} + U^{\varphi}$.

But $(U^{g'})^{\wedge}(x) = ix\hat{g}(x)\hat{K}_0(x) \in L^1(\mathbf{R})$ since $\hat{K}_0(x) = 1/|x| + O(|x|^{-N})$ for any N as $x \to \infty$ and hence $U^{g'}$ is a bounded function. This is also true for U^{φ} and S is thus a distribution on E with bounded logarithmic potential.

By the corollary this leads to a contradiction unless $S = 0$ and hence the theorem follows.

Our next theorem shows that this result is the best possible in the sense that logarithmic capacity cannot be replaced by capacity with respect to any larger kernel.

Before we state the theorem we give the following lemma.

LEMMA 2. *Let* $E = \{x \in \mathbf{R};\ x = \sum_1^\infty \varepsilon_i r_i,\ \ \varepsilon_i = 0, 1, \ldots, m_i\}$ *where* $\{r_i\}_1^\infty$ *and* $\{m_i\}_1^\infty$ *are some given sequences. Write* $l_n = \sum_{n+1}^\infty m_i r_i$ *and assume that* $(m_n + 1)l_n \le \frac{1}{2} l_{n-1}$ *for* $n = 1, 2, \ldots$.

Then E *has capacity zero with respect to a kernel* K *if*

$$\sum_{n=1}^{\infty} \prod_{1}^{n} (m_i + 1)^{-1} K(l_n) = \infty .$$

If $\dfrac{1}{x}\displaystyle\int_0^x K(t)\,dt \le CK(x),\ \ x \neq 0$ *then the condition is also necessary.*

Proof. The proof of the lemma is, apart from minor modifications, identical to the proof given in e.g. [2] of the corresponding theorem concerning the ordinary Cantor set (i.e. the case when $m_i = 1,\ i = 1, 2, \ldots$) and is therefore omitted here.

The last condition in the lemma is clearly satisfied for all kernels K_α, $0 \le \alpha < 1$.

THEOREM 3. *Let K be a kernel such that* $\lim_{x\to 0} K(x)(\log 1/|x|)^{-1} = \infty$. *Then there exists a compact subset of* $\mathbf{R}$ *with capacity zero with respect to* K *and a non-zero measure carried by the set whose primitive function is in* A^{loc}.

Proof. Let E be a set of the form $\{x = \sum_1^\infty \varepsilon_i r_i, \ \ \varepsilon_i = 0, 1, \ldots, m_i\}$ and assume that E has positive logarithmic capacity and choose a measure μ on E with finite energy with respect to the logarithmic kernel. Let $\nu = \mu * \mu$ and let $f(x) = \int_0^x d\nu$. The support of ν is obviously a subset of

$$F = E + E = \Big\{\sum_1^\infty \eta_i r_i, \ \ \eta_i = 0, 1, \ldots, 2m_i\Big\}.$$

That the energy of μ is finite implies that

$$\int_{|y|\geq 1} |\hat{\mu}(y)|^2 \frac{dy}{|y|} < \infty$$

But $\hat{f}(y) = \hat{\nu}(y)/iy = (\hat{\mu}(y))^2/iy$ and hence $f \in A^{\text{loc}}$.

We now claim that we can choose $\{r_i\}$ and $\{m_i\}$ in such a way that

$$\sum_1^\infty \Big(\prod_1^n (m_i + 1)\Big)^{-1} \log 1/l_n < \infty \tag{2}$$

$$\sum_1^\infty \Big(\prod_1^n (2m_i + 1)\Big)^{-1} K(2l_n) = \infty \tag{3}$$

where $l_n = \sum_{n+1}^\infty m_i r_i$.

Both these relations are satisfied if we e.g. choose $\{r_i\}$ and $\{m_i\}$ so that

(i) $\log 1/l_n = n^{-2} \prod_1^n (m_i + 1), \ \ n = 1, 2, \ldots$

(ii) $K(2l_n) \geq 3^n \log 1/l_n, \ \ n = 1, 2, \ldots$

(iii) $m_i \to \infty, \ \ i \to \infty$

(iv) $(2m_n + 1)l_n \leq l_{n-1}/2\,, \ \ n = 1, 2, \ldots$

Since $l_{n+1} = l_n - m_{n+1} r_{n+1}$ it is clear that this choice can be done.

Using Lemmas 2 and 3 it now follows from (2) and (3) that E (and thus F) has positive logarithmic capacity and that F has capacity zero with respect to K which proves our theorem.

The next theorem gives a necessary condition in terms of capacity for a set to be in our class of thin sets.

THEOREM 4. *Let* $E \subset \mathbf{R}$ *be a compact set of positive capacity with respect to the kernel* $(\log^+ 1/|x|)^{2+\delta}$ *for some* $\delta > 0$. *Then* E *carries a (positive) measure* μ *whose primitive function is in* A^{loc}.

Proof. Put $K(x) = (\log^+ 1/|x|)^{2+\delta}$ and let μ be a measure on E with finite energy with respect to K.

It is easy to show that $(\log |y|)^{1+\delta}|y|^{-1}(\hat{K}(y))^{-1}$ tends to a constant as $|y| \to \infty$ and it therefore follows that

$$\int_{|y|\geq 1} \frac{|\hat{\mu}(y)|^2}{|y|} (\log |y|)^{1+\delta} dy < \infty$$

By Schwarz's inequality this implies that $\int_{|y|\geq 1} |\hat{\mu}(y)|\,|y|^{-1} dy < \infty$ and hence that $f(x) = \int_0^x d\mu \in A^{\mathrm{loc}}$ which proves the theorem.

It is possible that any set E of positive logarithmic capacity carries a non-zero distribution with a primitive function in A^{loc}. The following result shows that such a distribution could not always be chosen as a positive measure and the necessary construction would therefore probably have to be complicated.

THEOREM 5. *There exists a $\delta > 0$ and a compact set $E_\delta \subset \mathbf{R}$ of positive capacity with respect to the kernel $K(x) = (\log^+ 1/|x|)^{1+\delta}$ such that if $f \in A^{\mathrm{loc}}$ is non-decreasing and constant on each interval of the complement of E_δ then f equals a constant.*

Proof. Let E_δ be the Cantor set $\{x \in \mathbf{R};\ x = \sum_1^\infty \varepsilon_i r_i,\ \varepsilon_i = 0 \text{ or } 1\}$ where $\sum_{n+1}^\infty r_i = \exp(-2^{n/(1+\delta)})$, $n = 1, 2, \ldots$ and where $\delta > 0$ is a number to be fixed later. E_δ has positive capacity with respect to a kernel $(\log^+ 1/|x|)^{1+t}$ if and only if $0 \leq t < \delta$.

Assume that f is a function that fulfills the hypothesis. Its derivative in the sense of distributions is a positive measure μ with support in E_δ. Consider the convolution $\nu = \mu * \mu$ which is a measure on $F_\delta = E_\delta + E_\delta$ and suppose we know that ν has bounded energy with respect to the kernel $(\log^+ 1/|x|)^{1+3\delta'}$ for some number $\delta' > \delta$.

This assumption is equivalent to

$$\int_{|y|\geq 1} \frac{|\hat{\mu}(y)|^4}{|y|} (\log |y|)^{3\delta'} dy < \infty$$

and, since $\int_0^x d\mu \in A^{\mathrm{loc}}$, we also know that

$$\int_{|y|\geq 1} \frac{|\hat{\mu}(y)|}{|y|} dy < \infty .$$

From these two inequalities it follows by means of Hölder's inequality that

$$\int\limits_{|y|\geq 1} \frac{|\hat{\mu}(y)|^2}{|y|} (\log |y|)^{\delta'} dy < \infty$$

i.e. that μ has finite energy with respect to the kernel $(\log^+ 1/|x|)^{1+\delta'}$ where $\delta' > \delta$.

But this is impossible unless $\mu = 0$ and hence the theorem follows as soon as we have proved the following lemma.

LEMMA 3. *Let E_δ be a Cantor set as above and let $F_\delta = E_\delta + E_\delta$. Let μ be a positive measure on E_δ with bounded logarithmic potential. Then, if $\delta > 0$ is sufficiently small, there exists $\delta' > \delta$ such that the measure $\nu = \mu * \mu$ has finite energy with respect to the kernel* $(\log^+ 1/|x|)^{1+3\delta'}$.

Proof. By means of a simple estimate we see that $\mu(I) \leq C(\log 1/|I|)^{-1}$ for all intervals I of length $|I|$ less than 1.

We also observe that $F_\delta = \bigcap_1^\infty F_n$ where each set F_n is the union of 3^n intervals $I_k^{(n)}$ of length $l_n = \sum_{n+1}^\infty 2r_i$ and with left endpoints $x_k^{(n)}$, $k = 1, \ldots, 3^n$.

Each point $x_k^{(n)}$ can be written $x_k^{(n)} = \sum_1^\infty (\varepsilon_i + \varepsilon_i')r_i$, with ε_i, $\varepsilon_i' = 0$ or 1, in $N_k^{(n)}$ different ways. Let q be the number of indices i for which $\eta_i = \varepsilon_i + \varepsilon_i' = 1$. It then follows that $x_k^{(n)}$ can be obtained in $N_k^{(n)} = 2^q$ different ways as a sum of two points in E_δ. For a fixed q there are $\binom{n}{q}2^{n-q}$ such points $x_k^{(n)}$ in F_δ. We find that $\sum_k (N_k^{(n)})^2 = \sum_{q=0}^n \binom{n}{q}2^{n-q} \cdot 2^{2q} = 6^n$.

Let $\nu^{(n)}$ be the measure whose restriction to any interval $I_k^{(n)}$ is uniformly distributed and whose mass on any interval $I_k^{(n)}$ equals $\nu(I_k^{(n)})$. It is easy to see that $\nu^{(n)}$ converges weakly to ν as $n \to \infty$ and it is therefore sufficient to prove that the energy of $\nu^{(n)}$ is bounded uniformly in n.

The energy of $\nu^{(n)}$ with respect to the kernel $(\log^+ 1/|x|)^{1+3\delta'}$ is by definition

$$E(\nu^{(n)}) = \sum_{k,l} \iint\limits_{I_k^{(n)} \times I_l^{(n)}} \left(\log^+ \frac{1}{|x-y|}\right)^{1+3\delta'} d\nu^{(n)}(x) d\nu^{(n)}(y) . \tag{4}$$

Let us define

$$D_n = \sum_k \iint\limits_{I_k^{(n)} \times I_k^{(n)}} \left(\log^+ \frac{1}{|x-y|}\right)^{1+3\delta'} d\nu^{(n)}(x) d\nu^{(n)}(y)$$

We claim that

$$E(\nu^{(n)}) \leq C \sum_0^n D_m . \tag{5}$$

To prove this, let Q_m, $1 \leq m \leq n$, denote the sum of all terms in the right hand member of (4) that correspond to pairs of intervals with a mutual distance less than l_{m-1} and greater than $\frac{1}{3} l_{m-1}$.

In other words, Q_m is the sum over the set T_m consisting of all indices (k, l) such that $\eta_i^{(k)} = \eta_i^{(l)}$ for $0 \leq i \leq m-1$ and $\eta_m^{(k)} \neq \eta_m^{(l)}$ (where $x_p^{(m)} = \sum_1^m \eta_i^{(p)} r_i$).

Then

$$Q_m \leq C\left(\log \frac{1}{l_{m-1}}\right)^{1+3\delta'} \sum_{T_m} \nu^{(n)}(I_k^{(n)})\nu^{(n)}(I_l^{(n)}) \leq$$

$$\leq C\left(\log \frac{1}{l_{m-1}}\right)^{1+3\delta'} \sum_k (\nu^{(m-1)}(I_k^{(m-1)}))^2 \leq CD_{m-1}$$

It is obvious that we get all terms in (4) by summing over m and adding D_n and hence (5) follows.

But since $\nu^{(n)}(I_k^{(n)}) \leq CN_k^{(n)}(\log 1/l_n)^{-2}$ and since

$$\frac{1}{l_n^2} \int_0^{l_n} \int_0^{l_n} \left(\log \frac{1}{|x-y|}\right)^{1+3\delta'} dxdy \leq C\left(\log \frac{1}{l_n}\right)^{1+3\delta'}$$

we have that

$$D_n \leq C\left(\log \frac{1}{l_n}\right)^{-3+3\delta'} \sum_k (N_k^{(n)})^2 = C\left(\log \frac{1}{l_n}\right)^{-3+3\delta'} 6^n \leq C\gamma^n$$

where $\gamma = 6 \cdot 2^{-(3-3\delta')/(1+\delta)}$ (since $l_n = 2 \exp(-2^{n/(1+\delta)})$).

Choose now $\delta > 0$ so that $3(1-\delta)/(1+\delta) > {}^2\log 6$ (i.e. $\delta < 0.074\ldots$) and choose $\delta' > \delta$ so that $\gamma < 1$. Since $E(\nu^{(n)}) \leq C \sum_0^n \gamma^n$ the lemma follows.

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Received June 7, 1971

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