Fatou's theorem for symmetric spaces

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1. Introduction

With a symmetric space of non-compact type there are associated certain spaces called boundaries and to each boundary there corresponds a Poisson integral. A natural question is then to ask for convergence theorems of Fatou's type. Helgason and Korányi [8] have proved »radial» convergence for Poisson integrals of L^∞-functions, and for symmetric spaces of rank one Knapp [9] has proved »radial» convergence for Poisson integrals of measures. These results have been extended in [10], [11], [12] and [13] to convergence with respect to generalized »non-tangential» domains (admissible convergence). The purpose of this paper is to prove Fatou's theorem for L^p-functions, $p_0 < p \leq \infty$, where p_0 depends on the symmetric space and the boundary (Theorem 6.2). This result is still unsatisfactory since $p_0 > 1$ and p_0 tends to infinity with the rank. For the maximal boundary of $SL(l\,;\mathbf{R})/SO(l\,;\mathbf{R})$ we obtain $p_0 = l-2$ (Theorem 7.1).

We now sketch the proof of our general result in the simplest case when the boundary is the maximal one. We represent the symmetric space as G/K, where G is a semisimple Lie group and $K \subset G$ a maximal compact subgroup. Let $G = KAN$ be the Iwasawa decomposition, let θ be the Cartan involution, write $\bar{N} = \theta N$ and let M be the centralizer of A in K. The Poisson integral is defined as an integral over K/M but our first step consists of transferring it to an integral over $\bar{N}$. This leads us to consider the integral

$$F(\bar{n}_0 aK) = \int_{\bar{N}} f(\bar{n}_0 a\bar{n}a^{-1})\,\psi(\bar{n})d\bar{n}\,, \qquad f \in L^p(\bar{N})\,, \tag{1.1}$$

where ψ is a certain Jacobian. Set $M^*f(\bar{n}) = \sup_{a\in A} |F(\bar{n}aK)|$. Fatou's theorem will follow from the estimate

$$\|M^*f\|_p \leq C_p \|f\|_p \quad \text{for} \quad p > p_0. \tag{1.2}$$

To prove this inequality we need an estimate of ψ. In § 5 we shall prove that

$$\psi(\bar{n}) \leq C\,|\bar{n}|^{-2} \quad \text{and} \quad \int_{\bar{N}} \psi(\bar{n})^{\frac{1}{2}+\varepsilon}\, d\bar{n} < \infty \quad (\varepsilon > 0)\,. \tag{1.3}$$

Here $|\cdot|$ is a certain »norm» on $\bar{N}$; the ball of radius R will be denoted by $B(R)$. In the maximal boundary case (1.3) follows from some results of Harish-Chandra [6]. We shall also need a result due to Knapp and Williamson [10] and Korányi [13] which states that a certain maximal function Mf on $\bar{N}$ satisfies $\|Mf\|_p \leq C_p\|f\|_p$, $p > 1$. To prove (1.2) we now split the integral (1.1) into a sum of integrals over the sets A_j where $2^{-j} < \psi(\bar{n}) \leq 2^{-j+1}$, then use Hölder's inequality and (1.3). This gives

$$|F(\bar{n}aK)| \leq \text{Const.} \sum_{j=1}^{\infty} \left((2^{-j+1})^{q-1/2-\varepsilon}\right)^{1/q} (\text{meas } B(2^{j/2}))^{1/p}\, (M(f^p)(\bar{n})\,)^{1/p}\,.$$

When $p > p_0$ the sum is convergent and we obtain

$$M^*f(\bar{n}) \leq C_p\,(M(f^p)(\bar{n})\,)^{1/p}\,, p > p_0\,,$$

from which (1.2) follows. The details are given in § 6.

In § 7 we consider $SL(l\,;\mathbf{R})/SO(l\,;\mathbf{R})$. Using the explicit formula for ψ we shall obtain a better result than Theorem 6.2 by covering the sets A_j with finite unions of rectangles.

I am indebted to A. W. Knapp and E. M. Stein for the value of p_0 in Theorem 7.1. My original value was worse, but during the final preparation of this manuscript they communicated to me that they had obtained Theorem 6.2 for maximal boundaries and Theorem 7.1 some two years ago. Their results were never published, however. This inspired me to rewrite my original proof of Theorem 7.1 by inserting Lemma 3.3 thus being able to get their value of p_0. This also simplified notationally the proof of Theorem 6.1.

I would also like to express my thanks to L. Carleson and S. Helgason for helpful conversations in connection with this paper.

2. Notation

Let G be a connected semisimple Lie group with finite center, K a maximal compact subgroup and $X = G/K$ the corresponding symmetric space. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition and let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. We denote by $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$ and consider the complexifications $\mathfrak{g}^c$ and $\mathfrak{h}^c$ of $\mathfrak{g}$ and $\mathfrak{h}$. The set of non-zero roots of $\mathfrak{g}^c$ with respect to $\mathfrak{h}^c$ will be denoted by Δ and the corresponding root spaces by $\mathfrak{g}^c_\lambda$ $(\lambda \in \Delta)$. The roots are real-valued on $\mathfrak{a} \oplus \mathfrak{h}^+$, where $i\mathfrak{h}^+ = \mathfrak{h} \cap \mathfrak{k}$, so we fix a lexicographic ordering of Δ by choosing a basis in $\mathfrak{a}$ and in $\mathfrak{h}^+$. Denote by Δ^+ the set of positive roots and by Σ the set of simple roots. Let $r\colon (\mathfrak{h}^c)^* \to \mathfrak{a}^*$ (* denotes »the dual of») be the restriction map $\alpha \to \alpha|_{\mathfrak{a}}$ and set $R = r(\Delta)\setminus\{0\}$, $S = r(\Sigma)\setminus\{0\}$. The elements of R are called the restricted roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$. The ordering of Δ induces an ordering of R under which S is the system of simple restricted roots.

For each $E \subset S$ we set

$$\mathfrak{a}(E) = \{H \in \mathfrak{a}\,;\alpha(H) = 0 \quad \forall\, \alpha \in E\} \quad \text{and}$$

$$\mathfrak{a}^+(E) = \{H \in \mathfrak{a}(E)\,;\alpha(H) > 0 \quad \forall\, \alpha \in S\setminus E\}\,.$$

We consider the adjoint representation of $\mathfrak{a}(E)$ on $\mathfrak{g}$ and denote by $R(E)$ the set of all non-zero weights, by $\mathfrak{g}_\alpha(E)$ $(\alpha \in R(E)\,)$ the corresponding weight spaces and by $\mathfrak{g}_0(E)$ the weight space for 0. Then $\mathfrak{g}$ is the direct sum of the weight spaces $\mathfrak{g}_\alpha(E)$, $\alpha \in R(E) \cup \{0\}$, and $\alpha \in R(E)$ if and only if there exists a $\beta \in R$ such that $\alpha = \beta|_{\mathfrak{a}(E)} \neq 0$. Denote by $R^+(E)$ and $R^-(E)$ the set of all positive respectively negative weights. Set

$$\mathfrak{n}(E) = \sum_{\alpha\in R^+(E)} \mathfrak{g}_\alpha(E) \ , \ \bar{\mathfrak{n}}(E) = \sum_{\alpha\in R^-(E)} \mathfrak{g}_\alpha(E) \ ;$$

these are nilpotent subalgebras of $\mathfrak{g}$.

$\mathfrak{g}_0(E)$ is a reductive subalgebra of $\mathfrak{g}$; we denote its semisimple part by $\mathfrak{g}^E$. Set $\mathfrak{k}^E = \mathfrak{g}^E \cap \mathfrak{k}$, $\mathfrak{p}^E = \mathfrak{g}^E \cap \mathfrak{p}$, $\mathfrak{a}^E = \mathfrak{g}^E \cap \mathfrak{a}$ and $\mathfrak{m}(E) = \mathfrak{g}_0(E) \cap \mathfrak{k}$. Then $\mathfrak{g}^E = \mathfrak{k}^E + \mathfrak{p}^E$ is a Cartan decomposition, $\mathfrak{a}^E$ is a maximal abelian subspace of $\mathfrak{p}^E$ and $\mathfrak{k}^E$ is contained in $\mathfrak{m}(E)$ (the centralizer of $\mathfrak{a}(E)$ in $\mathfrak{k}$).

Denote by 2Λ the sum of the roots in Δ^+, by $2\Lambda_E$ the sum of all $\lambda \in \Delta^+$ such that $\lambda|_{\mathfrak{a}(E)} \in R^+(E)$, and put $\varrho = \Lambda|_{\mathfrak{a}}$ and $\varrho_E = \Lambda_E|_{\mathfrak{a}}$.

The analytic subgroups of G corresponding to $\mathfrak{a}$, $\mathfrak{a}(E)$, $\mathfrak{n}(E)$, $\bar{\mathfrak{n}}(E)$, $\mathfrak{k}^E$ and $\mathfrak{a}^E$ will be denoted by A, $A(E)$, $N(E)$, $\bar{N}(E)$, K^E and A^E. Finally, let $M(E)$ be the centralizer of $\mathfrak{a}(E)$ in K, i.e., $M(E) = \{k \in K\,; \mathrm{Ad}\, k(H) = H \quad \forall\, H \in \mathfrak{a}\,(E)\}$. $M(E)$ centralizes $A(E)$, normalizes $N(E)$ and contains K^E.

If $E = \emptyset$ we have $\mathfrak{a}(E) = \mathfrak{a}$, $R(E) = R$, $K^E = A^E = \{e\}$, $\varrho_E = \varrho$, and in this case we shall write $\mathfrak{g}_\alpha$, $\mathfrak{n}$, $\bar{\mathfrak{n}}$, N, $\bar{N}$, M instead of $\mathfrak{g}_\alpha(E)$, $\mathfrak{n}(E)$, $\bar{\mathfrak{n}}(E)$, $N(E)$, $\bar{N}(E)$, $M(E)$.

By the Iwasawa decomposition theorem there exists a uniquely defined continuous function $G \to K \times \mathfrak{a} \times N$, $g \to (\varkappa(g),\ H(g),\ \nu(g)\,)$, such that $g = \varkappa(g)\,(\exp H(g))\,\nu(g)$ for all $g \in G$.

The spaces $K/M(E)$ $(E \subset S)$ are the *boundaries* of X in the sense of Furstenberg [4] und Satake [14]. Denote the cosets $k\,M(E)$ by $\dot{k}$ and the identity coset $\{K\}$ of $X = G/K$ by o. G acts on $K/M(E)$ by $g(\dot{k}) = \widehat{\varkappa(gk)}^{\bullet}$. The *Poisson integral* $P_E f$ of a function $f \in L^1(K/M(E)\,)$ is defined by

$$P_E f(g\cdot o) = \int_{K/M(E)} f(g(\dot{k}))d\mu_E(\dot{k}) \ ,$$

where $d\mu_E$ denotes the normalized K-invariant measure on $K/M(E)$.

3. Semirestricted admissible convergence

We shall consider the behaviour of the Poisson integral $P_E f(x)$ when x tends to a boundary point. Following Korányi [13] we define the following notion of convergence.

Let $C \subset X$ be compact with non-empty interior and invariant under $M(E)$, let $T \in \mathfrak{a}(E)$ and put

$$\mathscr{A}_C^T(\dot{k}) = \{ka\cdot x\,;\, a \in A(E),\ \log a \geq T,\ \ x \in C\}\ ,$$

where $\log a \geq T$ means $\log a - T \in \overline{\mathfrak{a}^+(E)}$. $\mathscr{A}_C^T(\dot{k})$ is called a *truncated semi-restricted admissible domain* at $\dot{k} \in K/M(E)$.

A function F on X is said to converge to the number r at $\dot{k} \in K/M(E)$ *admissibly and semirestrictedly* if for all compact $M(E)$-invariant sets $C \subset X$ with non-empty interior and all $\varepsilon > 0$ there exists a $T \in \mathfrak{a}(E)$ such that $x \in \mathscr{A}_C^T(\dot{k})$ implies $|F(x) - r| < \varepsilon$. We say that F *converges to a function f on $K/M(E)$ admissibly and semirestrictedly a.e.* if F converges to $f(\dot{k})$ in the sense just described at almost all $\dot{k} \in K/M(E)$.

In the case of the maximal boundary, i.e., $E = \emptyset$, semirestricted admissible convergence coincides with *unrestricted admissible convergence* as defined in [11].

By the Bruhat lemma the map $\tau\colon \bar{N}(E) \to K/M(E)$ defined by $\tau(\bar{n}) = \dot{\widehat{\varkappa(\bar{n})}}$ is an injective analytic map of $\bar{N}(E)$ onto an open dense subset of $K/M(E)$ whose complement has measure zero (see [13]). This allows us to transfer the Poisson integral to an integral over $\bar{N}(E)$, i.e., there is a function ψ_E on $\bar{N}(E)$ such that

$$P_E f(g\cdot o) = \int_{\bar{N}(E)} f(g(\tau(\bar{n})))\ \psi_E(\bar{n})\,d\bar{n}\ .$$

If the Haar measure $d\bar{n}$ on $\bar{N}(E)$ is normalized so that $\int_{\bar{N}(E)} e^{-2\varrho_E(H(\bar{n}))}\,d\bar{n} = 1$ then

$$\psi_E(\bar{n}) = e^{-2\varrho_E(H(\bar{n}))} \tag{3.1}$$

(see [11]).

For any $g, h \in G$ we denote by g^h the element hgh^{-1}. If $\bar{n}_0 \in \bar{N}(E)$, $m \in M(E)$

and $a \in A$ we have $\varkappa(\bar{n}_0 ma\varkappa(\bar{n})) = \varkappa(\bar{n}_0\bar{n}^{ma}m)$. It follows that $(\bar{n}_0 ma)\tau(\bar{n}) = \tau(\bar{n}_0\bar{n}^{ma})$. Hence

$$P_E f\,(\bar{n}_0 ma \cdot o) = \int_{\bar{N}(E)} f \circ \tau(\bar{n}_0\bar{n}^{ma})\, \psi_E(\bar{n})\, d\bar{n}\ . \tag{3.2}$$

The map τ amounts to transforming the Poisson integral from the disc to the upper halfplane in the classical case. The main reason for going over to the group $\bar{N}(E)$ is that the action of the group A on the points $\tau(\bar{n})$ is so simple and this makes all computations manageable. We now define the substitute for the non-tangential domains of the halfplane and then state a lemma which connects semi-restricted admissible convergence with these new domains.

For any $\bar{n} \in \bar{N}(E)$, $T \in \mathfrak{a}(E)$ and for any compact sets $U \subset \bar{N}(E)$, $V \subset A^E$, let $\Gamma^T_{U,V}(\bar{n}) = \{\bar{n}a\bar{n}_1 ma' \cdot o\,;\ a \in A(E),\ \ \log a\ \geq T,\ \ \bar{n}_1 \in U,\ m \in K^E,\ a' \in V\}$, $\Gamma_{U,V}(\bar{n}) = \bigcup \{\Gamma^T_{U,V}(\bar{n})\,;\, T \in \mathfrak{a}(E)\}$.

Lemma 3.1. (Korányi [13]). *Let $\bar{n} \in \bar{N}(E)$. A function F on X converges to the number r at $\tau(\bar{n})$ semirestrictedly and admissibly if and only if for all $\varepsilon > 0$ and all compact $U \subset \bar{N}(E)$, $V \subset A^E$ with non-empty interiors there exists $T \in \mathfrak{a}(E)$ such that $x \in \Gamma^T_{U,V}(\bar{n})$ implies $|F(x) - r| < \varepsilon$.*

We shall also need the following result from [13].

Lemma 3.2. *Let $1 \leq p \leq \infty$. If for all f such that $f \circ \tau \in L^p(\bar{N}(E))$ the Poisson integral of f converges admissibly and semirestrictedly a.e. to f, then the same is true for all $f \in L^p(K/M(E))$.*

In the classical case there is no difference between radial convergence and non-tangential convergence of Poisson integrals. We shall now give an analogous result for the general case.

For compact sets $U \subset \bar{N}(E)$, $V \subset A^E$ we define the operators $M^*_{U,V}$ by

$$M^*_{U,V}f(\bar{n}) = \sup_{x \in \Gamma_{U,V}(\bar{n})} |P_E f(x)|, \qquad f \in L^1(K/M(E)\,),\ \bar{n} \in \bar{N}(E).$$

Put $M^* = M^*_{\{e\},\{e\}}$, where e is the identity of G.

Lemma 3.3. *For all compact sets $U \subset \bar{N}(E)$, $V \subset A^E$ there is a constant C such that*

$$M^*_{U,V}f\,(\bar{n}) \leq C \cdot (M^*\,|f|\,(\bar{n})\,)$$

for all $f \in L^1(K/M(E)\,)$ and $\bar{n} \in \bar{N}(E)$.

Proof. Let $\bar{n}, \bar{n}_0 \in \bar{N}(E)$, $a \in A(E)$, $\bar{n}_1 \in U$, $a' \in V$ and $m \in K^E$. Set $\bar{n}_2 = a'^{-1}\, m^{-1}\, \bar{n}_1^{-1}\, m\bar{n}a'$. Since the map $n \to n^{a'}$ has Jacobian $e^{-2\varrho_E(\log a')}$, we obtain from (3.2) after a change of variables:

$$P_E f(\bar{n}_0 a \bar{n}_1 m a' \cdot o) = P_E f(\bar{n}_0 \bar{n}_1^a m a a' \cdot o) = \\ = \int_{\bar{N}(E)} f \circ \tau(\bar{n}_0 \bar{n}^{ma})\, \psi_E(\bar{n}_2)\, e^{2\varrho_E(\log a')}\, d\bar{n} \, .$$

Set $g = a'^{-1} m^{-1} \bar{n}_1^{-1} m$. Then $\bar{n}_2 = g \bar{n} a'$ and it follows that

$$H(\bar{n}_2) = H(g\varkappa(\bar{n})) + H(\bar{n}) + \log a' \, .$$

By (3.1)

$$\psi_E(\bar{n}_2)\, e^{-2\varrho_E(\log a')} = \psi_E(\bar{n})\, e^{-2\varrho_E(H(g\varkappa(\bar{n})))} \, .$$

When $\bar{n}_1, a', m$ and $\bar{n}$ run through U, V, K^E and $\bar{N}(E)$, respectively, $g\varkappa(\bar{n})$ stays in a compact set. It follows that there is a constant C, depending on U and V, only, such that

$$\psi_E(\bar{n}_2)\, e^{2\varrho_E(\log a')} \leq C\, \psi_E(\bar{n}) \, .$$

Hence

$$|P_E f(\bar{n}_0 a \bar{n}_1 m a' \cdot o)| \leq C \int_{\bar{N}(E)} |f \circ \tau(\bar{n}_0 \bar{n}^{ma})|\, \psi_E(\bar{n}) d\bar{n} = C\, (P_E |f|\, (\bar{n}_0 m a \cdot o)) \, .$$

This proves the lemma.

4. A maximal theorem for $\bar{N}(E)$

We choose a basis $X_1, X_2, \ldots, X_r$ in $\bar{\mathfrak{n}}(E)$ such that

(i) for each $1 \leq j \leq r$ there exists an $\alpha_j \in R^+$ such that $X_j \in \mathfrak{g}_{-\alpha_j}$,

(ii) $[X_i, X_j] \in \sum_{k=1}^{j-1} \mathbf{R}\, X_k$ for all i, j.

Let $\mathfrak{n}_j$ be the linear space spanned by $X_1, X_2, \ldots, X_j$. Then $\mathfrak{n}_0 = \{0\}, \mathfrak{n}_1, \mathfrak{n}_2, \ldots, \mathfrak{n}_r = \bar{\mathfrak{n}}(E)$ forms an increasing sequence of nilpotent ideals in $\bar{\mathfrak{n}}(E)$ with $[\bar{\mathfrak{n}}(E), \mathfrak{n}_j] \subset \mathfrak{n}_{j-1}$. We conclude (see [2, p. 513]) that the map

$$\varphi\colon\ (x_1, x_2, \ldots, x_r) \to (\exp x_r X_r) \cdot \ldots \cdot (\exp x_2 X_2)\, (\exp x_1 X_1)$$

is an analytic homeomorphism of $\mathbf{R}^r$ onto $\bar{N}(E)$.

Let $H_0 \in \mathfrak{a}^+(E)$ be the element such that $\alpha(H_0) = 1$ for all $\alpha \in S \setminus E$. We introduce a norm on $\bar{N}(E)$ by putting for $\bar{n} = \varphi(x_1, \ldots, x_r)$

$$|\bar{n}| = \max_{1 \leq j \leq r} |x_j|^{1/\alpha_j(H_0)} \, .$$

For every $R > 0$ we define the sets $B(R) = \{\bar{n} \in \bar{N}(E)\, ; |\bar{n}| < R\}$. This norm has the following properties.

LEMMA 4.1. (i) $|\bar{n}^{\exp tH_0}| = e^{-t}\,|\bar{n}|$ *for all* $t \in \mathbf{R}$, $\bar{n} \in \bar{N}(E)$.
(ii) *There is a constant* C *such that*
$|\bar{n}_1\bar{n}_2| \leq C\,(|\bar{n}_1| + |\bar{n}_2|)$ *for all* $\bar{n}_1, \bar{n}_2 \in \bar{N}(E)$.
(iii) *There is a constant* C *such that*
$|\bar{n}^m| \leq C\,|\bar{n}|$ *for all* $m \in M(E), \bar{n} \in \bar{N}(E)$.
(iv) meas $B(R) = R^{2\varrho_E(H_0)}$ meas $B(1)$ *for all* $R > 0$.

Proof. Let $\bar{n} = \varphi(x_1, \ldots, x_r)$. For $a \in A$ we have

$$\begin{aligned}\bar{n}^a &= (a(\exp x_rX_r)a^{-1})\cdot \ldots \cdot (a(\exp x_2X_2)a^{-1})(a(\exp x_1X_1)a^{-1})\\ &= \varphi(\mathrm{Ad}\,a\,(x_1X_1)\,, \ldots, \mathrm{Ad}\,a\,(x_rX_r)\,)\\ &= \varphi(x_1e^{-\alpha_1(\log a)}, \ldots, x_re^{-\alpha_r(\log a)}).\end{aligned}$$

This gives (i), and (ii) is an easy consequence of (i) (see [11, Lemma 2.3]). The map $(m, \bar{n}) \to \bar{n}^m$, $M(E) \times \bar{N}(E) \to \bar{N}(E)$, maps compact sets onto compact sets. Hence there is a C such that $|\bar{n}| \leq 1$ implies $|\bar{n}^m| \leq C$ for all $m \in M(E)$. If $\bar{n} \in \bar{N}(E)$ is arbitrary we choose t such that $e^t = |\bar{n}|$. Then $|\bar{n}^{\exp tH_0}| = 1$ and $e^{-t}|\bar{n}^m| = |(\bar{n}^m)^{\exp tH_0}| = |(\bar{n}^{\exp tH_0})^m| \leq C$, i.e., $|\bar{n}^m| \leq C\,|\bar{n}|$. This proves (iii). Since $B(e^t) = B(1)^{\exp -tH_0}$, (iv) follows from the fact that for $a \in A$ the map $\bar{n} \to \bar{n}^a$ of $\bar{N}(E)$ onto $\bar{N}(E)$ has Jacobian $e^{-2\varrho_E(\log a)}$.

Let Ω be the family of all sets $\omega = \varphi(I_1 \times I_2 \ldots \times I_r)$ where $I_j \subset \mathbf{R}$ are open symmetric intervals around 0. We note that $B(R)^a \in \Omega$ for each $a \in A$ and each $R > 0$. For $f \in L^1_{\mathrm{loc}}(\bar{N}(E)\,)$ we define the maximal function Mf by

$$Mf(\bar{n}) = \sup_{\omega \in \Omega} \frac{1}{\mathrm{meas}(\omega)} \int_{\omega} |f(\bar{n}\,\bar{n}')|\,d\bar{n}', \quad \bar{n} \in \bar{N}(E)\,.$$

The following maximal theorem is a special case of Theorem 3.1 in [13] (cf. also [10]).

THEOREM 4.2. *For each* $p > 1$ *there exists a constant* C_p *such that*

$$\|Mf\|_p \leq C_p\|f\|_p \quad \textit{for all} \quad f \in L^p(\bar{N}(E)\,).$$

The proof in [13] gives the estimate $C_p = O\,(\,(p-1)^{-r})$ as $p \to 1$.
We shall also need the maximal operator M' defined by

$$M'f\,(\bar{n}) = \sup_{R>0} \frac{1}{\mathrm{meas}\,B(R)} \int_{B(R)} |f(\bar{n}\,\bar{n}')|\,d\bar{n}', \quad \bar{n} \in \bar{N}(E).$$

It is well known that M' is of weak type (1,1) (see e.g. [3]).

5. The behaviour of ψ_E at infinity

In this section we shall prove some results on the behaviour of ψ_E at infinity. In the case $E = \emptyset$ they are essentially due to Harish-Chandra [6], and the extensions to the general case are straight-forward. We first recall some facts about representations (see e.g. [15, Ch. VII] for details).

Let σ be an irreducible representation of $\mathfrak{g}^c$ on a finite-dimensional vector space V. Since the symmetric space X is uniquely determined by $\mathfrak{g}$ we may assume that G is imbedded in the complex simply connected Lie group that corresponds to the Lie algebra $\mathfrak{g}^c$. The representation $\sigma|_{\mathfrak{g}}$ then lifts to a homomorphism $\tilde{\sigma}$: $G \to SL(V)$ given by $\tilde{\sigma}(\exp X) = e^{\sigma(X)}$, $X \in \mathfrak{g}$. It is possible to introduce an inner product on V so that $\tilde{\sigma}(k)$ becomes unitary for each $k \in K$ and $\tilde{\sigma}(a)$ becomes self-adjoint for each $a \in A$. Vectors belonging to different weight spaces of $\tilde{\sigma}(a)$ are then orthogonal. If ω denotes the highest weight of σ and ξ is a unit vector belonging to the corresponding weight space then, using the Iwasawa decomposition, we obtain

$$|\tilde{\sigma}(g)\xi| = e^{\omega(H(g))} \quad \text{for all} \quad g \in G,$$

because $\sigma(X)\xi = 0$ for all $X \in \mathfrak{n}$, $\sigma(H)\xi = \omega(H)\xi$ for all $H \in \mathfrak{a}$ and $|\tilde{\sigma}(k)\xi| = |\xi|$ for all $k \in K$. Let for each $\alpha \in \Delta$ the vector $H_\alpha \in [\mathfrak{g}^c_\alpha, \mathfrak{g}^c_{-\alpha}]$ be determined by $\alpha(H_\alpha) = 2$. There exists an irreducible finite-dimensional representation of highest weight ω if and only if $\omega(H_\alpha)$ is a non-negative integer for each $\alpha \in \Sigma$.

For the purpose of this section we introduce the following notation.

$$\Delta^{++}(E) = \{\lambda \in \Delta^+;\ \lambda|_{\mathfrak{a}(E)} \in R^+(E)\},\ \Delta^{+0}(E) = \{\lambda \in \Delta^+;\ \lambda|_{\mathfrak{a}(E)} = 0\},$$

$$\Sigma(E) = \Sigma \cap \Delta^{++}(E),\quad \Sigma^0(E) = \Sigma \cap \Delta^{+0}(E).$$

Obviously, $\Delta^{+0}(E) = \Delta^+ \setminus \Delta^{++}(E)$ and $\Sigma^0(E) = \Sigma \setminus \Sigma(E)$.

LEMMA 5.1. *There exist irreducible finite-dimensional representations of highest weight Λ and $2\Lambda_E$ for each $E \subset S$.*

Proof. Denote by s_α ($\alpha \in \Sigma$) the Weyl symmetry given by $s_\alpha\lambda = \lambda - \lambda(H_\alpha)\alpha$. s_α leaves $\Delta^+ \setminus \{\alpha\}$ invariant whereas $s_\alpha\alpha = -\alpha$. Hence $s_\alpha\Lambda = \Lambda - \alpha$, which implies $-\Lambda(H_\alpha) = s_\alpha\Lambda(H_\alpha) = \Lambda(H_\alpha) - \alpha(H_\alpha)$, i.e., $\Lambda(H_\alpha) = 1$.

Suppose $\alpha \in \Sigma^0(E)$. Then $s_\alpha\lambda|_{\mathfrak{a}(E)} = \lambda|_{\mathfrak{a}(E)}$ for each $\lambda \in \Delta$. Consequently, s_α leaves the set $\Delta^{++}(E)$ invariant. Hence $s_\alpha\Lambda_E = \Lambda_E$, i.e., $\Lambda_E(H_\alpha) = 0$.

Suppose $\alpha \in \Sigma(E)$. If $\lambda \in \Delta^{+0}(E)$ then $\lambda = \sum_{\beta \in \Sigma^0(E)} n_\beta\,\beta$ with non-negative integers n_β, and since $\beta(H_\alpha)$ is a non-positive integer if $\alpha, \beta \in \Sigma$, $\alpha \neq \beta$, we obtain $s_\alpha\lambda = \lambda + m_\lambda\alpha$ where m_λ is a non-negative integer. Hence $s_\alpha(\Lambda - \Lambda_E) = (\Lambda - \Lambda_E) + \frac{1}{2}\left(\sum_{\lambda \in \Delta^{+0}(E)} m_\lambda\right)\alpha$, which yields $2\Lambda_E(H_\alpha) = 2\Lambda(H_\alpha) + \sum m_\lambda = 2 + \sum m_\lambda$. Thus $2\Lambda_E(H_\alpha)$ is a positive integer in this case.

LEMMA 5.2. *Let* ω *be the highest weight of an irreducible finite-dimensional representation* σ. *Then:*
(i) $\omega(H(\bar{n})) \geq 0$ *for all* $\bar{n} \in \bar{N}(E)$.
(ii) *If* $\omega(H_\alpha) > 0$ *for all* $\alpha \in \Sigma(E)$, *then there is a constant* $\delta > 0$ *such that*

$$e^{\omega(H(\bar{n}))} \geq \delta\, |\bar{n}| \text{ for all } \bar{n} \in \bar{N}(E).$$

Proof. Let ξ be a unit vector belonging to the weight ω. Let $\lambda_1 < \lambda_2 < \ldots < \lambda_r$ be the elements in $\Delta^{++}(E)$; then $\bar{\mathfrak{n}}(E) = \sum_{j=1}^{r} (\mathfrak{g}^c_{-\lambda_j} \cap \mathfrak{g})$. Fix a basis $X_1, \ldots, X_r$ in $\bar{\mathfrak{n}}(E)$ by choosing $X_j \in \mathfrak{g}^c_{-\lambda_j} \cap \mathfrak{g}$; then each $\bar{n} \in \bar{N}(E)$ can be written $\bar{n} = (\exp x_1X_1) \cdot \ldots \cdot (\exp x_rX_r)$ and

$$\begin{aligned}\tilde{\sigma}(\bar{n})\xi &= \Big(\prod_{j=1}^{r} e^{x_j\sigma(X_j)}\Big)\xi = \\ &= \sum_{j_1,\ldots,j_r=0}^{\infty} \frac{1}{j_1!} \cdot \ldots \cdot \frac{1}{j_r!}\, x_1^{j_1} \cdot \ldots \cdot x_r^{j_r}\, \sigma(X_1)^{j_1} \cdot \ldots \cdot \sigma(X_r)^{j_r}\, \xi\,,\end{aligned} \tag{5.1}$$

where the sum is actually finite.

Let P_k, $0 \leq k \leq r$, denote the orthogonal projection of V onto the weight space corresponding to the weight $\omega - \lambda_k$, where we put $\lambda_0 = 0$. From (5.1) we get

$$P_0(\tilde{\sigma}(\bar{n})\xi) = \xi \text{ and}$$

$$P_k(\tilde{\sigma}(\bar{n})\xi) = \sum \frac{1}{j_1!} \cdot \ldots \cdot \frac{1}{j_r!}\, x_1^{j_1} \cdot \ldots \cdot x_r^{j_r}\, \sigma(X_1)^{j_1} \cdot \ldots \cdot \sigma(X_r)^{j_r}\, \xi \quad (1 \leq k \leq r)\,,$$

where the sum is taken over all r-tuples $(j_1, \ldots, j_r)$ such that $\sum_{n=1}^{r} j_n\lambda_n = \lambda_k$; from the ordering of the roots it follows that $j_{k+1} = \ldots = j_r = 0$ and $j_k = 0$ or 1. Obviously,

$$e^{\omega(H(\bar{n}))} = |\tilde{\sigma}(\bar{n})\xi| \geq |P_k(\tilde{\sigma}(\bar{n})\xi)|\,.$$

For $k = 0$ we obtain $e^{\omega(H(\bar{n}))} \geq |\xi| = 1$, and this proves (i).

The norm $|\cdot|$ on $\bar{N}(E)$ defined in § 4 depends on the choice of basis elements in $\bar{\mathfrak{n}}(E)$ but different choices give rise to equivalent norms. (This follows easily from Lemma 4.1 (i).) Therefore, we may assume that the basis is the one chosen above and then Lemma 5.2 (ii) amounts to proving

$$e^{\lambda_j(H_0)\omega(H(\bar{n}))} \geq \delta\, |x_j|\,,\ 1 \leq j \leq r\,. \tag{5.2}$$

We first note that $\sigma(X_j)\xi \neq 0$ for each j. Indeed, the assumption about $\omega(H_\alpha)$ implies that $\omega(H_\lambda) > 0$ for all $\lambda \in \Delta^{++}(E)$ so if we choose $Y_j \in \mathfrak{g}^c_{\lambda_j}$ such that $[Y_j, X_j] = H_{\lambda_j}$ then

$$\sigma(Y_j)\sigma(X_j)\xi = \sigma(H_{\lambda_j})\xi + \sigma(X_j)\sigma(Y_j)\xi = \omega(H_{\lambda_j})\xi \neq 0\,.$$

We now prove (5.2) by induction. Assume (5.2) proved for $j < k$. Then

$$|P_k(\tilde{\sigma}(\bar{n})\xi)| \geq |x_k \sigma(X_k)\xi| - \left|\sum \frac{1}{j_1!} \cdot \ldots \cdot \frac{1}{j_{k-1}!} x_1^{j_1} \cdot \ldots \cdot x_{k-1}^{j_{k-1}} \sigma(X_1)^{j_1} \ldots \sigma(X_{k-1})^{j_{k-1}} \xi\right|$$
$$\geq \text{Const.}\ |x_k| - \text{Const.} \sum |x_1|^{j_1} \cdot \ldots \cdot |x_{k-1}|^{j_{k-1}}$$
$$\geq \text{Const.}\ |x_k| - \text{Const.} \sum e^{(j_1\lambda_1(H_0) + \cdots + j_{k-1}\lambda_{k-1}(H_0))\omega(H(\bar{n}))},$$

where all sums are taken over the $(k-1)$-tuples $(j_1, \ldots, j_{k-1})$ such that $\sum_{n=1}^{k-1} j_n \lambda_n = \lambda_k$. Hence

$$|P_k(\tilde{\sigma}(\bar{n})\xi)| \geq \text{Const.}\ |x_k| - \text{Const.}\ e^{\lambda_k(H_0)\omega(H(\bar{n}))}$$

and it follows that

$$|x_k| \leq \text{Const.}\ (e^{\omega(H(\bar{n}))} + e^{\lambda_k(H_0)\omega(H(\bar{n}))}) \leq \text{Const.}\ e^{\lambda_k(H_0)\omega(H(\bar{n}))},$$

and this completes the proof of (5.2).

LEMMA 5.3. (i) $0 < \psi_E(\bar{n}) \leq 1$ *for all* $\bar{n} \in \bar{N}(E)$.
(ii) *There is a constant* C *such that*

$$\psi_E(\bar{n}) \leq C\, |\bar{n}|^{-1} \quad \textit{for all} \quad \bar{n} \in \bar{N}(E)$$

and

$$\psi(\bar{n}) \leq C\, |\bar{n}|^{-2} \quad \textit{for all} \quad \bar{n} \in \bar{N}.$$

Proof. Lemma 5.3 is nothing but Lemma 5.2 with $\omega = \Lambda$ and $\omega = 2\Lambda_E$; it follows from the proof of Lemma 5.1 that the hypothesis of Lemma 5.2 (ii) is then fulfilled.

Remark 1. Lemma 5.2 (i) occurs in [6] as Lemmas 2, 35 and 43. For $E = \emptyset$ Lemma 5.3 (ii) can also be deduced from Lemma 40 and an inequality on p. 290 in [6].

Remark 2. For symmetric spaces of rank one Helgason [7] has obtained an explicit formula for ψ from which it follows that $\psi(\bar{n}) \leq C\, |\bar{n}|^{-4\varrho(H_0)}$. However, in general the estimate for ψ of Lemma 5.3 is best possible as can be seen by considering $SL(3;\mathbf{R})/SO(3;\mathbf{R})$.

LEMMA 5.4. *For each* $E \subset S$ *there is a constant* $\gamma_E < 1$ *such that*

$$\int_{\bar{N}(E)} (\psi_E(\bar{n}))^\gamma \, d\bar{n} < \infty \quad \textit{if} \quad \gamma > \gamma_E.$$

When $E = \emptyset$, $\gamma_E = \frac{1}{2}$.

Proof. Obviously, there is an open halfspace Q in $\mathfrak{a}$ which is bounded by a hyperplane passing through $0 \in \mathfrak{a}$ and such that $\{\alpha \in R^+;\ \alpha|_{\mathfrak{a}(E)} \in R^+(E)\} = \{\alpha \in R^+;\ H_\alpha \in Q\}$ ($= R_Q^+$ say). Therefore, by a theorem of Gindikin and Karpelevič [5], if ν is a real-valued linear functional on $\mathfrak{a}$ then

$$I(\nu) = \int_{\bar N(E)} e^{-(\nu+\varrho)(H(\bar n))}\, d\bar n < \infty$$

if and only if $\nu(H_\alpha) > 0$ for all $\alpha \in R_Q^+$. Now consider $\nu_\gamma = 2\gamma\varrho_E - \varrho$. For $\gamma = 1$, $I(\nu_1) = \int \psi_E(\bar n)\, d\bar n = 1$. Hence $\nu_1(H_\alpha) > 0$ for all $\alpha \in R_Q^+$. By continuity there exists a $\gamma_E < 1$ such that $\nu_\gamma(H_\alpha) > 0$ for all $\alpha \in R_Q^+$ and $\gamma > \gamma_E$. Hence $I(\nu_\gamma) < \infty$ for $\gamma > \gamma_E$. When $E = \emptyset$ we can take $\gamma_E = \frac{1}{2}$ since $\varrho(H_\alpha) \geq 1$ for all $\alpha \in R^+$.

When $E = \emptyset$ another proof of the above lemma can be found in [6, Lemma 45].

6. The boundary behaviour of Poisson integrals

Let $H_0 \in \mathfrak{a}^+(E)$ be the element such that $\alpha(H_0) = 1$ for all $\alpha \in S \setminus E$ and set

$$p_E = \begin{cases} 2\,\varrho(H_0) - 1\,, & \text{if } E = \emptyset \\ \dfrac{2\,\varrho_E(H_0) - \gamma_E}{1 - \gamma_E}\,, & \text{otherwise,} \end{cases}$$

where γ_E is the constant of Lemma 5.4.

THEOREM 6.1. *Let $U \subset \bar N(E)$ and $V \subset A^E$ be compact. For each $p > p_E$ there exists a constant C_p (depending on U and V) such that*

$$\|M^*_{U,V} f\|_p \leq C_p\, \|f \circ \tau\|_p$$

for all $f \in L^1(K/M(E))$.

Proof. In view of Lemma 3.3 it suffices to prove the estimate of Theorem 6.1 with M^* instead of $M^*_{U,V}$. Assume $f \circ \tau \in L^p(\bar N(E))$, otherwise there is nothing to prove. Set $A_j = \{\bar n \in \bar N(E);\ 2^{-j} < \psi_E(\bar n) \leq 2^{-j+1}\}$, $j = 1, 2, \ldots$. Choose p' such that $p_E < p' < p$ and let $1/q' + 1/p' = 1$. Let $\bar n_0 \in \bar N(E)$, $a \in A(E)$, $m \in K^E$; by (3.2)

$$|P_E f(\bar n_0 am \cdot o)| \leq \int_{\bar N(E)} |f \circ \tau(\bar n_0 \bar n^{ma})|\, \psi_E(\bar n)\, d\bar n = \sum_{j=1}^{\infty} \int_{A_j} \leq$$

$$\leq \sum_{j=1}^{\infty} \left[\int_{A_j} |f \circ \tau(\bar n_0 \bar n^{ma})|^{p'}\, d\bar n\right]^{1/p'} \left[\int_{A_j} (\psi_E(\bar n))^{q'}\, d\bar n\right]^{1/q'}.$$

By Lemma 5.4 there is, for each $\gamma > \gamma_E$, a constant C_γ such that

$$\int_{A_j} (\psi_E(\bar n))^{q'}\, d\bar n \leq (2^{-j+1})^{q'-\gamma} \int_{\bar N(E)} (\psi_E(\bar n))^{\gamma}\, d\bar n = C_\gamma\, (2^{-j+1})^{q'-\gamma}.$$

By a change of variables we obtain

$$\int_{A_j} |f \circ \tau\,(\bar{n}_0\bar{n}^{ma})|^{p'}\,d\bar{n} = e^{2\varrho_E(\log a)} \int_{A_j^{ma}} |f \circ \tau\,(\bar{n}_0\bar{n})|^{p'}\,d\bar{n}\,. \tag{6.1}$$

By Lemma 5.3, $A_j \subset B(C2^{j\delta_E})$ where $\delta_E = \frac{1}{2}$ if $E = \emptyset$ and $\delta^E = 1$ otherwise. Using Lemma 4.1 (iii) we then see that there is a constant C_1 such that $A_j^m \subset B(C_1\,2^{j\delta_E})$, $= B_j$ say, for all $m \in K^E$. Thus the right hand side of (6.1) is majorized by

$$e^{2\varrho_E(\log a)} \int_{B_j^a} |f \circ \tau\,(\bar{n}_0\bar{n})|^{p'}\,d\bar{n} \leq$$

$$\leq e^{2\varrho_E(\log a)}\,\text{meas}\,(B_j^a)\,M(f \circ \tau)^{p'}(\bar{n}_0) =$$

$$= \text{meas}\,(B_j)\,M(f \circ \tau)^{p'}(\bar{n}_0) \leq C_2\,2^{2j\delta_E\varrho_E(H_0)}\,M(f \circ \tau)^{p'}(\bar{n}_0)\,.$$

We conclude that

$$M^*f\,(\bar{n}_0) \leq 2\,C_\gamma\,C_2\,(\sum_{j=1}^{\infty} 2^{-\eta j})\,(M(f \circ \tau)^{p'}(\bar{n}_0)\,)^{1/p'},$$

where $\eta = 1 - \gamma/q' - 2\,\delta_E\varrho_E(H_0)/p'$. Since $p' > p_E$ we can choose $\gamma > \gamma_E$ so that the sum is convergent. Hence

$$M^*f\,(\bar{n}_0) \leq C_3\,(M(f \circ \tau)^{p'}(\bar{n}_0)\,)^{1/p'}.$$

Applying Theorem 4.2 we obtain

$$\|M^*f\|_p \leq C_3\|\,(M(f \circ \tau)^{p'})^{1/p'}\|_p = C_3\,(\|M(f \circ \tau)^{p'}\|_{p/p'})^{1/p'} \leq$$

$$\leq C_4\,(\|(f \circ \tau)^{p'}\|_{p/p'})^{1/p'} = C_4\,\|f \circ \tau\|_p\,.$$

This finishes the proof.

THEOREM 6.2. *If $p > p_E$ and $f \in L^p(K/M(E))$ then the Poisson integral of f converges admissibly and semirestrictedly to f a.e.*

Proof. By virtue of Lemma 3.2 it is enough to consider the case where $f \circ \tau \in L^p(\bar{N}(E))$ and since the theorem holds for continuous functions, this case follows from Theorem 6.1 and Lemma 3.1 by classical methods (cf. [9]).

Remark. For symmetric spaces of *rank one* we use Remark 2 following Lemma 5.3 instead of Lemma 5.3. This gives the estimate

$$M^*f(\bar{n}) \leq \text{Const.}\cdot(M'(f \circ \tau)(\bar{n}))\,.$$

Since M' is of weak type (1,1), it follows that M^* is of weak type (1,1) and we conclude that Fatou's theorem holds for $p \geq 1$ in this case. (This proof is due to Korányi [12].)

Even for spaces of arbitrary rank we can sometimes get sharper results than Theorems 6.1 and 6.2 by looking at the explicit formula for ψ_E. We shall illustrate this with $X = SL(l;\mathbf{R})/SO(l;\mathbf{R})$ in the next section.

7. SL(l ; R)/SO(l ; R)

We shall first consider the Poisson integral corresponding to the maximal boundary of $SL(l;\mathbf{R})/SO(l;\mathbf{R})$. $\bar{N}$ consists then of all lower triangular matrices with units in the diagonal

$$\bar{n} = (x_{ij})_{i,j=1}^{l}, \; x_{ii} = 1 \text{ and } x_{ij} = 0 \text{ if } i < j.$$

This parametrization of $\bar{N}$ is in accordance with the decomposition of $\bar{N}$ in § 4. Let $\xi_1, \xi_2, \ldots, \xi_l$ denote the column vectors of $\bar{n}$ and let D_j, $1 \leq j \leq l$, be the Gram determinant formed by the vectors $\xi_1, \xi_2, \ldots, \xi_j$, i.e.,

$$D_j = \det \begin{pmatrix} (\xi_1, \xi_1) & \ldots & (\xi_1, \xi_j) \\ \vdots & & \vdots \\ (\xi_j, \xi_1) & \ldots & (\xi_j, \xi_j) \end{pmatrix}.$$

Then

$$\psi(\bar{n}) = (D_1 D_2 \cdot \ldots \cdot D_{l-1})^{-1}$$

(cf. [1]). From the interpretation of $\sqrt{D_j}$ as the volume of the parallelepiped spanned by $\xi_1, \xi_2, \ldots, \xi_j$ it follows easily that $D_j \geq |\xi_j|^2 = 1 + x_{j+1,j}^2 + \ldots + x_{l,j}^2$. Hence

$$\psi(\bar{n}) \leq (|\xi_1|\,|\xi_2| \cdot \ldots \cdot |\xi_{l-1}|)^{-2},$$

and we conclude that $\psi(\bar{n}) > R^{-2}$ implies the inequalities

$$\begin{cases} |x_{ij}| < R\,, \quad 1 \leq j < i \leq l\,, \\ |\prod_{j=1}^{l-k} x_{j+k,j}| < R\,, \quad k = 1, 2, \ldots, l-1\,. \end{cases} \tag{7.1}$$

Let $\varepsilon > 0$ be given and fix a natural number $s > l^2/\varepsilon$. Let $\mathscr{A}$ be the finite collection of all tuples $\alpha = (\alpha_{ij})$, where the α_{ij} are chosen from among the numbers $1/s, 2/s, \ldots, 1$ and such that $\sum_{j=1}^{l-k} \alpha_{j+k,j} = 1 + (l-k-1)/s$, $k = 1, 2, \ldots, l-1$. Let $\omega(\alpha, R) \subset \bar{N}$ be the subset given by $|x_{ij}| \leq R^{\alpha_{ij}}$. It follows from (7.1) that

$$\{\bar{n} \in \bar{N}\,;\, \psi(\bar{n}) > R^{-2}\} \subset \bigcup_{\alpha \in \mathscr{A}} \omega(\alpha, R)\,,$$

and, obviously, meas $\omega(\alpha, R) \leq R^{l-1+\varepsilon}$. Thus, if

$$A_j = \{\bar{n} \in \bar{N};\, 2^{-j} < \psi(\bar{n}) \leq 2^{-j+1}\}\,, \text{ then}$$

$$\int_{A_j} |f \circ \tau\, (\bar{n}_0 \bar{n}^a)|^p\, d\bar{n} \leq \sum_{\alpha \in \mathscr{A}} \int_{\omega(\alpha, 2^{j/2})} |f \circ \tau\, (\bar{n}_0 \bar{n}^a)|^p\, d\bar{n} \leq$$

$$\leq (\text{card}\, \mathscr{A})\, (2^{j/2})^{l-1+\varepsilon}\, M(f \circ \tau)^p(\bar{n}_0)\,.$$

Using this we conclude as in § 6 that for every $\gamma > \frac{1}{2}$ there is a constant $C_{\gamma,\varepsilon}$ such that

$$|Pf(\bar{n}_0 a \cdot o)| \leq C_{\gamma,\varepsilon}\, (\sum_{j=1}^{\infty} (2^{-j})^{1-\gamma/q-(l-1+\varepsilon)/2p})\, (M(f \circ \tau)^p(\bar{n}_0)\,)^{1/p}$$

$(1/p + 1/q = 1)$. If $p > l - 2$ we can choose ε and γ so that the sum is convergent, i.e., there is a constant C_p such that $M^*f(\bar{n}_0) \leq C_p\, (M(f \circ \tau)^p(\bar{n}_0)\,)^{1/p}$. By Theorem 4.2 it follows that M^* is of strong type (p, p) for $p > l - 2$. Lemmas 3.1, 3.2 and 3.3 now give the following

THEOREM 7.1. *For the maximal boundary of $SL(l;\mathbf{R})/SO(l;\mathbf{R})$ the Poisson integral of an L^p-function converges unrestrictedly and admissibly a.e. whenever* $p > l - 2$.

Remark. For $SL(3;\mathbf{R})/SO(3;\mathbf{R})$ the above proof gives the estimate $\|M^*f\|_p \leq C_p\, \|f \circ \tau\|_p$ with $C_p = O\,(\,(p-1)^{-5})$ as $p \to 1$. This allows us to conclude that Fatou's theorem holds for functions belonging to the class $L(\log^+ L)^5$ (cf. [16, Ch. XII]).

We shall finally consider the $l - 1$ boundaries of $SL(l;\mathbf{R})/SO(l;\mathbf{R})$ which correspond to those sets E for which $S \setminus E$ consists of one element.

THEOREM 7.2. *For these boundaries Fatou's theorem holds for L^1-functions* (*and for measures*) (*semirestricted admissible convergence*).

Proof. Take for instance the set E for which $\bar{N}(E)$ consists of the matrices $\bar{n}$ with column vectors $\xi_1 = (1, x_1, x_2\,, \ldots, x_{l-1})$, $\xi_2 = (0, 1, 0, \ldots, 0)$, $\xi_3 = (0, 0, 1, \ldots, 0)\,, \ldots, \xi_l = (0, 0, 0, \ldots, 1)$. Then $|\bar{n}| = \max_{1 \leq j \leq l-1} |x_j|$, $\varrho_E(H_0) = (l-1)/2$ and $\psi_E(\bar{n}) = |\xi_1|^{-l} = (1 + x_1^2 + \ldots + x_{l-1}^2)^{-l/2}$. Hence $\psi_E(\bar{n}) \leq |\bar{n}|^{-l}$ and it follows easily that

$$M^*f(\bar{n}) \leq \text{Const.} \left(M'(f \circ \tau)\, (\bar{n})\,\right) \quad \text{for all } \bar{n} \in \bar{N}(E).$$

Thus M^* is of weak type (1, 1) and this proves the assertion for L^1-functions. (For measures cf. [9].)

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