The regularity of growth of entire functions whose zeros are hyperplanes

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Let $f(z)$ be an entire function (of n variables) of finite order ϱ and normal type σ. We then define $h_r(z) = \overline{\lim}_{r\to\infty} r^{-\varrho} \ln|f(rz)|$, $r > 0$ (resp. $h_c(z) = \overline{\lim}_{|u|\to\infty} |u|^{-\varrho} \ln|f(uz)|$, $u \in \mathbf{C}$) and the smallest upper-semicontinuous majorant $h_r^*(z) = \overline{\lim}_{z'\to z} h_r(z')$ (resp. $h_c^*(z) = \overline{\lim}_{z'\to z} h_c(z)$). This is plurisubharmonic and satisfies the condition $h_r^*(tz) = t^\varrho h_r^*(z)$, $t > 0$, (resp. $h_c^*(uz) = |u|^\varrho h_c^*(z)$, $u \in \mathbf{C}$); it is called the radial (resp. circular) indicator function of f.

For $n = 1$, the function $h_r(z)$ is continuous, and so $h_r^*(z) = h_r(z)$ (see [4] or Lemma 1 below), but this is no longer necessarily the case for either $h_r^*(z)$ or $h_c^*(z)$ for $n \geq 2$, [3]. In [1], we undertook a study of the relationship between the distribution of the zeros of $f(z)$ and the local continuity of the function $h_r^*(z)$. We investigate here a condition on the zeros which implies the global continuity of $h_r^*(z)$.

If the function $f(z)$ as a function of several variables depends only upon a single variable, say z_1, and $f(0) \neq 0$, then $h_r^*(z) = h_r(z)$ and the two are continuous. The zeros are then presented by hyperplanes parallel to the hyperplane $z_1 = 0$. We generalize this result in the following way:

THEOREM. *Let $f(z)$ be an entire function of order ϱ and normal type σ such that $f(0) \neq 0$ and the zeros of $f(z)$ are hyperplanes. Then $h_r^*(z) = h_r(z)$ and there are constants T (depending only on σ and ϱ) and α (depending only on ϱ) such that $|h_r(w) - h_r(w')| \leq T\|w - w'\|^\alpha$ for $\|w\| = \|w'\| = 1$. In particular, $h_r^*(z)$ is continuous.*

Remark 1. We will assume, without loss of generality, that we use the Euclidean norm. The value of T depends upon the choice of the norm, but α is independent of the norm chosen.

Remark 2. If $f(z)$ is a function of only one of the variables, then elementary considerations show that the exponent we get is $\min(1, \varrho)$. The α we construct can be chosen to be $\alpha > \min(1/3, \varrho/(2\varrho + 1)) - \gamma$ for any $\gamma > 0$.

Remark 3. The function $h_r^*(z)$ does not depend on what point in $\mathbf{C}^n$ we choose as origin [2], so the assumption that $f(0) \neq 0$ does not effect the conclusion that $h_r^*(z)$ is continuous and satisfies the Lipschitz condition above; but it does of course effect the fact that $h_r(z) = h_r^*(z)$.

The proof will be established by classical methods pertaining to functions of a single complex variable. We shall first need the following results:

LEMMA 1. *Let $f(u)$ be an entire function of a single complex variable and $h_r(u)$ its indicator function. Then there exists a constant K_0 (depending on ϱ and σ) such that*

$$|h(e^{i\theta}) - h(e^{i\theta'})| \leq K_0 |e^{i\theta} - e^{i\theta'}| .$$

Proof. Let $K = \max |(\varrho/2)\, h(e^{i\theta}) \sec^2 (\varrho q)/2|$ with $|\theta_2 - \theta_1| \leq q < \pi/\varrho$ and $\theta_1 < \theta < \theta_2$. Then [4, p. 54]

$$\frac{h(e^{i\theta}) - h(e^{i\theta_1})}{\sin \varrho(\theta - \theta_1)} \leq \frac{h(e^{i\theta_2}) - h(e^{i\theta_1})}{\sin \varrho(\theta_2 - \theta_1)} + K(\theta_2 - \theta) .$$

A similar inequality exists for θ and θ_2. Choosing $|\theta_2 - \theta_1| \leq \pi/4\varrho$, we get the desired result. Q.E.D.

LEMMA 2. *Let $f(u)$ be holomorphic in the circle $|u| \leq 2eR$, u complex, with $f(0) = 1$, and let η be an arbitrary positive number not exceeding $3e/2$. Then inside the circle $|u| \leq R$ but outside a family of excluded circles the sum of whose radii is not greater than $4\eta R$, posing $M(R) = \max_{|u|=R} |f(u)|$, we have*

$$\ln |f(u)| > -\left(2 + \ln \frac{3e}{2\eta}\right) \ln M(2eR) .$$

Proof. The proof is to be found in Levin [4, p. 21].

LEMMA 3. *Let the function $f(u)$ be of order ϱ and type σ. Then there exists δ_0 (depending only on ϱ and σ) such that for each choice of the positive numbers δ and ω (with $\delta \leq \delta_0$ and $0 < \omega < 1$), there corresponds on each fixed ray $\arg u = \theta$ a sequence of intervals $r_n \leq r \leq (1 + \delta) r_n$ $(r_n \to \infty)$ on each of which, for suitable constants T_1 and T_2 (depending only on ϱ and σ) the inequality*

$$\frac{\ln |f(re^{i\theta})|}{r^{\varrho}} > \left[h(e^{i\theta}) - T_1\delta - T_2\delta\left(2 + \ln \frac{2e}{\omega}\right)(1 + 2e\delta)^{\varrho}\right] = h(e^{i\theta}) - g(\delta, \omega) \qquad (1)$$

is satisfied except perhaps on a set of measure not exceeding $\omega\delta r_n$.

Proof. Without loss of generality, we may assume $\theta = 0$. There is a sequence of $r_n \to \infty$ such that $\ln |f(r_n)| > [h(1) - \delta]r_n^{\varrho}$. Assume $\delta < 1/2e$. There exists an R_δ such that for $r \geq R_\delta$, $\ln |f(re^{i\phi})| < [h(e^{i\phi}) + \delta]r^{\varrho}$ [4, p. 71].

By Lemma 1, for $|\phi| \leq \sin^{-1}(2e\delta) \leq K'\delta$, $\ln |f(re^{i\phi})| < [h(1) + (K_0K' + 1)\delta]r^{\varrho}$. Let $\psi_n(u) = f(r_n + u)/f(r_n)$. Then $\psi_n(0) = 1$ and for $|u| \leq 2e\delta r_n$,

$$\ln |\psi_n(u)| \leq (K_0K' + 2)\delta(r_n + |u|)^{\varrho}.$$

Applying Lemma 2, we see that for $|u| \leq \delta r_n$,

$$\ln |\psi_n(u)| > -(K_0K' + 2)\delta\big(2 + \ln (2e/\omega)\big)(r_n + 2e\delta r_n)^{\varrho}$$

outside exceptional circles the sum of whose radii is less than $\omega\delta r_n/2$. Returning to the function $f(u)$, wee see that asymptotically

$$\ln |f(r)| > [h(1) - \delta - (K_0K' + 2)\delta\big(2 + \ln (2e/\omega)\big)(1 + 2e\delta)^{\varrho}]r_n^{\varrho} \qquad (2)$$

is satisfied for $(1 - \delta)r_n \leq r \leq (1 + \delta)r_n$ except perhaps for intervals the sum of whose lengths is less than $\omega\delta r_n$. Since $f(u)$ is of type σ, for δ sufficiently small (depending only on σ),

$$\sigma\left[1 - \frac{1}{(1 + \delta)^{\varrho}}\right] < (\varrho + 1)\sigma\delta$$

and hence

$$\frac{\ln |f(r)|}{r^{\varrho}} > [h(1) - \sigma\delta(\varrho + 1) - \delta - (K_0K' + 2)\delta\big(2 + \ln (2e/\omega)\big)(1 + 2e\delta)^{\varrho}]$$

holds wherever (2) holds. Q.E.D.

LEMMA 4. *If* $f(u)$ *is holomorphic in the circle* $|u| \leq er$ *with* $f(0) = 1$ *and if* $n(r)$ *is the number of zeros of* $f(u)$ *of modulus less than* r, *then* $n(r) \leq M(er)$.

Proof. This is an easy consequence of Jensen's formula (cf. [4, p. 15]).

LEMMA 5 (Cartan estimate). *Given any number* $H > 0$ *and complex numbers* $a_1, \ldots, a_N$, *there is a system of circles in the complex plane the sum of whose radii is* $2H$ *such that for all* u *lying outside these circles,* $\prod_{i=1}^{N} |u - a_i| \geq (H/e)^N$.

Proof. See [4, p. 19].

LEMMA 6 (Carathéodory inequality for the circle). *If* $f(u)$ *is any function holomorphic on the circle* $|u| \leq R$ *and*

$$A(r) = \max_{|u|=r} \operatorname{Re} f(u), \quad \textit{then} \quad M(r) \leq [A(R) - \operatorname{Re} f(0)] \frac{2r}{R-r} + |f(0)| \qquad (r < R)\,.$$

Proof. See [4, p. 17].

Proof of theorem. The proof, which is quite long, will be divided into several parts.

(i) Let $\zeta > 2$ be some fixed number to be specified later and let $\varepsilon = \|w - w'\|$ be so small that

$$\varepsilon^{1/\zeta} < \min\left(\frac{1}{12}\,,\ \delta_0\right) \tag{3}$$

where Lemma 3 is satisfied for $\delta \leq \delta_0$. Then by Lemma 3, by choosing $\delta = \omega = \varepsilon^{1/\zeta}$, we can find a sequence $r_n \to \infty$ such that $h_r(w) - g(\varepsilon^{1/\zeta}, \varepsilon^{1/\zeta}) \leq r^{-\varrho} \ln |f(rw)|$ for $r_n \leq r \leq (1 + \varepsilon^{1/\zeta})r_n$ except perhaps on a set of measure at most $\varepsilon^{2/\zeta} r_n$. Thus, for r_n sufficiently large, we have for $r_n \leq r \leq (1 + \varepsilon^{1/\zeta})r_n$

$$h_r(w) - h_r(w') - g(\varepsilon^{1/\zeta}, \varepsilon^{1/\zeta}) - \varepsilon \leq \frac{\ln |f(rw)|}{r^\varrho} - \frac{\ln |f(rw')|}{r^\varrho} \tag{4}$$

except perhaps for a set of measure at most $\varepsilon^{2/\zeta} r_n$.

(ii) Since $f(z)$ is of type σ, there is a constant $C \geq 1$ such that

$$|f(z)| \leq C \exp{(\sigma + \tfrac{1}{2})\|z\|^\varrho}.$$

For $\|\xi\| = 1$, we define the functions $n_\xi(r)$ to be the number of zeros of $f(u\xi)$ for $|u| < r$. By Lemma 4, $n_\xi(r) \leq \ln M(er) \leq \ln C + (\sigma + \frac{1}{2})e^\varrho r^\varrho \leq (\sigma + 1)e^\varrho r^\varrho$ for r sufficiently large. In what follows, we shall always assume that r_n is so large that this inequality holds for $r \geq r_n$.

In the complex line (uw'), we construct concentric circles C_{in}, centered at the origin, of radial increment $6\varepsilon^{2/\zeta} r_n$, with the radius of C_{0n} being r_n and all the radii less than or equal to $(1 + \varepsilon^{1/\xi})r_n$. This defines a set of annuli, and at least one of the annuli will not contain "too many" zeros of the function $f(uw')$. The number of annuli is $\left[\dfrac{\varepsilon^{1/\zeta} r_n}{6\varepsilon^{2/\zeta} r_n}\right]$ (where [] means "greatest integer in"), and since by (3), $\dfrac{1}{6\varepsilon^{1/\zeta}} > 2$, we have $\left[\dfrac{1}{6\varepsilon^{1/\zeta}}\right] > \dfrac{1}{12\varepsilon^{1/\zeta}}$. Since there are at most $(\sigma + 1)e^\varrho(1 + \varepsilon^{1/\zeta})^\varrho r_n^\varrho$ zeros inside the circle $(1 + \varepsilon^{1/\zeta})r_n$, at least one of the annuli has no more than $(\sigma + 1)e^\varrho(1 + \varepsilon^{1/\zeta})^\varrho r_n^\varrho 12\varepsilon^{1/\zeta} < 12(\sigma + 1)(2er_n)^\varrho/\varepsilon^{1/\zeta} = T_3\varepsilon^{1/\zeta} r_n^\varrho$ zeros of the function $f(uw')$. We shall select one such annulus and designate it Ω_n.

(iii) Let the zeros of $f(z)$ be the hyperplanes $(1 - \sum_{i=1}^N c_{im} z_i)$ and let $A_m =$

$\sum_{i=1}^{N} c_{im} w_i'$. Let $\Pi'(1 - A_m u)$ be the product over those indices m for which $f(uw')$ has a zero in Ω_n (at most $T_3 \varepsilon^{1/\zeta} r_n^{\varrho}$). By Lemma 5, there is a set of circles the sum of whose radii is $\varepsilon^{2/\zeta} r_n$ such that for all n lying outside these circles, $\Pi' |u - 1/A_m| \geq (\varepsilon^{2/\zeta} r_n / 2e)^{\lambda}$ (where λ is the number of zeros). Since $1/|A_m| \leq r_n(1 + \varepsilon^{1/\zeta}) \leq 2r_n$, for u lying outside these circles

$$|\Pi'(1 - A_m u)| = \Pi' |A_m| \Pi' \left| u - \frac{1}{A_m} \right| \geq \left(\frac{\varepsilon^{2/\zeta}}{4e} \right)^{\lambda} \geq \left(\frac{\varepsilon^{2/\zeta}}{4e} \right)^{T_3 \varepsilon^{1/\zeta} r_n^{\varrho}} . \tag{5}$$

Thus, we can find an r, $r_n \leq r \leq r_n(1 + \varepsilon^{1/\zeta}) r_n$ for which $rw' \in \Omega_n$ such that (4) and (5) hold simultaneously and such that the circle Q_n centered at r (in the complex line (uw')) of radius $\varepsilon^{2/\zeta} r_n$ is contained in Ω_n.

Let us now consider such an r.

(iv) We have $f(rw) = f(rw' + (w - w')r)$. Let $\phi(u) = f(rw' + u(w - w')/\varepsilon)$. Then $\phi(\varepsilon r) = f(rw)$ and $\phi(0) = f(rw')$, and $\phi(u)$ is a holomorphic function of the single complex variable u, so $\ln |\phi(u)|$ is subharmonic; thus

$$\ln |f(rw)| - \ln |f(rw')| = \ln |\phi(\varepsilon r)| - \ln |\phi(0)| \leq \max_{|u| = \varepsilon r} \{\ln |\phi(u)| - \ln |\phi(0)|\} . \tag{6}$$

We also have

$$|\phi(u)| = \left| \phi\left(rw' + u \frac{(w - w')}{\varepsilon} \right) \right| \leq C \exp\left(\sigma + \frac{1}{2} \right) 2^{\varrho} r^{\varrho} \tag{7}$$

for $|u| \leq r$.

(v) Let $D_m = \sum_{i=1}^{N} c_{im}(w_i - w_i')/\varepsilon$ and let $H(u) = \Pi''(1 - A_m r - D_m u)/(1 - A_m r)$, where the product Π'' is taken over all indices for which $(1 - A_m r - D_m u)$ has a zero for $|u| \leq \varepsilon^{2/\zeta} r$. If $f_0(v) = f(v(w - w')/\varepsilon)$ is the function f restricted to the complex line $(v(w - w'))$, then the numbers D_m are just the reciprocals of the zeros of $f_0(v)$. We shall use this fact to get estimates on the numbers D_m. We assume, without loss of generality, that the subscripts are so arranged that $|D_m| \leq |D_{m'}|$ for $m \geq m'$.

Let $\psi(u) = \phi(u)/H(u)$. Then $\psi(u)$ has no zeros in $|u| \leq \varepsilon^{2/\zeta} r$; hence, for $|u| \leq \varepsilon^{2/\zeta} r$, it can be written $\psi(u) = \exp \mu(u)$, where $\mu(u)$ is holomorphic in $|u| \leq \varepsilon^{2/\zeta} r$. We have

$$\max_{|u| = \varepsilon^{2/\zeta} r} |\psi(u)| \leq \max_{|u| = r} |\psi(u)| \leq \max_{|u| = r} \frac{1}{|H(u)|} \max_{|u| = r} |\phi(u)| .$$

Since $(1 - A_m r - D_m u)$ has a zero for $|u| \leq \varepsilon^{2/\zeta} r$, $|1 - A_m r| - |D_m| \varepsilon^{2/\zeta} r \leq 0$ or $1/\varepsilon^{2/\zeta} r \leq |D_m|/|1 - A_m r|$, so $|1 - D_m u/(1 - A_m r)| \geq |D_m| r/|1 - A_m r| - 1 \geq r/\varepsilon^{2/\zeta} r - 1 \geq 1$ for $|u| = r$ by (3). Thus $\max_{|u| = \varepsilon^{2/\zeta} r} |\psi(u)| \leq \max_{|u| = r} |\phi(u)| \leq \exp(\sigma + 1) 2^{\varrho} r^{\varrho}$ by (7). Since $\ln |\psi(u)| = \operatorname{Re} \mu(u)$ for $|u| \leq \varepsilon^{2/\zeta} r$, we have by Lemma 6,

$$\max_{|u|=\varepsilon r}\{\ln|\psi(u)| - \ln|\psi(0)|\} \leq A_{\mu-\mu(0)}(\varepsilon^{2/\zeta}r)\,\frac{2\varepsilon r}{\varepsilon^{2/\zeta}r - \varepsilon r} \leq \tag{8}$$
$$\leq \frac{4(\sigma+1)2^{\varrho}r^{\varrho}\varepsilon^{1-2/\zeta}}{(1-\varepsilon^{1-2/\zeta})} = T_4 r^{\varrho}\varepsilon^{1-2/\zeta}\,.$$

(vi) Since $\phi(u) = H(u)\psi(u)$,

$$\max_{|u|=\varepsilon r}\ln|\phi(u)| \leq \max_{|u|=\varepsilon r}\ln|H(u)| + \max_{|u|=\varepsilon r}\ln|\psi(u)|$$

and since $H(0) = 1$,

$$\max_{|u|=\varepsilon r}\ln|\phi(u)| - \ln|\phi(0)| \leq \max_{|u|=\varepsilon r}\ln|\psi(u)| - \ln|\psi(0)| + \max_{|u|=\varepsilon r}\ln|H(u)|\,. \tag{9}$$

It remains to estimate $\max_{|u|=\varepsilon r}\ln|H(u)| = \max_{|u|=\varepsilon r}\ln|\Pi''(1-A_m r-D_m u)/(1-A_m r)|$. If m is such that $(1-A_m u)$ has a zero in Ω_n, we have by (5)

$$|\Pi'''(1-A_m r)| \geq \left(\frac{\varepsilon^{2/\zeta}}{4e}\right)^{T_3\varepsilon^{1/\zeta}r_n^{\varrho}}$$

(where Π''' is taken over all indices in Π'' for which this is true). Hence

$$\ln\left|\frac{1}{\Pi'''(1-A_m r)}\right| \leq T_3\varepsilon^{1/\zeta}r_n^{\varrho}\ln\left(\frac{4e}{\varepsilon^{2/\zeta}}\right) \leq T_3\varepsilon^{1/\zeta}r^{\varrho}\ln\left(\frac{4e}{\varepsilon^{2/\zeta}}\right)\,. \tag{10}$$

Since $(1-A_m r-D_m u)$ has a zero in $|u| \leq \varepsilon^{2/\zeta}r$, say at q_m,

$$|(1-A_m r-D_m u)| = |D_m||u-q_m| \leq 2\varepsilon^{2/\zeta}r|D_m| \leq (1+2\varepsilon^{2/\zeta}r|D_m|)$$

and

$$\ln|\Pi'''(1-A_m r-D_m u)| \leq \sum{}''' \ln(1+2\varepsilon^{2/\zeta}r|D_m|)\,.$$

For all other m in Π'', either $1/|A_m| \leq r-\varepsilon^{2/\zeta}r_n$ or $1/|A_m| \geq r+\varepsilon^{2/\zeta}r_n$.

In the first case, $|1-A_m r| \geq |A_m|r - 1 \geq \varepsilon^{2/\zeta}r_n|A_m|$ and since

$$\frac{1}{|A_m|} \leq r_n(1+\varepsilon^{1/\zeta}) \leq 2r_n,\quad |1-A_m r| \geq \frac{\varepsilon^{2/\zeta}}{2}\,.$$

In the second case, if $|A_m| \leq 1/2r_n$

$$|1-A_m r| \geq 1-|A_m|r \geq 1-\frac{r}{2r_n} \geq 1-\frac{(1+\varepsilon^{1/\zeta})}{2} \geq \frac{\varepsilon^{2/\zeta}}{2}$$

by (3) and if $|A_m| \geq 1/2r_n$,

$$|1-A_m r| \geq 1-r|A_m| \geq \varepsilon^{2/\zeta}r_n|A_m| \geq \frac{\varepsilon^{2/\zeta}}{2}\,.$$

In any case, for $|u| \leq \varepsilon r$

$$\ln\left|\frac{(1 - A_m r - D_m u)}{(1 - A_m r)}\right| \leq \ln\left|1 + \frac{\varepsilon|D_m|r}{|1 - A_m r|}\right| \leq \ln(1 + 2\varepsilon^{1-2/\zeta}|D_m|r)$$

for these m.

By Lemma 4, there are at most $(\sigma + 1)e^{\varrho}2^{\varrho}r^{\varrho}$ values of m for which $1/|A_m| \leq 2r$ and at most $(\sigma + 1)e^{\varrho}2^{\varrho}r^{\varrho}$ values of m for which $1/|D_m| \leq 2r$. If $|A_m| \leq 1/2r$ and $|D_m| \leq 1/2r$,

$$|1 - A_m r - D_m u| \geq 1 - |A_m|r - |D_m|\varepsilon^{2/\zeta}r > 0 \quad \text{for} \quad |u| \leq \varepsilon^{2/\zeta}r ,$$

so there are at most $2(\sigma + 1)e^{\varrho}2^{\varrho}r^{\varrho} = T_5 r^{\varrho}$ values of m such that $(1 - A_m r - D_m u)$ has a zero for $|u| \leq \varepsilon^{2/\zeta}r$. Hence

$$\max_{|u|=\varepsilon r} \ln|H(u)| \leq T_3\varepsilon^{1/\zeta}r^{\varrho}\ln\left(\frac{4e}{\varepsilon^{2/\zeta}}\right) + \sum_{m=1}^{[T_5 r^{\varrho}]}\ln(1 + 2\varepsilon^{1-2/\zeta}r|D_m|) . \tag{11}$$

(vii) Let $A(r) = \sum_{m=1}^{[T_5 r^{\varrho}]}\ln(1 + 2\varepsilon^{1-2/\zeta}r|D_m|)$. We now estimate this sum. We choose m_0 so large that $\ln C < m/2$ for $m \geq m_0$. Since $1/|D_m|$, $m = 1, 2, \ldots$ represents the zeros of $f_0(v)$, we have by Lemma 4

$$\ln C + (\sigma + \tfrac{1}{2})\, e^{\varrho}\, \frac{1}{|D_m|^{\varrho}} \geq m$$

and for $m \geq m_0$,
$$(\sigma + \tfrac{1}{2})e^{\varrho}\, \frac{1}{|D_m|^{\varrho}} \geq \frac{m}{2}$$

or
$$[2(\sigma + \tfrac{1}{2})e^{\varrho}]^{1/\varrho}\, m^{-1/\varrho} = T_6 m^{-1/\varrho} \geq |D_m| .$$

Then

$$A(r) \leq \sum_{m=1}^{m_0}\ln(1 + 2\varepsilon^{1-2/\zeta}|D_m|r) + \sum_{m=m_0+1}^{[T_5 r^{\varrho}]}\ln(1 + 2\varepsilon^{1-2/\zeta}|D_m|r) \leq o(r^{\varrho}) + A_1(r) , \tag{12}$$

where

$$A_1(r) = \int_{m_0}^{T_5 r^{\varrho}}\ln(1 + 2\varepsilon^{1-2/\zeta}x^{-1/\varrho}T_6 r)dx .$$

Let $y = rx^{-1/\varrho}$. Integrating by parts, we have

$$A_1(r) = r^{\varrho}\int_{rm_0^{-1/\varrho}}^{T_5^{-1/\varrho}}\ln(1 + 2\varepsilon^{1-2/\zeta}T_6 y)d(y^{-\varrho})$$

$$= r^{\varrho}\left\{\ln(1 + 2\varepsilon^{1-2/\zeta}T_6 y)\cdot y^{-\varrho}\right]_{rm_0^{-1/\varrho}}^{T_5^{-1/\varrho}}\Big\} + r^{\varrho}\int_{T_5^{-1/\varrho}}^{rm_0^{-1/\varrho}}\frac{2\varepsilon^{1-2/\zeta}T_6 y^{-\varrho}}{(1 + 2\varepsilon^{1-2/\zeta}T_6 y)}\,dy$$

and since $\ln(1 + 2\varepsilon^{1-2/\zeta}T_6T_5^{-1/\varrho}) \leq 2\varepsilon^{1-2/\zeta}T_6T_5^{-1/\varrho}$, we have

$$A(r) \leq r^{\varrho}2\varepsilon^{1-2/\zeta}T_6T_5^{\frac{\varrho-1}{\varrho}} + r^{\varrho}\int_{T_5^{-1/\varrho}}^{rm_0^{-1/\varrho}} \frac{2\varepsilon^{1-2/\zeta}T_6y^{-\varrho}}{(1 + 2\varepsilon^{1-2/\zeta}T_6y)}\,dy + o(r^{\varrho})\,. \tag{13}$$

For $\varrho < 1$,

$$\begin{aligned}\int_{T_5^{-1/\varrho}}^{rm_0^{-1/\varrho}} \frac{2\varepsilon^{1-2/\zeta}T_6y^{-\varrho}}{(1 + 2\varepsilon^{1-2/\zeta}T_6y)}\,dy &\leq \int_0^{\infty} \frac{2\varepsilon^{1-2/\zeta}T_6y^{-\varrho}}{(1 + 2\varepsilon^{1-2/\zeta}T_6y)}\,dy \\ &\leq \varepsilon^{\varrho(1-2/\zeta)}(2T_6)^{\varrho}\int_0^{\infty}\frac{w^{-\varrho}}{(1+w)}\,dw \leq \varepsilon^{\varrho(1-2/\zeta)}(2T_6)^{\varrho}\pi \operatorname{cosec} \varrho\pi\end{aligned} \tag{14}$$

since $\int_0^{\infty} w^{-\varrho}/(1+w)dw = \pi \operatorname{cosec} \varrho\pi$ for $\varrho < 1$.

For $\varrho = 1$,

$$\begin{aligned}\int_{T_5^{-1/\varrho}}^{rm_0^{-1/\varrho}} \frac{2\varepsilon^{1-2/\zeta}T_6}{y(1 + 2\varepsilon^{1-2/\zeta}T_6y)}\,dy &= 2\varepsilon^{1-2/\zeta}T_6\int_{T_5^{-1/\varrho}}^{rm_0^{-1/\varrho}}\left\{\frac{1}{y} - \frac{2\varepsilon^{1-2/\zeta}T_6}{(1 + 2\varepsilon^{1-2/\zeta}T_6y)}\right\}dy \\ &= 2\varepsilon^{1-2/\zeta}T_6\left\{\ln\left(\frac{y}{1 + 2\varepsilon^{1-2/\zeta}T_6y}\right)\Bigg]_{T_5^{-1/\varrho}}^{rm_0^{-1/\varrho}}\right\} \\ = 2\varepsilon^{1-2/\zeta}T_6&\left\{\ln\left(\frac{rm_0^{-1/\varrho}}{1 + 2\varepsilon^{1-2/\zeta}T_6rm_0^{-1/\varrho}}\right) - \ln\left(\frac{T_5^{-1/\varrho}}{1 + 2\varepsilon^{1-2/\zeta}T_6T_5^{-1/\varrho}}\right)\right\} \\ \leq 2\varepsilon^{1-2/\zeta}T_6&\left\{\ln^+\left(\frac{1}{3\varepsilon^{1-2/\zeta}T_6}\right) + \frac{1}{\varrho}\ln T_5 + \ln(1 + 2\varepsilon^{1-2/\zeta}T_6T_5^{-1/\varrho})\right\}\end{aligned} \tag{14'}$$

for r sufficiently large.

For $\varrho > 1$,

$$\begin{aligned}\int_{T_5^{-1/\varrho}}^{rm_0^{-1/\varrho}} \frac{2\varepsilon^{1-2/\zeta}T_6}{(1 + 2\varepsilon^{1-2/\zeta}T_6y)y^{\varrho}}\,dy &\leq 2\varepsilon^{1-2/\zeta}T_6\int_{T_5^{-1/\varrho}}^{rm_0^{-1/\varrho}}\frac{dy}{y^{\varrho}} \leq 2\varepsilon^{1-2/\zeta}T_6\left\{\left[\frac{y^{-\varrho+1}}{1-\varrho}\right]_{T_5^{-1/\varrho}}^{rm_0^{-1/\varrho}}\right\} \\ &\leq \frac{2\varepsilon^{1-2/\zeta}}{\varrho - 1}T_6\{T_5^{\frac{\varrho-1}{\varrho}} - r^{1-\varrho}m_0^{\frac{\varrho-1}{\varrho}}\} \leq \frac{2\varepsilon^{1-2/\zeta}}{\varrho-1}T_6T_5^{\frac{1-\varrho}{\varrho}}\,.\end{aligned} \tag{14''}$$

By collecting the estimates (4), (6), (8), (9), (11), (12), (13), and (14), (14′), or (14″) (as the case may be), we have

$$h_r(w) - h_r(w') \leq k(\varepsilon)\,,$$

where $k(\varepsilon)$ involves terms in $\varepsilon^{1/\zeta}$, $\varepsilon^{1-2/\zeta}$ and for $\varrho < 1$, $\varepsilon^{\varrho(1-2/\zeta)}$ (times logarithmic terms). Thus, for $\varrho \geq 1$, we choose $\zeta = 3$, and for $\varrho < 1$, we choose $\zeta = 2 + 1/\varrho$. Then

$$h_r(w) - h_r(w') \leq T\varepsilon^\beta \ln \frac{1}{\varepsilon}$$

where

$$\beta = \min\left(\frac{1}{3}, \frac{\varrho}{2\varrho + 1}\right).$$

By reversing the roles of w and w', we get

$$|h_r(w) - h_r(w')| \leq T\|w - w'\|^{\beta-\gamma} \text{ for any } \gamma > 0 .$$

Q.E.D.

COROLLARY. *Under the same hypotheses as in the theorem, we have*

$$h_c^*(z) = h_c(z)$$

and

$$|h_c(w) - h_c(w')| \leq T\|w - w'\|^\alpha \text{ for } \|w\| = \|w'\| = 1 .$$

Proof. $h_c(z) = \sup_\theta h(ze^{i\theta})$ [3, p. 288].

One is interested to ask what kind of a function can have a non-continuous indicator. It is clear, at any rate, that such a function cannot be constructed by taking the product of functions depending on one variable.

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