The asymptotic distribution of the eigenvalues of a degenerate elliptic operator

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1. Introduction

Let R be a Riemannian manifold of dimension $n > 1$ and class C^2, let $\varphi \in C^2(R)$ be real and such that $\varphi = 0 \Rightarrow \operatorname{grad} \varphi \neq 0$ and such that $\varphi \geq 0$ defines a compact part R_φ of R. Let $\Sigma g_{jk} dx^j dx^k$ be the metric of R and $dV = g^{\frac{1}{2}} dx$ ($g = \det(g_{jk})$) its volume element. Let $L^2(R_\varphi)$ be the real Hilbert space on R_φ with norm square $\int_{R_\varphi} u^2 dV$. Let us interpret the degenerate differential operator

$$\Delta_\varphi = -\sum g^{-\frac{1}{2}} \partial_j \varphi g^{\frac{1}{2}} g^{jk} \partial_k, \quad \partial_j = \partial / \partial x^j \quad (g^{jk}) = (g_{jk})^{-1}$$

as the Friedrichs extension associated with the two quadratic forms

$$a(u) = \int_{R_\varphi} \varphi \sum g^{jk} \partial_j u \partial_k u dV, \quad b(u) = \int_{R_\varphi} u^2 dV$$

and the real space $C^1(R_\varphi)$. According to Baouendi and Goulaouic [1], $A = \Delta_\varphi$ is a non-negative selfadjoint operator on $L^2(R_\varphi)$ and $(I + A)^{-1}$ is compact. Let $\{\lambda_j\}_0^\infty$ be the eigenvalues of A associated with a complete set of eigenfunctions and let $N(\lambda)$ be the number of those eigenvalues which are $\leq \lambda$. We are going to give an asymptotic formula for $N(\lambda)$ as $\lambda \to \infty$. Let dv be the volume element on $S = \partial R_\varphi$ with respect to the induced metric and let $\partial / \partial \nu$ be the unit interior derivative on S. Let ω_n be the volume of the unit ball in R^n and put

$$c_{n-1} = (2\pi)^{1-n} \omega_{n-1} \int_S (\partial \varphi / \partial \nu)^{(1-n)/2} dv \,. \tag{1}$$

Finally, let

$$d_2 = c_1/4 \quad \text{and} \quad d_n = (n-1)c_{n-1} \int_1^\infty [(t+1)/2]t^{-n}dt \quad \text{when} \quad n > 2\,, \tag{2}$$

where $[x]$ is the greatest integer $\leq x$. Then we have the following theorem which generalizes earlier results by Baouendi and Goulaouic [1] and N. Shimakura [4]. The first two authors obtain only the order of growth of $N(\lambda)$, while Shimakura, who considers a case where the eigenvalues are known explicitly, does not have the correct factor d_n when $n > 2$.

THEOREM. *When* $\lambda \to \infty$, *then*

$$n = 2 \Rightarrow N(\lambda) \sim d_2 \lambda \log \lambda, \qquad n > 2 \Rightarrow N(\lambda) \sim d_n \lambda^{n-1}\,.$$

Here and throughout the paper, the sign $\sim$ means that the quotient of the two sides tends to 1 as λ increases to ∞.

Note. It follows easily from the proof that this result holds also, if Δ_φ is replaced by $\Delta_\varphi + \psi$, where ψ is a real function, bounded on R_φ.

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2. Quadratic forms and the Weyl-Courant principle

To simplify the notations we now put $R = R_\varphi$ and consider R as a Riemannian manifold with boundary $S = \partial R_\varphi$. Then $0 \leq \varphi \in C^2(R)$, $\varphi = 0$ only on S and $\varphi_\nu = \partial\varphi/\partial\nu$ is positive and continuous on S. By definition

$$A = \Delta_\varphi = -\sum g^{-\frac{1}{2}} \partial_j \varphi g^{\frac{1}{2}} g^{jk} \partial_k$$

is the Friedrichs extension associated with the two quadratic forms

$$a(u) = \int_R \varphi \sum g^{jk} \partial_j u \partial_k u dV, \qquad b(u) = \int_R u^2 dV$$

and the class $C^1(R)$. Let $H^k = H^k(R)$ $(0 \leq k \leq 2)$ be the space of all functions whose derivatives of order $\leq k$ are square integrable over R, topologized in the obvious way. According to Baouendi and Goulaouic, ([1], Théorème 1^{bis}), $A + I$ is a topological isomorphism between the space of all $f \in H^1$ such that $\varphi f \in H^2$ and the space H^0. In particular, there is a constant C such that

$$\int_R (u^2 + \sum g^{jk} \partial_j u \partial_k u) dV \leq C \int_R ((A + I)u)^2 dV$$

whenever u is in the domain of A. Since the imbedding of H^1 into H^0 is compact this shows that A has discrete spectrum.

When V is a subset of R, let $C_0^1(V)$ denote the space of all real continuously differentiable functions with compact supports in V. Note that if V is open then $C_0^1(V)$ consists of all elements of $C_0^1(\bar{V})$ that vanish close to the boundary of V and that $C_0^1(V)$ increases when V is open and increases. Put

$$\lambda_k(a/b, V) = \sup_{\mathcal{L}\subset D} \inf_{0\neq u\in\mathcal{L}} a(u)/b(u) , \tag{3}$$

where $D = C_0^1(V)$ and $\mathcal{L}$ ranges over all linear subspaces of D of codimension $k-1$. By the Weyl-Courant principle, $\{\lambda_k(a/b, R)\}_1^\infty$ are all the eigenvalues of A with the correct multiplicities. Also, every λ_k increases if a/b increases or if V is open and decreases. The function $N(a/b, V) = N(a/b, V, \lambda)$ which counts the number of solutions j of the inequality $\lambda_j(a/b, V) \leq \lambda$ then has the opposite properties. It is also well known that

$$N(R_1) + N(R_2) \leq N(\mathring{R}) \leq N(R) \leq N(\bar{R}_1) + N(\bar{R}_2) \tag{4}$$

where we have left out the arguments a/b and λ and R_1, R_2 are disjoint open subsets of R such that $R = \bar{R}_1 \cup \bar{R}_2$. We shall use these properties of the counting function to get successive reductions of our problem.

3. Reduction to a boundary strip

Close to S we may parametrise R as follows. To every x there is a geodesic $l = l(x)$, passing through x and normal to S. Let $y \in S$ be the point where l reaches S and let t be the geodesic distance from x to y. Then t, y are C^2-functions of x and can be used as coordinates. We notice in passing that in these coordinates, the metric is

$$dt^2 + \sum_2^n g_{jk}(t, y)dy^j dy^k$$

and $g_{jk}(0, y) = \gamma_{jk}(y)$ is the metric induced on S. Let $\varepsilon > 0$ be small and consider the boundary strip $R_\varepsilon : 0 < t < \varepsilon$ and its open complement $R_\varepsilon^* : t > \varepsilon$. By (4) we have

$$N(R_\varepsilon) + N(R_\varepsilon^*) \leq N(R) \leq N(\bar{R}_\varepsilon) + N(\bar{R}_\varepsilon^*) .$$

The result we want to prove is that

$$N(R) = N(a/b, R, \lambda) \sim \sigma_n(\lambda), \quad \lambda \to \infty ,$$

where $\sigma_2(\lambda) = d_2\lambda \log \lambda$ and $\sigma_n(\lambda) = d_n\lambda^{n-1}$ when $n > 2$, the constants d_n being given by (2). Now it is well known that

$$N(\bar{R}_\varepsilon^*) = O_\varepsilon(\lambda^{n/2}), \quad \lambda \to \infty ,$$

so that it suffices to prove that

$$\underline{\lim}\ \sigma_n(\lambda)^{-1}N(a/b, R_\varepsilon, \lambda) \quad \text{and} \quad \overline{\lim}\ \sigma_n(\lambda)^{-1}N(a/b, \bar{R}_\varepsilon, \lambda) \tag{5}$$

are both arbitrarily close to 1 when ε is small. In the next step we shall replace the quotient a/b by another one where the variables t, y are separated.

4. Separation of variables

Let us now put

$$a_1(u) = \int_{R_\varepsilon} t\varphi_\nu((\partial u/\partial t)^2 + \sum_2^n \gamma^{jk}(y)\partial_j u \partial_k u)\gamma^{\frac{1}{2}} dydt$$

and

$$b_1(u) = \int_{R_\varepsilon} u^2 \gamma^{\frac{1}{2}} dydt ,$$

where $y_2, \ldots, y_n$ are coordinates on S and $\varphi_\nu = \partial\varphi/\partial\nu$. Since

$$\varphi(x(t, y)) = t\varphi_\nu(y)(1 + O(t))$$

and

$$\sum_2^n g^{jk}(t, y)\partial_j u \partial_k u = (1 + O(t)) \sum_2^n \gamma^{jk}(y)\partial_j u \partial_k u, \quad g^{\frac{1}{2}} = \gamma^{\frac{1}{2}}(1 + O(t)) ,$$

it is obvious that

$$N(a/b, \bar{R}_\varepsilon, \lambda) \leq N(a_1/b_1, \bar{R}_\varepsilon, \lambda(1 + o(\varepsilon))) ,$$
$$N(a/b, R_\varepsilon, \lambda) \geq N(a_1/b_1, R_\varepsilon, \lambda(1 - o(\varepsilon))) .$$

Hence it suffices to show that, for every $\varepsilon > 0$,

$$N(a_1/b_1, T, \lambda) \sim \sigma_n(\lambda), \quad \lambda \to \infty, \quad T = R_\varepsilon \text{ or } \bar{R}_\varepsilon \tag{6}$$

In fact, this implies (5). Next, let us introduce the function $w = u\sqrt{\varphi_\nu}$ instead of u. Then

$$a_2(w) = a_1(w/\sqrt{\varphi_\nu}) = \int_{R_\varepsilon} t((\partial w/\partial t)^2 + \sum_2^n \varphi_\nu \gamma^{jk}(\partial_j w/\sqrt{\varphi_\nu})(\partial_k w/\sqrt{\varphi_\nu}))\gamma^{\frac{1}{2}} dydt ,$$

$$b_2(w) = b_1(w/\sqrt{\varphi_\nu}) = \int_{R_\varepsilon} w^2 \varphi_\nu^{-1} \gamma^{\frac{1}{2}} dydt ,$$

where the first equations are definitions. We now have a true separation of variables and we can rewrite (6) as

$$N(a_2/b_2, T, \lambda) \sim \sigma_n(\lambda), \quad \lambda \to \infty, \quad T = R_\varepsilon \text{ or } \bar{R}_\varepsilon . \tag{7}$$

5. The spectrum of a second order selfadjoint elliptic operator on S

Let

$$a_0(w) = \int_S \varphi_\nu \sum \gamma^{jk} \partial_j(w/\sqrt{\varphi_\nu}) \partial_k(w/\sqrt{\varphi_\nu}) dv ,$$

$$b_0(w) = \int_S w^2 \varphi_\nu^{-1} dv$$

be the forms on S that correspond to a_2, b_2. It is well known that the Friedrichs extension corresponding to the forms a_0, b_0 and the class $C^2(S)$ is the operator

$$A_0 w = - \sum (\varphi_\nu/\gamma)^{\frac{1}{2}} \partial_j \gamma^{\frac{1}{2}} \varphi_\nu \gamma^{jk} \partial_k(w/\sqrt{\varphi_\nu}) ,$$

which has the property that

$$a_0(w_1, w_2) = b_0(A_0 w_1, w_2) ,$$

where $a_0(.\,,.)$ and $b_0(.\,,.)$ are the bilinear forms associated with the forms a_0 and b_0. Moreover, $A_0 \geq 0$ is selfadjoint and has a discrete spectrum, the lowest eigenvalue being 0 and the corresponding eigenfunction $w = \sqrt{\varphi_\nu}$. Let $\{h_k\}_0^\infty$ with eigenvalues $\{\mu_k\}_0^\infty$ be a complete orthonormal set of eigenvalues and eigenfunctions of A_0, and let $N_0(\mu) = N(a_0/b_0, S, \mu)$ be the corresponding counting function. It is wellknown (cf. e.g. Hörmander [3]), that[1]

$$N_0(\mu) \sim c_{n-1} \mu^{(n-1)/2}, \quad \mu \to \infty , \tag{8}$$

where, as stated in the introduction,

$$c_{n-1} = (2\pi)^{1-n} \omega_{n-1} \int_S \varphi_\nu^{(1-n)/2} dv .$$

6. Expansions in eigenfunctions

When $w \in C_0^1(R_\varepsilon)$ or $C_0^1(\bar{R}_\varepsilon)$, let us expand w in terms of the eigenfunctions h_j. We get

$$w = \sum_0^\infty w_j(t) h_j(y) .$$

[1] Actually, supposing that everything is C^∞, Hörmander proves in [3] this formula with the error term $O(\mu^{(n-2)/2})$.

Hence, in view of the orthogonality properties of the h_j,

$$a_2(w) = \sum_0^\infty f(w_j, \mu_j) \quad \text{and} \quad b_2(w) = \sum_0^\infty g(w_j)\,.$$

Here

$$f(\mu) = f(\mu, u) = \int_0^\varepsilon t(u'(t)^2 + \mu u(t)^2)dt \quad \text{and} \quad g(u) = \int_0^\varepsilon u(t)^2 dt\,,$$

are forms involving just one variable and all w_j belong either to $C_0^1(I_\varepsilon)$ or $C_0^1(\bar{I}_\varepsilon)$, where I_ε is the interval $0 \leq t < \varepsilon$. Since all w_j are independent of each other, this gives

$$N(a_2/b_2, T, \lambda) = \textstyle\sum_0^\infty N(f(\mu_j)/g, J, \lambda), \quad T = R_\varepsilon \text{ or } \bar{R}_\varepsilon, \quad J = I_\varepsilon \text{ or } \bar{I}_\varepsilon\,.$$

Hence our theorem follows if we can show that the right side is $\sim \sigma_n(\lambda)$ in both cases. Now, from the Weyl-Courant principle

$$N(f(\mu)/g, \bar{I}_\varepsilon, \lambda) \leq N(f'/g, \bar{I}_\varepsilon, \lambda)$$

where

$$f'(u) = \int_0^\varepsilon t(1 - \varepsilon^{-1}t)u'^2 dt$$

and hence, according to Goulaouic ([2], p. 360—11) we have

$$N(f(\mu)/g, \bar{I}_\varepsilon, \lambda) = O(\sqrt{\lambda}), \quad \lambda \to \infty\,, \tag{9}$$

uniformly when $\mu \geq 0$. Since $\sqrt{\lambda} = o(\sigma_n(\lambda))$, $\lambda \to \infty$, this means that we are reduced to showing e.g. that

$$\int_1^\infty N(f(\mu)/g, J, \lambda)dN_0(\mu) \sim \sigma_n(\lambda), \quad \lambda \to \infty, \quad J = I_\varepsilon \text{ or } \bar{I}_\varepsilon\,. \tag{10}$$

Here, instead of a sum over the μ_j we have written a Stieltjes integral, the region of integration being $1 \leq \mu < \infty$.

7. A one-dimensional case with a parameter

Together with the forms f, g, consider the forms

$$F(\varrho, v) = \int_0^\varrho x(v'^2 + v^2)dx, \quad G(\varrho, v) = \int_0^\varrho v^2 dx\,, \tag{11}$$

depending on the parameter $\varrho > 0$. Putting $v(x) = u(x/\sqrt{\mu})$ we then have

$$f(\mu, u)/g(u) = \sqrt{\mu}F(\varepsilon\sqrt{\mu}, v)/G(\varepsilon\sqrt{\mu}, v)\,,$$

when $\mu > 0$. Hence putting for simplicity

$$M(\lambda, \varrho) = N(F(\varrho)/G(\varrho), I_\varrho, \lambda)$$

and writing $\bar{M}\,(\lambda,\varrho)$ when I_ϱ is replaced by $\bar{I}_\varrho$, we have

$$N(f(\mu)/g, I_\varepsilon, \lambda) = M(\lambda/\sqrt{\mu}, \varepsilon\sqrt{\mu})\,,$$

where I_ε, M may be replaced by $\bar{I}_\varepsilon, \bar{M}$ and it suffices to show that

$$\int_1^\infty m(\lambda/\tau, \varepsilon\tau)dN_0(\tau^2) \sim \sigma_n(\lambda), \quad m = M \text{ or } \bar{M}\,. \tag{12}$$

In order to proceed further, we now need detailed information about the functions M and $\bar{M}$. It is given in the following lemma where it is understood that $\lambda > 0$.

LEMMA. *Let* $m = M$ *or* $\bar{M}$. *Then*

a) $1 \leq \lambda < \varrho \Rightarrow m(\lambda, \varrho) = \lambda/2 + O(\lambda^{3/4})$

b) *For every* $\varrho_0 > 0$ *holds* $\lambda \geq \varrho \geq \varrho_0 \Rightarrow m(\lambda, \varrho) = O(\sqrt{\lambda\varrho})$

c) *Given an even integer* $A > 0$ *and* $0 < \delta < 1$, *there is a* $\varrho_0 > A$ *such that if* $\lambda \leq A$ *and* $\varrho \geq \varrho_0$, *then*

$$m(\lambda, \varrho) = [(\lambda + 1)/2]$$

except for symmetric intervals of length 2δ *around the odd integers* $1, 3, \ldots,$ *in these intervals the difference of the two expressions is at most 1 in absolute value.*

Proof. Let $M(\lambda, \varrho_1, \varrho_2)$ and $\bar{M}(\lambda, \varrho_1, \varrho_2)$ be the counting functions associated with the forms

$$\int_{\varrho_1}^{\varrho_2} x(v'^2 + v^2)dx, \qquad \int_{\varrho_1}^{\varrho_2} v^2 dx$$

and the classes $C_0^1(I)$ and $C_0^1(\bar{I})$ respectively, where $I = (\varrho_1, \varrho_2)$. When the first of these forms is replaced by $c\int_{\varrho_1}^{\varrho_2}(v'^2 + v^2)dx$ $(c > 0)$ the eigenvalues are λ_k and $\bar{\lambda}_k$, $k = 1, 2, \ldots,$ respectively, where

$$\lambda_k c^{-1} - 1 = \pi^2 k^2(\varrho_2 - \varrho_1)^{-2}\,,$$

$$\bar{\lambda}_1 = 0, \quad \bar{\lambda}_k = \lambda_{k-1}, \quad k \geq 2\,.$$

It follows easily from this that

$$-1+\pi^{-1}(\varrho_2-\varrho_1)(\lambda\varrho_2^{-1}-1)_+^{\frac{1}{2}} \le m(\lambda,\varrho_1,\varrho_2) \le \pi^{-1}(\varrho_2-\varrho_1)(\lambda\varrho_1^{-1}-1)_+^{\frac{1}{2}}+1\,, \tag{13}$$

where $m = M$ or $\bar{M}$ and x_+ denotes the positive part of x.

To prove a) let

$$1=\varrho_0<\varrho_1<\ldots<\varrho_{\nu-1}=\lambda<\varrho_\nu=\varrho$$

be a partition of $[1, \varrho]$, such that the partition of $[1, \lambda]$ is equidistant. By (4)

$$\sum_0^{\nu-1} M(\lambda,\varrho_k,\varrho_{k+1}) \le m(\lambda,1,\varrho) \le \sum_0^{\nu-1} \bar{M}(\lambda,\varrho_k,\varrho_{k+1})\,,$$

which combined with (13) gives

$$-\nu+\sum_0^{\nu-3} f(\varrho_{k+1})(\varrho_{k+1}-\varrho_k) \le m(\lambda,1,\varrho) \le \sum_0^{\nu-2} f(\varrho_k)(\varrho_{k+1}-\varrho_k)+\nu\,,$$

where

$$f(x)=\pi^{-1}\sqrt{\lambda/x-1} \quad \text{when} \quad 1\le x\le\lambda\,.$$

Now f is decreasing, and hence

$$-\lambda^{3/2}\nu^{-1}-\nu+\int_1^\lambda f(x)dx \le m(\lambda,1,\varrho) \le \int_1^\lambda f(x)dx+\lambda^{3/2}\nu^{-1}+\nu$$

Here

$$\int_1^\lambda f(x)dx = \lambda/2+O(\lambda^{3/4})$$

which is seen by an easy calculation. Choosing e.g. $\nu=[\lambda^{3/4}+3]$ we get

$$m(\lambda,1,\varrho)=\lambda/2+O(\lambda^{3/4})$$

and a) follows from (4) and (9).

To prove b) let

$$\varrho_0<\varrho_1<\varrho_2<\ldots<\varrho_\nu=\varrho$$

be an equidistant partition of $[\varrho_0, \varrho]$. By (4) and (13) we get, as in the proof of a)

$$m(\lambda,\varrho_0,\varrho) \le \sum_0^{\nu-1} f(\varrho_k)(\varrho_{k+1}-\varrho_k)$$

and hence

$$m(\lambda,\varrho_0,\varrho) \le \varrho_0^{-\frac{1}{2}}\nu^{-1}\varrho\lambda^{\frac{1}{2}}+\int_{\varrho_0}^{\varrho} f(x)dx+\nu\,.$$

Now, putting $x\lambda = t^2$

$$\int_{\varrho_0}^{\varrho} f(x)dx = 2\pi^{-1} \int_{\sqrt{\lambda\varrho_0}}^{\sqrt{\lambda\varrho}} \sqrt{1 - t^2\lambda^{-2}}dt \leq \sqrt{\lambda\varrho} .$$

Hence putting e.g. $\nu = [\varrho^{\frac{1}{2}} + 1]$, we get

$$m(\lambda, \varrho_0, \varrho) = O(\sqrt{\lambda\varrho}) ,$$

and b) follows from (4) and (9).

To prove c) observe that

$$u(t, \lambda) = (2\pi i)^{-1} \int_{\mathrm{Re}z=d>1} e^{zt}(z - 1)^{\frac{1}{2}(\lambda-1)}(z + 1)^{-\frac{1}{2}(\lambda+1)}dz$$

is a solution of

$$-(tu')' + tu = \lambda u , \tag{14}$$

which is regular at the origin. Every solution w of (14) with tw'^2 integrable near the origin is a multiple of u since the equation (14) has a basis of solutions

$$u_0(t) = 1 + tf_0(t), \quad u_1(t) = (1 + tf_1(t)) \log t$$

where f_0 and f_1 are regular. Hence if $\lambda_\nu(\varrho)$, $\nu = 1, 2, \ldots,$ are the eigenvalues of the Friedrichs extension associated with the forms $F(\varrho)$, $G(\varrho)$ and the class $C_0^1(I_\varrho)$ then they are the zeros of

$$\lambda \to u(\varrho, \lambda)$$

and the zeros of

$$\lambda \to u_t'(\varrho, \lambda)$$

if $C_0^1(I_\varrho)$ is replaced by $C_0^1(\bar{I}_\varrho)$. A change of variables shows that

$$u(t, \lambda) = (2t)^{-\frac{1}{2}(\lambda+1)}e^t v(t, \lambda) ,$$

where

$$v(t, \lambda) = (2\pi i)^{-1} \int_{\mathrm{Re}z=c>0} e^z z^{\frac{1}{2}(\lambda-1)}(1 + z/2t)^{-\frac{1}{2}(\lambda+1)}dz .$$

It is easy to verify that

$$v_t'(t, \lambda) = 4^{-1}(\lambda + 1)t^{-2}v(t, \lambda + 2)$$

and hence

$$u_t'(t, \lambda) = (2t)^{-\frac{1}{2}(\lambda+1)}e^t(- \tfrac{1}{2}(\lambda + 1)t^{-1} + 1)v(t, \lambda) + 4^{-1}(\lambda + 1)t^{-2}v(t, \lambda + 2)) .$$

Hence the zeros of $\lambda \to u(t, \lambda)$ are the same as the zeros of $\lambda \to v(t, \lambda)$ and the zeros of $\lambda \to u_t'(t, \lambda)$ are the same as the zeros of $\lambda \to w(t, \lambda)$, where

$$w(t, \lambda) = (1 - \tfrac{1}{2}(\lambda + 1)t^{-1})v(t, \lambda) + 4^{-1}(\lambda + 1)t^{-2}v(t, \lambda + 2) .$$

We also have

$$v(t, \lambda) \to v(\infty, \lambda), \quad t \to \infty$$
$$w(t, \lambda) \to w(\infty, \lambda), \quad t \to \infty ,$$

where

$$v(\infty, \lambda) = w(\infty, \lambda) = -\pi^{-1} \sin (\pi/2)(\lambda - 1) \int_0^\infty e^{-x} x^{\frac{1}{2}(\lambda-1)} dx .$$

The convergence is uniform on every compact subset of $\operatorname{Re} \lambda > 0$ and the limit function is analytic in $\operatorname{Re} \lambda > 0$ with simple zeros only at the points $1, 3, 5, \ldots$. Hence, if $0 < \delta < 1$ and an even integer $A = 2p$ are given, there exists a $\varrho_0 > A$ such that $\lambda \to v(\varrho, \lambda)$ $(\lambda \to w(\varrho, \lambda))$ for $\varrho \geq \varrho_0$ has precisely p zeros in the strip $0 < \operatorname{Re} \lambda < A$, one in each disc $|\lambda - (2k - 1)| < \delta$, $k = 1, \ldots, p$. The fact that $\overline{u(t, \bar{\lambda})} = u(t, \lambda)$ shows that the zeros are real and the proof of the lemma is finished.

8. End of the proof

By (8) and b) of the lemma we have

$$\int_1^{\sqrt{\lambda/\varepsilon}} m(\lambda/\tau, \varepsilon\tau) dN_0(\tau^2) = O(\lambda^{n/2}) = o(\sigma_n(\lambda)), \quad \lambda \to \infty ,$$

and hence, according to (10), we are reduced to proving that

$$I(\lambda) = \int_{\sqrt{\lambda/\varepsilon}}^\infty m(\lambda/\tau, \varepsilon\tau) dN_0(\tau^2) \sim \sigma_n(\lambda), \quad \lambda \to \infty . \tag{15}$$

Now let $0 < \delta < \frac{1}{2}$ be given. By (8) and c) of the lemma we can choose λ' so big that

$$\lambda \geq \lambda' \Rightarrow \begin{cases} N_0(\tau^2) = (c^{n-1} + O(1)\delta)\tau^{n-1}, & \text{all} \quad \tau \geq \sqrt{\lambda/\varepsilon} \\ m(\lambda/\tau, \varepsilon\tau) = 0, & \text{all} \quad \tau \geq \lambda/(1 - \delta) . \end{cases}$$

c) of the lemma, an integration by parts and an estimation of the product $m(\sqrt{\lambda\varepsilon}, \sqrt{\lambda\varepsilon})N_0(\lambda/\varepsilon)$ by b) of the lemma then shows that

$$I(\lambda) = O(\lambda^{n/2}) - \int_{\sqrt{\lambda/\varepsilon}}^{\lambda/(1-\delta)} N_0(\tau^2) dm(\lambda/\tau, \varepsilon\tau), \quad \lambda \geq \lambda' ,$$

and hence

$$I(\lambda) = O(\lambda^{n/2}) - \int_{\sqrt{\lambda/\varepsilon}}^{\lambda/(1-\delta)} (c_{n-1} + O(1)\delta)\tau^{n-1} dm(\lambda/\tau, \varepsilon\tau), \quad \lambda \geq \lambda' .$$

Now, another integration by parts gives

$$I(\lambda) = O(\lambda^{n/2}) + ((n-1)c_{n-1} + O(1)\delta) \int_{\sqrt{\lambda/\varepsilon}}^{\lambda/(1-\delta)} m(\lambda/\tau, \varepsilon\tau)\tau^{n-2} d\tau, \quad \lambda \geq \lambda' . \qquad (16)$$

When $n = 2$

$$I(\lambda) = O(\lambda) + (c_1 + O(1)\delta) \int_{\sqrt{\lambda/\varepsilon}}^{\lambda/(1-\delta)} m(\lambda/\tau, \varepsilon\tau) d\tau, \quad \lambda \geq \lambda' ,$$

and by a) of the lemma and the definition of d_2

$$I(\lambda) = (d_2 + O(1)\delta)\lambda \log \lambda, \quad \lambda \geq \lambda' ,$$

and hence

$$I(\lambda) \sim \sigma_2(\lambda), \quad \lambda \to \infty ,$$

which finishes the proof in the case $n = 2$.

When $n > 2$, choose an even integer $A = 2p$ so big that

$$A^{-1} < \delta \quad \text{and} \quad \int_A^\infty [(t+1)/2] t^{-n} dt < \delta .$$

Put

$$\int_{\sqrt{\lambda/\varepsilon}}^{\lambda/(1-\delta)} m(\lambda/\tau, \varepsilon\tau)\tau^{n-2} d\tau = I_1(\lambda) + I_2(\lambda) ,$$

where the region of integration is $(\sqrt{\lambda/\varepsilon},\ \lambda/A)$ in I_1 and $(\lambda/A, \lambda/(1-\delta))$ in I_2. By a) of the lemma

$$I_1(\lambda) = O(1)A^{-n+2}\lambda^{n-1} = O(1)\delta\lambda^{n-1} .$$

By c) of the lemma

$$I_2(\lambda) = \int_{\lambda/A}^{\lambda} [(\lambda/\tau + 1)/2]\tau^{n-2}d\tau + O(1) \sum_1^p \int_{\lambda/(2k-1+\delta)}^{\lambda/(2k-1-\delta)} \tau^{n-2}d\tau$$

where $O(1)$ refers to $\lambda \to \infty$.

Since $\int_A^\infty [(t + 1)/2]t^{-n}dt < \delta$, putting $\lambda/\tau = t$ we get

$$\int_{\lambda/A}^{\lambda} [(\lambda/\tau + 1)/2]\tau^{n-2}d\tau = \lambda^{n-1} \int_1^\infty [(t + 1)/2]t^{-n}dt + O(1)\delta\lambda^{n-1} .$$

Also,

$$\sum_1^p \int_{\lambda/(2k-1+\delta)}^{\lambda/(2k-1-\delta)} \tau^{n-2}d\tau = O(1)\delta\lambda^{n-1}$$

which follows from the mean-value theorem and trivial estimates. Hence, by (16) and the definitions of the constants d_n

$$I(\lambda) = (d_n + O(1)\delta)\lambda^{n-1}, \quad \lambda \geq \lambda' ,$$

which shows that

$$I(\lambda) \sim \sigma_n(\lambda), \quad \lambda \to \infty .$$

This finishes the proof.

Added in proof. The asymptotic formula of the theorem is not quite correct. To get the correct formula, replace the exponent $(1 - n)/2$ in (1) by $1 - n$ getting

$$c_{n-1} = (2\pi)^{1-n}\omega_{n-1} \int_S (\partial\varphi/\partial\nu)^{1-n}dv . \tag{1'}$$

The error occurs in section 6 and it was pointed out to me by Mme J. Fleckinger and G. Métivier. The eigenfunctions h_j are in general not orthonormal in the inner product $\int_S pq\, dv$ so that the formula for $f(\mu)$ is not correct unless $\varphi_\nu \equiv 1$.

To deduce the correct theorem from this special case, note that it holds when φ_ν is a constant. More generally, it holds when $N(\lambda)$ refers to a pair of quadratic forms $a_1(u)$, $b_1(u)$ as given in section 4 with $\varphi_\nu > 0$ constant and with $R_\varepsilon = S_0 \times \{t : 0 < t < \varepsilon\}$, where S_0 is an open nicely bounded part of $S = \partial R_\varphi$. The Weyl-Courant principle applied to fine partitions of S into such pieces and majorants and minorants of φ_ν^{1-n} in each piece finishes the proof.

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