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Representation of measures with simultaneous polynomial denseness in $L_p(\mathbf{R}, d\mu)$, $1\le p<\infty$

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Abstract. We give characterisations of certain positive finite Borel measures with unbounded support on the real axis so that the algebraic polynomials are dense in all spaces $L_p(\mathbf{R}, d\mu)$, $p\ge 1$. These conditions apply, in particular, to the measures satisfying the classical Carleman conditions.

1. Introduction

1.1. Background

Let $\mathcal{M}(\mathbf{R})$ denote the space of finite Borel measures μ on $\mathbf{R}$, $\mathcal{M}^+(\mathbf{R})$ the cone of positive measures in $\mathcal{M}(\mathbf{R})$, and $\mathcal{M}^*(\mathbf{R})$ the subset of measures in $\mathcal{M}^+(\mathbf{R})$ with all moments

$$\mu_n := \int_{\mathbf{R}} x^n \, d\mu(x), \quad n\ge 0,$$

finite. Denote by $\mathcal{P}$ the set of all real algebraic polynomials, i.e. polynomials with real coefficients, and let $\overline{\mathcal{P}}$ be the set of topological linear spaces of real functions on $\mathbf{R}$ which have $\mathcal{P}$ as a dense subset. For $\mu\in\mathcal{M}^*(\mathbf{R})$ we look at the real spaces $L_p(\mathbf{R}, d\mu)$, $1\le p<\infty$, and $L(\mathbf{R}, d\mu) := \bigcap_{p\ge 1} L_p(\mathbf{R}, d\mu)$. The aim of this note is to characterize the set([1])

$$\mathcal{M}^*_\infty(\mathbf{R}) := \bigcap_{p\ge 1} \mathcal{M}^*_p(\mathbf{R}),$$

where

$$\mathcal{M}^*_p(\mathbf{R}) := \{\mu\in\mathcal{M}^*(\mathbf{R}) \mid L_p(\mathbf{R}, d\mu)\in\overline{\mathcal{P}}\}, \quad 1\le p<\infty.$$

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([1]) This problem is due to Prof. Ajit Iqbal Singh who raised it during a discussion with A. Bakan in May 1999.

The background of this question are properties of measures from $\mathcal{M}^*(\mathbf{R})$ satisfying the so-called Carleman conditions. The definition of these conditions given below is slightly more general than the one in T. Carleman's original paper [14] (see also [15] and [16]). The following notation will be useful: let $m\in\mathbf{N}$ and $n\in\mathbf{Z}\setminus\{0\}$: a sequence of real numbers $a:=\{a_k\}_{k\geq 0}$ will be called *(m,n)-divergent* if

$$\sum_{k\geq 1} |a_{mk}|^{-1/nk} = \infty.$$

For $\mu\in\mathcal{M}^*(\mathbf{R})$ we shall write $S_\mu:=\{\mu_n\}_{n\geq 0}$, and set $b(\mu):=1$ if $\operatorname{supp}\mu:=\{x\in\mathbf{R}|\ \mu((x-\varepsilon,x+\varepsilon))>0$ for all $\varepsilon>0\}$ is unbounded in both directions and $b(\mu):=2$ otherwise.

Definition 1. A measure $\mu\in\mathcal{M}^*(\mathbf{R})$ is said to satisfy *Carleman's condition* (or is called a *Carleman measure*) if S_μ is $(2,2b(\mu))$-divergent.

The set of Carleman measures will be denoted by $\mathcal{M}^*_C(\mathbf{R})$. They have originally been introduced in the context of the so-called Hamburger and Stieltjes determinacy problem (see [1], [17–22], [25], [28] and [29]), and it has been known for some time (see [6] and [27]) that the determinacy of measures in $\mathcal{M}^*(\mathbf{R})$ is related to the polynomial density in certain L_2 spaces. For us the following result by C. Berg and J. P. R. Christensen [6] (see also [5]) is the most relevant.

Theorem A. *Every Carleman measure belongs to* $\mathcal{M}^*_\infty(\mathbf{R})$.

In the sequel we shall study general characterizations and representations for measures in $\mathcal{M}^*_\infty(\mathbf{R})$. In the case of the Carleman measures these will be even more explicit and constructive. It should be observed that every measure μ in $\mathcal{M}^+(\mathbf{R})$ with bounded support is a Carleman measure and belongs to $\mathcal{M}^*_\infty(\mathbf{R})$ automatically. Similar observations are valid also for all further results in this paper. Therefore, from now on, we shall study only measures with unbounded support.

1.2. The main results

Let $\mathcal{B}(\mathbf{R})$ denote the family of Borel subsets of $\mathbf{R}$ and $\mathcal{W}^*(\mathbf{R})$ the set of upper semi-continuous functions $w:\mathbf{R}\to\mathbf{R}^+$, satisfying $\|x^n\|_w<\infty$ for all $n\geq 0$, where $\|f\|_w:=\sup_{x\in\mathbf{R}} w(x)|f(x)|$. For $w\in\mathcal{W}^*(\mathbf{R})$ the space C^0_w is defined as the set of all $f:\mathbf{R}\to\mathbf{R}$, continuous on $\mathbf{R}$ and with $\lim_{|x|\to\infty} w(x)f(x)=0$. We endow C^0_w with the semi-norm $\|\cdot\|_w$. In this context a function $w\in\mathcal{W}^*(\mathbf{R})$ is also called a *weight.* For $0<\tau<\infty$ denote by $\mathcal{W}^*_\tau(\mathbf{R})$ the set of $w\in\mathcal{W}^*(\mathbf{R})$ with $C^0_{w^\tau}\in\overline{\mathcal{P}}$. Furthermore, set $\mathcal{W}^*_0(\mathbf{R}):=\bigcap_{\tau>0}\mathcal{W}^*_\tau(\mathbf{R})$. Observe, that the Weierstrass polynomial approximation theorem implies that $\chi_A \in \mathcal{W}_0^*(\mathbf{R})$ for every compact set $A \subset \mathbf{R}$. Here χ_A denotes the characteristic function of the set A.

Our main result is the following theorem.

Theorem 1. *A measure $\mu \in \mathcal{M}^*(\mathbf{R})$ with unbounded support belongs to the space $\mathcal{M}_\infty^*(\mathbf{R})$ if and only if any one (and therefore all) of the following three properties hold:*

(a) *There exists $w \in \mathcal{W}_0^*(\mathbf{R})$ such that $1/w \in L_1(\mathbf{R}, d\mu)$.*
(b) *There exists $w \in \mathcal{W}_1^*(\mathbf{R})$ such that $1/w \in L(\mathbf{R}, d\mu)$.*
(c) *There exists $w \in \mathcal{W}_0^*(\mathbf{R})$ such that $1/w \in L(\mathbf{R}, d\mu)$.*

Remark 1. Lebesgue integration theory implies that for $\mu \in \mathcal{M}^*(\mathbf{R})$ and $w \in \mathcal{W}^*(\mathbf{R})$ there exists $\nu \in \mathcal{M}^+(\mathbf{R})$ such that

$$\mu(A) = \int_A w(x)\, d\nu(x), \quad A \in \mathcal{B}(\mathbf{R}),$$

(in other words $d\mu(x) = w(x)\, d\nu(x)$) if and only if $1/w \in L_1(\mathbf{R}, d\mu)$. Here we have set $1/w := \lim_{N \to +\infty} (w + 1/N)^{-1}$. Note that, using this property, one can replace the integrability conditions on $1/w$ in Theorem 1 by the existence of representations for μ in specific forms. We omit the details.

The functions w found in Theorem 1(c) have particularly strong connections with the measure μ in question.

Corollary 1. *Let $\mu \in \mathcal{M}^*(\mathbf{R})$ have unbounded support and satisfy $L_p(\mathbf{R}, d\mu) \in \overline{\mathcal{P}}$ for all $1 \le p < \infty$. Then any function w as in Theorem 1(c) has the following additional properties:*

(a) *$L_p(\mathbf{R}, w^{-q}(x)\, d\mu(x)) \in \overline{\mathcal{P}}$ for all $1 \le p, q < \infty$;*
(b) *if f is continuous on $\mathbf{R}$ with $f(x) = \mathcal{O}(|x|^m)$ for some $m \in \mathbf{N}$ as $x \to \pm\infty$ then*

$$\|f\|_{L_p(\mathbf{R}, d\mu)} \le \|1/w\|_{L_p(\mathbf{R}, d\mu)} \|f\|_w, \quad 1 \le p < \infty. \tag{1}$$

Remark 2. If we approximate a function f as in Corollary 1(b) by a polynomial sequence $\{P_n\}_{n \ge 1}$ in the space C_w^0 and replace f by $f - P_n$ in (1), then we see that the rate of simultaneous polynomial approximation to f in the spaces $L_p(\mathbf{R}, d\mu)$ (independently of p) is not slower than the one of f in the $\|\cdot\|_w$-norm.

Corollary 1(a) will be instrumental in the proof of the following result.

Corollary 2.

$$\mathcal{M}^*_\infty(\mathbf{R}) = \{\mu \mid d\mu = w\,d\nu,\ w \in \mathcal{W}^*_0(\mathbf{R}),\ \nu \in \mathcal{M}^+(\mathbf{R})\}. \tag{2}$$

It is an open question which smaller set can replace $\mathcal{W}^*_0(\mathbf{R})$ on the right-hand side of (2) without violating the conclusion. The corresponding question arises also (and is open) for the representation

$$\mathcal{M}^*_p(\mathbf{R}) = \{\mu \mid d\mu = w\,d\nu,\ w \in \mathcal{W}^*_{1/p}(\mathbf{R}),\ \nu \in \mathcal{M}^+(\mathbf{R})\},\qquad 1 \le p < \infty, \tag{3}$$

(see [3, Theorem 2.1, p. 38]). However, in

$$\mathcal{M}^*(\mathbf{R}) = \{\mu \mid d\mu = w\,d\nu,\ w \in \mathcal{W}^*(\mathbf{R}),\ \nu \in \mathcal{M}^+(\mathbf{R})\}, \tag{4}$$

the set $\mathcal{W}^*(\mathbf{R})$ can be replaced by a much smaller one, namely the reciprocals of the even entire functions with all positive and decreasing Taylor coefficients, without affecting the validity (see Section 6).

If one wants more and better information about $\mathcal{M}^*_\infty(\mathbf{R})$ it is clear from Theorem 1 and Corollary 2 that more has to be known about the sets $\mathcal{W}^*_1(\mathbf{R})$ and $\mathcal{W}^*_0(\mathbf{R})$. This we shall discuss in the next section.

1.3. The sets $\mathcal{W}^*_1(\mathbf{R})$ and $\mathcal{W}^*_0(\mathbf{R})$

The following statement is a sort of converse to (3).

Theorem 2. *Let $w \in \mathcal{W}^*(\mathbf{R})$. If $w \notin \mathcal{W}^*_1(\mathbf{R})$ then there exists $\nu \in \mathcal{M}^+(\mathbf{R})$ such that for all $1 \le p < \infty$ we have $L_p(\mathbf{R}, w^p d\nu) \notin \overline{\mathcal{P}}$. On the other hand, if there exist $\nu \in \mathcal{M}^+(\mathbf{R})$ and $p \in [1, \infty)$ such that $L_p(\mathbf{R}, w^p d\nu) \notin \overline{\mathcal{P}}$ then $w \notin \mathcal{W}^*_1(\mathbf{R})$.*

Already in 1924 S. Bernstein [8] asked for conditions on $w \in \mathcal{W}^*(\mathbf{R})$ to be in $\mathcal{W}^*_1(\mathbf{R})$. In 1959 L. de Branges [13] obtained a solution to this problem. A slightly improved version (see [11] and [31]) of his result is as follows: let $\mathcal{E}_0$ be the family of entire functions B of minimal exponential type having real and simple zeros only and let Λ_B denote the set of these zeros.

Theorem B. (1959, [13]) *For $w \in \mathcal{W}^*(\mathbf{R})$ assume that $S_w := \{x \in \mathbf{R} \mid w(x) > 0\}$ is unbounded. Then $w \in \mathcal{W}^*_1(\mathbf{R})$ if and only if for every function $B \in \mathcal{E}_0$ with $\Lambda_B \subseteq S_w$ we have*

$$\sum_{\lambda \in \Lambda_B} \frac{1}{w(\lambda)|B'(\lambda)|} = \infty.$$

Using Theorem B we can translate the conditions on w in Theorem 1 as follows. As usual $\log^+ x$ equals $\log x$ if $x \geq 1$ and 0 otherwise.

Theorem 3. *For $w \in \mathcal{W}^*(\mathbf{R})$ assume that $S_w := \{x \in \mathbf{R} \mid w(x) > 0\}$ is unbounded. Then $w \in \mathcal{W}_0^*(\mathbf{R})$ if and only if for every function $B \in \mathcal{E}_0$ with $\Lambda_B \subseteq S_w$ we have*

$$\overline{\lim_{\substack{|\lambda|\to\infty \\ \lambda\in\Lambda_B}}} \frac{\log 1/w(\lambda)}{\log^+ |B'(\lambda)|} = \infty. \tag{5}$$

We note that Theorem B is also crucial for the proof of Theorem 1(c), see Lemmas 5 and 6 below.

1.4. Properties and representations of Carleman measures

We denote by Ψ the set of functions

$$w(x) := \frac{1}{\sum_{n\geq 0} a_{2n} x^{2n}}$$

with $a_0 > 0$, $a_{2n} \geq 0$ for all $n \geq 1$, $a_{2n} > 0$ for infinitely many n and $\overline{\lim}_{n\to\infty} \sqrt[2n]{a_{2n}} = 0$. Note that $\Psi \subset \mathcal{W}^*(\mathbf{R})$, and for $w \in \Psi$ we set

$$\sigma_w := \left\{ \sup_{x\in\mathbf{R}} |x|^n w(x) \right\}_{n\geq 0}.$$

S. Bernstein in [8] and [9] (see also [26]) solved his own problem mentioned above for functions $w \in \Psi$. His result was as follows.

Theorem C. (Bernstein's conditions) *Let $w \in \Psi$. Then the following conditions are equivalent*([2])*:*

(a) $w \in \mathcal{W}_1^*(\mathbf{R})$;
(b) $w \in \mathcal{W}_0^*(\mathbf{R})$;
(c) $\int_{\mathbf{R}} (\log w(x))/(1+x^2)\, dx = -\infty$;
(d) *σ_w is (1,1)-divergent.*

Bernstein's Theorem C has been generalized by many authors in various directions. In particular, several papers are devoted to the investigation of the corresponding polynomial density on certain subsets of the real line, and analogues of

([2]) It is known [26] that Theorem C is valid for a much larger class of functions, but this is of no relevance in the present context. For the implication (a) $\Rightarrow$ (d) see the proof of Theorem 11 in [26].

the third condition in Theorem C were obtained for various cases (see, for example, [12], [23], [26] and [30]). A complete analogue of Theorem C for the positive half-line is

Theorem C$^+$. *Let $w\in\Psi$. Then the following conditions are equivalent:*
(a) $w\chi_{\mathbf{R}^+}\in\mathcal{W}_1^*(\mathbf{R})$;
(b) *for each interval* $J=(-\infty,a]$ *and* $J=[a,\infty)$, $a\in\mathbf{R}$, *we have* $w\chi_J\in\mathcal{W}_0^*(\mathbf{R})$;
(c) $\int_0^{+\infty}\log w(x)/(1+x^{3/2})\,dx=-\infty$;
(d) σ_w *is* (1,2)*-divergent.*

Note that the sequence $\{\sqrt[n]{\sup_{x\in\mathbf{R}}|x|^n w(x)/w(0)}\,\}_{n\geq 1}$ is increasing. This implies that the conditions (d) in Theorems C and C$^+$ are equivalent to

$$\sigma_w \text{ is } (N,N)\text{-divergent}, \quad\text{and } \sigma_w \text{ is } (N,2N)\text{-divergent}, \tag{6}$$

respectively, for each $N\in\mathbf{N}$.

Consider an arbitrary $w\in\Psi$. The inequality

$$\frac{1}{\sup\limits_{x\in\mathbf{R}} x^{2n}w(x)}=\inf_{x\in\mathbf{R}}\frac{\sum_{k\geq 0}a_{2k}x^{2k}}{x^{2n}}\geq a_{2n}, \tag{7}$$

together with (6), leads immediately to the following corollary of Theorems C and C$^+$.

Corollary 3. *Let*

$$w(x)=\frac{1}{\sum_{n\geq 0}b_{2n}x^{2n}}\in\Psi. \tag{8}$$

Then we have
(a) *If* $\{b_{2n}\}_{n\geq 1}$ *is* $(1,-2)$*-divergent then* $w\in\mathcal{W}_0^*(\mathbf{R})$;
(b) *If* $\{b_{2n}\}_{n\geq 1}$ *is* $(1,-4)$*-divergent then* $w\chi_J\in\mathcal{W}_0^*(\mathbf{R})$ *holds for each interval of the form* $J=(-\infty,a]$ *and* $J=[a,\infty)$, $a\in\mathbf{R}$.

In the sequel we shall use the abbreviation

$$\mathcal{R}(\mu):=\begin{cases}\{\mathbf{R}\}, & \text{if } b(\mu)=1,\\ \{[a,+\infty),(-\infty,a]\mid a\in\mathbf{R}\}, & \text{if } b(\mu)=2.\end{cases}$$

The following theorem provides a complete description of the weight functions in Theorem 1(a) if the measure in question is Carleman.

Theorem 4. (a) *If $\mu\in\mathcal{M}_C^*(\mathbf{R})$ has unbounded support then there exists $\nu\in\mathcal{M}^+(\mathbf{R})$ such that*

$$\mu(A)=\int_A w(x)\,d\nu(x),\quad A\in\mathcal{B}(\mathbf{R}),\tag{9}$$

where $w(x):=1/\sum_{n\geq 0}x^{2n}/2^n\mu_{2n}$.

(b) *Let $\nu\in\mathcal{M}^+(\mathbf{R})$ have unbounded support and for some $w\in\Psi$ as in (8) assume that $\{b_{2n}\}_{n\geq 1}$ is $(1,-2b(\nu))$-divergent. Then the measure μ as in (9) belongs to $\mathcal{M}_C^*(\mathbf{R})$.*

In both cases we have $w\chi_J\in\mathcal{W}_0^*(\mathbf{R})$ for each $J\in\mathcal{R}(\mu)$ and $b(\mu)=b(\nu)$ since $\operatorname{supp}\mu=\operatorname{supp}\nu$.

To obtain a representation of Carleman measures in terms of the function w of Theorem 1(c) (instead of (a)), we require a property of certain (m,n)-divergent sequences:

Lemma 1. *Let $a:=\{a_k\}_{k\geq 1}$ be a sequence with $|a_k|^{1/k}$ increasing and $m,n\in\mathbf{N}$. If a is (m,n)-divergent then there exists a sequence $\{n_k\}_{k\geq 1}$ of natural numbers with $k\leq n_k<n_{k+1}$ for $k\in\mathbf{N}$, and*

$$\lim_{k\to\infty}\frac{n_k}{k}=\infty,\tag{10}$$

such that $\{a_{n_k}\}_{k\geq 1}$ is (m,n)-divergent as well.

It will be shown that S_μ for a Carleman measure with unbounded support satisfies (essentially) the conditions of Lemma 1 with $(m,n)=(2,2b(\mu))$. Therefore we find a sequence n_k as in Lemma 1 and using this sequence we define

$$w_\mu(x):=\left(1+\sum_{k\geq 1}\frac{x^{2k}}{2^k\mu_{2n_k}^{k/n_k}}\right)^{-1}.\tag{11}$$

Theorem 5. *For $\mu\in\mathcal{M}_C^*(\mathbf{R})$ with unbounded support define w_μ as in (11). Then for every $1\leq p<\infty$ there exists a measure $\nu_p\in\mathcal{M}_C^*(\mathbf{R})$ such that*

$$\mu(A)=\int_A w_\mu(x)^p\,d\nu_p(x),\quad A\in\mathcal{B}(\mathbf{R}).$$

In addition, $1/w_\mu\in L(\mathbf{R},d\mu)$ and $w_\mu\chi_J\in\mathcal{W}_0^*(\mathbf{R})$ for all $J\in\mathcal{R}(\mu)$.

Corollary 4. *Let $\mu \in \mathcal{M}_C^*(\mathbf{R})$ have unbounded support and w_μ be as in Theorem 5. Then the following hold:*

(a) *For every $1 \le p, q < \infty$ we have*

$$L_p(\mathbf{R}, w_\mu(x)^{-q} d\mu(x)) \in \overline{\mathcal{P}}.$$

(b) *If f is continuous on $\mathbf{R}$ with $f(x) = \mathcal{O}(|x|^m)$ for some $m \in \mathbf{N}$ as $x \to \pm\infty$ then*

$$\|f\|_{L_p(\mathbf{R}, d\mu)} \le \|1/w_\mu\|_{L_p(\mathbf{R}, d\mu)} \|f\|_{w_\mu}, \quad 1 \le p < \infty. \tag{12}$$

Note that

$$\mathcal{M}_C^*(\mathbf{R}) = \{\mu \mid d\mu = w d\nu, \ w \in \Psi_{b(\nu)}, \ \nu \in \mathcal{M}^+(\mathbf{R})\}$$

is a direct consequence of Theorem 4, where $\Psi_{b(\nu)}$ denotes the class of weights defined in Theorem 4(b), compare Corollary 2 and (3) and (4).

Similarly to the observation made in Remark 2, we point out that inequality (12) contains information about the rate of simultaneous polynomial approximation in the spaces $L_p(\mathbf{R}, d\mu)$, $1 \le p < \infty$. Under certain conditions even quantitative conclusions can be obtained (see, for example, the survey [24]).

Corollary 4(1) gives a rather substantial improvement of Theorem A since the entire function $w_\mu(x)^{-1} = 1 + \sum_{k \ge 1} x^{2k} / (2^k \mu_{2n_k}^{k/n_k})$ is of order not less than $1/b(\mu)$.

To see this let first $b(\mu) = 1$. From the definition of the order ϱ of an entire function we have

$$\varrho = \varlimsup_{k \to \infty} \frac{2k \log 2k}{\log 2^k \mu_{2n_k}^{k/n_k}} = \varlimsup_{k \to \infty} \frac{2k \log 2k}{k \log 2 + (k/n_k) \log \mu_{2n_k}},$$

and the assumption $\varrho < 1$ implies that for sufficiently large k, $(2n_k \log 2k) / \log \mu_{2n_k} < \varrho < 1$, i.e., $\mu_{2n_k}^{1/2n_k} \ge (2k)^{1/\varrho}$. But then

$$\sum_{k \ge 1} \frac{1}{\mu_{2n_k}^{1/2n_k}} \le \sum_{k \ge 1} \frac{1}{(2k)^{1/\varrho}} < \infty,$$

a contradiction to the choice of the subsequence $\{n_k\}_{k \ge 1}$ in Lemma 1. The proof for the case $b(\mu) = 2$ is similar.

2. Proof of Theorem 1

2.1. Proof of the sufficiency part of Theorem 1

It is enough to prove sufficiency of (a) and (b), since (c) obviously implies (b). The method of the proof of Theorem 2.3.2 in [1] can be extended from $L_2(\mathbf{R}, d\mu)$ to $L_p(\mathbf{R}, d\mu)$ and provides us (see also [23] and [2, Proposition 2.2, p. 15]) with a suitable criterion: $L_p(\mathbf{R}, d\mu) \in \overline{\mathcal{P}}$, $p \in [1, \infty)$, if and only if two special functions (namely $1/(1+x^2)$ and $x/(1+x^2)$) can be approximated in $L_p(\mathbf{R}, d\mu)$ by polynomials from $\mathcal{P}$.

According to Remark 1(a) and (b) (for arbitrary $p \geq 1$) imply special representations of the measure μ, namely $d\mu(x) = w(x)\, d\nu(x)$ for (a) and $d\mu(x) = w(x)^p\, d\nu_p(x)$ for (b), along with $C^0_{w^{1/p}} \in \overline{\mathcal{P}}$ and $C^0_w \in \overline{\mathcal{P}}$, respectively. Then the obvious inequalities

$$\|f\|^p_{L_p(\mathbf{R}, w d\nu)} = \int_{\mathbf{R}} |f(x) w(x)^{1/p}|^p\, d\nu(x) \leq \|f\|^p_{w^{1/p}} \nu(\mathbf{R}),$$

$$\|f\|^p_{L_p(\mathbf{R}, w^p d\nu_p)} = \int_{\mathbf{R}} |f(x)|^p w(x)^p\, d\nu_p(x) \leq \|f\|^p_w \nu_p(\mathbf{R}),$$

allow us to approximate the two functions $x/(1+x^2)$ and $1/(1+x^2)$ by polynomials in $L_p(\mathbf{R}, d\mu)$, as requested.

2.2. Necessity of Theorem 1(a)

For every $p \in \mathbf{N}$ the relation (3) shows that our assumption implies a representation of the measure μ as

$$d\mu(x) = w_p(x)^p\, d\nu_p(x), \quad w_p \in \mathcal{W}^*_1(\mathbf{R}),\ \nu_p \in \mathcal{M}^+(\mathbf{R}).$$

Hence, according to Remark 1, $1/w_p(x)^p \in L_1(\mathbf{R}, d\mu)$, and

$$\nu_p(A) = \int_A \frac{1}{w_p(x)^p}\, d\mu(x), \quad A \in \mathcal{B}(\mathbf{R}).$$

By the Beppo–Levi theorem we find

$$\nu(A) := \sum_{p \geq 1} \frac{\nu_p(A)}{2^p \nu_p(\mathbf{R})} = \int_A \frac{1}{w(x)}\, d\mu(x), \quad A \in \mathcal{B}(\mathbf{R}),$$

where

$$\frac{1}{w(x)} := \sum_{p \geq 1} \frac{1}{2^p \nu_p(\mathbf{R}) w_p(x)^p} \in L_1(\mathbf{R}, d\mu).$$

The function $1/w$ is lower semi-continuous and, furthermore, $w(x) \leq 2^p \nu_p(\mathbf{R}) w_p(x)^p$, $p \geq 1$. Therefore $w \in \mathcal{W}^*(\mathbf{R})$ and $C^0_{w^{1/p}} \in \overline{\mathcal{P}}$ for any $p \geq 1$. So that $w(x)$ is indeed a function whose existence was asserted in Theorem 1(a).

2.3. Necessity of Theorem 1(c)

To finish the proof of Theorem 1, it suffices to prove the necessity of (c), since (c) implies (b). The crucial tool will be the following lemma which is of independent interest.

Lemma 2. *For every* $w \in \mathcal{W}_0^*(\mathbf{R})$ *there exist* $V \in \mathcal{W}_0^*(\mathbf{R})$ *and finite positive constants* C_τ, *independent of* x, *such that*

$$w(x) \leq C_\tau V(x)^\tau, \quad x \in \mathbf{R}, \ 0 < \tau < \infty. \tag{13}$$

The proof of Lemma 2 is rather lengthy and involved. We postpone it to Section 9. If $\mu \in \mathcal{M}_\infty^*(\mathbf{R})$ then by Theorem 1(a) (which has been established already) there exists $w \in \mathcal{W}_0^*(\mathbf{R})$ such that $1/w \in L_1(\mathbf{R}, d\mu)$. Lemma 2, applied to this w, asserts the existence of a function $V \in \mathcal{W}_0^*(\mathbf{R})$ satisfying (13). These inequalities imply $1/V \in L(\mathbf{R}, d\mu)$ and, hence, that V satisfies all the conditions of Theorem 1(c).

3. Proof of Corollaries 1 and 2

To prove Corollary 1 we need the representation of the measure in question, as described in Remark 1, namely $d\mu(x)/w(x)^q = w(x)^{p-q}\, d\nu_p(x)$, $q<p$, $\nu_p \in \mathcal{M}^+(\mathbf{R})$. Theorem 1 and the fact that $w \in \mathcal{W}_0^*(\mathbf{R})$ complete the proof of Corollary 1(a). The integrability of $1/w^p$ with respect to μ for every $1 \leq p < \infty$ together with the evident inequality

$$\int_{\mathbf{R}} |f(x)|^p \, d\mu(x) = \int_{\mathbf{R}} \frac{1}{w(x)^p} w(x)^p |f(x)|^p \, d\mu(x) \leq \|f\|_w^p \|1/w(x)\|_{L_p(\mathbf{R}, d\mu)}^p,$$

imply Corollary 1(b).

Corollary 2 follows easily from Theorem 1(a) and Remark 1.

4. Proof of Theorem 2

Assume first that $w \notin \mathcal{W}_1^*(\mathbf{R})$. Since $\mathcal{P}$ is not dense in C_w^0 there exists a non-trivial linear continuous functional L on C_w^0, vanishing on all monomials. In [4] (see also [2, Theorem 1.3, p. 9]) a general form of the linear continuous functionals on C_w^0 was determined, and that result implies, for our case, the existence of $\varrho \in \mathcal{M}(\mathbf{R})$ with

$$L(x^n) = \int_{\mathbf{R}} x^n w(x) \, d\varrho(x) = 0, \quad n \geq 0.$$

Let $\mathbf{R}_\varrho^+, \mathbf{R}_\varrho^- \in \mathcal{B}(\mathbf{R})$, $\mathbf{R}_\varrho^+ \cup \mathbf{R}_\varrho^- = \mathbf{R}$, $\mathbf{R}_\varrho^+ \cap \mathbf{R}_\varrho^- = \emptyset$, be a Hahn decomposition of the space $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$ with respect to the measure ϱ, and let $\varrho = \varrho_+ - \varrho_-$ be its decomposition in the sense of Jordan: $\varrho_\pm(A) := \varrho(A \cap \mathbf{R}_\varrho^\pm)$, $A \in \mathcal{B}(\mathbf{R})$, and $|\varrho| := \varrho_+ + \varrho_-$. Then

$$0 = \int_{\mathbf{R}} x^n [\chi_{\mathbf{R}_\varrho^+}(x) - \chi_{\mathbf{R}_\varrho^-}(x)] w(x)\, d|\varrho|(x), \quad n \geq 0. \tag{14}$$

Since $\chi_{\mathbf{R}_\varrho^+}(x) - \chi_{\mathbf{R}_\varrho^-}(x)$ is uniformly bounded on $\mathbf{R}$ we conclude from (14) that $L_1(\mathbf{R}, w\, d\nu) \notin \overline{\mathcal{P}}$, where $\nu := |\varrho| \in \mathcal{M}^+(\mathbf{R})$. For $1 < p < \infty$ the relation (14) reads

$$0 = \int_{\mathbf{R}} x^n \frac{\chi_{\mathbf{R}_\varrho^+}(x) - \chi_{\mathbf{R}_\varrho^-}(x)}{w(x)^{p-1}} w(x)^p\, d\nu(x), \quad n \geq 0. \tag{15}$$

The inequality

$$\left| \frac{\chi_{\mathbf{R}_\varrho^+}(x) - \chi_{\mathbf{R}_\varrho^-}(x)}{w(x)^{p-1}} \right|^q \leq \frac{2^q}{w(x)^p}, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

implies

$$\frac{\chi_{\mathbf{R}_\varrho^+}(x) - \chi_{\mathbf{R}_\varrho^-}(x)}{w(x)^{p-1}} \in L_q(\mathbf{R}, w^p d\nu),$$

and this together with (15) proves $L_p(\mathbf{R}, w^p\, d\nu) \notin \overline{\mathcal{P}}$.

For the converse suppose that $w \in \mathcal{W}_1^*(\mathbf{R})$. Then, for $p \in [1, +\infty)$, the relation (3) shows that the measure μ, defined as $d\mu := w^p d\nu$, belongs to $\mathcal{M}_p^*(\mathbf{R})$ for any $\nu \in \mathcal{M}^+(\mathbf{R})$. The proof of Theorem 2 is complete.

5. Proof of Theorem 3

Without loss of generality we can assume that $\|1\|_w \leq 1$. If $w \in \mathcal{W}_0^*(\mathbf{R})$ then Theorem B implies

$$\sum_{\lambda \in \Lambda_B} \frac{(1/w(\lambda))^\tau}{|B'(\lambda)|} = \infty \tag{16}$$

for all $\tau > 0$ and for all $B \in \mathcal{E}_0$ with $\Lambda_B \subset S_w$. Assume there exists $F \in \mathcal{E}_0$ and $\Lambda_F \subset S_w$ with

$$\overline{\lim_{\substack{|\lambda| \to \infty \\ \lambda \in \Lambda_F}}} \frac{\log 1/w(\lambda)}{\log^+ |F'(\lambda)|} \leq \beta < \infty.$$

Then, for some $C > 0$,

$$C \log \frac{1}{w(\lambda)} \leq \log^+ |F'(\lambda)|, \quad \lambda \in \Lambda_F.$$

or

$$|F'(\lambda)| \geq \frac{1}{w(\lambda)^C}, \quad \lambda \in \Lambda_F.$$

Therefore, by the definition of $\mathcal{W}^*(\mathbf{R})$,

$$\sum_{\lambda\in\Lambda_F} \frac{(1/w(\lambda))^{C/2}}{|F'(\lambda)|} \leq \sum_{\lambda\in\Lambda_F} w(\lambda)^{C/2} < \infty,$$

which contradicts (16). In the other direction, if there exists $F\in\mathcal{E}_0$ with $\Lambda_F \subset S_w$ and $\varrho>0$ such that

$$C := \sum_{\lambda\in\Lambda_F} \frac{(1/w(\lambda))^{\varrho}}{|F'(\lambda)|} < \infty,$$

then

$$\frac{1}{w(\lambda)^{\varrho}} \leq C|F'(\lambda)|, \quad \lambda \in \Lambda_F.$$

By definition of $\mathcal{W}^*(\mathbf{R})$ this implies

$$\varlimsup_{\substack{|\lambda|\to\infty \\ \lambda\in\Lambda_F}} \frac{\log 1/w(\lambda)}{\log^+ C|F'(\lambda)|} \leq \varlimsup_{\substack{|\lambda|\to\infty \\ \lambda\in\Lambda_F}} \frac{\log 1/w(\lambda)}{\log 1/C + \varrho \log 1/w(\lambda)} = 1/\varrho < \infty,$$

contradicting (5) and completing the proof.

6. Proofs of (4) and Theorem 4

The inclusion $\supset$ in (4) is obvious. On the other hand, since for every $\mu\in\mathcal{M}^*(\mathbf{R})$ the entire function $\sum_{n\geq 0} x^{2n}/(2^n\mu_{2n})$ belongs to $L_1(\mathbf{R}, d\mu)$, Remark 1 and the definition of $\mathcal{W}^*(\mathbf{R})$ imply that

$$d\mu(x) = \frac{1}{\displaystyle\sum_{n\geq 0} \frac{x^{2n}}{2^n \mu_{2n}}}\, d\nu(x), \quad \nu \in \mathcal{M}^+(\mathbf{R}), \tag{17}$$

and $\left(\sum_{n\geq 0} x^{2n}/2^n\mu_{2n}\right)^{-1} \in \mathcal{W}^*(\mathbf{R})$, as asserted. Observe that (17) also justifies the remark following (4).

We are now ready to prove Theorem 4. Indeed, Theorem 4(a) follows immediately from the representation (17). Using the assumption of Theorem 4(b), Definition 1, and the inequalities

$$\frac{1}{\mu_{2n}} = \frac{1}{\int_{\mathbf{R}} x^{2n} w(x)\, d\nu(x)} \geq \frac{b_{2n}}{\nu(\mathbf{R})}, \quad n \geq 0,$$

(which follow from (7)), we obtain $\mu\in\mathcal{M}_C^*(\mathbf{R})$, as asserted. In both cases Corollary 3 yields $C^0_{w^\tau \chi_J} \in \overline{\mathcal{P}}$ for every $J\in\mathcal{R}(\nu)$ and $\tau>0$ which completes the proof of Theorem 4.

7. Proof of Lemma 1

Since $b_k := |a_{mk}|^{1/nk}$, $k \geq 1$, is an increasing sequence the assumptions imply that

$$\sum_{k \geq 1} \frac{1}{b_{Nk}} = \infty$$

holds for any $N \in \mathbf{N}$. For $N=1$ this makes it possible to find $k_1 > 1$ such that

$$\sum_{k=1}^{k_1-1} \frac{1}{b_k} \geq 1.$$

Similarly, for $N=2$ we find a positive integer $k_2 > k_1$ such that:

$$\sum_{k=k_1}^{k_2-1} \frac{1}{b_{2k}} \geq 1,$$

and so on. Consequently, we obtain a strictly increasing sequence of positive integers $\{k_r\}_{r \geq 0}$ (setting $k_0 := 1$), which satisfies

$$\sum_{r \geq 1} \sum_{k=k_{r-1}}^{k_r} \frac{1}{b_{rk}} = \infty. \tag{18}$$

Rearranging the indices rk in (18) in increasing order we obtain a sequence $\{n_k\}_{k \geq 1}$ which satisfies

$$\sum_{k \geq 1} \frac{1}{b_{n_k}} = \infty. \tag{19}$$

Since the sequence $\{n_k\}_{k \geq 1}$ is strictly increasing we conclude that $n_k \geq k$ for every $k \geq 1$.

Now we shall prove (10). Fix an arbitrary integer $q \geq 1$ and find $m_{2q} \geq 1$ such that $n_{m_{2q}} = 2qk_{2q-1}$. Then for every $m \geq m_{2q}$ and $r \geq 1$ we have $n_{m+r} - n_m \geq 2qr$, and, hence,

$$n_{m_{2q}+r} \geq 2qk_{2q-1} + 2rq = q(2k_{2q-1} + 2r) \geq q(m_{2q} + r) \quad \text{for } r \geq m_{2q}.$$

Therefore $n_k \geq qk$ for all $k \geq 2m_{2q}$. These inequalities and (19) complete the proof of Lemma 1.

8. Proofs of Theorem 5 and Corollary 4

For Carleman measures we may assume that $\mu_0=1$. Then Hölder's inequality yields

$$m_s^{1/s} \le m_t^{1/t}, \quad 0 \le s \le t < \infty, \tag{20}$$

where

$$m_s := \int_{\mathbf{R}} |x|^s \, d\mu(x), \quad s \ge 0.$$

This shows that S_μ satisfies the assumption of Lemma 1 with $(m,n)=(2,2b(\mu))$ as has been remarked just before Theorem 5.

Using (20) and (10) we conclude that the series on the right of

$$\left\| \frac{1}{w_\mu(x)} \right\|_{L_p(\mathbf{R},d\mu)} \le \mu(\mathbf{R}) + \sum_{k\ge 1} \frac{m_{2kp}^{1/p}}{2^k \mu_{2n_k}^{k/n_k}} = \mu(\mathbf{R}) + \sum_{k \ge 1} \frac{1}{2^k} \left(\frac{m_{2kp}^{1/2kp}}{m_{2n_k}^{1/2n_k}} \right)^{2k}$$

is convergent and, hence, $1/w_\mu$ belongs to $L_p(\mathbf{R}, d\mu)$ for every $1 \le p < \infty$. As in Section 6 we find the representation $d\mu = w_\mu^p \, d\varrho_p$, with $\varrho_p \in \mathcal{M}^+(\mathbf{R})$ for $p \ge 1$. Hence $d\mu = w_\mu^{1+p} \, d\varrho_{1+p} = w_\mu^p \, d\nu_p$, where $d\nu_p = w_\mu \, d\varrho_p$. By Theorem 4(b), Lemma 1 and Corollary 3 the measure ν_p is Carleman for every $p \ge 1$ and $w_\mu \chi_J \in \mathcal{W}_0^*(\mathbf{R})$ for all $J \in \mathcal{R}(\mu)$. Furthermore, Theorem A implies that $\nu_p \in \mathcal{M}_\infty^*(\mathbf{R})$, and from Remark 1 we have $1/w_\mu \in L(\mathbf{R}, d\mu)$. Theorem 5 is proved.

Finally, Corollary 4 follows immediately from Corollary 1, since w_μ satisfies all conditions of Theorem 1(c).

9. Proof of Lemma 2

9.1. Some auxiliary lemmas

Let Φ be the following collection of functions

$$\phi(x) := \begin{cases} \left(\displaystyle\sum_{n \ge 1} \frac{a_n}{x^n} \right)^{-1}, & x > 0, \\ 0, & x = 0, \end{cases} \tag{21}$$

where $a_n \ge a_{n+1} > 0$, $n \in \mathbf{N}$, and $\overline{\lim}_{n\to\infty} \sqrt[n]{a_n} = 0$. Observe that ϕ and its inverse ϕ^{-1} are both strictly increasing and continuous on $\mathbf{R}^+$, $\phi^{-1}(0)=0$ and

$$\phi(x) \le x^n / a_n, \quad \phi^{-1}(x) \ge a_n^{1/n} x^{1/n}, \quad n \ge 1, \ x \ge 0. \tag{22}$$

Lemma 3. *For every $\phi\in\Phi$ there exists a $\psi\in\Phi$ such that*

$$\psi(\psi(x)) \geq \phi(x), \quad x \geq 0. \tag{23}$$

Proof. We have $\phi(x)=1/\varphi(1/x)$, where $\varphi(x)=\sum_{n\geq 1} a_n x^n$ is an entire function with positive and non-increasing coefficients a_n, $n\geq 1$. Let $b_n:=a_n/\max\{1,a_1\}$, $n\geq 1$, and

$$\omega(x) := \sum_{n\geq 1} \beta_n x^n, \quad \beta_n := \frac{b_{n^2}}{2^{2n^2}}, \; n \geq 1.$$

To prove the first inequality in

$$\omega(\omega(x)) \leq \sum_{n\geq 1} b_n x^n \equiv \frac{\varphi(x)}{\max\{1, a_1\}} \leq \varphi(x), \quad x \geq 0, \tag{24}$$

we write for $n\in\mathbf{N}$,

$$\omega(x)^n = \Big(\sum_{m\geq 1} \beta_m x^m\Big)^n = \sum_{k\geq n} \beta_k(n) x^k,$$

where

$$\beta_k(n) = \sum_{\substack{m_1,\dots,m_n\geq 1\\ m_1+\dots+m_n=k}} \beta_{m_1}\beta_{m_2}\dots\beta_{m_n}, \quad k\geq n\geq 1. \tag{25}$$

Note that the number of terms in (25) is $\binom{k-1}{n-1}\leq 2^k$.

Let $k\geq n^2$. Then each term in (25) contains at least one factor with index m_j not less than k/n, so that $m_j^2\geq k^2/n^2\geq k$ and

$$\beta_{m_j} = \frac{b_{m_j^2}}{2^{2m_j^2}} \leq \frac{b_k}{2^{2m_j^2}} \leq \frac{b_k}{2^{2k}}.$$

Since all other factors of this same term are not greater than 1 we find

$$\beta_k(n) \leq \binom{k-1}{n-1} \frac{b_k}{2^{2k}} \leq \frac{b_k}{2^k}, \quad 1 \leq n^2 \leq k. \tag{26}$$

On the other hand,

$$\beta_k(n) \leq \binom{k-1}{n-1} \frac{1}{2^k} \leq 1, \quad 1 \leq n \leq k, \tag{27}$$

since $\beta_n\leq 1/2^n$ for all $n\geq 1$.

We have

$$\omega(\omega(x))=\sum_{n\geq 1}\beta_n\omega(x)^n=\sum_{n\geq 1}\beta_n\sum_{k\geq n}\beta_k(n)x^k=\sum_{k\geq 1}x^k\left(\sum_{n=1}^{k}\beta_n\beta_k(n)\right),$$

and from (26) and (27) we get for fixed k

$$\begin{aligned}\sum_{n=1}^{k}\beta_n\beta_k(n)&=\sum_{1\leq n^2\leq k}\beta_n\beta_k(n)+\sum_{k<n^2\leq k^2}\beta_n\beta_k(n)\leq\sum_{1\leq n^2\leq k}\beta_k(n)+\sum_{k<n^2\leq k^2}\beta_n\\&\leq\sum_{1\leq n^2\leq k}\frac{b_k}{2^k}+\sum_{k<n^2\leq k^2}\frac{b_{n^2}}{2^{2n^2}}\leq\frac{b_k\sqrt{k}}{2^k}+b_k\sum_{n\geq 8}\frac{1}{2^n}\leq b_k\left(\frac{1}{2}+\frac{1}{2^7}\right)<b_k,\end{aligned}$$

so that (24) follows. Setting $\psi(x):=1/\omega(1/x)$ we obtain (23). □

From now on, for a function $F\colon\mathbf{R}\to\mathbf{R}^+$, we shall write $S_F:=\{x\in\mathbf{R}\,|\,F(x)>0\}$.

Lemma 4. *Let $w,v\in\mathcal{W}^*(\mathbf{R})$ and assume that*

$$w(x)\leq C_n v(x)^n,\quad x\in\mathbf{R},\ n\geq 1, \tag{28}$$

with positive finite constants C_n, $n\geq 1$. Then there exists a function $\theta\in\mathcal{W}^*(\mathbf{R})$ with $S_v\subset S_\theta$ and

$$w(x)\leq D_n(v(x)\theta(x))^n,\quad x\in\mathbf{R},\ n\geq 1, \tag{29}$$

with positive finite constants D_n, $n\geq 1$.

Proof. If S_w is bounded then one can find $a>0$ such that $S_w\subset[-a,a]$ and set $\theta(x):=\chi_{[-a,a]}(x)+e^{-(a-|x|)^2}\chi_{\mathbf{R}\setminus[-a,a]}(x)\in\mathcal{W}^*(\mathbf{R})$ with $S_\theta=\mathbf{R}\supset S_v$.

Let now S_w be unbounded. From (28) it is obvious that $\underline{\lim}_{n\to\infty}C_n^{1/n}=+\infty$, and

$$w(x)\leq\phi(v(x)),\quad x\in\mathbf{R},$$

where $\phi(x):=(\sum_{k\geq 1}2^{-k}d_k^{-1}x^{-k})^{-1}$, $d_k:=\max_{1\leq i\leq k}C_i$, $k\geq 1$, and $\underline{\lim}_{k\to\infty}d_k^{1/k}=+\infty$. Since $\phi\in\Phi$ Lemma 3 gives the existence of

$$\psi(x)=\left(\sum_{n\geq 1}\frac{c_n}{x^n}\right)^{-1}\in\Phi$$

such that

$$\phi(x)\leq\psi(\psi(x)),\quad x\in\mathbf{R}^+,$$

and therefore

$$w(x) \leq \psi(\psi(v(x))), \quad x \in \mathbf{R}. \tag{30}$$

Writing

$$u(x) := \psi(v(x)), \quad x \in \mathbf{R},$$

we get from (22) and (30) that

$$w(x) \leq \psi(u(x)) \leq \frac{u(x)^n}{c_n}, \quad x \in \mathbf{R}, \ n \geq 1. \tag{31}$$

Using formula (21) it is easy to verify that the function $\psi(x)/x$ is non-negative and strictly increasing on $\mathbf{R}^+$, equals zero at $x=0$ and is bounded from above by $1/c_1$ uniformly on $\mathbf{R}^+$. Furthermore, by (22), $\psi(x)/x \leq x^n/c_{n+1}$, $x \in \mathbf{R}^+$, $n \in \mathbf{N}$. Thus, the function

$$\theta(x) := \frac{\psi(v(x))}{v(x)}, \quad x \in \mathbf{R},$$

belongs to $\mathcal{W}^*(\mathbf{R})$ and

$$u(x) = \theta(x)v(x), \quad x \in \mathbf{R}.$$

Substitution of the last expression for $u(x)$ in (31) proves (29) and completes the proof of Lemma 4. □

The following Lemma 5 is instrumental for the proof of Lemma 6, which supplies a suitable application of Theorem B. Lemma 5 is a special case of Lemma 4.1 in [3], but to make the paper more self-contained a proof of Lemma 5 is given in the appendix below.

Lemma 5. *For any entire function $B \in \mathcal{E}_0$ with zeros $\Lambda_B = \{b_n\}_{n \geq 1}$ there exists a constant $C > 0$ and a sequence of real positive numbers $\{\delta_n\}_{n \geq 1}$, such that for any sequence of real numbers $\{d_n\}_{n \geq 1}$ satisfying*

$$|b_n - d_n| \leq \delta_n, \quad n \geq 1, \tag{32}$$

it is possible to find an entire function $D \in \mathcal{E}_0$ such that $\Lambda_D = \{d_n\}_{n \geq 1}$ and

$$|B'(b_n)| \leq C|D'(d_n)|, \quad n \geq 1. \tag{33}$$

We recall the definition of the so-called *upper Baire function* M_F of $F: \mathbf{R} \to \mathbf{R}$ as $M_F(x) := \lim_{\delta \downarrow 0} \sup_{y \in (x-\delta, x+\delta)} F(y)$. If F is locally bounded from above then M_F is an upper semi-continuous function.

Lemma 6. *For $w\in\mathcal{W}^*(\mathbf{R})$ with unbounded S_w let $S\subset S_w$ be such that $w(x)=M_{w\chi_S}(x)$, $x\in\mathbf{R}$. If for any countable set $A\subset S$ without finite accumulation point we have $C^0_{w\chi_A}\in\overline{\mathcal{P}}$ then also $C^0_w\in\overline{\mathcal{P}}$.*

Proof. The assumptions of Lemma 6, together with Theorem B, show that

$$\sum_{\lambda\in\Lambda_F}\frac{1}{w(\lambda)|F'(\lambda)|}=\infty \tag{34}$$

holds for every $F\in\mathcal{E}_0$ with $\Lambda_F\subset S$.

Assume that $\mathcal{P}$ is not dense in C^0_w. Then by Theorem B one can find an entire function $B\in\mathcal{E}_0$ such that $\Lambda\subset S_w$ and

$$\sum_{n\geq 1}\frac{1}{w(b_n)|B'(b_n)|}<\infty,\qquad \Lambda_B=\{b_n\}_{n\geq 1}. \tag{35}$$

For this function B Lemma 5 gives the existence of a sequence of positive numbers $\{\delta_n\}_{n\geq 1}$ valid for the conclusions of that lemma. We now choose a sequence of real numbers $\{d_n\}_{n\geq 1}$ satisfying (32): if $b_n\in S$ then let $d_n=b_n$. If $b_n\in\Lambda_B\setminus S(\subset S_w\setminus S)$, then the definition of S in Lemma 6 assures the existence of a sequence $s_k\in S$, $k\geq 1$, such that $\lim_{k\to\infty}s_k=b_n$ and $\lim_{k\to\infty}w(s_k)=w(b_n)$. Thus, it is possible to find an index $k\geq 1$ for which $|s_k-b_n|<\delta_n$ and $w(s_k)\geq\frac{1}{2}w(b_n)$. In this case we set $d_n=s_k$. With this choice we have $\{d_n\}_{n\geq 1}\subset S$ and a combination of Lemma 5 and (35) gives

$$\sum_{n\geq 1}\frac{1}{w(d_n)|D'(d_n)|}<\infty,$$

for a suitable $D\in\mathcal{E}_0$. This contradicts (34) with $F=D$ and finishes the proof. □

9.2. The proof of Lemma 2

Below we shall work with the complex space $CC^0_w=C^0_w+iC^0_w$ and the set $\mathcal{P}[\mathbf{C}]$ of polynomials with complex coefficients. Reasoning similarly to that at the beginning of Subsection 2.1 shows that the denseness of $\mathcal{P}$ in the real space C^0_w is equivalent to the denseness of $\mathcal{P}[\mathbf{C}]$ in the complex space CC^0_w and this, in turn, to the possibility of approximating the single function $1/(x+i)$ in CC^0_w by polynomials from $\mathcal{P}[\mathbf{C}]$.

If S_w is bounded, i.e. if there exists $a>0$ such that $S_w\subset[-a,a]$, then one can set $V=\chi_{[-a,a]}$. Assume that S_w is unbounded. Then the proof of Lemma 2 consists of four steps.

9.2.1. Without loss of generality we assume that w is bounded by 1 from above on $\mathbf{R}$. According to Theorem 1(a), we find a measure $\nu\in\mathcal{M}^+(\mathbf{R})$ and a function $w\in\mathcal{W}_0^*(\mathbf{R})$ such that

$$d\mu(x)=w(x)\,d\nu(x).$$

For every $k\geq 1$ one can approximate $1/(x+i)$ by polynomials from $\mathcal{P}[\mathbf{C}]$ in the space $C^0_{w^{1/k^2}}$. Consequently we find a sequence of polynomials $P_k\in\mathcal{P}[\mathbf{C}]$, $k\in\mathbf{N}$, such that

$$\left\|\frac{1}{x+i}-P_k(x)\right\|_{w^{1/k^2}}\leq\frac{1}{2^k},\quad k\geq 1,$$

or, which is the same,

$$\left|\frac{1}{x+i}-P_k(x)\right|^k\leq\frac{1}{2^{k^2}w(x)^{1/k}},\quad k\geq 1,\ x\in S_w. \tag{36}$$

Next we look at

$$\alpha_N(x):=1+\sum_{k=1}^{N}\left|\frac{1}{x+i}-P_k(x)\right|^k,\quad N\geq 1,$$

and set

$$\gamma_k(x):=\left|\frac{1}{x+i}-P_k(x)\right|^k\quad\text{for } k\geq 1,$$

and $\alpha_0(x):=\gamma_0(x)\equiv 1$.

For every $N\geq 1$ the function α_N is continuous, non-negative on the real axis and majorized at infinity by some power function (and the same holds for γ_k for every $k\geq 1$). Therefore there exist positive finite constants D_N, $N\geq 1$, such that

$$\|\alpha_{N-1}\|_{w^{1/N}}\leq D_N,\quad N\geq 1. \tag{37}$$

Since α_N increases with N the limit (finite or infinite)

$$\alpha(x):=\lim_{N\to\infty}\alpha_N(x)=1+\sum_{k\geq 1}\left|\frac{1}{x+i}-P_k(x)\right|^k$$

is lower semi-continuous on $\mathbf{R}$. Moreover, from (36) and $w(x)\in[0,1]$ it follows that

$$\alpha(x)\leq 1+\sum_{k\geq 1}\frac{1}{2^{k^2}w(x)^{1/k}}\leq 1+\frac{1}{w(x)},\quad x\in S_w.$$

Therefore,

$$\alpha(x) < \infty, \quad x \in S_w.$$

The right-hand side of (36) tends to zero for every $x \in S_w$ as $k \to \infty$. Hence, the polynomials P_k converge to $1/(x+i)$ at every point of S_w. Therefore

$$\lim_{k \to \infty} \deg P_k = \infty. \tag{38}$$

Since

$$\alpha(x) \geq \left| \frac{1}{x+i} - P_k(x) \right|^k$$

the relation (38) implies that $\alpha(x)$ grows faster than any power function as $x \to \pm\infty$.

Setting $1/\infty := 0$, we define

$$\omega(x) := \frac{1}{\alpha(x)}, \quad x \in \mathbf{R},$$

and note that in view of the above $\omega \in \mathcal{W}^*(\mathbf{R})$ and $\omega(x) \leq 1$, $x \in \mathbf{R}$.

Fix an integer $p > 1$. Then Hölder's inequality yields for $1/p + 1/q = 1$,

$$\sum_{k \geq p} \gamma_k(x) \leq \left(\sum_{k \geq p} \frac{1}{2^{kq}} \right)^{1/q} \left(\sum_{k \geq p} 2^{kp} \gamma_k(x)^p \right)^{1/p} \leq \left(\sum_{k \geq p} 2^{kp} \gamma_k(x)^p \right)^{1/p}.$$

Hence, using (36),

$$\left(\sum_{k \geq p} \gamma_k(x) \right)^p \leq \sum_{k \geq p} 2^{kp} \gamma_k(x)^p \leq \sum_{k \geq p} \frac{1}{2^{k^2 - kp} w(x)^{p/k}}, \quad x \in S_w,$$

and finally

$$w(x) \left(\sum_{k \geq p} \gamma_k(x) \right)^p \leq \sum_{k \geq p} \frac{w(x)^{1-p/k}}{2^{k(k-p)}} \leq 2, \quad x \in S_w. \tag{39}$$

Since $\alpha(x) = \alpha_{p-1}(x) + \sum_{k \geq p} \gamma_k(x)$, (37) and (39) give

$$w(x)\alpha(x)^p \leq 2^p w(x) \alpha_{p-1}(x)^p + 2^p w(x) \left(\sum_{k \geq p} \gamma_k(x) \right)^p \leq 2^p D_p^p + 2^{p+1}, \quad x \in S_w.$$

Renaming $2^p D_p^p + 2^{p+1}$ by D_p, we get $w(x) \leq D_p \omega(x)^p$ for all $x \in S_w$ and since this inequality is also valid for $x \in \mathbf{R} \setminus S_w$ it follows that

$$w(x) \leq D_p \omega(x)^p, \quad p \geq 1, \ x \in \mathbf{R}. \tag{40}$$

9.2.2. Lemma 4 and (40) imply the existence of $\theta\in\mathcal{W}^*(\mathbf{R})$ such that $S_\omega\subset S_\theta$ and after a suitable renaming of the constants we find that

$$w(x)\le D_p(\omega(x)\theta(x))^p,\quad x\in\mathbf{R},\ p\ge 1.$$

Define(3)

$$V(x):=M_{\omega\theta\chi_{S_w}}(x),\quad x\in\mathbf{R}, \tag{41}$$

and observe that because every function in $\mathcal{W}^*(\mathbf{R})$ is uniformly bounded on $\mathbf{R}$, the inequalities (13) are non-trivial only for $\tau\in[1,+\infty)$ or, rather, only for $\tau\in\mathbf{N}$. Since the function $\omega\theta$ is upper semi-continuous, taking limits $\delta\to 0$ in

$$\omega(x)\theta(x)\le \sup_{y\in(x-\delta,x+\delta)\cap S_w}\omega(y)\theta(y)\le \sup_{y\in(x-\delta,x+\delta)}\omega(y)\theta(y),\quad x\in S_w,$$

gives $V\chi_{S_w}\equiv\omega\theta\chi_{S_w}$ and that, together with (41), implies (13) and

$$V(x)=M_{V\chi_{S_w}}(x),\quad x\in\mathbf{R}. \tag{42}$$

That $V\in\mathcal{W}^*(\mathbf{R})$ follows easily from (41) and from $\omega,\theta\in\mathcal{W}^*(\mathbf{R})$.

9.2.3. The next step is to prove that $C^0_{V^{1/m}}\in\overline{\mathcal{P}}$, $m\in\mathbf{N}$. To establish this property, we use Lemma 6 and (42): it is enough to show that $C^0_{V^{1/m}\chi_A}\in\overline{\mathcal{P}}$ for arbitrary countable sets $A\subset S_w$ without finite accumulation points, where, obviously,

$$V^{1/m}(x)\chi_A(x)\equiv\omega(x)^{1/m}\theta(x)^{1/m}\chi_A(x),\quad x\in A.$$

This will follow from

$$\lim_{k\to\infty}\left\|P_k(x)-\frac{1}{x+i}\right\|_{\omega^{1/m}\theta^{1/m}\chi_A}=0. \tag{43}$$

(3) The set S_ω is not smaller than S_w and so it is generally impossible to draw conclusions about the density of $\mathcal{P}$ in C^0_ω. To avoid this we restrict $\omega\theta$ to S_w and adjust the restriction to become an upper semi-continuous function V. Then $S_V\subset\bar{S}_w$.

9.2.4. We now prove (43). For arbitrary $T>0$, $x\in A$, $|x|\ge T$ and $k\ge m$ we have

$$\omega(x)^{1/m}\theta(x)^{1/m}\left|\frac{1}{x+i}-P_k(x)\right| = \theta(x)^{1/m}\frac{\left|\frac{1}{x+i}-P_k(x)\right|}{\left(1+\sum_{n\ge1}\left|\frac{1}{x+i}-P_n(x)\right|^n\right)^{1/m}}$$

$$\le \theta(x)^{1/m}\frac{\left|\frac{1}{x+i}-P_k(x)\right|}{\left(1+\left|\frac{1}{x+i}-P_k(x)\right|^k\right)^{1/m}}$$

$$\le \theta(x)^{1/m}$$

$$\le \sup_{|x|\ge T}\theta(x)^{1/m},$$

i.e.,

$$\sup_{\substack{x\in A\\ |x|\ge T}}\omega(x)^{1/m}\theta(x)^{1/m}\left|\frac{1}{x+i}-P_k(x)\right|\le\sup_{|x|\ge T}\theta(x)^{1/m},\quad k\ge m. \tag{44}$$

Fix an arbitrary $\varepsilon>0$. Since $\theta\in\mathcal{W}^*(\mathbf{R})$ one can find $T_\varepsilon>0$ such that

$$\sup_{|x|\ge T_\varepsilon}\theta(x)^{1/m}<\tfrac{1}{2}\varepsilon. \tag{45}$$

On the other hand, the interval $(-T_\varepsilon,T_\varepsilon)$ contains only finitely many points from $A\subset S_w$ and according to (36),

$$\lim_{k\to\infty}\left|P_k(x)-\frac{1}{x+i}\right|=0,\quad x\in S_w.$$

Thus there exists a positive integer $k_\varepsilon\ge m$ such that

$$\left|P_k(x)-\frac{1}{x+i}\right|\le\frac{\varepsilon}{2\sup_{x\in\mathbf{R}}\theta(x)^{1/m}},\quad x\in A\cap(-T_\varepsilon,T_\varepsilon). \tag{46}$$

The relation (44) (with $T=T_\varepsilon$), (45) and (46) imply that

$$\left\|P_k(x)-\frac{1}{x+i}\right\|_{\omega^{1/m}\theta^{1/m}\chi_A}<\varepsilon,\quad k\ge k_\varepsilon.$$

This proves (43) and thereby $V\in\mathcal{W}_0^*(\mathbf{R})$. The proof of Lemma 2 is complete.

10. An example

In this section we construct a discrete measure μ and a non-discrete measure ϱ in $\mathcal{M}^*_\infty(\mathbf{R})$ which do not satisfy Carleman's condition. The idea of the construction is similar to the one used in the proof of Theorem 17 in [26].

Let $\lambda_k := e^k$, $k \geq 1$, and

$$\Lambda := \{\lambda_k\}_{k\geq 1}, \quad Q_n(x) := \prod_{k=1}^{n} \left(1 - \frac{x}{\lambda_k}\right), \quad n \geq 1, \ x \in \mathbf{R}.$$

Define a discrete measure $\nu \in \mathcal{M}^+(\mathbf{R})$ and a discrete function v by

$$d\nu(x) := \sum_{k\geq 1} \frac{\delta(x-\lambda_k)}{1+\lambda_k^2}, \quad v(x) := \sum_{k\geq 1} v_k \chi_{\{\lambda_k\}}(x), \quad x \in \mathbf{R},$$

where $\delta(x)$ is the Dirac measure at zero,

$$v_1 = 1 \quad \text{and} \quad v_k = \min_{1\leq n\leq k-1} \left\{1, \frac{e^{-n^2}}{|Q_n(\lambda_k)|^n}\right\}, \ k \geq 2. \tag{47}$$

For any integer $m > N \geq 1$ and $x \in \Lambda \setminus \{\lambda_k\}_{k=1}^m$ we obtain

$$\begin{aligned} v(x)^{1/N} |Q_m(x)| &= \min_{n\geq 1} \left\{ |Q_m(x)|, \frac{|Q_m(x)| e^{-n^2/N}}{|Q_n(x)|^{n/N}} \right\} \\ &\leq \min \left\{ |Q_m(x)|, \frac{e^{-m^2/N}}{|Q_m(x)|^{(m-N)/N}} \right\} \leq e^{-m}. \end{aligned}$$

Here we used the inequality

$$\min\{y, a/y^\lambda\} \leq a^{1/(1+\lambda)}, \quad y \geq 0, \ a, \lambda > 0,$$

and the convention $1/0 := +\infty$. Hence

$$\|Q_n\|_{v^{1/N}} \leq e^{-n}, \quad n > N \geq 1, \tag{48}$$

and, in particular, $v \in \mathcal{W}^*(\mathbf{R})$.

According to a known criterion (see [26, Theorem 1]), the denseness of $\mathcal{P}$ in C^0_w, $w \in \mathcal{W}^*(\mathbf{R})$, is equivalent to $M(w/(1+|x|), i) = +\infty$, where

$$M(w, z) := \sup\{|P(z)| \mid \|P\|_w \leq 1 \text{ and } P \in \mathcal{P}\}, \quad z \in \mathbf{C},$$

is the so-called Holl–Mergelyan majorant. Since $w\in\mathcal{W}_0^*(\mathbf{R})$ yields $(1+|x|)^r w\in \mathcal{W}_0^*(\mathbf{R})$ for arbitrary $r>0$, we observe that $w\in\mathcal{W}_0^*(\mathbf{R})$ if and only if $(1+|x|)w^{1/N}\in \mathcal{W}_1^*(\mathbf{R})$ for every $N\geq 1$. Therefore this criterion for the membership in $\mathcal{W}_0^*(\mathbf{R})$ can be reformulated as

$$(49)\quad w\in\mathcal{W}_0^*(\mathbf{R}) \iff \inf\left\{\|P\|_{w^{1/N}} \,\middle|\, |P(i)|=1 \text{ and } P\in\mathcal{P}\right\}=0 \quad \text{for all } N\geq 1.$$

From $|Q_n(i)|^2\geq 1+e^{-2}$ combined with (48) we obtain

$$|P_n(i)|=1, \quad \|P_n\|_{v^{1/N}}\leq \frac{e^{-n}}{\sqrt{1+e^{-2}}}, \; n>N\geq 1,$$

where $P_n(x)=Q_n(x)/|Q_n(i)|$. From (49) we deduce that the weight v as well as $(1+x^2)v(x)$ belong to $\mathcal{W}_0^*(\mathbf{R})$. This fact and Theorem 1 allow us to claim that the measure μ defined as

$$(50)\qquad d\mu(x):=(1+x^2)v(x)\,d\nu(x)=\sum_{k\geq 1} v_k\delta(x-\lambda_k), \quad x\in\mathbf{R},$$

belongs to $\mathcal{M}_\infty^*(\mathbf{R})$.

Next we show that $\mu\notin\mathcal{M}_C^*(\mathbf{R})$, i.e.

$$(51)\qquad \sum_{p\geq 1}\mu_{2p}^{-4p}<\infty,$$

where

$$(52)\qquad \mu_{2p}=\int_{\mathbf{R}} x^{2p}\,d\mu(x)=\sum_{k\geq 1} v_k\lambda_k^{2p}, \quad p\geq 1.$$

To this end we need to derive some estimates. For the integers $1\leq n<k$ we have

$$|Q_n(\lambda_k)|\leq \frac{\lambda_k^n}{\lambda_1\lambda_2\ldots\lambda_n}=e^{n(k-(n+1)/2)}.$$

It is easy to verify that $n\big(k-\frac12(n+1)\big)\leq\frac12 k(k-1)$ holds for all $1\leq n\leq k-1$, $k\geq 2$. Therefore

$$|Q_n(\lambda_k)|\leq e^{k(k-1)/2}, \quad 1\leq n\leq k-1,$$

and

$$(53)\qquad |Q_n(\lambda_k)|^n\leq e^{k(k-1)^2/2}, \quad 1\leq n\leq k-1.$$

Substituting (53) into (47) we get

$$v_k \geq \min_{1\leq n\leq k-1} \{1, e^{-n^2} e^{-k(k-1)^2/2}\} \geq e^{-k^2-k^3/2}, \quad k\geq 2,$$

and then, by (52),

$$\mu_{2p} \geq \sum_{k\geq 1} e^{2pk-k^2/2-k^3/2}, \quad p\geq 1. \tag{54}$$

To further estimate the expression on the right of (54) observe that the function $\psi(x):=2px-\frac{1}{2}x^3$ attains its unique maximum over $[0,+\infty)$ at $x_p=\sqrt{4p/3}$ and that $\psi(x_p)=(8/\sqrt{27})p^{3/2}$. The interval $[x_p, x_p+1)$ contains the unique integer $k_p\geq 2$. Since $\psi(x)$ decreases on $[x_p,+\infty)$ we find that

$$\psi(x_p)-\psi(k_p)\leq \psi(x_p)-\psi(x_p+1) = -2p+\tfrac{1}{2}(3x_p^2+3x_p+1) = \sqrt{3p}+\tfrac{1}{2}\leq \tfrac{7}{3}p,$$

and thus

$$2pk_p-\frac{k_p^3}{2}\geq \frac{8}{3\sqrt{3}}p^{3/2}-\frac{7}{3}p\geq p^{3/2}-\frac{7}{3}p. \tag{55}$$

Combining (55) with the estimate

$$k_p^2\leq (x_p+1)^2 = \frac{4p}{3}+\frac{4\sqrt{p}}{\sqrt{3}}+1\leq \left(1+\frac{4}{3}+\frac{4}{\sqrt{3}}\right)p\leq \frac{14}{3}p,$$

we get

$$2pk_p-\tfrac{1}{2}k_p^3-k_p^2\geq p^{3/2}-7p,$$

so that

$$\mu_{2p}\geq e^{2pk_p-k_p^3/2-k_p^2}\geq e^{p^{3/2}-7p}, \quad p\geq 1. \tag{56}$$

These inequalities prove (51) and establish that $\mu\in\mathcal{M}_\infty^*(\mathbf{R})\setminus\mathcal{M}_C^*(\mathbf{R})$ which was our aim here.

Theorem 1 in [30, p. 294] along with the evident inequalities

$$((1+x^2)v(x)+e^{-x^2})^{1/N}\leq ((1+x^2)v(x))^{1/N}+e^{-x^2/N}, \quad N\geq 1,$$

imply that $(1+x^2)v(x)+e^{-x^2}\in\mathcal{W}_0^*(\mathbf{R})$ and therefore, using Theorem 1, we can conclude that $\varrho\in\mathcal{M}_\infty^*(\mathbf{R})$, where

$$d\varrho(x) := ((1+x^2)+e^{-x^2})\left(\frac{dx}{1+x^2}+d\nu(x)\right).$$

The definition of ϱ implies that $\operatorname{supp}\varrho=\mathbf{R}$ and the obvious inequalities

$$\int_{\mathbf{R}} x^{2p}\, d\varrho(x)\geq \mu_{2p}, \quad p\geq 1,$$

establish that $\varrho\in\mathcal{M}_\infty^*(\mathbf{R})\setminus\mathcal{M}_C^*(\mathbf{R})$.

Appendix: Proof of Lemma 5

Since the statement of Lemma 5 is invariant under the translation $x \mapsto x+a$, $a \in \mathbf{R}$, we may assume that $0 \notin \Lambda_B$, and also that $0<|b_k|\le|b_{k+1}|$ for $k\ge1$.

Let

$$\varrho_k := \min\Big\{1, |b_k|, \min_{\substack{\lambda\in\Lambda_B\\ |\lambda|\neq|b_k|}} \big||b_k|-|\lambda|\big|\Big\}, \quad k\ge1. \tag{57}$$

It is easy to verify that for any real constants α, β, ϱ, Δ and x, satisfying

$$0<2\Delta\le\varrho<|\alpha|, \quad |\alpha-\beta|\le\Delta \quad \text{and} \quad |x-\beta|\ge\varrho, \tag{58}$$

we have

$$\left|\left(1-\frac{x}{\alpha}\right)\left(1-\frac{x}{\beta}\right)^{-1}\right|\le 1+\frac{4\Delta}{\varrho}. \tag{59}$$

For every $k\ge1$ the function $B_k(x):=(1-x/b_k)^{-1}B(x)$ is continuously extendable to $\mathbf{R}$ with $B_k(b_k)=-b_kB'(b_k)$. Thus, there exists $\alpha_k>0$ such that

$$|B_k(x)|\ge\tfrac12|b_k|\,|B'(b_k)|, \quad |x-b_k|\le\alpha_k. \tag{60}$$

Let

$$\delta_k := \min\Big\{\alpha_k, \frac{\varrho_k}{4(1+b_k^2)}\Big\}, \quad k\ge1, \tag{61}$$

and consider an arbitrary sequence $\{d_k\}_{k\ge1}$ satisfying (32). Since $B\in\mathcal{E}_0$ its zeros satisfy one of the two possible conditions for entire functions of minimal exponential type in Lindelöf's theorem (see [10]). By (57) and (61) we find $|1/b_k-1/d_k|\le2/b_k^2$ for $k\ge1$, so that the same Lindelöf condition is true for the numbers d_k, $k\ge1$. In both cases the function

$$D(z) := \lim_{R\to+\infty} \prod_{\substack{k\ge1\\ |d_k|<R}} \left(1-\frac{z}{d_k}\right)$$

is an entire function of minimal exponential type. Moreover, due to our choice of numbers ϱ_k and δ_k it is possible to find a sequence of positive numbers R_n, $n\ge1$, tending to infinity, such that for every n the interval $(-R_n, R_n)$ contains the same number N_n of zeros of the functions $D(z)$ and $B(z)$. Then, obviously,

$$\frac{B_k(d_k)}{(-d_k)D'(d_k)} = \lim_{n\to\infty} \prod_{\substack{m=1\\ m\neq k}}^{N_n} \left(1-\frac{d_k}{b_m}\right)\left(1-\frac{d_k}{d_m}\right)^{-1}, \quad k\ge1. \tag{62}$$

Using (59) with $x=d_k$, $\alpha=b_m$, $\beta=d_m$, $\Delta=\delta_m$ and $\varrho=\varrho_m$, which satisfy (58), we get

$$\left|\left(1-\frac{d_k}{b_m}\right)\left(1-\frac{d_k}{d_m}\right)^{-1}\right| \le 1+\frac{4\delta_m}{\varrho_m} \le 1+\frac{1}{1+b_m^2}, \quad k,m\ge 1,\ m\ne k,$$

which by (60), (61) and (62) imply (note that $|d_k|\le|b_k|+\delta_k\le|b_k|+\frac{1}{4}\varrho_k\le\frac{5}{4}|b_k|$)

$$\frac{|B'(b_k)|}{|D'(d_k)|} \le \frac{2}{|b_k|}\frac{|B_k(d_k)|}{|D'(d_k)|} = \frac{2|d_k|}{|b_k|}\frac{|B'(d_k)|}{(-d_k)|D'(d_k)|} \le \frac{5}{2}\lim_{n\to\infty}\prod_{\substack{m=1\\ m\ne k}}^{N_n}\left(1+\frac{1}{1+b_m^2}\right)$$

$$\le \frac{5}{2}\lim_{n\to\infty}\exp\left(\sum_{m=1}^{N_n}\log\left(1+\frac{1}{1+b_m^2}\right)\right) \le \frac{5}{2}\exp\left(\sum_{m\ge 1}\frac{1}{1+b_m^2}\right) =: C$$

for all $k\ge1$. This estimate implies (33) and hence completes the proof of Lemma 5.

References

1. Akhiezer, N. I., *The Classical Moment Problem and Some Related Questions in Analysis*, Hafner Publ., New York, 1965.
2. Bakan, A. G., Polynomial approximation in $L_p(R^1, d\mu)$ I. General results and representation theorem, *Preprint,* Nats. Akad. Nauk Ukraïn., Inst. of Math., Kiev, 1998, No. 7.
3. Bakan, A. G., Polynomial density in $L_p(R^1, d\mu)$ and representation of all measures which generate a determinate Hamburger moment problem, in *Approximation, Optimization and Mathematical Economics* (*Pointe-à-Pitre, 1999*), pp. 37–46, Physica, Heidelberg, 2001.
4. Bakan, A. G., A criterion for denseness of polynomials and the general form of a linear continuous functional in the space C_w^0, *Ukraïn. Mat. Zh.* **54** (2002), 610–622 (Russian). English transl.: *Ukrainian Math. J.* **54** (2002), 750–762.
5. Berg, C., Recent results about moment problems, in *Probability Measures on Groups and Related Structures, XI* (*Oberwolfach 1994*), pp. 1–13, World Sci. Publishing, River Edge, NJ, 1995.
6. Berg, C. and Christensen, J. P. R., Exposants critiques dans le problème des moments, *C. R. Acad. Sci. Paris* **296** (1983), 661–663.
7. Berg, C. and Thill, M., Rotation invariant moment problems, *Acta Math.* **167** (1991), 207–227.
8. Bernstein, S. N., Le problème de l'approximation des fonctions continues sur tout l'axe réel et l'une de ses applications, *Bull. Soc. Math. France* **52** (1924), 399–410.
9. Bernstein, S. N., On weight functions., *Dokl. Akad. Nauk SSSR* **77** (1951), 549–552 (Russian).
10. Boas, R. P., Jr., *Entire Functions*, Academic Press, New York, 1954.

11. Borichev, A. and Sodin, M., The Hamburger moment problem and weighted polynomial approximation on discrete subsets of the real line, *J. Anal. Math.* **76** (1998), 219–264.
12. Borichev, A. and Sodin, M., Krein's entire functions and the Bernstein approximation problem, *Illinois J. Math.* **45** (2001), 167–185.
13. de Branges, L., The Bernstein problem, *Proc. Amer. Math. Soc.* **10** (1959), 825–832.
14. Carleman, T., Sur le problème des moments, *C. R. Acad. Sci. Paris* **174** (1922), 1680–1682.
15. Carleman, T., Sur les équations intégrales singulières à noyau réel et symétrique, in *Uppsala Univ. Årsskrift*, pp. 1–228, 1923.
16. Carleman, T., *Les fonctions quasi-analytiques*, Gauthier-Villars, Paris, 1926.
17. Fuchs, W. H. J., On a generalization of the Stieltjes moment problem, *Bull. Amer. Math. Soc.* **52** (1956), 1057–1059.
18. Greenstein, D. S., On a criterion for determinate moment sequences, *Proc. Amer. Math. Soc.* **13** (1962), 120–121.
19. Hamburger, H., Über eine Erweiterung des Stieltjesschen Momentenproblems, *Math. Ann.* **81** (1920), 235–319; *Math. Ann.* **82** (1921), 120–164; 168–187.
20. Hamburger, H., Hermitian transformations of deficiency index (1,1), Jacobi matrices and undetermined moment problems, *Amer. J. Math.* **66** (1944), 489–522.
21. Heyde, C. C., Some remarks on the moment problem. I, *Quart. J. Math. Oxford Ser.* **14** (1963), 91–96.
22. Korenblum, B., Mascuilli, A. and Panariello J., A generalization of Carleman's uniqueness theorem and a discrete Phragmén–Lindelöf theorem, *Proc. Amer. Math. Soc.* **126** (1998), 2025–2032.
23. Levin, B. Ya., Completeness of systems of functions, quasi-analyticity and subharmonic majorants, *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), Issled. Linein. Oper. Teorii Funktsii* **170** (1989), 102–156, 322–323 (Russian). English transl.: *J. Soviet Math.* **63** (1993), 171–201.
24. Lubinsky, D. S., Ideas of weighted polynomial approximation on $(-\infty, \infty)$, in *Approximation Theory VIII, Vol.* 1 (*College Station, TX, 1995*), pp. 371–396, World Sci. Publishing, River Edge, NJ, 1995.
25. Mandelbrojt, S., Théorèmes d'approximation et problèmes des moments, *C. R. Acad. Sci. Paris* **232** (1951), 1054–1056.
26. Mergelyan, S. N., Weighted approximation by polynomials, *Uspekhi Mat. Nauk* **11** (1956), 107–152 (Russian). English transl.: *Amer. Math. Soc. Transl.* **10** (1958), 59–106.
27. Riesz, M., Sur le problème des moments et le théorème de Parseval correspondant, *Acta Sci. Math.* (*Szeged*) **1** (1923), 209–225.
28. Shohat, J. A. and Tamarkin, J. D., *The Problem of Moments,* Amer. Math. Soc. Math. Surveys **1**, Amer. Math. Soc., New York, 1943.
29. Sjödin, T., A note on the Carleman condition for determinacy of moment problems, *Ark. Mat.* **25** (1987), 289–294.
30. Sodin, M., Which perturbations of quasianalytic weights preserve quasianalyticity? How to use de Branges' theorem, *J. Anal. Math.* **69** (1996), 293–309.

31. SODIN, M. and YUDITSKII, P., Another approach to de Branges' theorem on weighted polynomial approximation, in *Proceedings of the Ashkelon Workshop on Complex Function Theory (1996)*, Israel Math. Conf. Proc. **11**, pp. 221–227, Bar-Ilan Univ., Ramat Gan, 1997.

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