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Linear two-phase Venttsel problems

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Dedicated to the memory of A. D. Aleksandrov.

Abstract. A priori estimates are established for the two-phase boundary value problems with Venttsel interface conditions for linear nondivergent parabolic and elliptic equations. By these estimates, the existence and uniqueness theorems in Sobolev and Hölder spaces are proved.

The boundary value problems with Venttsel type conditions describe various physical processes in media containing a thin film of a material having high permeability. Some examples of the physical background can be found in [CM], [K1], [K2] and [S].

If such a thin film covers a boundary surface then we deal with the one-phase Venttsel problem which has been treated extensively during the last decade. For survey of results on the one-phase problem and a complete bibliography we refer to [AN5].

In this paper we start the study of the two-phase Venttsel problem where a thin film separates a medium into two parts. The condition on the interface in this case is specified by an equation of the second order with the principal term being a parabolic (elliptic) operator in tangential variables and with the first order term being a "jump" operator across the separating film.

The purpose of this paper is to establish the solvability results in Sobolev and Hölder spaces for the linear two-phase Venttsel problems under the condition that the separating film does not intersect the exterior boundary of a medium.

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Our arguments are based on a version of the Aleksandrov type maximum principle. The history of this excellent idea takes the beginning about forty years ago, and, over the last twenty years, it is used as a powerful tool for the study of boundary value problems for elliptic and parabolic equations. In the pioneering papers of A. D. Aleksandrov [A1], [A2], [A3], [A4] such a maximum principle was established for solutions of the elliptic Dirichlet problem. In the simplest case, a similar result is presented also in the paper of I. Ya. Bakel′man [B]. Further, the estimates from [A1], [A2], [A3], [A4] were improved in [A5] and [A6]. For solutions of the parabolic Dirichlet problem the Aleksandrov type maximum estimates were obtained by N. V. Krylov [Kr1], [Kr2], [Kr3] via probabilistic-analytic methods. Developing the geometric approach due to Aleksandrov, A. I. Nazarov and N. N. Ural′tseva [NU] and K. Tso [T] generalized these parabolic estimates to the case where the lower-order coefficients of the equation under investigation have summable singularities as had been done for the elliptic problem. A local maximum principle for solutions of the linear elliptic oblique derivative problem was proved by N. S. Nadirashvili [Na] and was extended to the case of unbounded coefficients by A. I. Nazarov [N2]. A corresponding estimate for solutions of the parabolic oblique derivative problem was announced in [N1] and proved in [N2]. For the one-phase elliptic Venttsel problem a local Aleksandrov type maximum principle was obtained by Y. Luo and N. S. Trudinger [LT] in two cases—in the nondegenerate case, i.e. when the boundary operator almost everywhere does not degenerate with respect to the second order terms, and in the degenerate one, i.e. when the second order terms in the boundary operator may vanish on some subsets of the separating film having nonzero surface measure. Later, these results were generalized by the authors [AN2], [AN3] to the case where the coefficients of the boundary equation may have summable singularities. Corresponding estimates for the one-phase parabolic Venttsel problem were established by D. E. Apushkinskaya [Ap]. Finally, the first section of this paper may be considered as a continuation of the results mentioned above to solutions of the two-phase Venttsel problems.

This paper is organized as follows. The major part of our paper deals with the nonstationary problem since that requires more special techniques. In Section 1, a local estimate of the maximum is established for solutions of the nondegenerate and degenerate linear parabolic problems. In Section 2 we concentrate our attention on the nondegenerate case only. Namely, we pose the linear two-phase problem in an arbitrary domain and obtain the global maximum estimate for solutions. Then, in the case of $\Sigma\cap\partial\Omega=\emptyset$, the coercive estimates are established and the existence and uniqueness theorems in Sobolev and Hölder spaces are obtained. For reason of space we provide in Section 2 just formulations of the theorems. We refer the reader to the preprint [AN4], where full proofs are given, and to the paper [AN1], where similar statements are obtained for the one-phase Venttsel problem. Corresponding a priori estimates for solutions of the stationary linear problem are formulated in Section 3.

Notation

By $x=(x',x_n)=(x_1,\dots,x_{n-1},x_n)$ we denote a vector in $\mathbf{R}^n$ with Euclidean norm $|x|$; (x,t) is a point in $\mathbf{R}^{n+1}$; (x',t) is a point in $\mathbf{R}^n_t$.

By Ω we denote a bounded domain in $\mathbf{R}^n$ and $\partial\Omega$ is its boundary; Σ is a sufficiently smooth hypersurface separating Ω into two subdomains: $\Omega^{(\mathrm{in})}$ and $\Omega^{(\mathrm{ex})}$; $\Sigma_T=\Sigma\times]0,T[$; $\mathbf{n}(x)=(\mathbf{n}_i(x))$ is the unit vector of the outward (with respect to $\Omega^{(\mathrm{in})}$) normal to Σ at the point x.

By Q we denote a cylinder in $\mathbf{R}^{n+1}$. For a cylinder $Q=\Omega\times]0,T[$ we denote by $\partial''Q=\partial\Omega\times]0,T[$ its lateral surface and by $\partial'Q=\partial''Q\cup(\bar{\Omega}\times\{0\})$ its parabolic boundary. We set $Q^{(\mathrm{ex})}=\Omega^{(\mathrm{ex})}\times]0,T[$ and $Q^{(\mathrm{in})}=\Omega^{(\mathrm{in})}\times]0,T[$. Various letters with indices (ex) and (in) stand for the functions defined in $Q^{(\mathrm{ex})}$ and in $Q^{(\mathrm{in})}$, respectively. We use the index (hh) if we would like to emphasize that some assumptions hold both for a function with index (in) and for one with index (ex).

We also use the following notation: $Q^{(\mathrm{r})}$ and $Q^{(\mathrm{l})}$ ($\Omega^{(\mathrm{r})}$ and $\Omega^{(\mathrm{l})}$) stand for the parts of Q (of Ω) lying in the right halfspace $x_n>0$ and in the left halfspace $x_n<0$, respectively; $\Gamma(Q)$ ($\Gamma(\Omega)$) denotes the part of Q (of Ω) lying on the hyperplane $x_n=0$. To denote functions defined in $Q^{(\mathrm{r})}$ and in $Q^{(\mathrm{l})}$ we use various letters with indices (r) and (l), respectively. We use the index (h) if some assumptions hold both for a function with index (r) and for one with index (l). We let

$$\begin{aligned} B_R &= \{x\in\mathbf{R}^n : |x|<R\},\\ Q_{R,T} &= B_R\times]0,T[,\\ \Gamma_{R,T} &= \Gamma(Q_{R,T}). \end{aligned}$$

The indices i and j always run from 1 to n, whereas the indices s and m run from 1 to $n-1$. We use the convention that repeated indices indicate summation.

Below we use D_i to denote the operator of differentiation with respect to x_i; $Du=(D_iu)=(D'u,D_nu)$ is the gradient of u.

Let d_i be the tangent differential operator on the manifold Σ. Then $du=(d_iu)$ is the tangential gradient of u; in particular, $du=(D'u,0)$ on $\Gamma_{R,T}$. Also $u_t=\partial u/\partial t$.

We denote by $\|\cdot\|_{p,Q}$ the norm in the space $L_p(Q)$. We also introduce the following spaces:

$W^{2,1}_p(Q)$ with the norm

$$\|u\|_{W^{2,1}_p(Q)}=\|u_t\|_{p,Q}+\|D(Du)\|_{p,Q}+\|u\|_{p,Q};$$

$W_p^{2,1}(\Sigma_T)$ with the norm

$$\|u\|_{W_p^{2,1}(\Sigma_T)} = \|u_t\|_{p,\Sigma_T} + \|d(du)\|_{p,\Sigma_T} + \|u\|_{p,\Sigma_T};$$

$V_p(Q) = W_p^{2,1}(Q^{(\mathrm{in})} \cup Q^{(\mathrm{ex})}) \cap W_{p-1}^{2,1}(\Sigma_T)$ with the norm

$$\|u\|_{V_p(Q)} = \|u\|_{W_p^{2,1}(Q^{(\mathrm{in})})} + \|u\|_{W_p^{2,1}(Q^{(\mathrm{ex})})} + \|u\|_{W_{p-1}^{2,1}(\Sigma_T)}.$$

By $C(\overline{Q})$ we denote the space of continuous functions; $\|\cdot\|_Q$ is the norm on $C(\overline{Q})$. By $C^{2,1}(\overline{Q})$ we denote the space of all functions continuous in Q together with their first derivatives with respect to (x,t) and second derivatives in x.

By $C^\gamma(\overline{Q})$ and $C^{2+\gamma}(\overline{Q})$, $0<\gamma<1$, we denote the Hölder spaces with the norms

$$\|u\|_{C^\gamma(\overline{Q})} = \|u\|_Q + [u]_{\gamma,Q},$$
$$\|u\|_{C^{2+\gamma}(\overline{Q})} = \|u\|_Q + \|D(Du)\|_Q + \|u_t\|_Q + [D(Du)]_{\gamma,Q} + [u_t]_{\gamma,Q},$$

respectively, where $[\cdot]_{\gamma,Q}$ stands for the Hölder constant, with Hölder exponent γ with respect to the parabolic distance

$$d_{\mathrm{par}}((x^1,t^1),(x^2,t^2)) = |x^1-x^2| + |t^1-t^2|^{1/2}.$$

We set $C_\Sigma^{2+\gamma}(\overline{Q}) = C^{2+\gamma}(\overline{Q^{(\mathrm{in})}}) \cap C^{2+\gamma}(\overline{Q^{(\mathrm{ex})}})$,

$$\|u\|_{C_\Sigma^{2+\gamma}(\overline{Q})} = \|u\|_{C^{2+\gamma}(\overline{Q^{(\mathrm{in})}})} + \|u\|_{C^{2+\gamma}(\overline{Q^{(\mathrm{ex})}})}.$$

We use the notation $\widehat{x,y}$ for the angle between the vectors x and y. We set $g_+ = \max\{g,0\}$, while $g_- = \max\{-g,0\}$, and denote by $\mathrm{tr}(a)$ the trace of the matrix (a). The exponent q satisfies $n<q<\infty$.

We use the letters M, N, C (with or without indices) to denote various constants. To indicate that, say, N depends on some parameters, we list them in the parentheses: $N(...)$.

1. The local maximum estimates

Let u be a function defined in $\overline{Q_{R,T}}$, $u|_{\Gamma_{R,T}} = u_0$, and suppose $u \le 0$ on $\partial' Q_{R,T}$. We introduce two upper convex-monotone hulls: z for the function u_+ in $Q_{R,T}$ and z' for the function $(u_0)_+$ on $\Gamma_{R,T}$. We recall that the upper convex-monotone hull of u is the least function which is concave with respect to space variables, increases in

t and majorizes u. For detailed descriptions of the properties of convex-monotone hulls we refer to [NU]. We define the contact set of u to be the set

$$Z=\{(x,t)\in Q_{R,T}:z(x,t)=u(x,t)\}.$$

In a similar manner, the contact set of u_0 is determined as the set

$$Z'=\{(x,t)\in \Gamma_{R,T}:z'(x,t)=u_0(x,t)\}.$$

It is evident that $Z\cap\Gamma_{R,T}\subset Z'$.

We consider the linear parabolic operators $L^{(\mathrm{h})}$,

$$L^{(\mathrm{h})}u\equiv\sigma_{(\mathrm{h})}(x,t)u_t-a^{ij}_{(\mathrm{h})}(x,t)D_iD_ju+b^i_{(\mathrm{h})}(x,t)D_iu+c^{(\mathrm{h})}(x,t)u,\quad (x,t)\in Q^{(\mathrm{h})},$$
$$a^{ij}_{(\mathrm{h})}=a^{ji}_{(\mathrm{h})}.$$

Suppose B is the linear parabolic interface operator

$$Bu\equiv\tau(x,t)u_t-\alpha^{sm}(x,t)D_sD_mu+\beta^s(x,t)D_su+\gamma(x,t)u,\quad (x,t)\in\Gamma(Q),$$
$$\alpha^{sm}=\alpha^{ms},$$

while J is the "jump" operator

$$Ju\equiv\beta_{(\mathrm{l})}(x',t)\lim_{x_n\to 0-}D_nu(x',x_n,t)-\beta_{(\mathrm{r})}(x',t)\lim_{x_n\to 0+}D_nu(x',x_n,t),\quad (x',t)\in\Gamma(Q).$$

Let us write

$$\Delta_{(\mathrm{h})}=(\sigma_{(\mathrm{h})}\det(a^{ij}_{(\mathrm{h})}))^{1/(n+1)},\quad g^{(\mathrm{h})}=\frac{|(b^i_{(\mathrm{h})})|}{\Delta_{(\mathrm{h})}};$$
$$\Delta'=(\tau\det(\alpha^{sm}))^{1/n},\qquad g'=\frac{|(\beta^s)|}{\Delta'}.$$

Theorem 1.1. *Suppose that* $\Omega\subset B_R$ *and* u *is a function such that*

$$u\in W^{2,1}_{n+1}(Q^{(\mathrm{r})}\cup Q^{(\mathrm{l})})\cap W^{2,1}_n(\Gamma(Q))\cap C(\overline{Q}),$$

$$L^{(\mathrm{r})}u\le f^{(\mathrm{r})}\quad in\ Q^{(\mathrm{r})},\tag{1.1}$$

$$L^{(\mathrm{l})}u\le f^{(\mathrm{l})}\quad in\ Q^{(\mathrm{l})},\tag{1.2}$$

$$Bu+Ju\le\varphi\quad on\ \Gamma(Q).\tag{1.3}$$

Assume also that the coefficients in (1.1)–(1.3) *satisfy the conditions*

(I) $a^{ij}_{(\mathrm{h})}\xi_i\xi_j\ge 0$ *for* $\xi\in\mathbf{R}^n$ *and* $\sigma_{(\mathrm{h})}\ge 0$, $c^{(\mathrm{h})}\ge 0$, $\sigma_{(\mathrm{h})}+\mathrm{tr}(a^{ij}_{(\mathrm{h})})>0$ *a.e. in* $Q^{(\mathrm{h})}$;

(II) $\alpha^{sm}\xi_s\xi_m\ge 0$ *for* $\xi\in\mathbf{R}^{n-1}$ *and* $\tau\ge 0$, $\gamma\ge 0$, $\beta_{(\mathrm{r})}\ge 0$, $\beta_{(\mathrm{l})}\ge 0$, $\tau+\mathrm{tr}(\alpha^{sm})>0$ *a.e. on* $\Gamma(Q)$.

If, in addition, $u \le 0$ on $\partial' Q$, then

$$u \le C_1\left(\left\|\frac{f_+^{(\mathrm{r})}}{\Delta_{(\mathrm{r})}}\right\|_{n+1,Z\cap Q^{(\mathrm{r})}}+\left\|\frac{f_+^{(\mathrm{l})}}{\Delta_{(\mathrm{l})}}\right\|_{n+1,Z\cap Q^{(\mathrm{l})}}\right)+C_2\left\|\frac{\varphi_+}{\Delta'}\right\|_{n,Z\cap\Gamma(Q)}, \tag{1.4}$$

where

$$C_1=C_0(n)\big(R^{n/(n+1)}+\|g^{(\mathrm{r})}\|^n_{n+1,Z\cap Q^{(\mathrm{r})}}+\|g^{(\mathrm{l})}\|^n_{n+1,Z\cap Q^{(\mathrm{l})}}+\|g'\|^{n^2/(n+1)}_{n,Z\cap\Gamma(Q)}\big),$$
$$C_2=C_0(n)\big(R^{(n-1)/n}+\|g^{(\mathrm{r})}\|^{(n^2-1)/n}_{n+1,Z\cap Q^{(\mathrm{r})}}+\|g^{(\mathrm{l})}\|^{(n^2-1)/n}_{n+1,Z\cap Q^{(\mathrm{l})}}+\|g'\|^{n-1}_{n,Z\cap\Gamma(Q)}\big)$$

and we set $\frac{0}{0}=0$ if such an indeterminacy arises.

Remark 1.1. Without loss of generality we may assume that $L^{(\mathrm{r})}$, $L^{(\mathrm{l})}$ and B are uniformly parabolic operators, $u\in C^{2.1}(\overline{Q^{(\mathrm{r})}})\cap C^{2.1}(\overline{Q^{(\mathrm{l})}})$ and $u|_{\partial' Q}<0$. All these restrictions could be removed by using the same argument as in [NU], Section 4.

In addition, we extend the function u_+ by zero to the set $Q_{R.T}\setminus Q$ and preserve the notation u for the extended function.

Proof. Let $M=\sup_{Q_{R.T}} z=\sup_{Q_{R.T}} u=u(x^0,t^0)$.

Next, consider some modification of the $(n+1)$-dimensional Legendre transformation Φ used in [T]. We define $\Phi\colon Z\to\mathbf{R}^{n+1}$ by

$$\Phi(x,t)=(Du,u-Du\cdot(x'-(x^0)',x_n)),$$

where $u=u(x,t)$. The determinant of the Jacobian of Φ is equal to $u_t\det(DDu)$. In the same way, we define the n-dimensional Legendre transformation $\Phi'\colon Z'\to\mathbf{R}^n_t$ by

$$\Phi'(x',t)=(D'u,u-D'u\cdot(x'-(x^0)')),$$

where $u=u(x',0,t)$.

Let

$$K=\{(p,q):p\in\mathbf{R}^n,\ q\in\mathbf{R}^1,\ 2R|p|<q<M\}.$$

For every $(p,q)\in K$ we introduce a hyperplane Π' that is the graph of the function $\pi'(x)=p'\cdot(x'-(x^0)')+q$. Since $\pi'(x)>0\ge u(x,0)$ for all $x\in B_R$, whereas $\pi'(x^0)=q<u(x^0,t^0)=M$, the hyperplane Π' will touch the graph of $u(\cdot,t)$ tangentially for some $t<t^0$. Below we will denote by (x^*,t^*) the point where Π' and the graph of u touch the first time. It is obvious that $(x^*,t^*)\in Z$ and $(p',0,q)=\Phi(x^*,t^*)$.

Consider the sets

$$K_{[r]}=\{(p,q)\in K:(p',0,q)\in\Phi(Z\cap Q^{(\mathrm{r})}_{R.T})\text{ and }p_n<0\};$$
$$K_{[l]}=\{(p,q)\in K:(p',0,q)\in\Phi(Z\cap Q^{(\mathrm{l})}_{R.T})\text{ and }p_n>0\};$$
$$K_{[\Gamma]}=\{(p,q)\in K:(p',0,q)\in\Phi(Z\cap\Gamma_{R.T})\}.$$

By definition, if $(p,q)\in K_{[\Gamma]}$ then the point (x^*,t^*) described above belongs to $Z\cap\Gamma_{R,T}$ and at this point we have

$$\lim_{x_n\to 0+} D_n u\le 0 \quad \text{and} \quad \lim_{x_n\to 0-} D_n u\ge 0. \tag{1.5}$$

Denote by $\widetilde{Z}$ the collection of all points belonging to $Z\cap\Gamma_{R,T}$ where the inequalities (1.5) are valid. Then, from (1.3), (1.5) and the assumptions on $\beta_{(\mathrm{h})}$ it follows that

$$Bu\le\varphi \quad \text{on } \widetilde{Z}. \tag{1.6}$$

We are now in a position to define a special subset $\mathcal{K}$ of K with properties to be investigated in the following. Let

$$\mathcal{K}=K_{[r]}\cup K_{[l]}\cup K_{[\Gamma]}.$$

Lemma 1.2. *We have*

$$\mathcal{K}\subset\Phi(Z\cap Q^{(\mathrm{r})}_{R,T})\cup\Phi(Z\cap Q^{(\mathrm{l})}_{R,T})\cup\left(\Phi'(\widetilde{Z})\times\left[-\frac{M}{2R},\frac{M}{2R}\right]\right). \tag{1.7}$$

Proof. For arbitrary $(p,q)\in\mathcal{K}$ we introduce the second hyperplane Π that is the graph of the function $\pi(x)=p\cdot(x'-(x^0)',x_n)+q=\pi'(x)+p_nx_n$. Next, our reasoning falls naturally into three parts.

1. Let $(p,q)\in K_{[r]}$. By definition, $p_n<0$ and $(x^*,t^*)\in Z\cap Q^{(\mathrm{r})}_{R,T}$. Hence, $\pi(x^*)<\pi'(x^*)=u(x^*,t^*)$. On the other hand, $\pi(x)>0\ge u(x,0)$ for all $x\in B_R$. Therefore, the support hyperplane Π will touch the graph of u for some $t<t^*$. Denote by $(\hat{x},\hat{t})$ the point where Π and the graph of u touch the first time.

We are going to show that $(\hat{x},\hat{t})\in Z\cap Q^{(\mathrm{r})}_{R,T}$. If this would not be the case, then

$$u(\hat{x},\hat{t})=\pi(\hat{x})\ge\pi'(\hat{x})\ge u(\hat{x},\hat{t}).$$

Therefore, the only possible case is the case with equalities throughout. But the relation $\pi'(\hat{x})=u(\hat{x},\hat{t})$ contradicts our assumption that t^* is the first time where Π' and the graph of u touch. Hence, $(\hat{x},\hat{t})\in Z\cap Q^{(\mathrm{r})}_{R,T}$ and, consequently, $(p,q)=\Phi(\hat{x},\hat{t})$. Therefore

$$K_{[r]}\subset\Phi(Z\cap Q^{(\mathrm{r})}_{R,T}). \tag{1.8}$$

2. Precisely as in part 1, we conclude

$$K_{[l]} \subset \Phi(Z \cap Q^{(l)}_{R,T}). \tag{1.9}$$

3. Consider now an arbitrary $(p,q) \in K_{[\Gamma]}$. By definition $(x^*,t^*) \in Z \cap \Gamma_{R,T}$. Consequently, it follows that $\pi(x^*) = \pi'(x^*) = u(x^*,t^*)$, while $\pi(x) > 0 \geq u(x,0)$ for all $x \in B_R$. But this means, that there exists $(\hat{x},\hat{t})$, $\hat{t} \leq t^*$ where Π and the graph of u touch the first time. It should be emphasized that the last inequality is a nonstrict one in contrast to above.

There are only three possibilities:

(1) $(\hat{x},\hat{t}) \in Z \cap Q^{(r)}_{R,T}$;

(2) $(\hat{x},\hat{t}) \in Z \cap Q^{(l)}_{R,T}$;

(3) $(\hat{x},\hat{t}) \in Z \cap \Gamma_{R,T}$.

It is evident that $(p,q) \in \Phi(Z \cap Q^{(r)}_{R,T})$ in the case (1) and $(p,q) \in \Phi(Z \cap Q^{(l)}_{R,T})$ in the case (2). In the case (3), we note that the support hyperplane Π' also would touch $u(\cdot,\hat{t})$ at the point $\hat{x}$. Therefore, $\hat{t} = t^*$ and $(\hat{x},\hat{t}) \in \widetilde{Z}$. Further, it is obvious that $-M/2R < p_n < M/2R$ and an observation shows that $(p',q) \in \Phi'(\widetilde{Z})$.

Hence we get

$$K_{[\Gamma]} \subset \Phi(Z \cap Q^{(r)}_R) \cup \Phi(Z \cap Q^{(l)}_R) \cup \left(\Phi'(\widetilde{Z}) \times \left[-\frac{M}{2R}, \frac{M}{2R} \right] \right). \tag{1.10}$$

Now, combining (1.8)–(1.10) we obtain (1.7). □

We continue the proof of Theorem 1.1, and introduce the weight function

$$\psi(p) = ((|p|\lambda)^2 + \Lambda^2)^{-(n+1)/2}$$

with some positive constants λ and Λ to be specified later. We know from (1.7) that

$$\begin{aligned} \int_{\mathcal{K}} \psi(p)\,dp\,dq \leq & \int_{\Phi(Z \cap Q^{(r)}_R)} \psi(p)\,dp\,dq + \int_{\Phi(Z \cap Q^{(l)}_R)} \psi(p)\,dp\,dq \\ & + \int_{\Phi'(\widetilde{Z}) \times [-M/2R, M/2R]} \psi(p)\,dp\,dq. \end{aligned} \tag{1.11}$$

First, we are going to obtain the lower bound for the integral on the left-hand side of (1.11). For this purpose we view the cone K as a union of perpendiculars to $\Gamma(K) = \{p \in K : p_n = 0\}$. For every such perpendicular, as is easily seen, at least one half of it is contained in the set $\mathcal{K}$. By the radial symmetry of the function ψ in the p variables, it follows that

$$\int_{\mathcal{K}} \psi(p)\,dp\,dq \geq \frac{1}{2} \int_K \psi(p)\,dp\,dq. \tag{1.12}$$

By an argument completely analogous to that given in [N2], it follows that the inequality $\lambda M/2R\Lambda > 1$ implies

$$\int_K \psi(p)\,dp\,dq \geq N_1(n)\frac{M}{\lambda^n\Lambda}. \tag{1.13}$$

We pass to estimate the integrals on the right-hand side of (1.11). Similarly to [N2], we obtain that

$$\begin{aligned}\int_{\Phi(Z\cap Q^{(\mathrm{r})}_{R,T})} \psi(p)\,dp\,dq + \int_{\Phi(Z\cap Q^{(\mathrm{l})}_{R,T})} \psi(p)\,dp\,dq \\ \leq N_2(n)\int_{Z\cap Q^{(\mathrm{r})}}\left(\left(\frac{g^{(\mathrm{r})}}{\lambda}\right)^{n+1} + \left(\frac{f_+^{(\mathrm{r})}}{\Lambda\Delta_{(\mathrm{r})}}\right)^{n+1}\right)dx\,dt \\ + N_2(n)\int_{Z\cap Q^{(\mathrm{l})}}\left(\left(\frac{g^{(\mathrm{l})}}{\lambda}\right)^{n+1} + \left(\frac{f_+^{(\mathrm{l})}}{\Lambda\Delta_{(\mathrm{l})}}\right)^{n+1}\right)dx\,dt.\end{aligned} \tag{1.14}$$

To estimate the third term

$$\int_{\Phi'(\tilde{Z})\times[-M/2R,M/2R]} \psi(p)\,dp\,dq = \int_{\Phi'(\tilde{Z})}\int_{-M/2R}^{M/2R} \psi(p)\,dp_n\,dp'\,dq,$$

we introduce the new variable

$$\varrho = \frac{\lambda p_n}{((|p'|\lambda)^2+\Lambda^2)^{1/2}}.$$

Then

$$\begin{aligned}\int_{-M/2R}^{M/2R} \psi(p)\,dp_n &\leq \frac{1}{\lambda}((|p'|\lambda)^2+\Lambda^2)^{-n/2}\int_{-\infty}^{+\infty}(\varrho^2+1)^{-(n+1)/2}\,d\varrho \\ &\leq \frac{N_3(n)}{\lambda}((|p'|\lambda)^2+\Lambda^2)^{-n/2}.\end{aligned}$$

Thus, taking into account relation (1.6) and arguing in the same way as in [Ap], we find

$$\begin{aligned}\int_{\Phi'(\tilde{Z})\times[-M/2R,M/2R]} \psi(p)\,dp\,dq &\leq \frac{N_3(n)}{\lambda}\int_{\Phi'(\tilde{Z})}((|p'|\lambda)^2+\Lambda^2)^{-n/2}\,dp'\,dq \\ &\leq \frac{N_4(n)}{\lambda}\int_{\tilde{Z}}\left(\left(\frac{g'}{\lambda}\right)^n + \left(\frac{\varphi_+}{\Lambda\Delta'}\right)^n\right)dx'\,dt.\end{aligned} \tag{1.15}$$

It is evident that the right-hand side of (1.15) will increase if we extend the integration to the set $Z\cap\Gamma(Q_{R,T})$.

Combining (1.11)–(1.15), we obtain that the estimate

$$M \le \frac{N_5(n)\Lambda}{\lambda}\bigg(\|g^{(\mathrm{r})}\|^{n+1}_{n+1,Z\cap Q^{(\mathrm{r})}} + \left(\frac{\lambda}{\Lambda}\right)^{n+1}\left\|\frac{f_+^{(\mathrm{r})}}{\Delta_{(\mathrm{r})}}\right\|^{n+1}_{n+1,Z\cap Q^{(\mathrm{r})}}$$
$$+\|g^{(\mathrm{l})}\|^{n+1}_{n+1,Z\cap Q^{(\mathrm{l})}} + \left(\frac{\lambda}{\Lambda}\right)^{n+1}\left\|\frac{f_+^{(\mathrm{l})}}{\Delta_{(\mathrm{l})}}\right\|^{n+1}_{n+1,Z\cap Q^{(\mathrm{l})}} \tag{1.16}$$
$$+\|g'\|^{n}_{n,Z\cap\Gamma(Q)} + \left(\frac{\lambda}{\Lambda}\right)^{n}\left\|\frac{\varphi_+}{\Delta'}\right\|^{n}_{n,Z\cap\Gamma(Q)}\bigg)$$

holds if $M > 2\Lambda R/\lambda$.

We now take

$$\Lambda = \lambda^{1/(n+1)}\left(\left\|\frac{f_+^{(\mathrm{r})}}{\Delta_{(\mathrm{r})}}\right\|_{n+1,Z\cap Q^{(\mathrm{r})}} + \left\|\frac{f_+^{(\mathrm{l})}}{\Delta_{(\mathrm{l})}}\right\|_{n+1,Z\cap Q^{(\mathrm{l})}}\right) + \left\|\frac{\varphi_+}{\Delta'}\right\|_{n+1,Z\cap\Gamma(Q)} + \varepsilon$$

for an arbitrary $\varepsilon > 0$. Then from (1.16) it follows that the inequality

$$M \le \frac{N_6(n)\Lambda}{\lambda}\left(R + \lambda^n + \|g^{(\mathrm{r})}\|^{n+1}_{n+1,Z\cap Q^{(\mathrm{r})}} + \|g^{(\mathrm{l})}\|^{n+1}_{n+1,Z\cap Q^{(\mathrm{l})}} + \|g'\|^{n}_{n,Z\cap\Gamma(Q)}\right)$$

is true in all cases. Hence, choosing

$$\lambda = \max\{R^{1/n}, \|g^{(\mathrm{r})}\|^{(n+1)/n}_{n+1,Z\cap Q^{(\mathrm{r})}}, \|g^{(\mathrm{l})}\|^{(n+1)/n}_{n+1,Z\cap Q^{(\mathrm{l})}}, \|g'\|_{n,Z\cap\Gamma(Q)}\}$$

and letting $\varepsilon \to 0$, we arrive at (1.4). □

When the interface operator B can totally degenerate, we assume that the jump operator J does not degenerate. In this case we have the following theorem.

Theorem 1.3. *Let $\Omega \subset B_R$ and let $u \in W^{2,1}_{n+1}(Q^{(\mathrm{r})} \cup Q^{(\mathrm{l})}) \cap W^{2,1}_{\infty}(\Gamma(Q)) \cap C(\overline{Q})$ satisfy inequalities* (1.1)–(1.3). *Suppose that the coefficients in* (1.1) *and* (1.2) *satisfy conditions* (I) *of Theorem* 1.1.

Assume also that the coefficients in (1.3) *satisfy the conditions*

(II′) *$\alpha^{sm}\xi_s\xi_m \ge 0$ for $\xi \in \mathbf{R}^{n-1}$ and $\tau \ge 0$, $\gamma \ge 0$, $\beta_{(\mathrm{r})} \ge 0$, $\beta_{(\mathrm{l})} \ge 0$, $\beta_{(\mathrm{r})} + \beta_{(\mathrm{l})} \ge \chi$ a.e. on $\Gamma(Q)$, $\chi = \mathrm{const} > 0$.*

If, in addition, $u \le 0$ on $\partial' Q$, then

$$u \le C_3\left(\left\|\frac{f_+^{(\mathrm{r})}}{\Delta_{(\mathrm{r})}}\right\|_{n+1,Z\cap Q^{(\mathrm{r})}} + \left\|\frac{f_+^{(\mathrm{l})}}{\Delta_{(\mathrm{l})}}\right\|_{n+1,Z\cap Q^{(\mathrm{l})}}\right) + C_4\left\|\frac{\varphi_+}{\chi}\right\|_{\infty,Z\cap\Gamma(Q)}, \tag{1.17}$$

where

$$C_3 = C(n)(1+\|g_0\|^{n-1}_{\infty,Z\cap\Gamma(Q)})\big(R^{n/(n+1)}+\|g^{(\mathrm{r})}\|^{n}_{n+1,Z\cap Q^{(\mathrm{r})}}+\|g^{(\mathrm{l})}\|^{n}_{n+1,Z\cap Q^{(\mathrm{l})}}\big),$$
$$C_4 = C(n)(1+\|g_0\|^{n-1}_{\infty,Z\cap\Gamma(Q)})\big(R+\|g^{(\mathrm{r})}\|^{n+1}_{n+1,Z\cap Q^{(\mathrm{r})}}+\|g^{(\mathrm{l})}\|^{n+1}_{n+1,Z\cap Q^{(\mathrm{l})}}\big),$$

$g_0=|(\beta^s)|/\chi$ *and we set* $\frac{0}{0}=0$ *if such an indeterminacy arises.*

Proof. For the beginning we note that we can suppose that all assumptions from Remark 1.1 hold, except the uniform parabolicity of the operator B. Also we extend the function u_+ to the whole set $Q_{R.T}$.

Let $M=\sup_{Q_{R,T}} z=\sup_{Q_{R,T}} u=u(x^0,t^0)$ and $\delta=(1+\|g_0\|^2_{\infty,Z\cap\Gamma(Q)})^{-1/2}$.

It should be noted that the notation we are going to use to the end of this section is the same as in the proof of Theorem 1.1 unless otherwise specified.

1. Suppose initially that $\varphi\leq 0$ on $Z\cap\Gamma(Q)$.

As a preliminary, we intend to define a set $K^{\delta}_{[\Gamma]}\subset K_{[\Gamma]}$ which satisfies

$$K^{\delta}_{[\Gamma]}\subset\Phi(Z\cap Q^{(\mathrm{r})}_{R.T})\cup\Phi(Z\cap Q^{(\mathrm{l})}_{R.T}). \tag{1.18}$$

Let us fix $(p',0,q)$ belonging to $K_{[\Gamma]}$. As is known from the proof of Lemma 1.2, for all possible values of p_n such that $(p',p_n,q)\in\Phi(Z\cap\Gamma_{R.T})$, the support hyperplanes Π corresponding to (p,q) would touch the graph of u the first time at one and the same point $(\hat{x},\hat{t})\in\widetilde{Z}$.

Since $\beta_{(\mathrm{r})}+\beta_{(\mathrm{l})}\geq\chi$, it follows that at least one of these functions has to be greater than or equal to $\frac{1}{2}\chi$. If $\beta_{(\mathrm{r})}(\hat{x},\hat{t})\geq\frac{1}{2}\chi$ then we consider the set

$$\big\{(p',p_n,q)\in K_{[\Gamma]} : p_n<0 \text{ and } |tg(\widehat{p,p_n})|<\tfrac{1}{2}\delta\big\}$$

(in the case $\beta_{(\mathrm{l})}(\hat{x},\hat{t})\geq\frac{1}{2}\chi$ it is necessary only to change the sign of p_n) and define $K^{\delta}_{[\Gamma]}$ as a union of such sets for all $(p',0,q)\in K_{[\Gamma]}$.

Assume $\beta_{(\mathrm{r})}(\hat{x},\hat{t})\geq\frac{1}{2}\chi$ for definiteness. If $(p,q)\in\Phi(Z\cap\Gamma_{R.T})$ then, taking into account (1.5), we get at the point $(\hat{x},\hat{t})$,

$$D'u=p',\qquad \lim_{x_n\to 0+} D_n u\leq p_n<0,\qquad \lim_{x_n\to 0-} D_n u\geq 0.$$

Thus, we obtain at the point $(\hat{x},\hat{t})$ the estimate

$$\begin{aligned}\beta^s D_s u+Ju &\geq \beta^s D_s u-\beta_{(\mathrm{r})}\lim_{x_n\to 0+} D_n u\\ &\geq -\tfrac{1}{2}\chi p_n-|(\beta^s)|\,|p'|>\tfrac{1}{2}\chi|p_n|(1-\delta\|g_0\|_{\infty.Z\cap\Gamma(Q)})>0.\end{aligned} \tag{1.19}$$

On the other hand, $(D'D'u)\leq 0$, $u_t\geq 0$ on Z' and, consequently

$$\beta^s D_s u + Ju \leq Bu + Ju \leq \varphi \leq 0 \quad \text{on } Z'$$

which contradicts (1.19). This implies $K^{\delta}_{[\Gamma]}\cap\Phi(Z\cap\Gamma_{R,T})=\emptyset$, and, hence, the constructed set $K^{\delta}_{[\Gamma]}$ satisfies (1.18).

Now let us define

$$\mathcal{K}_\delta = K^{\delta}_{[\Gamma]} \cup \{(p,q)\in K_{[r]}\cup K_{[l]} : |tg(\widehat{p,p_n})| < \tfrac{1}{2}\delta\}.$$

From (1.8), (1.9) and (1.18) it follows that

$$\mathcal{K}_\delta \subset \Phi(Z\cap Q^{(\mathrm{r})}_{R,T})\cup\Phi(Z\cap Q^{(\mathrm{l})}_{R,T})$$

which guarantees the inequality

$$\int_{\mathcal{K}_\delta}\psi(p)\,dp\,dq \leq \int_{\Phi(Z\cap Q^{(\mathrm{r})}_R)}\psi(p)\,dp\,dq + \int_{\Phi(Z\cap Q^{(\mathrm{l})}_R)}\psi(p)\,dp\,dq. \tag{1.20}$$

Applying the same reasoning as in the proof of Theorem 1.1, we obtain

$$\int_{\mathcal{K}_\delta}\psi(p)\,dp\,dq \geq \frac{1}{2}\int_{K_\delta}\psi(p)\,dp\,dq \geq N_7(n)\delta^{n-1}\int_K \psi(p)\,dp\,dq. \tag{1.21}$$

Here K_δ denotes the set $\{(p,q)\in K : |tg(\widehat{p_n,p})| < \frac{1}{2}\delta\}$.

Now, choosing Λ in the function ψ as

$$\Lambda = \left\| \frac{f_+^{(\mathrm{r})}}{\Delta_{(\mathrm{r})}} \right\|_{n+1,Z\cap Q^{(\mathrm{r})}} + \left\| \frac{f_+^{(\mathrm{l})}}{\Delta_{(\mathrm{l})}} \right\|_{n+1,Z\cap Q^{(\mathrm{l})}} + \varepsilon$$

for an arbitrary $\varepsilon>0$, and combining (1.13), (1.14), (1.20) and (1.21), we get that the estimate

$$\begin{aligned} M &\leq \frac{N_8(n)\Lambda}{\delta^{n-1}\lambda}\Bigg(\|g^{(\mathrm{r})}\|^{n+1}_{n+1,Z\cap Q^{(\mathrm{r})}} + \left(\frac{\lambda}{\Lambda}\right)^{n+1} \left\| \frac{f_+^{(\mathrm{r})}}{\Delta_{(\mathrm{r})}} \right\|^{n+1}_{n+1,Z\cap Q^{(\mathrm{r})}} \\ &\qquad + \|g^{(\mathrm{l})}\|^{n+1}_{n+1,Z\cap Q^{(\mathrm{l})}} + \left(\frac{\lambda}{\Lambda}\right)^{n+1} \left\| \frac{f_+^{(\mathrm{l})}}{\Delta_{(\mathrm{l})}} \right\|^{n+1}_{n+1,Z\cap Q^{(\mathrm{l})}} \Bigg) \\ &\leq \frac{N_9(n)\Lambda}{\delta^{n-1}\lambda}(\lambda^{n+1} + \|g^{(\mathrm{r})}\|^{n+1}_{n+1,Z\cap Q^{(\mathrm{r})}} + \|g^{(\mathrm{l})}\|^{n+1}_{n+1,Z\cap Q^{(\mathrm{l})}}) \end{aligned}$$

holds if $M>2\Lambda R/\lambda$. Therefore, the inequality

$$M \leq \frac{N_{10}(n)\Lambda}{\delta^{n-1}\lambda}(R\delta^{n-1} + \lambda^{n+1} + \|g^{(\mathrm{r})}\|^{n+1}_{n+1,Z\cap Q^{(\mathrm{r})}} + \|g^{(\mathrm{l})}\|^{n+1}_{n+1,Z\cap Q^{(\mathrm{l})}})$$

is true in all cases and we see that the estimate

$$\sup_{Q_{R,T}} u = M \le \frac{N_{11}(n)}{\delta^{(n-1)/(n+1)}}\left(R^{n/(n+1)} + \frac{\|g^{(\mathrm{r})}\|^n_{n+1,Z\cap Q^{(\mathrm{r})}} + \|g^{(\mathrm{l})}\|^n_{n+1,Z\cap Q^{(\mathrm{l})}}}{\delta^{(n-1)n/(n+1)}}\right)$$

$$\times\left(\left\|\frac{f_+^{(\mathrm{r})}}{\Delta_{(\mathrm{r})}}\right\|_{n+1,Z\cap Q^{(\mathrm{r})}} + \left\|\frac{f_+^{(\mathrm{l})}}{\Delta_{(\mathrm{l})}}\right\|_{n+1,Z\cap Q^{(\mathrm{l})}}\right) \tag{1.22}$$

follows if $\varepsilon \to 0$ and

$$\lambda = \max\{(R\delta^{n-1})^{1/(n+1)}, \|g^{(\mathrm{r})}\|_{n+1,Z\cap Q^{(\mathrm{r})}}, \|g^{(\mathrm{l})}\|_{n+1,Z\cap Q^{(\mathrm{l})}}\}.$$

2. Now we remove the assumption about nonpositivity of φ on $Z\cap\Gamma(Q)$ and consider the general case.

Let us introduce the auxiliary function

$$v(x) = (R-|x_n|)\left\|\frac{\varphi_+}{\chi}\right\|_{\infty,Z\cap\Gamma(Q)}.$$

For the function $w=u-v$ an elementary computation shows that

$$\begin{aligned} L^{(\mathrm{h})}w &\le f^{(\mathrm{h})} + |b^n_{(\mathrm{h})}|\left\|\frac{\varphi_+}{\chi}\right\|_{\infty,Z\cap\Gamma(Q)} && \text{in } Q^{(\mathrm{h})}_{R,T}, \\ Bw + Jw &\le 0 && \text{on } Z\cap\Gamma(Q). \end{aligned} \tag{1.23}$$

Therefore, applying inequality (1.22) to w, taking into account the value of δ, and using (1.23), we immediately get

$$\begin{aligned} \sup_{Q_{R,T}} w \le{}& C_3\left(\left\|\frac{f_+^{(\mathrm{r})}}{\Delta_{(\mathrm{r})}}\right\|_{n+1,Z\cap Q^{(\mathrm{r})}} + \left\|\frac{f_+^{(\mathrm{l})}}{\Delta_{(\mathrm{l})}}\right\|_{n+1,Z\cap Q^{(\mathrm{l})}}\right) \\ &+ N_{12}(n)(1+\|g_0\|^{n-1}_{\infty,Z\cap\Gamma(Q)})(R^{n/(n+1)} + \|g^{(\mathrm{r})}\|^n_{n+1,Z\cap Q^{(\mathrm{r})}} + \|g^{(\mathrm{l})}\|^n_{n+1,Z\cap Q^{(\mathrm{l})}}) \\ &\times(\|g^{(\mathrm{r})}\|_{n+1,Z\cap Q^{(\mathrm{r})}} + \|g^{(\mathrm{l})}\|_{n+1,Z\cap Q^{(\mathrm{l})}})\left\|\frac{\varphi_+}{\chi}\right\|_{\infty,Z\cap\Gamma(Q)} \end{aligned} \tag{1.24}$$

Since

$$\sup_{Q_{R,T}} v \le R\left\|\frac{\varphi_+}{\chi}\right\|_{\infty,Z\cap\Gamma(Q)},$$

by applying Young's inequality to the terms containing $g^{(\mathrm{r})}$ and $g^{(\mathrm{l})}$ on the right-hand side of (1.24), we arrive at (1.17). □

Remark 1.2. Since $u>0$ on Z, the statements of Theorems 1.1 and 1.3 remain true if the intersections of all sets with Z are replaced by their intersections with $\{(x,t):u(x,t)>0\}$.

2. The solvability of the two-phase Venttsel problem

Suppose $\mathcal{L}^{(\mathrm{hh})}$ are the linear parabolic operators

$$\mathcal{L}^{(\mathrm{hh})}u \equiv u_t - a^{ij}_{(\mathrm{hh})}(x,t)D_iD_ju + b^i_{(\mathrm{hh})}(x,t)D_iu + c^{(\mathrm{hh})}(x,t)u, \quad (x,t)\in Q^{(\mathrm{hh})},$$
$$a^{ij}_{(\mathrm{hh})} = a^{ji}_{(\mathrm{hh})},$$
$$\nu|\xi|^2 \le a^{ij}_{(\mathrm{hh})}\xi_i\xi_j \le \nu^{-1}|\xi|^2 \quad \text{for } \xi\in\mathbf{R}^n,\ \nu=\mathrm{const}>0.$$

Suppose $\mathcal{B}$ is the linear parabolic interface operator

$$\mathcal{B}u \equiv u_t - \alpha^{ij}(x,t)d_id_ju + \beta^i(x,t)d_iu + \gamma(x,t)u, \quad (x,t)\in\Sigma_T,$$
$$\alpha^{ij} = \alpha^{ji},$$
$$\nu|\xi|^2 \le \alpha^{ij}\xi_i\xi_j \le \nu^{-1}|\xi|^2 \quad \text{for } \xi\in\mathbf{R}^n \text{ such that } \xi\perp\mathbf{n}(x).$$

Suppose, finally, that $\mathcal{J}$ is the jump operator

$$\mathcal{J}u \equiv \beta_{(\mathrm{in})}(x,t)\lim_{\varepsilon\to0-}\frac{\partial u}{\partial\mathbf{n}}(x+\varepsilon\mathbf{n}(x),t) - \beta_{(\mathrm{ex})}(x,t)\lim_{\varepsilon\to0+}\frac{\partial u}{\partial\mathbf{n}}(x+\varepsilon\mathbf{n}(x),t),$$

where $\beta_{(\mathrm{in})}(x,t)\ge0$ and $\beta_{(\mathrm{ex})}(x,t)\ge0$ for $(x,t)\in\Sigma_T$. (Here $\partial u/\partial\mathbf{n}$ denotes the normal derivative of u).

We consider solutions of the problem

$$\mathcal{L}^{(\mathrm{in})}u = f^{(\mathrm{in})} \quad \text{in } Q^{(\mathrm{in})}, \tag{2.1}$$
$$\mathcal{L}^{(\mathrm{ex})}u = f^{(\mathrm{ex})} \quad \text{in } Q^{(\mathrm{ex})}, \tag{2.2}$$
$$\mathcal{B}u + \mathcal{J}u = \varphi \quad \text{on } \Sigma_T, \tag{2.3}$$

satisfying the initial-boundary condition

$$u|_{\partial'Q} = 0. \tag{2.4}$$

Remark 2.1. The null initial-boundary condition (2.4) is used for simplicity only. Obviously, it can be replaced by $u|_{\partial''Q}=\theta_1$, $u|_{t=0}=\theta_2$ in Ω, where θ_1 and θ_2 belong to the appropriate functional spaces dictated by the embedding theorems. In addition, some "smooth pasting" conditions at the points of $\partial\Omega\times\{0\}$ should be fulfilled in this case.

Remark 2.2. Once either of the two coefficients of the jump operator $\mathcal{J}$ vanishes, we can solve, at first, the one-phase Venttsel problem in the corresponding subdomain, and then study the Dirichlet problem in the rest part of the medium.

Theorem 2.1. *Let* $\Sigma\in W^2_{n+1}$. *Assume that a function* $u\in V_{n+1}(Q)\cap C(\overline{Q})$ *is a solution of the problem* (2.1)–(2.3). *Suppose, in addition, that*

$$f_+^{(\mathrm{hh})}, |\mathbf{b}_{(\mathrm{hh})}|, c_-^{(\mathrm{hh})} \in L_{n+1}(Q^{(\mathrm{hh})}) \quad \textit{and} \quad \varphi_+, |\beta|, \gamma_-, \beta_{(\mathrm{in})}, \beta_{(\mathrm{ex})} \in L_n(\Sigma_T),$$

(*we set* $\mathbf{b}_{(\mathrm{hh})}(x,t)=(b^i_{(\mathrm{hh})}(x,t))$ *and* $\beta(x,t)=(\beta^i(x,t))$).

Then u *satisfies the estimate*

$$\sup_Q u_+ \le C_5\Big(\sup_{\partial' Q} u_+ + \|f_+^{(\mathrm{in})}\|_{n+1,Q^{(\mathrm{in})}} + \|f_+^{(\mathrm{ex})}\|_{n+1,Q^{(\mathrm{ex})}} + \|\varphi_+\|_{n,\Sigma_T}\Big), \tag{2.5}$$

where C_5 *depends only on* n, ν, T, $\operatorname{diam}\Omega$, *the characteristics of* Σ, *the numbers* $\|b^i_{(\mathrm{hh})}\|_{n+1,Q^{(\mathrm{hh})}}$, $\|\beta^i\|_{n,\Sigma_T}$, *and on the moduli of absolute continuity of* $b^i_{(\mathrm{hh})}(x,t)$, $c_-^{(\mathrm{hh})}(x,t)$ *in the space* $L_{n+1}(Q^{(\mathrm{hh})})$ *and of* $\beta^i(x,t)$, $\gamma_-(x,t)$, $\beta_{(\mathrm{hh})}(x,t)$ *in the space* $L_n(\Sigma_T)$.

Proof. This statement is proved along the same lines as Theorem 2.1 in [AN1]. The differences are only that Theorem 1.1 is used instead of Theorem 1 in [Ap] and Theorem 4 in [NU] is applied twice in the cylinders $Q^{(\mathrm{in})}_{[1]}$ and $Q^{(\mathrm{ex})}_{[1]}$, respectively. The full proof can be found in [AN4], Theorem 3.1. We should also say that the formulation, as well as the proof, of Theorem 2.1 in [AN1] contains several misprints. Namely, the functions u and v were used instead of u_+ and v_+, and sometimes the signs of plus and minus were confused. These misprints are avoided in [AN4]. □

Remark 2.3. It is evident that the estimate

$$\sup_Q u_- \le C_5'\Big(\sup_{\partial' Q} u_- + \|f_-^{(\mathrm{in})}\|_{n+1,Q^{(\mathrm{in})}} + \|f_-^{(\mathrm{ex})}\|_{n+1,Q^{(\mathrm{ex})}} + \|\varphi_-\|_{n,\Sigma_T}\Big) \tag{2.5$'$}$$

could be obtained in the same way as (2.5). Here C_5' is determined by almost the same quantities as C_5. The only difference is that $c_-^{(\mathrm{in})}(x,t)$, $c_-^{(\mathrm{ex})}(x,t)$ and $\gamma_-(x,t)$ are replaced by $c_+^{(\mathrm{in})}(x,t)$, $c_+^{(\mathrm{ex})}(x,t)$ and $\gamma_+(x,t)$, respectively.

Theorem 2.2. *Let* Σ *and* $\partial\Omega$ *belong to* W^2_{q+2}, $\Sigma\cap\partial\Omega=\emptyset$. *Assume that a function* $u\in V_{q+2}(Q)$ *is a solution of the problem* (2.1)–(2.4).

If, in addition,

$$\begin{aligned} a^{ij}_{(\mathrm{hh})} &\in C(\overline{Q^{(\mathrm{hh})}}), & f^{(\mathrm{hh})}, |\mathbf{b}_{(\mathrm{hh})}|, c^{(\mathrm{hh})} &\in L_{q+2}(Q^{(\mathrm{hh})}); \\ \alpha^{ij} &\in C(\overline{\Sigma_T}), & \varphi, |\beta|, \gamma, \beta_{(\mathrm{in})}, \beta_{(\mathrm{ex})} &\in L_{q+1}(\Sigma_T), \end{aligned}$$

then

$$\|u\|_{V_{q+2}(Q)} \le C_6(\|f^{(\mathrm{in})}\|_{q+2,Q^{(\mathrm{in})}} + \|f^{(\mathrm{ex})}\|_{q+2,Q^{(\mathrm{ex})}} + \|\varphi\|_{q+1,\Sigma_T}),$$

where C_6 *depends only on* n, ν, q, T, $\operatorname{diam}\Omega$, *the characteristics of* Σ *and* $\partial\Omega$, $\operatorname{dist}(\Sigma, \partial\Omega)$, *the numbers* $\|b^i_{(\mathrm{hh})}\|_{q+2,Q^{(\mathrm{hh})}}$, $\|c^{(\mathrm{hh})}\|_{q+2,Q^{(\mathrm{hh})}}$, $\|\beta^i\|_{q+1,\Sigma_T}$, $\|\gamma\|_{q+1,\Sigma_T}$, $\|\beta_{(\mathrm{hh})}\|_{q+1,\Sigma_T}$, *and on the moduli of continuity of* $a^{ij}_{(\mathrm{hh})}(x,t)$ *and of* $\alpha^{ij}(x,t)$.

Proof. For the detailed proof the reader is referred to [AN4], Theorem 3.1, where we have refined our arguments from the proof of Theorem 2.2 in [AN1]. □

Theorem 2.3. *Let* Σ *and* $\partial\Omega$ *belong to* $C^{2+\delta}$, $\delta\in]0,1[$, $\Sigma\cap\partial\Omega=\emptyset$. *Assume that a function* $u\in C^{2+\delta}_{\Sigma}(\overline{Q})$ *is a solution of the problem* (2.1)–(2.4).

Let the coefficients and the functions on the right-hand sides of equations (2.1), (2.2) *and* (2.3) *belong to* $C^{\delta}(\overline{Q^{(\mathrm{in})}})$, $C^{\delta}(\overline{Q^{(\mathrm{ex})}})$ *and* $C^{\delta}(\overline{\Sigma_T})$, *respectively.*

If, in addition,

$$f^{(\mathrm{hh})}=\varphi \quad \text{on } \Sigma\times\{0\},$$
$$f^{(\mathrm{ex})}=0 \quad \text{on } \partial\Omega\times\{0\}$$

then

$$\|u\|_{C^{2+\delta}_{\Sigma}(\overline{Q})} \le C_7(\|f^{(\mathrm{in})}\|_{C^{\delta}(\overline{Q^{(\mathrm{in})}})}+\|f^{(\mathrm{ex})}\|_{C^{\delta}(\overline{Q^{(\mathrm{ex})}})}+\|\varphi\|_{C^{\delta}(\overline{\Sigma_T})}),$$

where C_7 *depends only on* n, ν, δ, T, $\operatorname{diam}\Omega$, *the characteristics of* Σ *and* $\partial\Omega$, *and on the norms of the coefficients of the operators* $\mathcal{L}^{(\mathrm{in})}$, $\mathcal{L}^{(\mathrm{ex})}$ *and* $\mathcal{B}+\mathcal{J}$ *in the spaces* $C^{\delta}(\overline{Q^{(\mathrm{in})}})$, $C^{\delta}(\overline{Q^{(\mathrm{ex})}})$, *and* $C^{\delta}(\overline{\Sigma_T})$, *respectively.*

Proof. This statement is proved in the same way as Theorem 2.2. □

Corollary 2.4. (1) *If* Σ, $\partial\Omega$, *the coefficients and the functions on the right-hand sides of* (2.1)–(2.3) *satisfy the assumptions of Theorem* 2.2, *then the initial-boundary value problem* (2.1)–(2.4) *has a unique solution* $u\in V_{q+2}(Q)$.

(2) *If* Σ, $\partial\Omega$, *the coefficients and the functions on the right-hand sides of* (2.1)–(2.3) *satisfy the assumptions of Theorem* 2.3, *then the initial-boundary value problem* (2.1)–(2.4) *has a unique solution* $u\in C^{2+\delta}_{\Sigma}(\overline{Q})$.

Proof. This corollary can be proved via the standard method of extending by continuity with respect to the parameter (see, for example, Theorem 2.3 in [AN1] where it was made for the one-phase Venttsel problem). □

3. Appendix. The stationary case

In this section we formulate results for elliptic equations corresponding to the statements from Sections 1 and 2.

Throughout this section we will assume that all coefficients of the operators J and $\mathcal{J}$ as well as the function u do not depend on the variable t.

Theorem 1.1*. *Suppose that $\Omega\subset B_R$ and u is a function such that*

$$u\in W_n^2(\Omega^{(\mathrm{in})}\cup\Omega^{(\mathrm{ex})})\cap W_{n-1}^2(\Gamma(\Omega))\cap C(\overline{\Omega}),$$

$$-a_{(\mathrm{r})}^{ij}(x)D_iD_ju+b_{(\mathrm{r})}^i(x)D_iu+c^{(\mathrm{r})}(x)u\le f^{(\mathrm{r})}(x) \quad \text{in } \Omega^{(\mathrm{r})}, \tag{3.1}$$

$$-a_{(\mathrm{l})}^{ij}(x)D_iD_ju+b_{(\mathrm{l})}^i(x)D_iu+c^{(\mathrm{l})}(x)u\le f^{(\mathrm{l})}(x) \quad \text{in } \Omega^{(\mathrm{l})}, \tag{3.2}$$

$$-\alpha^{sm}(x)D_sD_mu+\beta^s(x)D_su+\gamma(x)u+Ju\le\varphi(x) \quad \text{on } \Gamma(\Omega). \tag{3.3}$$

Assume also that the coefficients in (3.1)–(3.3) *satisfy the conditions*

(I) $a_{(\mathrm{h})}^{ij}=a_{(\mathrm{h})}^{ji}$, $a_{(\mathrm{h})}^{ij}\xi_i\xi_j\ge0$ *for* $\xi\in\mathbf{R}^n$ *and* $c^{(\mathrm{h})}\ge0$, $\mathrm{tr}(a_{(\mathrm{h})}^{ij})>0$ *a.e. in* $\Omega^{(\mathrm{h})}$;

(II) $\alpha^{sm}=\alpha^{ms}$, $\alpha^{sm}\xi_s\xi_m\ge0$ *for* $\xi\in\mathbf{R}^{n-1}$ *and* $\gamma\ge0$, $\beta_{(\mathrm{r})}\ge0$, $\beta_{(\mathrm{l})}\ge0$, $\mathrm{tr}(\alpha^{sm})>0$ *a.e. on* $\Gamma(\Omega)$.

If, in addition, $u\le0$ *on* $\partial\Omega$, *then*

$$u\le C_1'R\left(\left\|\frac{f_+^{(\mathrm{r})}}{\Delta_{(\mathrm{r})}}\right\|_{n,Z\cap\Omega^{(\mathrm{r})}}+\left\|\frac{f_+^{(\mathrm{l})}}{\Delta_{(\mathrm{l})}}\right\|_{n,Z\cap\Omega^{(\mathrm{l})}}+\left\|\frac{\varphi_+}{\Delta'}\right\|_{n-1,Z\cap\Gamma(\Omega)}\right),$$

where

$$C_1'=C_0'(n)\exp\left(\frac{1}{n}(\|g^{(\mathrm{r})}\|_{n,Z\cap\Omega^{(\mathrm{r})}}^n+\|g^{(\mathrm{l})}\|_{n,Z\cap\Omega^{(\mathrm{l})}}^n+\|g'\|_{n-1,Z\cap\Gamma(\Omega)}^{n-1})\right),$$

$$\Delta_{(\mathrm{h})}=\det(a_{(\mathrm{h})}^{ij})^{1/n},\quad g^{(\mathrm{h})}=\frac{|(b_{(\mathrm{h})}^i)|}{\Delta_{(\mathrm{h})}},\quad \Delta'=\det(\alpha^{sm})^{1/(n-1)},\quad g'=\frac{|(\beta^s)|}{\Delta'},$$

and we set $\frac{0}{0}=0$ *if such an indeterminacy arises.*

Remark 3.1. In the simplest case, this theorem was established by I. V. Denisova in a diploma thesis written at the St. Petersburg State University in 1999.

Theorem 1.3*. *Let* $\Omega\subset B_R$ *and let* $u\in W_n^2(\Omega^{(\mathrm{r})}\cup\Omega^{(\mathrm{l})})\cap W_\infty^2(\Gamma(\Omega))\cap C(\overline{\Omega})$ *satisfy inequalities* (3.1)–(3.3). *Suppose that the coefficients in* (3.1) *and* (3.2) *satisfy conditions* (I).

Assume also that the coefficients in (3.3) *satisfy the conditions*

(II′) $\alpha^{sm}=\alpha^{ms}$, $\alpha^{sm}\xi_s\xi_m\ge0$ *for* $\xi\in\mathbf{R}^{n-1}$ *and* $\gamma\ge0$, $\beta_{(\mathrm{r})}\ge0$, $\beta_{(\mathrm{l})}\ge0$, $\beta_{(\mathrm{r})}+\beta_{(\mathrm{l})}\ge\chi$ *a.e. on* $\Gamma(\Omega)$, $\chi=\mathrm{const}>0$.

If, in addition, $u\le0$ *on* $\partial\Omega$, *then*

$$u\le C_3'R\left(\left\|\frac{f_+^{(\mathrm{r})}}{\Delta_{(\mathrm{r})}}\right\|_{n,Z\cap\Omega^{(\mathrm{r})}}+\left\|\frac{f_+^{(\mathrm{l})}}{\Delta_{(\mathrm{l})}}\right\|_{n,Z\cap\Omega^{(\mathrm{l})}}\right)+C_4'R\left\|\frac{\varphi_+}{\chi}\right\|_{\infty,Z\cap\Gamma(\Omega)},$$

where

$$C_3' = C'(n)\exp\left(\frac{1+\|g_0\|^{n-1}_{\infty,Z\cap\Gamma(\Omega)}}{n}(\|g^{(\mathrm{r})}\|^n_{n,Z\cap\Omega^{(\mathrm{r})}}+\|g^{(\mathrm{l})}\|^n_{n,Z\cap\Omega^{(\mathrm{l})}})\right),$$

$$C_4' = C_3'(\|g^{(\mathrm{r})}\|_{n,Z\cap\Omega^{(\mathrm{r})}}+\|g^{(\mathrm{l})}\|_{n,Z\cap\Omega^{(\mathrm{l})}}),$$

$g_0=|(\beta^s)|/\chi$ *and we set* $\frac{0}{0}=0$ *if such an indeterminacy arises.*

By analogy with the parabolic case we define the Sobolev space

$$V_q(\Omega)=W^2_q(\Omega^{(\mathrm{in})}\cup\Omega^{(\mathrm{ex})})\cap W^2_{q-1}(\Sigma)$$

with the norm

$$\|u\|_{V_q(\Omega)}=\|u\|_{W^2_q(\Omega^{(\mathrm{in})})}+\|u\|_{W^2_q(\Omega^{(\mathrm{ex})})}+\|u\|_{W^2_{q-1}(\Sigma)}.$$

Theorem 2.2*. *Let* Σ *and* $\partial\Omega$ *belong to* W^2_q, $\Sigma\cap\partial\Omega=\emptyset$, *and* $u\in V_q(\Omega)$ *be a function such that*

$$-a^{ij}_{(\mathrm{in})}(x)D_iD_ju+b^i_{(\mathrm{in})}(x)D_iu+c^{(\mathrm{in})}(x)u=f^{(\mathrm{in})}(x) \quad \text{in } \Omega^{(\mathrm{in})}, \tag{3.4}$$

$$-a^{ij}_{(\mathrm{ex})}(x)D_iD_ju+b^i_{(\mathrm{ex})}(x)D_iu+c^{(\mathrm{ex})}(x)u=f^{(\mathrm{ex})}(x) \quad \text{in } \Omega^{(\mathrm{ex})}, \tag{3.5}$$

$$-\alpha^{ij}(x)d_id_ju+\beta^i(x)d_iu+\gamma(x)u+\mathcal{J}u=\varphi(x) \quad \text{on } \Sigma. \tag{3.6}$$

Assume also that

$$\beta_{(\mathrm{in})}(x)\geq 0 \text{ and } \beta_{(\mathrm{ex})}(x)\geq 0, \quad x\in\Sigma, \tag{3.7}$$

while $(a^{ij}_{(\mathrm{hh})})$, (α^{ij}) *are symmetric matrices satisfying the ellipticity condition, i.e.*

$$\nu|\xi|^2\leq a^{ij}_{(\mathrm{hh})}\xi_i\xi_j\leq\nu^{-1}|\xi|^2, \quad \nu=\mathrm{const}>0, \tag{3.8}$$

for any $\xi\in\mathbf{R}^n$ *and* $x\in\Omega^{(\mathrm{hh})}$;

$$\nu|\xi|^2\leq\alpha^{ij}\xi_i\xi_j\leq\nu^{-1}|\xi|^2 \quad \text{for } x\in\Sigma \text{ and } \xi\in\mathbf{R}^n \text{ such that } \xi\perp\mathbf{n}(x). \tag{3.9}$$

If, in addition,

$$a^{ij}_{(\mathrm{hh})}\in C(\overline{\Omega^{(\mathrm{hh})}}), \qquad f^{(\mathrm{hh})}, |\mathbf{b}_{(\mathrm{hh})}|, c^{(\mathrm{hh})}\in L_q(\Omega^{(\mathrm{hh})});$$
$$\alpha^{ij}\in C(\overline{\Sigma_T}), \qquad \varphi, |\beta|, \gamma, \beta_{(\mathrm{in})}, \beta_{(\mathrm{ex})}\in L_{q-1}(\Sigma),$$

then

$$\|u\|_{V_{q+2}(\Omega)}\leq C_6'(\|f^{(\mathrm{in})}\|_{q,\Omega^{(\mathrm{in})}}+\|f^{(\mathrm{ex})}\|_{q,\Omega^{(\mathrm{ex})}}+\|\varphi\|_{q-1,\Sigma}+\|u\|_{q,\Omega}+\|u\|_{q-1,\Sigma}), \tag{3.10}$$

where C_6' *depends only on* n, ν, q, $\mathrm{diam}\,\Omega$, *the characteristics of* Σ *and* $\partial\Omega$, $\mathrm{dist}(\Sigma,\partial\Omega)$, *the numbers* $\|b^i_{(\mathrm{hh})}\|_{q,\Omega^{(\mathrm{hh})}}$, $\|c^{(\mathrm{hh})}\|_{q,\Omega^{(\mathrm{hh})}}$, $\|\beta^i\|_{q-1,\Sigma}$, $\|\gamma\|_{q-1,\Sigma}$ *and* $\|\beta_{(\mathrm{hh})}\|_{q-1,\Sigma}$, *and on the moduli of continuity of the coefficients* $a^{ij}_{(\mathrm{hh})}(x)$, $\alpha^{ij}(x)$.

Theorem 2.3*. *Let* Σ *and* $\partial\Omega$ *belong to* $C^{2+\delta}$, $\delta\in]0,1[$, $\Sigma\cap\partial\Omega=\emptyset$. *Assume that a function* $u\in C_\Sigma^{2+\delta}(\overline{\Omega})$ *is a solution of the problem* (3.4)–(3.6) *and that the inequalities* (3.7)–(3.9) *hold.*

If the coefficients and the function involved on the right-hand side of equations (3.4), (3.5) *and* (3.6) *belong to* $C^\delta(\overline{\Omega^{(\text{in})}})$, $C^\delta(\overline{\Omega^{(\text{ex})}})$ *and* $C^\delta(\overline{\Sigma})$, *respectively, then*

$$\|u\|_{C_\Sigma^{2+\delta}(\overline{\Omega})}\le C_7'(\|f^{(\text{in})}\|_{C^\delta(\overline{\Omega^{(\text{in})}})}+\|f^{(\text{ex})}\|_{C^\delta(\overline{\Omega^{(\text{ex})}})}+\|\varphi\|_{C^\delta(\overline{\Sigma})}+\|u\|_{C(\overline{\Omega})}), \tag{3.11}$$

where C_7' *depends only on* n, ν, δ, $\operatorname{diam}\Omega$, *the characteristics of* Σ *and* $\partial\Omega$, *and on the norms of the coefficients of the equations* (3.4), (3.5) *and* (3.6) *in the spaces* $C^\delta(\overline{\Omega^{(\text{in})}})$, $C^\delta(\overline{\Omega^{(\text{ex})}})$, *and* $C^\delta(\overline{\Sigma})$, *respectively.*

If the right-hand sides of the inequalities (3.10) and (3.11) will be without the last terms, then the existence and uniqueness theorems in Sobolev and Hölder spaces follow by a standard argument from Theorem 2.2* and Theorem 2.3*, respectively.

To remove the above-mentioned terms from the right-hand sides of (3.10) and (3.11), the global maximum estimate is required. But, unlike the parabolic case, the global maximum principle does not hold for solutions of the two-phase elliptic Venttsel problem without additional assumptions. As an illustration of this statement, let us consider the following example.

Example. Consider the problem

$$\begin{cases} -\Delta u=0, & \text{in } B_1,\\ -\Delta u=0, & \text{in } B_2\backslash\overline{B}_1,\\ -\Delta' u+Ju=0, & \text{on } \partial B_1,\\ u|_{\partial B_2}=0 & \end{cases} \tag{3.12}$$

(here Δ' stands for the Laplace–Beltrami operator). Suppose also that $\beta_{(\text{in})}(x)\ge 0$ and $\beta_{(\text{ex})}(x)\equiv 0$ on ∂B_1. Then the problem (3.12) has a solution identically equal to 1 in B_1, and thus, in spite of the homogeneity of all the equations in (3.12), the global maximum estimate for the solutions does not exist.

Moreover, if we add cu with some positive c to the left-hand side of the equation in $B_2\backslash\overline{B}_1$, and, in the case of $\beta_{(\text{in})}(x)\equiv 0$ on ∂B_1, add a similar term to the left-hand side of the equation in B_1 as well, things will not improve.

There are various types of additional requirements on the coefficients of the equations (3.4)–(3.6) providing the existence of the global maximum principle for solutions. For instance, the following statement holds true.

Theorem 3.1. *Let* $\Sigma\in C^2$, *let* $u\in C^2(\overline{\Omega^{(\text{in})}})\cap C^2(\overline{\Omega^{(\text{ex})}})$ *be a solution of the problem* (3.4)–(3.6), *and assume that the inequalities* (3.7) *and* (3.8) *hold. Suppose that*

$$f_+^{(\text{hh})}, |\mathbf{b}_{(\text{hh})}| \in L_n(\Omega^{(\text{hh})}) \quad \text{and} \quad \varphi_+, |\beta|, \beta_{(\text{in})}, \beta_{(\text{ex})} \in L_\infty(\Sigma).$$

If, in addition,

$$c^{(\text{hh})} \geq 0 \text{ in } \Omega^{(\text{hh})} \quad \text{and} \quad \gamma(x) \geq \gamma_0 \text{ on } \Sigma, \ \gamma_0 = \text{const} > 0,$$

then u *satisfies the estimate*

$$\sup_{\Omega} u_+ \leq \sup_{\partial\Omega} u_+ + \frac{\|\varphi_+\|_{\infty,\Sigma}}{\gamma_0} + C_8(\|f_+^{(\text{in})}\|_{n,\Omega^{(\text{in})}} + \|f_+^{(\text{ex})}\|_{n,\Omega^{(\text{ex})}}),$$

where C_8 *depends only on* n, ν, γ_0, $\operatorname{diam}\Omega$, *the numbers* $\|b^i_{(\text{hh})}\|_{n,\Omega^{(\text{hh})}}$, $\|\beta^i\|_{\infty,\Sigma}$ *and* $\|\beta_{(\text{hh})}\|_{\infty,\Sigma}$.

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