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Resultants and the Hilbert scheme of points on the line

Roy Mikael Skjelnes

Abstract. We present an elementary and concrete description of the Hilbert scheme of points on the spectrum of fraction rings $k[X]_U$ of the one-variable polynomial ring over a commutative ring k. Our description is based on the computation of the resultant of polynomials in $k[X]$. The present paper generalizes the results of Laksov–Skjelnes [7], where the Hilbert scheme on spectrum of the local ring of a point was described.

1. Introduction

In the present article we generalize the techniques and results of [7], where the Hilbert scheme of points on the spectrum of the local ring of a point on the affine line over a field was described. We will describe the Hilbert scheme of points on the spectrum of arbitrary fraction rings $k[X]_U$ of the polynomial ring in the variable X, over an arbitrary (commutative and unitary) base ring k.

It is well known that the Hilbert scheme of n-points on a smooth curve is given by the n-fold symmetric product ([1], [3], [4], see also Remark 4.3). When the multiplicatively closed set $U \subseteq k[X]$ is generated by a single element then $\operatorname{Spec}(k[X]_U) \to \operatorname{Spec}(k)$ is a family of curves, and our results are well known. For general closed subsets $U \subseteq k[X]$ the situation is new since the scheme $\operatorname{Spec}(k[X]_U)$ is not of finite type over the base $\operatorname{Spec}(k)$. Even though $\operatorname{Spec}(k[X]_U)$ is not generally a curve, it is not surprising that the Hilbert scheme of n-points on $\operatorname{Spec}(k[X]_U)$ is given by its n-fold symmetric product. However, the important thing to keep in mind when studying points on $\operatorname{Spec}(k[X]_U)$ is that one has to consider families of points. An argument based on the classification of the k-rational points of the Hilbert scheme will not suffice; the Hilbert scheme of points can be large even though the underlying set of k-rational points is small or even empty (see Examples 3.9 and 3.11). Furthermore, as $\operatorname{Spec}(k[X]_U)$ is not necessarily a sub-scheme of the line $\operatorname{Spec}(k[X])$, even the existence of the Hilbert scheme on $\operatorname{Spec}(k[X]_U)$ has to be established.

We shall give an elementary and concrete description of the Hilbert scheme of points on the spectrum of fraction rings $k[X]_U$ of the polynomial ring $k[X]$. Essential for our description is the classification of degree n, monic polynomials $F\in A[X]$, for an arbitrary k-algebra A, such that the induced map $A[X]/(F)\to A\otimes_k k[X]_U/(F)$ is an isomorphism.

The principal tool we use to study monic polynomials $F\in A[X]$ is the homomorphism $u_F\colon A[s_1,\dots,s_n]\to A$ determined by the coefficients of F (the source of u_F is the polynomial ring in the elementary symmetric functions s_1, ..., s_n in n-variables t_1, ..., t_n). The homomorphism u_F was introduced in [7] and used there to study roots of polynomials defined over fields. The approach presented here is different, emphasizing endomorphisms and characteristic polynomials. Our main innovation is Theorem 2.2 where we show that for any $f\in A[X]$ we have that $u_F(f(t_1)\dots f(t_n))\in A$ is the determinant of the A-linear endomorphism on $A[X]/(F)$ that sends $a\mapsto af$. In other words the image of the symmetric function $f(t_1)\dots f(t_n)$ under the morphism u_F is the resultant of f and F.

An interesting observation that follows from our description of the Hilbert scheme of $\operatorname{Spec}(k[X]_U)$ is that the Hilbert scheme of points on the line commutes with localization, see Remark 3.10. In a forthcoming paper [10] the localization property will be established for schemes in general, not only for smooth curves or, as discussed here, the line.

2. Symmetric operators on the polynomial ring

We recall the symmetric operators introduced in [7], and we introduce some notation. We also establish the elementary, but important Theorem 2.2.

2.1. Notation

Let $t_1,\dots,t_n$ be independent variables over a commutative ring A. Let $s_1,\dots,s_n$ denote the elementary symmetric functions in $t_1,\dots,t_n$. Denote the polynomial ring of symmetric functions as

$$S_A^n=A[s_1,\dots,s_n].$$

To each polynomial $f\in A[X]$ in one variable we define the symmetric functions $s_1(f),\dots,\ s_n(f)$ by the following identity in the polynomial ring $A[t_1,\dots,t_n,X]$:

$$\prod_{i=1}^{n}(X-f(t_i))=X^n-s_1(f)X^{n-1}+\dots+(-1)^n s_n(f). \tag{2.1}$$

We have that $s_1(X),\dots,s_n(X)$ are the elementary symmetric functions $s_1,\dots,s_n$. Let $\mathrm{S}_A^n[X]$ denote the polynomial ring in the variable X over the ring of symmetric functions. The element $\Delta_n(X)=\prod_{i=1}^n(X-t_i)$ is, when written out, a polynomial in the variable X, with coefficients in the ring of symmetric functions S_A^n.

2.2. The homomorphism u_F

Let $F(X)=X^n-u_1X^{n-1}+\dots+(-1)^n u_n\in A[X]$ be a monic polynomial of degree n. We define an A-algebra homomorphism

$$u_F\colon \mathrm{S}_A^n\longrightarrow A, \tag{2.2}$$

by sending $s_i\mapsto u_i$ for each $i=1,\dots,n$. The coefficients of $\Delta_n(X)$ are then mapped to the coefficients of $F(X)$ by the homomorphism u_F, hence

$$\mathrm{S}_A^n[X]/(\Delta_n(X))\otimes_{\mathrm{S}_A^n}A\cong A[X]/(F(X)).$$

Proposition 2.1. *Let M be a square matrix having coefficients in a commutative ring A. Assume that the characteristic polynomial $P_M(X)=\prod_{i=1}^n(X-a_i)$ of M splits into linear factors over A. Then for any polynomial $f(X)$ in $A[X]$ the matrix $f(M)$ has characteristic polynomial $P_{f(M)}(X)=\prod_{i=1}^n(X-f(a_i))$.*

When A is a field, the result is known as the Spectral Theorem. A proof of the theorem over general commutative rings can be found in [8].

Theorem 2.2. *Let F be a non-constant, monic polynomial in $A[X]$. Denote with n the degree of F. For any element f in $A[X]$ we let $\mu_F(f)$ be the A-linear endomorphism on $A[X]/(F)$ given as multiplication by the residue class of f modulo the ideal (F). Then the characteristic polynomial of $\mu_F(f)$ is*

$$X^n-u_F(s_1(f))X^{n-1}+\dots+(-1)^n u_F(s_n(f)).$$

In particular $u_F(f(t_1)\dots f(t_n))$ is the determinant of $\mu_F(f)$.

Proof. For any f in $A[X]$ we let $\mu(f)$ be the S_A^n-linear endomorphism on $E=\mathrm{S}_A^n[X]/(\Delta_n(X))$ given as multiplication by the residue class of f in E. Let $u_F\colon\mathrm{S}_A^n\to A$ be the A-algebra homomorphism determined by F in $A[X]$. We have that the induced A-linear endomorphism $\mu(f)\otimes\mathrm{id}$ on $E\otimes_{\mathrm{S}_A^n}A\cong A[X]/(F)$ is $\mu_F(f)$. Hence to prove our theorem it suffices to show that the endomorphism $\mu(f)$ has characteristic polynomial

$$X^n-s_1(f)X^{n-1}+\dots+(-1)^n s_n(f). \tag{2.3}$$

Note that $\mathrm{S}^n_A \subseteq A[t_1, \dots, t_n]$ is a subring and that

$$E \otimes_{\mathrm{S}^n_A} A[t_1, \dots, t_n] \cong A[t_1, \dots, t_n, X]/(\Delta_n(X)). \tag{2.4}$$

By ring extension we consider $\mu(f)$ as an $A[t_1, \dots, t_n]$-linear endomorphism on (2.4). Since $\Delta_n(X) = \prod_{i=1}^n (X - t_i)$ splits into linear factors over $A[t_1, \dots, t_n]$ the theorem is proven if we show that $\mu(X)$ has characteristic polynomial $\Delta_n(X)$. Indeed, it then follows from Proposition 2.1 that $\mu(f)$ has characteristic polynomial $\prod_{i=1}^n (X - f(t_i))$, which written out is (2.3). Thus what remains is to compute the characteristic polynomial of the particular endomorphism $\mu(X)$.

Let x be the residue class of X modulo the ideal $(\Delta_n(X))$ in $A[t_1, \dots, t_n, X]$. We have that $1, x, \dots, x^{n-1}$ form an $A[t_1, \dots, t_n]$ basis for (2.4). The matrix M representing the endomorphism $\mu(X)$, with respect to the given basis is easy to describe and is called the *companion matrix* of $\Delta_n(X)$. In general, if $F(X) = X^n - u_1 X^{n-1} - \dots - u_n$ is a monic polynomial, then the companion matrix of $F(X)$ is the matrix

$$M_F = \begin{pmatrix} 0 & 0 & \dots & 0 & u_n \\ 1 & 0 & \dots & 0 & u_{n-1} \\ 0 & 1 & \dots & 0 & u_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & u_1 \end{pmatrix}.$$

Note that the matrix obtained by deleting the first row and the first column of M_F, is the companion matrix of $G(X) = X^{n-1} - u_1 X^{n-2} - \dots - u_{n-1}$. It follows readily by induction on the size n of M_F, that the determinant $\det(XI - M_F) = F(X)$. Thus the matrix M representing the endomorphism $\mu(X)$ with respect to the basis $1, x, \dots, x^{n-1}$ has characteristic polynomial $\Delta_n(X)$. $\square$

Proposition 2.3. *Let F in $A[X]$ be a monic polynomial which has positive degree n. Let $U \subseteq A[X]$ be a multiplicatively closed subset. The following three assertions are equivalent:*

(1) *The canonical map $A[X]/(F) \to A[X]_U/(F)$ is an isomorphism.*

(2) *The resultant $\det(\mu_F(f)) \in A$ is a unit for all f in $U \subseteq A[X]$.*

(3) *The A-algebra homomorphism $u_F \colon \mathrm{S}^n_A \to A$ factors through the fraction ring $(\mathrm{S}^n_A)_{U(n)}$, where*

$$U(n) = \{f(t_1) \dots f(t_n) \mid f \in U \subseteq A[X]\}.$$

Proof. First we show that Assertion (1) is equivalent to Assertion (2). The fraction map $A[X]/(F) \to A[X]_U/(F)$ is an isomorphism if and only if the class of f in $A[X]/(F)$ is invertible for all f in the multiplicatively closed set $U \subseteq A[X]$.

The residue class of f modulo the ideal $(F)\subseteq A[X]$ is invertible if and only if the endomorphism $\mu_F(f)$ on $A[X]/(F)$ given as multiplication by the residue class of f is invertible. The endomorphism $\mu_F(f)$ is invertible if and only if its determinant $u_F(f)\in A$ is a unit. We have shown that Assertion (1) is equivalent to Assertion (2).

That Assertion (2) is equivalent to Assertion (3) follows from Theorem 2.2 and the universal property of fraction rings. Indeed, we have that $\det(\mu_F(f))= u_F(f(t_1)\dots f(t_n))$, and consequently the elements of $U(n)$ are mapped to invertible elements by u_F if and only if the resultants $\det(\mu_F(f))$ are invertible. $\square$

Corollary 2.4. *The fraction map*

$$(\mathrm{S}^n_A)_{U(n)}\otimes_A A[X]/(\Delta_n(X)) \longrightarrow (\mathrm{S}^n_A)_{U(n)}\otimes_A A[X]_U/(\Delta_n(X))$$

is an isomorphism.

Proof. The $(\mathrm{S}^n_A)_{U(n)}$-module $E=(\mathrm{S}^n_A)_{U(n)}\otimes_A A[X]/(\Delta_n(X))$ is free of rank n. For any element $f\in A[X]$ the multiplication map gives an endomorphism $\mu(f)$ of E. The endomorphism f is invertible if and only if the determinant $\det(\mu(f))$ is a unit in $(\mathrm{S}^n_A)_{U(n)}$. As shown in the proof of Theorem 2.2 the determinant of $\mu(f)$ is $f(t_1)\dots f(t_n)$. If $f\in U$ then the fraction map of Corollary 2.4 inverts the residue class of f in E, which however already must be invertible in E since the determinant has already been inverted—by the very definition of $(\mathrm{S}^n_A)_{U(n)}$. $\square$

3. Hilbert scheme of points on fraction rings

We will in this section use Theorem 2.2 to describe the Hilbert scheme parameterizing n points on $\mathrm{Spec}(k[X]_U)$, where $k[X]_U$ is any fraction ring of the polynomial ring in one variable X over a commutative and unitary base ring k.

Proposition 3.1. *Let $I\subseteq R_U$ be an ideal of a fraction ring R_U of an A-algebra R. Assume that the residue class ring R_U/I is finitely generated as an A-module. Then there exists a unique ideal $J\subseteq R$ such that the canonical map $R/J\to R_U/I$ is an isomorphism.*

Proof. Let $J=R\cap I$. Since any ideal in a fraction ring R_U is the extension of some ideal in R, it follows that $JR_U=I$. In particular the canonical map

$$R/J \longrightarrow R_U/I \tag{3.1}$$

is injective. We will show that (3.1) is surjective by passing to the stalks. Let P be a prime ideal of R/J. The map (3.1) is a fraction map hence the extension of P is either a prime ideal or the unit ideal. If the extension of P is a prime ideal then the induced map of stalks $(R/J)_P \to (R_U/I)_P$ is an isomorphism, and in particular surjective. If the extension of P was the whole ring R_U/I, then since R_U/I is a finitely generated A-module, it follows by Nakayama's lemma that the finitely generated $(R/J)_P$-module $(R_U/I)_P$ is zero. In either case we see that the map of stalks induced by (3.1) is surjective for all prime ideals P of R/I. $\square$

Corollary 3.2. *Let $R=A[X]$ be the polynomial ring in the variable X over A, and let $I \subseteq A[X]_U$ be an ideal such that the A-module $A[X]_U/I$ is free of rank n. Then there exists a unique monic polynomial $F \in A[X]$ such that*

$$A[X]/(F) \longrightarrow A[X]_U/I$$

is an isomorphism.

Proof. By the proposition there exists an ideal $J \subseteq A[X]$ such that the canonical map $A[X]/J \to A[X]_U/I$ is an isomorphism, and we need to show that J is generated by a monic polynomial F of degree n. The variable X gives an endomorphism of the free A-module $A[X]/J$. Let F be the characteristic polynomial of $a \mapsto aX$. By the Cayley–Hamilton theorem we have a surjective map $A[X]/(F) \to A[X]/J$ of free A-modules of rank n, hence an isomorphism. The uniqueness of F is clear. $\square$

Definition 3.3. Fix a fraction ring $k[X]_U$ of the polynomial ring in the variable X, over a commutative ring k. Let A be a k-algebra, and let $\mathrm{Hilb}^n(A)$ denote the set of ideals $I \subseteq A \otimes_k k[X]_U$ such that the residue class ring $A \bigotimes_k k[X]_U/I$ is locally free of finite rank n as an A-module. Then Hilb^n becomes in a natural way a covariant functor from the category of k-algebras to sets.

Remark 3.4. One could replace the definition above and consider the set of ideals $I \subseteq A \otimes_k k[X]_U$ such that the residue class ring $M = A \otimes_k k[X]_U/I$ is a flat A-module with constant fiber dimension $n = \dim_{\varkappa(P)} M \otimes_A \varkappa(P)$ for all prime ideals $P \subseteq A$. It can be shown (see e.g. [6], Theorem 3.5) that such a module must necessarily be finitely generated since $A \otimes_k k[X]_U$ is an A-algebra essentially of finite type, hence M is locally free of rank n.

Proposition 3.5. *The Hilbert functor* Hilb^n *of n-points on $k[X]_U$ is represented by the fraction ring $H = (\mathrm{S}_k^n)_{U(n)} = k[s_1, \dots, s_n]_{U(n)}$, where*

$$U(n) = \{f(t_1) \dots f(t_n) \mid f \in U \subseteq k[X]\}.$$

The universal family is the ideal generated by $\Delta_n(X) \in H \otimes_k k[X]_U$.

Proof. We have by Corollary 2.4 that the ideal generated by $\Delta_n(X)$ is an element of $\mathrm{Hilb}^n(H)$. To prove the claim we need to show that the morphism of functors $\mathrm{Hom}_{k\text{-alg}}(H,-)\to\mathrm{Hilb}^n$ induced by the H-valued point $(\Delta_n(X))$ is an isomorphism. Let A be a k-algebra, and let $I\in\mathrm{Hilb}^n(A)$ be an A-valued point of the Hilbert functor. We have that $A\otimes_k k[X]_U$ is the localization of $A[X]=A\otimes_k k[X]$ with respect to the image U_A of the multiplicatively closed set $U\subseteq k[X]$ by the natural map $k[X]\to A\otimes_k k[X]$. Then by Proposition 3.1 there exists an ideal $J\subseteq A[X]$ such that

$$A[X]/J\to A\otimes_k k[X]_U/I=A[X]_{U_A}/I \tag{3.2}$$

is an isomorphism. The A-module $A[X]/J$ is locally free of finite rank n, and it is easy to see that the A-module $A[X]/J$ must be free. Indeed, locally the $A'=A_f$-module $A[X]/J\otimes_A A'$ is free of rank n, and it follows by Corollary 3.2 that $1, x, \dots, x^{n-1}$ is an A'-module basis. Then we get that the classes $1, x, \dots, x^{n-1}$ form an A-module basis of $A[X]/J$, and consequently $A[X]/J$ is free. By Corollary 3.2 again, there exists a unique monic polynomial $F\in A[X]$ that generates J, and hence generates I. The element $F\in A[X]$ determines an A-algebra homomorphism $u_F\colon \mathrm{S}^n_A\to A$. The isomorphism (3.2) and Proposition 2.3(3) shows that the homomorphism u_F factors via the fraction ring $(\mathrm{S}^n_A)_{U_A(n)}$. Clearly we have that $A\otimes_k(\mathrm{S}^n_k)_{U(n)}=(\mathrm{S}^n_A)_{U_A(n)}$ and we obtain a map $H=(\mathrm{S}^n_k)_{U(n)}\to A$ that sends the coefficients of $\Delta_n(X)$ to the coefficients of $F\in A[X]$. It is clear that the constructed map of sets $\mathrm{Hilb}^n(A)\to\mathrm{Hom}_{k-\mathrm{alg}}(H,A)$ is functorial in A, and inverse to the natural transformation induced by $\Delta_n(X)$. $\square$

Remark 3.6. It is clear that the functor $\mathcal{H}$ from the category of k-schemes to sets sending a k-scheme T to the set of closed subschemes $Z\subseteq T\times_k\mathrm{Spec}(k[X]_U)$ such that the projection map $Z\to T$ is finite and locally trivial of finite rank n is represented by an affine scheme whose coordinate ring is $H=(\mathrm{S}^n_k)_{U(n)}$. We call the scheme $\mathrm{Spec}((\mathrm{S}^n_k)_{U(n)})$ the Hilbert scheme of n-points on $\mathrm{Spec}(k[X]_U)$.

Example 3.7. *The Hilbert scheme of points on the affine line.* When $U=\{1\}\subseteq k[X]$ is the trivial subset we have that $\mathrm{Spec}(k[X]_U)=\mathbf{A}^1_k$ is the affine line over k. The Hilbert scheme of n-points on $\mathbf{A}^1_k$ is then $\mathrm{Spec}(\mathrm{S}^n_k)$, where S^n_k the ring of symmetric functions of $k[t_1,\dots,t_n]$. Hence the Hilbert scheme of n-points on the affine line $\mathbf{A}^1_k$ is simply the affine n-space $\mathbf{A}^n_k$ over k. Note that the only assumptions on the base ring k is that k is commutative and unitary.

Example 3.8. *The Hilbert scheme of points on open subsets of the line.* Let the multiplicatively closed subset U be given by multiples of an element f in $k[X]$, that is $U=\{f^m\}_{m\geq 0}$. Then $\mathrm{Spec}(A[X]_U)=D(f)$ is a basic open subscheme of $\mathbf{A}^1_k$, the affine line over k.

The Hilbert scheme of n-points on $D(f)$ is then the spectrum of $(\mathrm{S}_k^n)_{U(n)}$, where $U(n)=\{(f(t_1)\dots f(t_n))^m\}_{m\geq 0}$. Hence we have that the Hilbert functor of n-points on a basic open subscheme $D(f)$ of the line is represented by a basic open subscheme $D(f(t_1)\dots f(t_n))$ of the Hilbert scheme of n-points on the line.

Example 3.9. *The Hilbert scheme parameterizing finite length subschemes of the line with support at the origin.* Let the base ring k be a field, and let $U\subseteq k[X]$ be the set of polynomials f such that $f(0)\neq 0$. Thus $k[X]_U=k[X]_{(X)}$ is the local ring of the origin on the line, and the Hilbert functor parameterizes the length n subschemes of $\mathrm{Spec}(k[X]_{(X)})$. There is only one closed subscheme of $\mathrm{Spec}(k[X]_{(X)})$ of length n, namely the scheme given by the ideal $(X^n)\subseteq k[X]_{(X)}$. The Hilbert scheme of n-points on $\mathrm{Spec}(k[X]_{(X)})$ is given as the spectrum of $(\mathrm{S}_k^n)_{U(n)}$, where $U(n)$ is the product $f(t_1)\dots f(t_n)$ for all $f(X)\in k[X]$ with $f(0)\neq 0$. See also [7].

Remark 3.10. *Inverse limits.* A consequence of the explicit description given in Proposition 3.5 is that the Hilbert functor parameterizing points on the line, commutes with inverse limits. Let $D(f_\alpha)$ be a basic open subscheme of the line $\mathrm{Spec}(k[X])$, and let Hilb_α^n denote the Hilbert scheme of n-points on $D(f_\alpha)$. Furthermore, let $\{D(f_\alpha)\}_{\alpha\in\mathcal{A}}$ be a collection of open sets that form a directed set by intersection. The inverse limit $\varprojlim_\alpha D(f_\alpha)$ is clearly the affine scheme given by the fraction ring $k[X]_{U_{\mathcal{A}}}$, where $U_{\mathcal{A}}$ is the multiplicatively closed set $\{f_\alpha\}_{\alpha\in\mathcal{A}}$. We have by Proposition 3.5 that the Hilbert scheme of n-points on the inverse limit $\varprojlim_\alpha D(f_\alpha)=\mathrm{Spec}(k[X]_{U(\mathcal{A})})$ is given by $\mathrm{Spec}((\mathrm{S}_k^n)_{U_{\mathcal{A}}(n)})$, which is the corresponding inverse limit of Hilbert schemes $\varprojlim_\alpha \mathrm{Hilb}_\alpha^n$.

Example 3.11. *A Hilbert scheme without rational points.* Assume that the base ring k is an integral domain. Let $U\subseteq k[X]$ be the set of non-zero polynomials. We have that $k[X]_U=k(X)$ is the function field of the line. Clearly, since $k(X)$ is a field, the Hilbert scheme of n-points on $\mathrm{Spec}(k(X))$ has no k-valued points. By Proposition 3.5 we get that the coordinate ring of the Hilbert scheme is the fraction ring of the symmetric functions S_k^n with respect to the set $U(n)$ of products $f(t_1)\dots f(t_n)$ for any non-zero $f\in k[X]$. We will below compute the relative dimension of the Hilbert scheme.

Assume that k is an algebraically closed field. Since the ring of symmetric functions S_k^n is the polynomial ring $k[s_1,\dots,s_n]$, and in particular finitely generated as a k-algebra, we get that the maximal ideals of S_k^n correspond uniquely to the set of k-rational points of S_k^n. Since the fraction ring $(\mathrm{S}_k^n)_{U(n)}$ has no k-rational points, being the coordinate ring of the Hilbert scheme of n-points on $\mathrm{Spec}(k(X))$, we get that the Krull dimension of $(\mathrm{S}_k^n)_{U(n)}$ is strictly less than n, the dimension of $\mathrm{S}_k^n=k[s_1,\dots,s_n]$.

Let $Q \subseteq \mathrm{S}_k^n$ be the prime ideal generated by the elementary symmetric functions $s_1, \dots, s_{n-1}$. The dimension of the local ring $(\mathrm{S}_k^n)_Q$ is $n-1$. To show that the fraction ring $(\mathrm{S}_k^n)_{U(n)}$ has dimension $n-1$ we need only show that Q does not meet $U(n)$. We will consider S_k^n as a graded ring by letting $\deg(s_i)=i$.

Let f be a non-zero polynomial in the variable X. Let m be the degree of f which we may write as $f=aX^m+g(X)$ with $a\neq 0$, where $\deg g(X)<m$. We may assume that $m>0$. Then we have that $f(t_1)\dots f(t_n)=a^n t_1^m \dots t_n^m+G$, where $G=G(t_1, \dots, t_n)$ is symmetric but of lower degree than the degree of $t_1^m \dots t_n^m = s_n(X^m) = (s_n)^m$. The elementary symmetric functions $s_1, \dots, s_n$ are algebraically independent over k. Hence the residue class of $(s_n)^m$ modulo the ideal Q is non-zero. Since G is of less degree than $(s_n)^m$ it follows that the residue class of $a^n(s_n)^m+G=f(t_1)\dots f(t_n)$ is non-zero modulo the ideal Q. We have shown that the intersection $Q\cap U(n)$ is empty.

The fact that we need an $(n-1)$-dimensional scheme to parameterize the empty set of closed subschemes of $\mathrm{Spec}(k(X))$ of finite length n, shows that one should take care when only considering rational points of the Hilbert functor.

Example 3.12. *Hilbert schemes of one point.* Let the fixed integer $n=1$. For any multiplicatively closed subset $U\subseteq k[X]$ we have that the scheme $\mathrm{Spec}(k[X]_U)$ itself is the Hilbert scheme of one-points on $\mathrm{Spec}(k[X]_U)$. See also [5] (Corollary 2.3 of Proposition 2.2, p. 109) where Kleiman proves that for any S-scheme X the functor $\mathrm{Hilb}^1_{X/S}$ is represented by the scheme X, and where the universal family is given by the diagonal in $X\times_S X$.

4. Symmetric products

The purpose of this last section is to show that the universal family of n-points on $C=\mathrm{Spec}(k[X]_U)$ is isomorphic to $\mathrm{Sym}_k^{n-1}(C)\times_k C$, as is the case when C is a smooth curve [4].

4.1. Set up

Let $t_1, \dots, t_n$ $(n>0)$ be independent variables over A, and denote the elementary symmetric functions in $t_1, \dots, t_n$ by $s_{1,n}, \dots, s_{n,n}$. The elementary symmetric functions in the $(n-1)$-variables $t_1, \dots, t_{n-1}$ we denote with $s_{1,n-1}, \dots, s_{n-1,n-1}$. Let X be an additional variable and define the A-algebra homomorphism

$$a_n\colon \mathrm{S}_A^n[X] \longrightarrow \mathrm{S}_A^{n-1}[X]$$

by sending $s_{i,n} \mapsto s_{i,n-1}+s_{i-1,n-1}X$ (here $s_{0,n}=1$ and $s_{n,n-1}=0$) for $i=1, \dots, n$, and $X \mapsto X$. We define recursively an $A[X]$-algebra homomorphism

$$p_n \colon \mathrm{S}_A^{n-1}[X] \longrightarrow \mathrm{S}_A^n[X]$$

by letting $p_n(s_{i,n-1})=s_{i,n}-p_n(s_{i-1,n-1})X$ for $i=1, \dots, n-1$ and where $p_n(s_{0,n-1})=p_n(1)=1$. The variable X is mapped to X.

Lemma 4.1. *For all positive integers $n>0$ the following three assertions hold*:

(1) *We have that $p_n \circ a_n(s_{i,n})=s_{i,n}$ for all $i=1, \dots, n-1$.*

(2) *The composite map $a_n \circ p_n$ is the identity map.*

(3) *The kernel of a_n is generated by $\Delta_n(X)=\prod_{i=1}^n (X-t_i)$.*

Proof. The first assertion follows immediately from the identity

$$\begin{aligned} p_n \circ a_n(s_{i,n}) &= p_n(s_{i,n-1})+p_n(s_{i-1,n-1})X \\ &= s_{i,n}-p_n(s_{i-1,n-1})X+p_n(s_{i-1,n-1})X = s_{i,n} \end{aligned}$$

that holds for each $i=1$, ..., $n-1$. To prove the second assertion it is enough to show that the composition $a_n \circ p_n$ acts as the identity on the elementary symmetric functions $s_{1,n-1}$, ..., $s_{n-1,n-1}$. We prove this by induction on i. For $i=1$ we get by definition that $a_n \circ p_n(s_{1,n-1})=a_n(s_{1,n}-X)=s_{1,n-1}$. Assume as the induction hypothesis that $a_n \circ p_n(s_{i,n-1})=s_{i,n-1}$ for $i \geq 1$. We then get that

$$\begin{aligned} a_n \circ p_n(s_{i+1,n-1}) &= a_n(s_{i+1,n}-p_n(s_{i,n-1})X) \\ &= s_{i+1,n-1}+s_{i,n-1}X-a_n \circ p_n(s_{i,n-1})X = s_{i+1,n-1}. \end{aligned}$$

Thus we have proven Assertion (2). To prove the last assertion we first show that $\Delta_n(X)$ is in the kernel of a_n. We have that

$$\begin{aligned} a_n(\Delta_n(X)) &= X^n+\sum_{i=1}^n (-1)^i a_n(s_{i,n})X^{n-i} \\ &= X^n+\sum_{i=1}^n (-1)^i (s_{i,n-1}+s_{i-1,n-1}X)X^{n-i}. \end{aligned} \tag{4.1}$$

By definition we have that $s_{0,n-1}=1$ and that $s_{n,n-1}=0$. Thus it follows from (4.1) that $\Delta_n(X)$ is in the kernel of a_n. It follows by Assertion (2) that we get an induced surjective map $\hat{a}_n \colon \mathrm{S}_A^n[X]/(\Delta_n(X)) \to \mathrm{S}_A^{n-1}[X]$. To prove Assertion (3) we need to show that $\hat{a}_n$ is an isomorphism, or equivalently that the induced map $\hat{p}_n \colon \mathrm{S}_A^{n-1}[X] \to \mathrm{S}_A^n[X]/(\Delta_n(X))$ is surjective. It follows by Assertion (1) that it suffices to show that the residue class of $s_{n,n}$ in $\mathrm{S}_A^n[X]/(\Delta_n(X))$ is in the image of $\hat{p}_n$. Since the residue class of $s_{n,n}$ modulo the ideal $(\Delta_n(X))$ can be expressed as

$$(-1)^{n-1}s_{n,n} = X^n-s_{1,n}X^{n-1}+\dots+(-1)^{n-1}s_{n-1,n}X,$$

the result follows. $\square$

Proposition 4.2. *The Hilbert scheme of n-points on $C=\operatorname{Spec}(k[X]_U)$ is isomorphic to the n-fold symmetric product $\operatorname{Sym}_k^n(C)$. The universal family of n-points on C is isomorphic to $\operatorname{Sym}_k^{n-1}(C)\times_k C$.*

Proof. By Proposition 3.5 the Hilbert scheme of n-points on C is affine and given as the spectrum of $H=k[s_{1,n},\dots,s_{n,n}]_{U(n)}$. Let $\bigotimes_k^n k[X]_U$ denote the n-fold tensor product of $k[X]_U$ over k. The symmetric group G of n-letters acts on $\bigotimes_k^n k[X]_U$ by permuting the factors. By definition we have that $\operatorname{Sym}_k^n(C)$ is the spectrum of the invariant ring $(\bigotimes_k k[X]_U)^{\mathrm{G}}$. To prove the first claim we must show that the invariant ring $(\bigotimes_k^n k[X]_U)^{\mathrm{G}}$ equals $H=k[s_1,\dots,s_n]_{U(n)}$. It is clear that we have a natural identification $(\bigotimes_k^n k[X]_U)=k[t_1,\dots,t_n]_{U(n)}$, and that $k[s_{1,n},\dots,s_{n,n}]_{U(n)}$ is a G-invariant subring of $k[t_1,\dots,t_n]_{U(n)}$. We must show that a G-invariant element F of $k[t_1,\dots,t_n]_{U(n)}$ is in $k[s_{1,n},\dots,s_{n,n}]_{U(n)}$. Let F be a G-invariant element, and write $F=f/g$, with $f\in k[t_1,\dots,t_n]$ and $g\in U(n)$. Since F and g are G-invariants we get for each $\sigma\in G$ that $F=F^\sigma=f^\sigma/g$. Thus $(f-f^\sigma)g_\sigma=0$ in $k[t_1,\dots,t_n]$, for some $g_\sigma\in U(n)$. Let $g_G=\prod_{\sigma\in G}g_\sigma$. It follows that the element fg_G in $k[t_1,\dots,t_n]$ is G-invariant, hence fg_G is in $k[s_{1,n},\dots,s_{n,n}]$. Then finally we have that $F=f/g=fg_G/gg_G$, hence F is in $k[s_{1,n},\dots,s_{n,n}]_{U(n)}$ and we have proven the first claim.

We next prove the second claim. By Lemma 4.1, Assertions (2) and (3), we get that the sequence of $\mathrm{S}_A^n[X]$-modules

$$0\longrightarrow(\Delta_n(X))\longrightarrow\mathrm{S}_A^n[X]\xrightarrow{a_n}\mathrm{S}_A^{n-1}[X]\longrightarrow 0 \tag{4.2}$$

is exact. Write $\mathrm{S}_A^n[X]=\mathrm{S}_A^n\otimes_A A[X]$. Clearly we have that $a_n(f(t_1)\dots f(t_n))=f(t_1)\dots f(t_{n-1})\otimes f(X)$ for all $f\in A[X]$. Thus when we localize the sequence (4.2) in $U(n)$ we get that

$$(\mathrm{S}_A^n)_{U(n)}\otimes_A A[X]/(\Delta_n(X))\cong(\mathrm{S}_A^{n-1})_{U(n-1)}\otimes_A A[X]_U.$$

Specializing to the case $A=k$, and recalling that we have already proven that $\operatorname{Spec}((\mathrm{S}_A^{n-1})_{U(n-1)})=\operatorname{Sym}_k^{n-1}(C)$, the result follows. $\square$

Remark 4.3. *The Hilbert scheme of smooth curves.* It was pointed out by Grothendieck ([3], p. 275) that by generalizing the concept of norm one could identify the Hilbert scheme of n-points on a family of smooth curves $C\to S$ with $\operatorname{Sym}_S^n(C)$. A generalized norm on a k-algebra A is a *homogeneous multiplicative polynomial law of degree n*, and parameterized by a gamma-algebra $\Gamma_k^n A$ (see [9] or [2]). In [1] (pp. 431–437) Deligne explains the connection between the Hilbert scheme of points on smooth curves $C\to S$ and the scheme parameterizing generalized norms $\operatorname{Sym}_S^n(C)$, where the latter scheme is given as the spectrum of the sheaf of gamma-algebras on C.

In a more restrictive setting, but more concretely in terms of coordinates, Iversen identifies in [4] the Hilbert scheme of n-points of a smooth family of irreducible curves as the n-fold symmetric product.

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Roy Mikael Skjelnes
Department of Mathematics
Royal Institute of Technology
SE-100 44 Stockholm
Sweden
email: skjelnes@math.kth.se