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Gabor analysis of the continuum model for impedance tomography

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Abstract. We give a sharp upper estimate for the response of boundary current-voltage measurements to perturbations of the admittivity in a body that are localized in space and frequency. We calculate the differential of the measurement mapping and study the *Gabor symbol* of this operator.

1. Introduction

The electric impedance imaging technique is used for determining the distribution of electric conductivity and permittivity in a body from a series of measurements of current and voltage on the body surface, see the surveys [2] and [4]. The electric potential u in a body Ω is governed by the Poisson equation

$$\nabla\cdot\gamma\nabla u=0, \tag{1.1}$$

where $\gamma=\gamma(x,\omega)=\sigma(x,\omega)+\omega\varepsilon(x,\omega)i$. The real part $\sigma>0$ is the electric conductivity, ε is the electric permittivity of the body and $\omega=\omega_0$ is the frequency of the applied current (it is taken to be constant). The coefficient γ is called the admittivity of the medium. We assume that the domain Ω is an open bounded set in the Euclidean space E of dimension 3 and that the boundary Γ is of the class C^2; ∂_n will denote the unit inwards normal derivative on Γ. The boundary values

$$u|_\Gamma=f, \quad \gamma\partial_n u|_\Gamma=g$$

are voltage and current, respectively. Any measurement of the current-voltage pair (g,f) gives a point in the graph of the operator $R_\gamma: g\mapsto f$. The function g has zero integral over Γ (conservation of charge), the voltage f is subject to the same condition and is uniquely defined. The inverse mapping $L_\gamma: f\mapsto g$ is called the Calderón (or "Dirichlet-to-Neumann") operator. Suppose that we know the function f for any

g, i.e. we know the operator R_γ. This assumption is called the continuum model for the electrical impedance imaging. The mathematical problem is to reconstruct the function $\gamma(x):=\gamma(x,\omega_0)$ in Ω from the knowledge of R_γ. A similar problem arises in geophysical prospecting (see, e.g. [13]). It is known that the admittivity γ is uniquely defined by this operator, that is, the mapping $R:\gamma\mapsto R_\gamma$ (and the mapping $L:\gamma\mapsto L_\gamma$) is one-to-one. This was proved in [11] and [9], see also the survey [12]. For the inverse mapping $L_\gamma\mapsto\gamma$ only a weak stability estimate is known [1].

The objective of this paper is to estimate the response of the operators R_γ and L_γ to perturbations of the function γ that are sharply localized both in space and in frequency. The logarithm of the response decreases as the product of the distance of the localization point to Γ and of the local frequency. This implies that a very limited volume of information on the function γ could be extracted from any real measurements. Roughly speaking, this volume grows at most as $O(\log N)$ where N denotes the volume of reliable measurement data.

2. Differential of the measurement mapping

We assume, as customary, that the voltage u belongs to the "finite energy" space $H^1(\Omega)$. (We use the notation $H^k(X)$ for the Sobolev space $W_2^k(X)$.) The boundary value f is then an element of the space $H^{1/2}(\Gamma)$. Vice versa, the Dirichlet problem for the equation (1.1) has a unique finite energy solution u for any function $f\in H^{1/2}(\Gamma)$. The normal derivative $\partial_n u$ is an element of the space $H_0^{-1/2}(\Gamma)$; this is the subspace of $H^{-1/2}(\Gamma)$ of functions in Γ with zero mean. The Calderón operator $L_\gamma: f\mapsto g=\gamma\partial_n u$ is an elliptic pseudodifferential first order operator $H^{1/2}(\Gamma)\to H_0^{-1/2}(\Gamma)$. Write the Poisson equation (1.1) in the slightly different form

$$\Delta u+(\nabla\log\gamma,\nabla u)=0. \tag{2.1}$$

The solution does not change if we replace γ by $c\gamma$. Therefore the measurement mapping L is homogeneous, that is $L_{c\gamma}=cL_\gamma$, $c\neq 0$.

Let $C^1(\Omega)$ be the space of complex-valued functions in $\bar\Omega$ whose first order derivatives are continuous up to Γ. It is endowed with the standard norm $\|\cdot\|^{(1)}$. Denote by $C_*^1(\Omega)$ the open subset of non-vanishing functions. The mapping $L:\gamma\mapsto L_\gamma$ is continuous from $C_*^1(\Omega)$ to the space of operators $\mathcal{L}:=\mathcal{L}(H^{1/2}(\Gamma),H^{-1/2}(\Gamma))$; we call it the *measurement* operator. The response of the measurement data to small perturbations of the admittivity can be evaluated by the differential of the mapping L. The differential at a point γ in the strong sense is a linear mapping $d_\gamma L: C^1(\Omega)\to\mathcal{L}$ such that

$$L_{\gamma+\delta}-L_\gamma=d_\gamma L(\delta)+\Lambda_\gamma(\delta),\quad \|\Lambda_\gamma(\delta)\|_{\mathcal{L}}=o(\|\delta\|^{(1)}),\quad \delta\in C^1(\Omega).$$

We will denote by $\|\cdot\|_X^k$ the standard norm in $H^k(X)$.

Proposition 2.1. *The differential $d_\gamma L$ exists for an arbitrary $\gamma \in C^1_*(\Omega)$ and for any $\delta \in C^1(\Omega)$ and $f \in H^{1/2}(\Gamma)$ we have*

$$d_\gamma L(\delta)\colon f \longmapsto (\gamma \partial_n v + \delta \partial_n u)|_\Gamma,$$

where u is the solution of the boundary value problem $u|_\Gamma = f$ for (1.1) and v is the unique solution of the boundary value problem

$$\nabla \cdot \gamma \nabla v = -\nabla \cdot \delta \nabla u, \quad v|_\Gamma = 0. \tag{2.2}$$

A similar statement was used in [10]. A proof can be done by standard arguments.

3. An estimate of the differential

We estimate here the value of the differential $d_\gamma L$ in the point $\delta = e_\lambda$, where

$$e_\lambda(x) := \exp(-\pi\sigma(x-p)^2 + 2\pi i \xi(x-p)), \quad \lambda := (p, \xi) \in E \times E^* \tag{3.1}$$

for some choice of the parameter $\sigma > 0$. A function of this form (in the case $\dim E = 1$) was called an *elementary signal* by D. Gabor [6]; since "it occupies the smallest possible area in the information diagram", i.e. in $E \times E^*$. Really, the function e_λ is sharply localized in the vicinity of $p \in E$, whereas the Fourier transform $\hat{e}_\lambda(\theta) := \int_E \exp(-2\pi i \theta x) e_\lambda(x)\, dx$ is sharply concentrated near the point ξ in the dual space E^*. We modify this construction for the bounded domain Ω. The optimal localization both in Ω and in E^* is attained if we fix the dispersion parameter σ as

$$\sigma := \frac{|\xi|}{d}, \quad d := \operatorname{dist}(p, \Gamma), \quad \xi \neq 0.$$

Really, for an arbitrary dimensionless parameter $s > 0$ we have

$$\int_{|x-p|>sd} |e_\lambda|^2\, dx = (2\sigma)^{-3/2} \exp(-\pi s^2 d |\xi|)$$

which characterizes the localization of the test function in the vicinity of p and

$$\int_{|\theta-\xi|>s|\xi|} |\hat{e}_\lambda|^2\, d\theta = (2\sigma)^{-3/2} \exp(-\pi s^2 d |\xi|)$$

which shows the localization of $\hat{e}_\lambda$ in the vicinity of ξ. Moreover, we have

$$\max_{q \in \Gamma} |e_\lambda(q)| \leq \exp(-\pi d |\xi|).$$

Consider the complexification $E_{\mathbf{C}} := E \otimes \mathbf{C}$ of the space E. Let Γ_r denote the closed r-neighbourhood of Γ in $E_{\mathbf{C}}$ for $r > 0$ and $\|A\|^{l,k}$ stands for the norm of a linear operator $A\colon H^k(\Gamma) \to H^l(\Gamma)$.

Theorem 3.1. *Suppose that γ has holomorphic non-vanishing continuation to the set Γ_r for some $r>0$. Then for some $s>0$ the operator function $\lambda \mapsto d_\gamma L(e_\lambda)$ has the estimate, in the domain $\{\lambda=(p,\xi):\pi d|\xi|\geq 1,\ d\leq s\}$,*

$$\|d_\gamma L(e_\lambda)\|^{-1/2,1/2} \leq (C|\xi|+l)(d|\xi|)^{3/2}\exp(-\pi d|\xi|), \tag{3.2}$$

where $l(\gamma):=\sup_{\Omega_r}|\nabla \log\gamma|$ and C is a positive constant.

A similar estimate holds for the measurement mapping $R:\gamma\mapsto R_\gamma$ in terms of the "current-to-voltage" operator $R_\gamma: H_0^{-1/2}(\Gamma)\to H_0^{1/2}(\Gamma)$.

Corollary 3.2. *Under the same conditions we have the inequality*

$$\|d_\gamma R(e_\lambda)\|^{1/2,-1/2} \leq (C|\xi|+l)(d|\xi|)^{3/2}|\exp(-\pi d|\xi|). \tag{3.3}$$

Proof. Really, we have $d_\gamma R=-R_\gamma\circ d_\gamma L\circ R_\gamma$; then (3.3) follows from (3.2) since R_γ is bounded. □

Proof of Theorem 3.1. Let $m:=\sup_{\Omega_r}|\varrho|\sup_{\Omega_r}|\varrho|^{-1}$ and take an arbitrary $s<(1+e\sqrt{3m})^{-1}r$. Consider the domain $P:=\{(z,q):(z-q)^2\in\mathbf{C}\setminus\mathbf{R}_-\}$ in $E_{\mathbf{C}}\times E_{\mathbf{C}}$ and define in P the holomorphic function $\sqrt{(z-q)^2}$ with positive real part. Set $G:=\{(z,q):q\in\Gamma_s,\ |z-q|\leq s\}$. □

Lemma 3.3. *There exists a function $\Phi(z,q)$ in the domain $G\cap P$ that satisfies the equation*

$$\nabla_z\cdot\gamma(z)\nabla_z\Phi(z,q)=\delta(z-q)$$

and has the structure

$$\Phi(z,q)=\frac{\phi(z,q)}{\sqrt{(z-q)^2}},$$

where ϕ is a holomorphic function in G.

Proof. We find the fundamental solution for (1.1) by Hadamard's method [7],

$$\Phi(z,q)=\varrho(z)\sum_{k=0}^{\infty}u_k(z,q)E_k(z-q),\quad E_k(z-q):=\frac{(-1)^{k+1}k!}{4\pi(2k)!}\frac{(z-q)^{2k}}{\sqrt{(z-q)^2}},\quad \varrho=\frac{1}{\gamma^{1/2}}.$$

We use here the abbreviated notation $w^{2k}=(w_1^2+w_2^2+w_3^2)^k$. The kernel $E_k(z-q)$ is the holomorphic continuation of the spherical symmetric fundamental solution for Δ^{k+1} to the domain P and u_k are holomorphic functions given by the recurrent formula

$$u_0=1,\quad u_{k+1}(z,q)=\int_0^1\frac{\Delta(\varrho u_k)(q+t(z-q))}{\varrho(q+t(z-q))}t^k\,dt,\quad k=0,1,\dots .$$

The functions ϱ and ϱ^{-1} are holomorphic and bounded in the ball $B_q:=\{z:|z-q|\le r\}$ for an arbitrary point $q\in\Gamma$. We show by means of induction that for any $k\ge 0$,

$$|u_k(z,q)|\le \frac{(3m_q)^k}{(r-|z-q|)^{2k}}\frac{(2k)!}{k!},\qquad m_q:=\sup_{B_q}|\varrho|\sup_{B_q}\frac{1}{|\varrho|}.$$

For this we estimate each derivative $\partial^2 u_k/\partial^2 z_j$, $j=1,2,3$, by means of the Cauchy inequality. It follows that

$$\frac{k!}{(2k)!}|u_k(z,q)(z-q)^{2k}|\le (3m_q)^k\left(\frac{e}{r-|z-q|}\right)^{2k}|z-q|^{2k};$$

therefore the series $A(z,q)=\sum_{k=0}^{\infty}u_k(z,q)(z-q)^{2k}k!/(2k)!$ converges in the ball $\{z: |z-q|<\left(1+e\sqrt{3m_q}\right)^{-1}r\}$. This ball contains the s-neighbourhood of q since $m_q\le m$. □

Lemma 3.4. *An arbitrary solution $h\in H^1(\Omega)$ of (1.1) has a holomorphic continuation $\tilde h$ to $G:=\{z=x+yi\in\Gamma_r:|y|<\operatorname{dist}(x,\Gamma)\}$ such that*

$$\|\tilde h\|_G^{5/2}\le C\|h\|_\Omega^1.$$

Proof. It is sufficient to prove this statement for the subset $G'=G\cap U$, where U stands for the s-neighbourhood of a point $q_0\in\Gamma$. Take the fundamental solution Φ found in the previous lemma. The boundary values of the kernel $2\partial_n\Phi(x,q)$ in $\Gamma':=\partial(U\cap\Omega)$ is equal $I+K$, where I is the identity operator and K is a pseudodifferential operator of order -1. Therefore $2\Phi(x,q)$ is a parametrix for the Neumann boundary value problem in $U\cap\Omega$. The function $\partial_n h|_{\Gamma'}$ belongs to $H^{-1/2}(\Gamma')$ according to the regularity theorem for elliptic boundary value problems. By solving the Fredholm equation

$$\int_{\Gamma'}\partial_n\Phi(x,q)b(q)\,dq=\partial_n h(x),\qquad x\in\Gamma',$$

we find a function $b\in H^{-1/2}(\Gamma')$ such that $\|b\|_{\Gamma'}^{-1/2}\le C\|\partial_n h\|_{\Gamma'}^{-1/2}\le C\|h\|_{\Omega'}^1$. It follows that

$$h(x)=\int_{\Gamma'}\Phi(x,q)b(q)\,dq+h_0$$

for a constant h_0. For any $q\in\Gamma$ the fundamental solution has a holomorphic continuation $\Phi(z,q)$ to the domain G since $\operatorname{Re}(z-q)^2=(x-q)^2-y^2\ge d^2-y^2>0$. Take the continuation of h to Γ' by means of

$$\tilde h(z)=\int_{\Gamma'}\Phi(z,q)b(q)\,dq+h_0.$$

The boundary Γ' has dimension 2 and codimension 4 in G; the inequality $\|\tilde h\|_G^{5/2}\le C\|b\|_{\Gamma'}^{-1/2}$ holds because of the structure of the kernel Φ. This together with the above estimate of the norm of b implies the lemma. □

Lemma 3.5. *Let $u\in H^1(\Omega)$ be an arbitrary solution to* (1.1) *and λ be as above. There exists a solution v_λ to the equation*

$$\nabla\cdot\gamma\nabla v_\lambda=-\nabla\cdot e_\lambda\nabla u \tag{3.4}$$

that satisfies

$$\|v_\lambda\|_\Gamma^{1/2}\le(C_0|\xi|+l)(d|\xi|)^{3/2}\exp(-\pi d|\xi|\|u\|_\Omega^1. \tag{3.5}$$

Proof. Write the right-hand side as

$$w:=-\nabla\cdot e_\lambda\nabla u=(-\nabla e_\lambda+e_\lambda\nabla\log\gamma,\nabla u)=w_p+w_0,$$

where w_p is supported in the d-neighbourhood U_p of the point p and w_0 vanishes in U_p. There is a unique solution v_0 of the selfadjoint boundary value problem $\nabla\cdot\gamma\nabla v_0=w_0$, $v_0|_\Gamma=0$. It satisfies the inequality

$$\|v_0\|_\Gamma^{3/2}\le C\|v_0\|_\Omega^2\le C\|w_0\|_\Omega^0\le(C|\xi|+l)\exp(-\pi d|\xi|), \tag{3.6}$$

where the last estimate follows from the obvious inequalities for e_λ and ∇e_λ. The integral

$$v_p(q)=\int_\Omega\Phi(q,x)w_p(x)\,dx=\int_{U_p}\Phi(q,x)w(x)\,dx$$

satisfies $\nabla\cdot\gamma\nabla v_p=w_p$. Write the right-hand side in a different way. The function e_λ is holomorphic in $E_{\mathbf{C}}$ and u has a holomorphic continuation $\tilde u$ to G by Lemma 3.4. The form $\Phi(q,x)w(x)\,dx$ has for an arbitrary $q\in\Gamma$ the holomorphic continuation $\alpha_q=\Phi(q,z)\widetilde w(z)\,dz$ to G, where $dz:=dz_1\wedge dz_2\wedge dz_3$ and $\widetilde w:=-\nabla\cdot e_\lambda\nabla\tilde u$. Consider the chains in $E_{\mathbf{C}}$,

$$\begin{aligned}Y&:=\{z=x+iy:x\in U_p,\ y=(1-\varepsilon)d(x)|\xi|^{-1}\xi\},\\ B&:=\{z=x+iy:x\in\partial U_p,\ y=td(x)|\xi|^{-1}\xi,\ 0\le t\le1-\varepsilon\},\end{aligned}$$

where $d(x):=\min(\operatorname{dist}(x,\Gamma),d)$ and $\varepsilon:=(\pi d|\xi|)^{-1}\le1$. We have $\partial(Y\cup B)=\partial\Omega$. Thus

$$v_p=v_Y+v_B,\quad v_Y(q)=\int_Y\alpha_q\,dx,\quad v_B(q)=\int_B\alpha_q\,dx \tag{3.7}$$

by Stokes' theorem. In the chain Y we have

$$\begin{aligned}\operatorname{Re}(-\pi\sigma(z-p)^2+2\pi i\xi(z-p))&=-\pi\sigma((x-p)^2-(1-\varepsilon)^2d^2(x))-2\pi(1-\varepsilon)d(x)|\xi|\\&=-\pi\sigma((x-p)^2-d^2(x)+2d(x)d\\&\quad+(2\varepsilon-\varepsilon^2)\,d^2(x)-2\varepsilon d(x)d)\\&\le-\pi\sigma(q(x)+d^2-2\varepsilon d(x)d)\\&=-\pi\sigma q(x)-\pi d|\xi|+2\pi\varepsilon d(x)|\xi|,\end{aligned}$$

where $q(x):=(x-p)^2-(d-d(x))^2\geq 0$ since $d\leq d(x)+|x-p|$. We have $2\pi\varepsilon d(x)|\xi|\leq 2$ since $x\in U_p$; hence

$$|e_\lambda|\leq \exp(-\pi\sigma q(x)+2)\exp(-\pi d|\xi|).$$

We have also $(x-p)^2\leq q(x)+d^2$; hence $|x-p|\leq q^{1/2}(x)+d$ and

$$|\sigma(z-p)-i\xi|\leq \sigma|x-p|+|\xi|\leq \sigma q(x)^{1/2}+2|\xi|.$$

Therefore

$$|\nabla e_\lambda(x)|\leq (\sigma q(x)^{1/2}+2|\xi|)\exp(-\pi\sigma q(x)+2)\exp(-\pi d|\xi|).$$

The first factor is estimated by $C|\xi|$, the second one is bounded by 1 and consequently

$$\sup_Y |e_\lambda\nabla\log\gamma-\nabla e_\lambda|\leq (C|\xi|+l)\exp(-\pi d|\xi|).$$

Therefore

$$\|\widetilde{w}\|_Y^0\leq \sup_Y |e_\lambda\nabla\log\gamma-\nabla e_\lambda|\,\|\tilde{u}\|_Y^1\leq (C|\xi|+l)\exp(-\pi d|\xi|)\|\tilde{u}\|_Y^1. \tag{3.8}$$

The kernel of the integral transformation $\tilde{u}|_Y\mapsto v_Y$ has weak singularity $O(|z-q|^{-1})$; hence we have to apply the inequality $\|v_Y\|_G^{5/2}\leq C(\varepsilon)\|\widetilde{w}\|_Y^0$ with a constant $C(\varepsilon)$ that may depend on the parameter ε. To find a bound for this constant we estimate the kernel $\nabla\Phi|_Y$ of the operator $b\mapsto\nabla\tilde{u}|_Y$:

$$\begin{aligned}\nabla_q\Phi(z,q)|_Y&=-((z-q)^2)^{-3/2}(z-q)A(z,q)+((z-q)^2)^{-1/2}\nabla A(z,q)\\&=\frac{B(z,q)}{((x-q)^2-(d')^2(x)+2(x-q)yi)^{3/2}},\end{aligned}$$

where the kernel $B(z,q)$ is bounded in P. The inequalities

$$\begin{aligned}|\Phi(z,q)|&\leq\frac{C}{\sqrt{(x-q)^2-(d')^2(x)}}\leq\frac{C}{\sqrt{d^2(x)-(d')^2(x)}}\leq\frac{C}{\sqrt{\varepsilon}\,d(x)}=\frac{C\sqrt{d|\xi|}}{d(x)},\\|\nabla\Phi(z,q)|^{2/3}&\leq\frac{C}{(x-q)^2-(d')^2(x)}\leq C\frac{d|\xi|}{d^2(x)}\end{aligned}$$

hold in Y with $d'=(1-\varepsilon)d$ and a constant C which does not depend on ε and λ. This numerator comes into the estimate

$$\|v_Y\|_\Gamma^{1/2}\leq C\|v_Y\|_G^{5/2}\leq C(d|\xi|)^{3/2}\,\|\widetilde{w}\|_Y^0.$$

Together with (3.8) it finally gives

$$\|v_Y\|_\Gamma^{1/2} \leq (C|\xi|+l)(d|\xi|)^{3/2}\exp(-\pi d|\xi|)\|u\|_\Omega^1.$$

(Note that even a sharper norm of v_Y can be estimated in this way.)

A similar estimate holds for the function v_B because the test function satisfies in B the inequality

$$|e_\lambda| = \exp(-\pi\sigma(d^2-t^2d^2(x))-2td(x)|\xi|) \leq \exp(-\pi\sigma d^2) = \exp(-\pi d|\xi|)$$

since $d(x)\leq d$ and $t<1$. These inequalities together with (3.6) imply (3.5) for the function $v_\lambda := v_p + v_0$. $\square$

Now we solve the Dirichlet problem $h_\lambda|_\Gamma = -v_\lambda|_\Gamma$ for the equation (2.1). The solution belongs to $H^2(\Omega)$, because the boundary value problem is elliptic and the boundary values fulfil (3.5). The function $v = v_\lambda + h_\lambda$ vanishes at the boundary; hence, by Proposition 2.1,

$$d_\gamma L(e_\lambda)\colon f \longmapsto g := \gamma\partial_n v + e_\lambda \partial_n u|_\Gamma = \gamma(\partial_n v_\lambda + \partial_n h_\lambda) + e_\lambda\partial_n u|_\Gamma.$$

Estimate the boundary function g as

$$\|g\|_\Gamma^{-1/2} \leq C\|\gamma\|_\Gamma^{(1)}\|\partial_n v_\lambda + \partial_n h_\lambda\|_\Gamma^{-1/2} + \|e_\lambda\|_\Gamma^{(1)}\|\partial_n u\|_\Gamma^{-1/2},$$

where we have used the inequality $\|af\|_\Gamma^{1/2} \leq C\|a\|_\Gamma^{(1)}\|f\|_\Gamma^{1/2}$ and $\|\cdot\|_\Gamma^{(1)}$ denotes the norm in the space $C^1(\Gamma)$. To estimate the first term we use the inequality $\|h_\lambda\|_\Omega^2 \leq C\|v_\lambda\|_\Gamma^{3/2}$ and inequality (3.5) together with the estimate $\|u\|_\Omega^1 \leq C\|f\|_\Gamma^{1/2}$ for the solution of the Dirichlet boundary value problem for the elliptic equation (2.1). For the second term we apply the estimates $\|e_\lambda\|_\Gamma^{(1)} \leq C(|\xi|+1)\exp(-\pi d|\xi|)$ and $\|\partial_n u\|_\Gamma^{-1/2} \leq C\|u\|_\Omega^1$. This gives

$$\|g\|_\Gamma^{-1/2} \leq (C|\xi|+l)(d|\xi|)^{3/2}\exp(-\pi d|\xi|)\|f\|_\Gamma^{1/2}$$

and (3.2) follows. $\square$

4. Remarks

Remark 4.1. In general terms, Theorem 3.1 says that the response of the differential of the measurement mapping to an elementary perturbation of the function γ exponentially decreases if the dimensionless product $d|\xi|$ grows. It is true also if we take the normalized elementary signal $\tilde{e}_\lambda := c_\lambda e_\lambda$. It has unit $L_2(X)$-norm if we take $c_\lambda = (2\sigma)^{3/4}(1+O(\exp(-2\pi d|\xi|)))$ (see Section 6). We get by (3.2),

$$\|d_\gamma L(\tilde{e}_\lambda)\|^{-1/2,1/2} \le C|\xi|^{5/2}(d|\xi|)^{3/4} \exp(-\pi d|\xi|).$$

The right-hand side decreases exponentially, as $d|\xi| \to \infty$, anyway. The function $\tilde{e}_\lambda$ does not vanish on the boundary, but it is exponentially small since $|\tilde{e}_\lambda(x)| \le C|\xi|^{3/2} \exp(-\pi d|\xi|)$ for $x \in \Gamma$.

Remark 4.2. We can conclude from (3.2) that the norm $\|d_\gamma L(\,\cdot\,)\|^{-1/2,1/2}$ does not dominate any reasonable norm $\|\cdot\|_?$. Really, suppose that a norm $\|\cdot\|_?$ is dominated by this norm. Apply this inequality to the test function e_λ and get by (3.2),

$$\|e_\lambda\|_? \le (C|\xi|+l)(d|\xi|)^{3/2} \exp(-\pi d|\xi|)$$

for any $\lambda \in T^*(\Omega)$ which is impossible for any standard norm. The conclusion holds true if we compare $\|\cdot\|_?$ with the norm $\|d_\gamma L(\,\cdot\,)\|^{k-1,k}$ for arbitrary real k.

Remark 4.3. In the n-dimensional case $n \ne 3$ a similar estimate can be proven in the same way.

Remark 4.4. It seems plausible that for an arbitrary smooth non-vanishing function γ a weaker estimate of the type (3.2) is valid with the exponential function replaced by a fast decreasing function of $d|\xi|$.

5. Response of the measurement mapping

Now we give an upper estimate for the responsibility of the measurement mapping to the perturbation $\gamma \mapsto \gamma + \delta$ of the admittivity by means of a function δ that is well-localized in a neighbourhood of a point $p \in \Omega$. We shall see that the sensitivity of the measurement mapping to the localized perturbation of the admittivity coefficient is exponentially small with respect to the parameter $d(p)\omega$, where ω is the effective scalar frequency of the perturbation.

Proposition 5.1. *We have for arbitrary* $\gamma_1, \gamma_2 \in C^1_*(\Omega)$,

$$L_{\gamma(1)} - L_{\gamma(0)} = \int_0^1 dL_{\gamma(t)}(\gamma'(t))\,dt,$$

where $\gamma(t)$, $0 \le t \le 1$, *is an arbitrary* C^1*-curve in* $C^1_*(\Omega)$.

A proof follows from the Newton–Leibniz theorem. Now we estimate the response of the measurement operator for the perturbation of the admittivity of the form

$$\delta(\omega) = \int_{B(\omega)} \alpha(\lambda) e_\lambda,$$

where α is an integrable density in the ball $B(\omega)=\{\xi:|\xi|\le\omega\}$ for some scalar frequency ω.

Proposition 5.2. *We have the estimate*

$$\|L_{\gamma+\delta(\omega)} - L_\gamma\|^{-1/2,1/2} \le (C\omega+l)|d(p)\omega|^{3/2} \exp(-\pi d(p)\omega),$$

where $\delta(\omega)$ is as above and the density α is so small that $|\gamma|>|\delta(\omega)|$ in Ω.

Proof. The interval $\{\gamma(t)=\gamma+t\delta(\omega),\ 0\le t\le 1\}$ is contained in $C^1_*(\Omega)$. We apply Proposition 5.1 and estimate the differential $d_{\gamma(t)}L(\delta(\omega))$ as in Theorem 3.1 taking in account that the constant C in (3.2) can be taken bounded in this interval. □

6. Gabor analysis on a manifold

Gabor's elementary signals appeared already in theoretical physics as "coherent states" that form a representation of the Weyl–Heisenberg group. Later D. Iagolnitzer and H. P. Stapp [8] proposed the "generalized Fourier transform" for microlocal analysis of distributions. A. Cordoba and C. Fefferman [5] introduced the "wave packet transform", whose kernel is given by (3.1) with more general quadratic phase function. They applied this transform to analysis of differential operators in $\mathbf{R}^n$.

In this section we develop a similar approach for analysis in an open bounded set in $\mathbf{R}^n$ and on a compact manifold. In the next section we apply this analysis to a detailed study of the measurement operator.

Definition 6.1. Let (X,g) be a smooth Riemannian manifold; we say that it satisfies the condition $(*)$ for a positive number r_0, if X is complete and for any point $p\in X$ the geodesic mapping $y_p: U_p \to B_0 \subset \mathbf{R}^n$ is a diffeomorphism from the ball U_p centered at p of radius r_0 to the ball B_0 of the same radius centered at the origin. Supposing that X satisfies $(*)$, we call a family of smooth halfdensities $\{h_\lambda \in L_2(X):\lambda\in T^*(X)\}$ a *Gabor* family in (X,g) if the following conditions are satisfied for any λ:

(i) $\langle h_\lambda, h_\lambda\rangle=1$, where $\langle\cdot,\cdot\rangle$ denotes the inner product in $L_2(X)$;

(ii) the density $|h_\lambda|^2$ is sharply localized at the point p where $\lambda=(p,\xi)$, i.e. for any r, $0<r\leq r_0$,

$$\int_{\operatorname{dist}(x,p)\geq r} |h_\lambda|^2 \leq \frac{C_k}{(r^2|\xi|_g)^k}, \quad k=0,1,2,\dots,$$

where $|\cdot|_g$ stands for the norm of a covector in X;

(iii) the density $|F(\varphi h_\lambda)|^2$ is sharply localized in the vicinity of the point ξ for an arbitrary $\varphi\in\mathcal{D}(U_p)$, i.e. for any $r>0$,

$$\int_{|\eta-\xi|\geq r|\xi|} |F(\varphi h_\lambda)|^2 \leq \frac{C_k}{(r^2|\xi|_g)^k}, \quad k=0,1,2,\dots,$$

where F denotes for the Fourier transform of halfdensities with respect to the geodesic chart y_p:

$$F(a)(\eta) := \int_{\mathbf{R}^n} a(y_p)\exp(-2\pi i\eta y_p)\sqrt{dy_p\,d\eta}\,.$$

The constants C_k in both inequalities do not depend on λ and r.

Denote by $T^*_\circ(X)$ the set of non-zero covectors ξ in the cotangent bundle $T^*(X)$ of the manifold X. Define the distance function on this set by

$$\operatorname{dist}(\lambda,\mu)^2 := \frac{\operatorname{dist}(p,q)^2|\xi|_g|\eta|_g+\operatorname{dist}_g(\xi,\eta)^2}{|\xi|_g+|\eta|_g}, \quad \lambda=(p,\xi),\ \mu=(q,\eta),$$

where we set $\operatorname{dist}_g(\xi,\eta):=|\xi-\eta_p|_g=|\xi_q-\eta|_g$ if $\operatorname{dist}(p,q)\leq r_0$. Here η_p stands for the parallel translation of the covector $\eta\in T^*_q(X)$ to the point p along the geodesic from q to p; ξ_q has the similar meaning and $|\xi|_g$ stands for the g-norm of the covector ξ. We set $\operatorname{dist}_g(\xi,\eta)=0$ if $\operatorname{dist}(p,q)>r_0$.

Proposition 6.2. *Let $\{h_\lambda\}$ be a Gabor family in (X,g). Then the function $\langle h_\lambda,h_\mu\rangle$ decreases fast off the diagonal $D\subset T^*_\circ(X)\times T^*_\circ(X)$, namely, it satisfies the inequalities, for* $\operatorname{dist}(p,q)\leq r_0$,

$$|\langle h_\lambda,h_\mu\rangle| \leq \frac{C_k}{\operatorname{dist}(\lambda,\mu)^k}, \quad k=0,1,2,\dots, \tag{6.1}$$

with some constants C_k.

These inequalities show how sharp the Gabor functions are localized in the cotangent space.

Proof. Assume first that $d^2|\xi|\,|\eta|\geq\operatorname{dist}(\xi,\eta)^2$, where we set $d:=\operatorname{dist}(p,q)$, and we omit subscripts g for brevity. By the Cauchy–Schwarz inequality and by (i), for any positive numbers s and t such that $s+t=d$ we have

$$\begin{aligned}|\langle h_\lambda,h_\mu\rangle| &\leq \int_{\operatorname{dist}(x,p)\geq s}|h_\lambda\bar h_\mu|+\int_{\operatorname{dist}(x,q)\geq t}|h_\lambda\bar h_\mu|\\ &\leq\left(\int_{\operatorname{dist}(x,p)\geq s}|h_\lambda|^2\right)^{1/2}+\left(\int_{\operatorname{dist}(x,q)\geq t}|h_\mu|^2\right)^{1/2}.\end{aligned}$$

Take $s=d|\eta|^{1/2}/(|\xi|^{1/2}+|\eta|^{1/2})$, $t=d|\xi|^{1/2}/(|\xi|^{1/2}+|\eta|^{1/2})$ and apply (ii) to both terms in the right-hand side. We get the inequalities

$$|\langle h_\lambda,h_\mu\rangle|\leq C_k'\left(\frac{d^2|\xi|\,|\eta|}{|\xi|+|\eta|}\right)^{-k},\quad k=0,1,2,\ldots\,.$$

This implies (6.1) in virtue of the assumption. In the opposite case we have $d^2|\xi|\,|\eta|<\operatorname{dist}(\xi,\eta)^2$, hence $d\leq r_0$. We assume that $|\xi|\geq|\eta|$. Choose a real function $\varphi\in\mathcal{D}(U_p)$ that is equal to 1 in a neighbourhood of p, and write

$$\langle h_\lambda,h_\mu\rangle=\int_X\varphi^2h_\lambda\bar h_\mu+\int_X(1-\varphi^2)h_\lambda\bar h_\mu.$$

By (ii) and the Cauchy–Schwarz inequality the second term is equal to $O((|\xi|+1)^{-k})$ for arbitrary k and $4|\xi|\geq 2\operatorname{dist}(\xi,\eta)\geq\operatorname{dist}(\lambda,\mu)$. Therefore the second term is equal to $O(\operatorname{dist}(\lambda,\mu)^{-k})$ for any k. The first term is equal to $\langle\hat h_\lambda,\hat h_\mu\rangle$ in virtue of the Parseval equation where we set $\hat h_\nu=F(\varphi h_\nu)$:

$$\int_X\varphi^2h_\lambda\bar h_\mu=\int_{T_p^*(X)}\hat h_\lambda\bar{\hat h}_\mu. \tag{6.2}$$

Choose the numbers s and t such that $s|\xi|+t|\eta|=|\xi-\eta|$ and estimate this quantity by means of (iii):

$$\begin{aligned}\left|\int_{T_p^*(X)}\hat h_\lambda\bar{\hat h}_\mu\right| &\leq\int_{|\theta-\xi|\geq s|\xi|}|\hat h_\lambda\bar{\hat h}_\mu|+\int_{|\theta-\eta|\geq t|\eta|}|\hat h_\lambda\bar{\hat h}_\mu|\\ &\leq\int_{|\theta-\xi|\geq s|\xi|}|\hat h_\lambda|^2+\int_{|\theta-\eta|\geq t|\eta|}|\hat h_\mu|^2\leq C_k\left(\frac{|\xi|+|\eta|}{\operatorname{dist}(\xi,\eta)}\right)^k\leq\frac{2^kC_k}{\operatorname{dist}(\lambda,\mu)^k},\end{aligned}$$

which implies an inequality like (6.1) for (6.2). This completes the proof. □

Definition 6.1 is, in fact, axiomatization of the following examples.

Example 6.3. In a Euclidean space $X=E$ the family of halfdensities $h_\lambda:= c_\lambda e_\lambda\sqrt{dV_E}$ is a Gabor family for $L_2(E)$, where the functions e_λ are given by (3.1) with the dispersion coefficient $\sigma=|\xi|+\delta$ for a constant $\delta>0$ and $c_\lambda=(2\sigma)^{n/4}$.

Example 6.4. Take an arbitrary smooth compact submanifold $S\subset E$ and consider the family of functions $e_\mu|_S$ for covectors $\mu=(p,\theta)\in S\times T^*(E)$ such that $\theta(n)=0$ for any vector n normal to S at p. The set of such covectors can be identified with $T^*(S)$. Consider the family of halfdensities $f_\mu:=c_\mu e_\mu|_S\sqrt{dS}$, $\mu\in T^*(S)$, where dS is the Euclidean area element in S. They form a Gabor family for $L_2(S)$ if c_μ are normalizing factors.

Example 6.5. Let X be an open bounded set in a Euclidean space E endowed with the conformal metric $g(p)=d^{-2}(p)\,ds^2$, where ds^2 is the Euclidean metric and d is a smooth positive function in X such that $d(p)=\operatorname{dist}(p,\partial X)$ in a neighbourhood X' of the boundary. We call this metric *hyperbolic*; if X is the unit disc, it is quasiconformal to the standard hyperbolic metric. The Riemannian manifold (X,g) satisfies Definition 6.1 for some $r_0>0$. Consider the halfdensities $h_\lambda=c_\lambda e_\lambda\sqrt{dV_E}$, where

$$(6.3)\qquad e_\lambda:=\exp(-\pi\sigma_\lambda(x-p)^2+2\pi i\xi(x-p)),\quad \sigma_\lambda=\frac{d(p)|\xi|+\delta}{d^2(p)}$$

and δ is a positive constant. This choice of Gabor functions is close to (3.1) but we blow up X with the centre at the boundary. The extra term δ helps to define the Gabor family for zero covectors.

Proposition 6.6. *The set* $\{h_\lambda:=c_\lambda e_\lambda\sqrt{dV}:\lambda\in T^*(X)\}$ *is a Gabor family in* X, *where* $c_\lambda=(2\sigma)^{n/4}(1+O(\exp(-2\pi|\xi|_g)))$.

Proof. The Riemannian norm $|\xi|_g$ of a covector ξ is equal to $d|\xi|$, where we write $d=d(p)$. Hence

$$(6.4)\qquad \int_{E\setminus X}|e_\lambda|^2\,dV\le\int_{|x-p|\ge d}\exp(-2\pi\sigma_\lambda(x-p)^2)\,dV\le C\frac{d^n\exp(-2\pi|\xi|_g)}{(|\xi|_g+\delta)^{n/2}}.$$

Therefore

$$\int_X|e_\lambda|^2\,dV=\int_E|e_\lambda|^2\,dV-\int_{E\setminus X}|e_\lambda|^2\,dV=\frac{1+\varrho}{(2\sigma_\lambda)^{n/2}}=\frac{Cd^n(1+\varrho)}{(|\xi|_g+\delta)^{n/2}},$$

where the remainder $\varrho=O(\exp(-2\pi|\xi|_g)$. Hence the factor $c_\lambda=(2\sigma)^{n/4}(1+\varrho')$ fulfils (i) with $\varrho'=-\varrho(1+\varrho)^{-1}$. To check the inequality (ii) we need to estimate an integral like (6.4) taken over the set $\operatorname{dist}_g(x,p)\ge r$, where $r\le r_0$. It is easy to show that this set is contained in the set $|x-p|\geq r'd$, where $r'=cr$ for a constant c. We replace d by $r'd$ and simultaneously ξ by $r'\xi$ in (6.4). Thus we get the estimate $O(\exp(-2\pi c^2 r^2 |\xi|_g))$ and (ii) follows. We check (iii) by means of the equation $F(\varphi h_\lambda)=F(\varphi)*F(h_\lambda)$. $\square$

Definition 6.7. Let X be a Riemannian manifold as in Definition 6.1, $\Phi(X)$ be a linear topological space of halfdensities and $\{h_\lambda\}\subset\Phi(X)$ be a Gabor family in X. If u is a linear continuous functional in $\Phi(X)$ we call the function $G_u(\lambda)=\langle u,h_\lambda\rangle$ the *Gabor transform* of u.

Let X and Y be Riemannian manifolds endowed with some Gabor families $\{g_\lambda\}$, $\{h_\mu\}$, and $\Phi(X)$, $\Psi(Y)$ be some spaces of halfdensities such that $\{g_\lambda\}\subset\Phi(X)$ and $\{h_\mu\}\subset\Psi'(Y)$. For a linear operator $A\colon\Phi(X)\to\Psi(Y)$ we define the function $G_A(\lambda,\mu):=\langle A(g_\lambda),h_\mu\rangle_Y$ in the bundle $T^*(X)\times T^*(Y)$. We call this function the *Gabor symbol* of the operator A. We say that a conic subset V of this bundle is non-essential for the operator A if $|G_A(\lambda,\mu)|\leq C_q(|\xi|+|\eta|+1)^{-k}$ in V for any natural k, where $\lambda=(x,\xi)$ and $\mu=(y,\eta)$. We call the complement $S(A)$ in $T^*(X)\times T^*(Y)$ of the union of all non-essential open conic subsets the *essential support* of A. The Gabor support is a closed conic subset of the bundle.

Example 6.8. Let I be the identity operator in $L_2(X)$ for a Riemannian manifold as in Definition 6.1 and $\{g_\lambda\}$ be a Gabor family in X. Proposition 2.1 implies that the essential support of I is equal to the diagonal $D\subset T^*(X)\times T^*(X)$.

Example 6.9. Let A be a differential operator of order $m\leq 0$ in X. The essential support of A is again contained in the diagonal D and its symbol is equal to $G_A(\lambda,\lambda)=a_m(p,2\pi\xi)+O(|\xi|^{m-1})$, as $|\xi|\to\infty$, where $\lambda=(p,\xi)$ and a_m is the principal symbol of A.

7. Gabor analysis of the measurement mapping

Theorem 3.1 is, in fact, the first step to the Gabor analysis of the operator

$$d_\gamma L\colon C^1(\Omega)\longrightarrow\mathcal{L}:=\mathcal{L}(H^{1/2}(\Gamma),H^{-1/2}(\Gamma)).$$

We now do the next step and study the Gabor symbol of this operator. The functions $c_\lambda e_\lambda$, where e_λ are as in Example 6.5, form a Gabor family in Ω for the hyperbolic metric $g=d^{-2}(p)\,ds^2$. The target space is $\Psi(Y):=\mathcal{L}$, where $Y:=\Gamma\times\Gamma$. The space $\Psi'(Y):=L(H^{-1/2}(\Gamma),H^{1/2}(\Gamma))$ contains the subspace $H^{1/2}(\Gamma)\otimes H^{1/2}(\Gamma)$. Consider the family of halfdensities $f_\mu:=c_\mu e_\mu|_\Gamma\sqrt{dS}$ as in Example 6.4 for $S=\Gamma$. They form a Gabor family in the Riemannian manifold Γ; this family is contained in $H^{1/2}(\Gamma)$. Consequently the products $f_\mu \otimes f_\nu$, $(\mu,\nu) \in T^*(\Gamma) \times T^*(\Gamma) = T^*(Y)$ form a Gabor family in $\Psi(Y)$. According to Definition 6.7,

$$G_{d_\gamma L}(\lambda;\mu,\nu) = \langle d_\gamma L(e_\lambda), f_\mu \otimes f_\nu \rangle = \int_\Gamma \bar{f}_\nu \, d_\gamma L(e_\lambda)(f_\mu)\, dx \tag{7.1}$$

is the Gabor symbol. Now we find the essential support of this operator in the bundle $T^*(\bar{\Omega})$.

Theorem 7.1. *Suppose that Ω has an analytic boundary, i.e. that $\Omega = \{x \in E : b(x) < 0\}$, where b is a real function that has analytic continuation to a complex neighbourhood $\tilde{\Omega}$ of $\bar{\Omega}$ in $E_{\mathbf{C}}$ and $db \neq 0$ in $\Gamma := \partial\Omega$. Suppose, moreover, that the function γ has analytic non-vanishing continuation to a complex neighbourhood of Γ. Then the symbol of the operator $d_\gamma L$ decreases exponentially in any closed conic set $K \subset T^*(\bar{\Omega}) \times T^*(\Gamma \times \Gamma) \setminus S$, where $S := S_0 \cup S_1 \cup S_2 \cup S_3$,*

$$\begin{aligned}
S_0 &:= \{(\lambda;\mu,\nu) : p = q_\mu = q_\nu,\ \xi|_{T_p(\Gamma)} + \theta_\mu - \theta_\nu = 0\},\\
S_1 &:= \{(\lambda;\mu,\nu) : p = q_\mu,\ \xi|_{T_p(\Gamma)} + \theta_\mu = 0,\ \theta_\nu = 0\},\\
S_2 &:= \{(\lambda;\mu,\nu) : p = q_\nu,\ \xi|_{T_p(\Gamma)} - \theta_\nu = 0,\ \theta_\mu = 0\},\\
S_3 &:= \{(\lambda;\mu,\nu) : q_\mu = q_\nu,\ \xi = 0,\ \theta_\mu - \theta_\nu = 0\}
\end{aligned}$$

and we use the notation $\lambda := (p,\xi) \in T^*(\bar{\Omega})$, $(\mu,\nu) \in T^*(\Gamma \times \Gamma)$, $\mu = (q_\mu, \theta_\mu)$ and $\nu = (q_\nu, \theta_\nu)$.

Corollary 7.2. *It follows that the essential support of $d_\gamma L$ is contained in $S_0 \cup S_1 \cup S_2 \cup S_3$.*

Remark. Identify $T^*(\bar{\Omega}) \times T^*(\Gamma \times \Gamma) = T^*(\bar{\Omega} \times \Gamma \times \Gamma)$ and write the varieties S_j in the form

$$\begin{aligned}
S_0 &:= \{(q,q,q;\xi,\eta,\zeta) \in T^*(\bar{\Omega} \times \Gamma \times \Gamma) : \xi|_{T_q(\Gamma)} + \eta - \zeta = 0\},\\
S_1 &:= \{(q,q,q';\xi,\eta,\zeta) \in T^*(\bar{\Omega} \times \Gamma \times \Gamma) : \xi|_{T_q(\Gamma)} + \eta = 0,\ \zeta = 0\},\\
S_2 &:= \{(q,q',q;\xi,\eta,\zeta) \in T^*(\bar{\Omega} \times \Gamma \times \Gamma) : \xi|_{T_q(\Gamma)} - \zeta = 0,\ \eta = 0\},\\
S_3 &:= \{(q,q',q';\xi,\eta,\zeta) \in T^*(\bar{\Omega} \times \Gamma \times \Gamma) : \xi = 0,\ \eta - \zeta = 0\}.
\end{aligned}$$

Endow the bundle $T^*(\bar{\Omega} \times \Gamma \times \Gamma)$ with the symplectic structure $\beta := d\xi \wedge dx + d\eta \wedge dy - d\zeta \wedge dy$. We denote the symplectic manifold by $T^*(\bar{\Omega} \times \Gamma \times \Gamma^\flat)$; the manifolds Γ and $\Gamma^\flat$ are considered as the source and the target manifolds for the operator space $\mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$. This difference implies the negative sign in the above formula for β. The varieties S_j, $j = 0,1,2,3$, are conic Lagrange manifolds in $T^*(\bar{\Omega} \times \Gamma \times \Gamma^\flat)$. Really, the form $\alpha := \xi \wedge dx + \eta \wedge dy - \zeta \wedge dy$ vanishes in S_j which follows from the above definitions. Since $\beta = d\alpha$, the form β vanishes too.

Proof of Theorem 7.1. First we replace the function b by the function $b':=(|\nabla b|^2-b)^{-1/2}b$ for convenience. This function is holomorphic in a neighbourhood of $\widetilde{\Omega}$. We keep the notation b for it and have $|\nabla b|=1$ in Γ. We can write the right-hand side of (7.1) in the form

$$\int_\Gamma \bar{f}_\nu\, d_\gamma L(e_\lambda)(f_\mu)\, dx = c_\lambda c_\mu c_\nu \int_\Gamma \bar{e}_\nu \gamma(\partial_b v_{\lambda\mu} + e_\lambda \partial_b u_\mu)\, \frac{dV}{db}, \tag{7.2}$$

where u_μ is the solution of the boundary value problem

$$\nabla\cdot\gamma\nabla u_\mu = 0, \quad u_\mu|_\Gamma = e_\mu,$$

$v_{\lambda\mu}$ is the solution of the problem

$$\nabla\cdot\gamma\nabla v_{\lambda\mu} = -\nabla\cdot e_\lambda \nabla u_\mu, \quad v_{\lambda\mu}|_\Gamma = 0, \tag{7.3}$$

and $\partial_b:=\langle\nabla b, \partial x\rangle$. We shall show that the right-hand side of (7.2) decreases exponentially in an open conic neighbourhood K of an arbitrary point $\omega:=(\lambda,\mu,\nu)\in T^*(\bar{\Omega})\times T^*(\Gamma\times\Gamma)\setminus S$. We check this statement in several steps.

Case I: $p\in\Omega$ *and* $\xi\neq 0$. We take a closed cone K that does not contain any point (λ',μ',ν') such that $p(\lambda')\in\Gamma$. The norm of the operator $d_\gamma L(e_\lambda)$ is estimated as in Theorem 3.1. Really, the inequality (3.2) can be proved for elementary signals of the form (6.3) with minor modifications. For the next steps we need the following lemma.

Lemma 7.3. *If u is a solution of the Poisson equation such that the boundary values $f:=u|_\Gamma$ have analytic continuation to a neighbourhood of Γ in $\Gamma_{\mathbf{C}}$. Then u admits analytic continuation to the domain*

$$\Omega_b := \{z=x+iy\in\widetilde{\Omega} : |y|<c,\ |\langle y,\nabla b\rangle| < cb(x)\},$$

where the number $c>0$ depends only on b.

Proof. Take an analytic field τ in $\widetilde{\Omega}$ that is tangent to $\Gamma_{\mathbf{C}}$ and is real in Ω. Consider the flow $F_{\tau,t}$, $0\le t\le t_0$, in $\widetilde{\Omega}$ generated by the field $i\tau$. The set $\Omega_t:=F_{\tau,t}(\Omega)$ is a real analytic manifold that is not characteristic for the Laplace operator $\Delta=\Sigma_j\partial^2/\partial z_j^2$ if $0\le t\le t_0$ for some small t_0. The boundary $\Gamma_t:=\partial\Omega_t$ is a real analytic submanifold in $\Gamma_{\mathbf{C}}$ that is homotopic to Γ. Consider the boundary value problem

$$\nabla\cdot\gamma\nabla u_t = 0 \text{ in } \Omega_t, \quad u_t|_{\Gamma_t} = e_\mu, \tag{7.4}$$

where $\nabla:=\partial/\partial z:=\frac{1}{2}(\partial/\partial x - i\langle\partial y_t/\partial x, \partial/\partial y\rangle)$ and $y=y_t(x)$ is the equation of Ω_t. Since $F_{\tau,t}\colon\Omega\to\Omega_t$ is an analytic mapping together with its inverse, we can consider

(7.4) as the Dirichlet problem for the Poisson equation with the coefficient $\gamma(x+iy_t)$. The kernel of this boundary value problem is equal to zero by virtue of the maximum principle. Therefore this problem is uniquely solvable since the index equals zero. We claim that the family of functions u_t defines an analytic continuation $F_\tau(u_\mu)$ of u_μ over the union $U_\tau := \bigcup_{0\le t\le t_0} \Omega_t$. To check this assertion we differentiate (7.4) with respect to t successively and estimate the derivatives of u_t in Ω_t. From the standard estimates for the elliptic boundary value problem we get the inequality $|\partial_t^k u_t| \le CB^k k!$ for some B and C as long as the functions γ and f are analytic and the boundary value problem (7.4) is elliptic. Therefore the analytic continuation can be performed for the step-size $1/B$. Continuing in this way we get the continuation $F_\tau(u_\mu)$ to U_τ.

The union of the sets U_τ taken over all fields τ as above covers the set Ω_b as above. It can be proved by means of Lemma 7.4. The continuations $F_\tau(u_\mu)$ are consistent with each other in Ω_b since they coincide with the analytic function u in Ω. It can be shown that the functions u_t define a single-valued holomorphic function in Ω_b. We shall not use this global conclusion and omit the details.

Lemma 7.4. *For any point $p\in\Gamma$, any neighbourhood $V\subset\widetilde{\Omega}$, any vector $\tau_0\in T_p(\Gamma)$ and any number $\varepsilon>0$, there exists an analytic field τ in Ω that is tangent to Γ such that $\tau(p)=\tau_0$ and $|\tau|<\varepsilon$ in $\Omega\setminus V$.*

Proof. Extend τ_0 to $\widetilde{\Omega}$ as a constant field and set

$$\tau_1 = |\nabla b|^2\tau_0 - \tau_0(b)\nabla b.$$

We then get $\tau_1(b)=0$ and $\tau_1(p)=\tau_0$ and set $\tau := \exp(-\varrho(x-p)^2)\tau_1$ for sufficiently large ϱ. □

Case II: *$p\in\Omega$, $\xi=0$, $q_\mu\neq q_\nu$ and $\theta_\mu\neq 0$.* Take a field τ as in Lemma 7.4 such that $\tau(\theta_\mu)(q_\mu)>0$ and τ is very small in a neighbourhood of q_ν. Consider the analytic continuation of $F_\tau(u_\mu)$ as in Lemma 7.3. Take the function $u_{\mu,s} := F_\tau(u_\mu)|_{\Omega_s}$ for some small $s>0$. This is the solution of the equation (7.4) with the exponentially small boundary value

$$\max_{\Gamma_s} |e_\mu| \le C\exp(-c|\theta_\mu|)$$

for some positive c. This inequality is seen from the structure of e_μ. By the maximum principle for the Poisson equation (7.4) the maximum of $|u_{\mu,s}|$ in Ω_s is estimated by the right-hand side. A similar estimate is valid for the normal derivative $\partial_b u_{\mu,s}|_{\Gamma_s}$ with c replaced by any $c'<c$ since the e_μ admits such an estimate in a neighbourhood of Γ_s. This estimate holds also for the solution $v_{\lambda\mu,s}$ of the boundary value problem like (7.3) in the domain Ω_s since $|e_\lambda|$ is bounded in Γ_s

by a constant that does not depend on θ_μ. This implies that the factor $a_{\lambda\mu,s}:=\partial_b v_{\lambda\mu,s}+e_\lambda\partial_b u_{\mu,s}$ in (7.2) is exponentially small in Γ_s, as $|\theta_\mu|\to\infty$. We can replace the chain Γ by Γ_s in (7.2) by Stokes' theorem since the integrand is a holomorphic differential form in a neighbourhood of Γ. The integral is equal to $O(\exp(-c|\theta_\mu|))$. The numerical coefficient admits the estimate $c_\mu c_\nu\le C|\theta_\mu|$, as the point (θ_μ,θ_ν) tends to infinity. Therefore the right-hand side of (7.2) decreases exponentially.

Case III: $p\in\Omega$, $\xi=0$, $q_\mu\neq q_\nu$ *and* $\theta_\nu\neq 0$. In this case we take a field τ such that $\tau(\theta_\nu)(q_\nu)<0$ and $|\tau|\le\varepsilon$ in a neighbourhood V of q_μ. Consider the flow F_τ and replace Γ by Γ_s in the right-hand side of (7.2). We have $\bar e_\nu=O(\exp(-c|\theta_\nu|))$ in Γ_s for some $s>0$. Take the analytic continuation $F_\tau(u_\mu)$ and consider the function $u_{\mu,s}:=F_\tau(u_\mu)|_{\Omega_s}$. We have $u_{\mu,s}=O(\exp(\varepsilon s'|\theta_\nu|))$, where $s'=s+o(s)$ for small s. Therefore the integrand in (7.2) is again exponentially small, as $|\theta_\nu|\to\infty$.

Case IV: $p\in\bar\Omega$, $\xi=0$, $q_\mu=q_\nu$ *and* $\theta_\mu-\theta_\nu\neq 0$. We take a field τ as in Lemma 7.4 such that $\tau(\theta_\mu-\theta_\nu)(q_\mu)>0$ and argue as above. We get the estimates

$$\max_{\Gamma_s}|u_{\mu,s}|+|v_{\lambda\mu,s}|\le C\exp(-s'\tau(\theta_\mu)),\quad \max_{\Gamma_s}|\bar e_\nu|\le C\exp(s''\tau(\theta_\nu)),$$

where $s'=s+o(s)$ and $s''=s+o(s)$. This implies the estimate $O(\exp(-c|\theta_\mu-\theta_\nu|))$ for the right-hand side in $\Gamma_{\tau,s}$ for some small s.

Case V: $p\in\Gamma$, $p=q_\mu$ *and* $p\neq q_\nu$. Suppose that $\xi|_{T_p(\Gamma)}+\theta_\mu\neq 0$ and take a field τ as above with the property $\tau(\xi+\theta_\mu)(p)>0$ which is very small in a neighbourhood of the point q_ν. Now we obtain the estimate

$$\max_{\Gamma_s}|u_{\mu,s}|+|v_{\lambda\mu,s}|\le C\exp(-s'\tau(\xi+\theta_\mu)),\quad \max_{\Gamma_s}|\bar e_\nu|\le C\exp(\varepsilon|\xi+\theta_\mu|)$$

for some small ε. This implies again that (7.2) is exponentially small if we replace Γ by Γ_s for some $s>0$. In the case $\theta_\nu\neq 0$ we choose a field τ such that $\tau(\theta_\nu)(q_\nu)<0$ and τ is small in a neighbourhood of q_μ. Then we argue as in Case IV.

Case VI: $p\in\Gamma$, $p=q_\nu$ *and* $p\neq q_\mu$. We show by similar arguments that (7.2) is again exponentially small in the case $\xi|_{T(\Gamma)}-\theta_\nu\neq 0$ and in the case $\theta_\mu\neq 0$.

Case VII: $p=q_\mu=q_\nu$ *and* $\xi|_{T(\Gamma)}+\theta_\mu-\theta_\nu\neq 0$. Here we take a field τ such that $\tau(\xi+\theta_\mu-\theta_\nu)(p)>0$ and use the above method. □

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