The asymptotic-norming and Radon—Nikodým properties for Banach spaces

Robert C. James* and Aggie Ho

Introduction

A Banach space has the Radon—Nikodým property (RNP) if it is isomorphic to a subspace of a separable dual. Until very recently, it was thought this might be a necessary condition for RNP. The asymptotic-norming property (ANP) is introduced. It is shown that ANP is satisfied by a larger class of Banach spaces than those that are isomorphic to subspaces of separable duals, and that ANP implies RNP. For product spaces and subspaces of duals, there are significant similarities between ANP and RNP. Different formulations of ANP are studied, as well as relations between ANP and Kadeć—Klee-type properties.

A Banach space X is said to have the *Radon—Nikodým property* if for each finite-measure space (S, Σ, μ) and each μ-continuous, X-valued measure m with finite norm, there is a Bochner-integrable function f from S to X such that $m(E)=\int_E f d\mu$ if E belongs to Σ. A Banach space has RNP if it is isomorphic to a subspace of a separable dual ([8] and [7, Theorem 1, page 79]). Until recently, it was thought this might be a necessary condition for RNP. However, there are now examples of Banach spaces with RNP that are not isomorphic to a subspace of any separable dual ([4] and [12]). In Section 1, we will introduce ANP and show it to be a sufficient condition for RNP. Also, ANP is satisfied by all subspaces of separable duals. In Section 2, it will be shown that ANP is satisfied by some spaces not isomorphic to a subspace of any separable dual, and is preserved for certain product-type spaces. However, it remains unknown whether ANP is necessary for RNP. In Section 3, ANP will be studied for subspaces of duals.

The spaces c_0 and $L_1[0, 1]$ do not have RNP, but l_1 does have RNP. A great deal is known about conditions for RNP (e.g., see [7, pages 217—219]). Frequently,

* Supported in part by NSF-MCS 78-01890, and by the Mittag-Leffler Institute.

we will need to use the fact that a Banach space has RNP if and only if it does not contain a bush [7, page 216]. A *bush* in a Banach space is a subset $B=\{x^{n,i}\colon 1\leqq i\leqq N(n),\ n\geqq 1\}$ of the unit ball that satisfies the conditions:

(a) For each $n\geqq 1$, the first $N(n+1)$ positive integers is the union of $N(n)$ consecutive sets $\{S_i^{n+1}\colon 1\leqq i\leqq N(n)\}$ such that each S_i^{n+1} has $r_i^{n+1}\geqq 2$ members (the bush is a *tree* if each $r_i^n=2$) and

$$x^{n,i} = \frac{1}{r_i^{n+1}}\Sigma\{x^{n+1,j}\colon j\in S_i^{n+1}\}.$$

(b) There is a positive *separation constant* ε such that

$$\|x^{n,i}-x^{n+1,j}\| > \varepsilon \quad \text{if} \quad j\in S_i^{n+1}.$$

A Banach space is said to have the *Krein—Milman property* (KMP) if each bounded closed convex subset is the closed convex span of its extreme points. A Banach space has KMP if it has RNP [14, Theorem 2], but it is not known whether the converse is true. As might be suggested by this situation, it sometimes is easier to establish KMP than RNP. This will be illustrated dramatically in Section 1 by Theorems 1.5 and 1.8, where the following basic facts are the crucial tools needed.

(1) The *Bishop—Phelps theorem* [3, Corollary 4, page 31], which states that if K is a bounded closed convex nonempty subset of a Banach space X, then the set of continuous linear functionals on X that attain their sups on K is norm-dense in X^*.

(2) Suppose K_1 is a bounded closed convex subset of a Banach space and K_{n+1} is *extremal* in K_n for each n, meaning that K_{n+1} is the supporting set for some supporting hyperplane of K_n. Then x is an extreme point of K_1 if x is an extreme point of some K_n or if x is an extreme point of $\bigcap_{n=1}^{\infty} K_n$.

(3) A Banach space has KMP if and only if each bounded closed convex nonempty subset has an extreme point [6, Prop. 1, page 230].

It will be convenient to agree that a *norming set* for a Banach space X is a subset Φ of the unit ball of X^* such that, for each x in X,

$$\|x\| = \sup\{\varphi(x)\colon \varphi\in\Phi\}.$$

1. The Asymptotic-Norming Properties

The next definition really contains three different definitions, superimposed as one formal statement. These definitions make use of three different "convergence criteria" for a bounded sequence $\{w_i\}$ in a Banach space:

(I) The sequence $\{w_i\}$ converges strongly.

(II) Some subsequence of $\{w_i\}$ converges strongly.

(III) $\bigcap_{n=1}^{\infty} K_n$ is nonempty, where $K_n=\mathrm{cl}\,[\mathrm{co}\,\{w_i\colon i\geqq n\}]$.

Definition 1.1. Let $\varkappa$ be I, II, or III. For a Banach space X to have the *asymptotic-norming property* $\varkappa$ (ANP-$\varkappa$) means that X has an equivalent norm for which there is a norming set Φ which has the property that $\{w_i\}$ satisfies ($\varkappa$) above if $\|w_i\|=1$ for each i and $\{w_i\}$ is *asymptotically normed* by Φ, meaning that, for each positive ε, there exist $\varphi\in\Phi$ and N such that

$$\varphi(w_i) > 1-\varepsilon \quad \text{if} \quad i > N. \tag{1}$$

Clearly, ANP-I implies ANP-II, and ANP-II implies ANP-III. It is easy to see that l_1 has ANP-II for its usual norm. We just let the norming set be the set of all members of l_∞ with only finitely many nonzero components, each $+1$ or -1. However, we cannot let the norming set be the set of all members of l_∞ with all components $+1$ or -1, since then (1) is satisfied if $\{w_i\}$ is the natural basis of l_1, ε is any positive number, and φ has all components $+1$. It follows from the next theorem that l_1 has ANP-I.

The next theorem establishes that all ANP's are equivalent for separable Banach spaces. Thus we will usually refer to *the* ANP when dealing with separable Banach spaces.

Theorem 1.2. *A separable Banach space has* ANP-I *if it has* ANP-III.

Proof. Suppose X has ANP-III and that Φ and $\|\ \|$ are as described in Definition 1.1. Since X is separable, there is no loss of generality if we take the norming set to be a sequence $\{\varphi_n\}$. Also, choose a sequence $\{E_n\}$ of increasing finite-dimensional subspaces of X for which $X=\operatorname{cl}\left(\bigcup_1^\infty E_n\right)$. Let

$$|||x||| = \left\{\|x\|^2+\sum_{n=1}^\infty 2^{-2(n+2)}[\operatorname{dist}(x, E_n)]^2+\sum_{n=1}^\infty 2^{-2(n+2)}[\varphi_n(x)]^2\right\}^{1/2}. \tag{2}$$

Then $|||\ |||$ can be defined by the norming set Ψ, which is the set of all functionals ψ of type

$$\psi = \lambda_0\varphi_p+\sum_{n=1}^\infty 2^{-(n+2)}\lambda_{2n-1}f_n^\psi+\sum_{n=1}^\infty 2^{-(n+2)}\lambda_{2n}\varphi_n, \tag{3}$$

where $\sum_{n=0}^\infty \lambda_n^2=1$ and the sequence $\{f_n^\psi\colon n\geqq 1\}$ is a sequence of linear functionals of norm one that are zero on E_n. Since $2\sum_{n=1}^\infty 2^{-(n+2)}=\frac{1}{2}$, there is no loss of generality if we also require $\lambda_0\geqq\frac{1}{2}$. Now let $\{w_i\}$ be an arbitrary sequence with $|||w_i|||=1$ for each i that is asymptotically normed by Ψ. For an arbitrary positive ε, suppose

$$\lambda_0\varphi_p(w_i)+\sum_{n=1}^\infty 2^{-(n+2)}\lambda_{2n-1}f_n^\psi(w_i)+\sum_{n=1}^\infty 2^{-(n+2)}\lambda_{2n}\varphi_n(w_i) > 1-\tfrac{1}{2}\varepsilon,$$

if $i>N$. The left member of this inequality cannot be greater than $|||w_i|||$, which is 1. Since $\lambda_0\geqq\frac{1}{2}$, it would increase by at least $\frac{1}{2}\varepsilon$ if $\varphi_p(w_i)$ were increased by ε. Therefore,

$$\varphi_p(w_i) \geqq \|w_i\|-\varepsilon \quad \text{if} \quad i > N.$$

This implies that $\{w_i/\|w_i\|\}$ is asymptotically normed by $\{\varphi_n\}$ with respect to $\|\ \|$. Therefore, there is a w such that

$$w \in \bigcap_{n=1}^{\infty} K_n, \quad \text{where} \quad K_n = \text{cl}\,[\text{co}\,\{w_i\colon\ i \geqq n\}]. \tag{4}$$

Note that $|||w_i|||=1$ for each i and $\{w_i\}$ being asymptotically normed by Ψ imply $|||w_i+w_j|||\to 2$ as i and j increase. Therefore, it follows from (2) that the following limits exist for each n:

$$\lim_{i\to\infty} \|w_i\|, \quad \lim_{i\to\infty} \text{dist}\,(w_i, E_n), \quad \lim_{i\to\infty} \varphi_n(w_i).$$

Since $\lim_{i\to\infty} \varphi_n(w_i)$ exists for each n, it follows from (4) that $\lim_{i\to\infty} \varphi_n(w_i)=\varphi_n(w)$ for each n, and therefore that $\lim_{i\to\infty} \|w_i\|=\|w\|$. That $|||w|||=1$ follows from (4) and the facts that $|||w_i|||=1$ for each i and $\{w_i\}$ is asymptotically normed by Ψ. These facts, together with (2), imply

$$\lim_{i\to\infty} \text{dist}\,(w_i, E_n) = \text{dist}\,(w, E_n).$$

Since $\text{dist}\,(w, E_n)\to 0$, this implies that for any $\varepsilon>0$ there are an E_n and an N such that

$$\text{dist}\,(w, E_n) < \varepsilon \quad \text{and} \quad \text{dist}\,(w_i, E_n) < \varepsilon \quad \text{if} \quad i > N.$$

Since $\{\varphi_n\}$ is norming for $\|\ \|$, E_n is finite-dimensional, and $\lim_{i\to\infty} \varphi_n(w_i)=\varphi_n(w)$ for each n, we can conclude that w is the strong limit of $\{w_i\}$.

Theorem 1.3. *If a Banach space X has* ANP-$\varkappa$, *where $\varkappa$ is* I, II, *or* III, *then there is a norm arbitrarily close to the given norm of X for which there is a norming set as described in Definition 1.1.*

Proof. Let $\|\ \|$ be the given norm for X, and let Ψ be a norming set. E.g., Ψ could be the unit ball of X^*. Suppose there is an equivalent norm $\|\ \|_\Phi$ for which there is a norming set Φ as described in Definition 1.1. For an arbitrary positive Δ, define a norm $|||\ |||$ by letting

$$|||x||| = \|x\| + \Delta\|x\|_\Phi \quad \text{if} \quad x \in X.$$

Then $F=\{f\colon f=\psi+\Delta\varphi,\ \psi\in\Psi,\ \varphi\in\Phi\}$ is a norming set for X with the norm $|||\ |||$. Suppose $|||w_i|||=1$ for each i and $\{w_i\}$ is asymptotically normed by F. Let $\omega=\inf\{\|w_i\|_\Phi\}$. For an arbitrary positive ε, choose $f=\psi+\Delta\varphi$ in F and an N such that

$$\psi(w_i)+\Delta\varphi(w_i) > 1-\Delta\varepsilon\omega \quad \text{if} \quad i > N. \tag{5}$$

Then the left member of (5) cannot increase by as much as $\Delta\varepsilon\omega$ if $\varphi(w_i)$ is replaced by $\|w_i\|_\Phi$, so we have $\Delta[\|w_i\|_\Phi-\varphi(w_i)]<\Delta\varepsilon\omega$ and

$$\varphi(w_i) > \|w_i\|_\Phi - \varepsilon\omega \quad \text{if} \quad i > N.$$

Thus $\varphi(w_i/\|w_i\|_\Phi)>1-\varepsilon$ if $i>N$. If X has ANP-I with respect to Φ, then $\{w_i/\|w_i\|_\Phi\}$ and $\{\|w_i\|/\|w_i\|_\Phi\}$ converge, so it follows from $\|w_i\|+\|w_i\|_\Phi=1$ that $\{\|w_i\|_\Phi\}$ converges and therefore that $\{w_i\}$ converges. If X has ANP-II [or ANP-III] with respect to Φ, we can use a subsequence of $\{w_i\}$ for which $\{\|w_i\|_\Phi\}$ converges to see that X has ANP-II [or ANP-III] with respect to F.

The next corollary follows directly from Theorems 1.2 and 1.3.

Corollary 1.4. *If a separable Banach space has* ANP, *then there is a norm arbitrarily close to the given norm of X with respect to which X has* ANP-I.

The next three theorems show that ANP is a more general criterion for RNP and KMP than was previously available. The fact that separable duals have KMP was proved [2, Theorem 1, page 262] long before it was known that RNP implies KMP. It also is true that Theorem 1.6 is a consequence of Theorem 1.9. Its proof is included because of its relative simplicity. In fact, before Theorem 1.9 was known, ANP-type properties were introduced as sufficient conditions for KMP, and it was proved by a slight modification of the proof given here for Theorem 1.6 that ANP-III implies KMP if the corresponding norming set is countable [9, Theorem 3.10, page 12]. Theorem 1.5 is a strengthening of a known result [9, Corollary 3.12, page 17].

Theorem 1.5. *If a Banach space B is isomorphic to a subspace of a separable dual, then B has* ANP. *In fact, B is isomorphic to a subspace of a dual X^* for which X^* has* ANP-I *with respect to its dual norm.*

Proof. It is known that if X^* is separable, then X has an equivalent norm for which X^* is locally uniformly convex ([1] and [6, Theorem 2, page 118]). Also, all subspaces of a Banach space have ANP if the space has ANP. Thus it is sufficient to consider only separable duals that are locally uniformly convex. Let X^* be separable and locally uniformly convex. We choose as a norming set Φ the image of the unit ball of X in X^{**}. Suppose $\{w_i\}$ is a sequence in X^* for which $\|w_i\|=1$ for each i and $\{w_i\}$ is asymptotically normed by Φ. Let w be a w^*-accumulation point of $\{w_i\}$. If $\|x\|\leqq 1$ and

$$w_i(x)>1-\varepsilon \quad \text{if} \quad i>N,$$

then $w(x)\geqq 1-\varepsilon$. Since $\|w\|\leqq 1$ and X^* is locally uniformly convex, it follows that $\|w\|=1$ and $\{w_i\}$ converges to w.

Theorem 1.6. *A Banach space has* KMP *if it has* ANP-I.

Proof. Suppose X has ANP-I and that Φ is the corresponding norming set. To show that X has KMP, it is sufficient to show that K has an extreme point if K is a bounded closed convex nonempty subset of X and $0\notin K$. Let K_0 be such a subset, and let $\{\varepsilon_i\}$ be a null sequence of positive numbers. We will determine $\{K_n\}$ inductively. Suppose K_n has been determined for $n \leqq p$. Let

$$M_p = \sup \{\|\xi\|: \xi \in K_p\}.$$

Then $M_p \neq 0$ and there exists φ_{p+1} in Φ such that

$$\sup \{\varphi_{p+1}(\xi): \xi \in K_p\} > \left(1 - \tfrac{1}{3} \varepsilon_{p+1}\right) M_p.$$

By the Bishop—Phelps theorem, there is a linear functional f_{p+1} that assumes its sup on K_p and satisfies $\|\varphi_{p+1} - f_{p+1}\| < \frac{1}{3} \varepsilon_{p+1}$. Let $\mu_p = \sup \{f_{p+1}(\xi): \xi \in K_p\}$ and then let

$$K_{p+1} = \{x: x \in K_p \text{ and } f_{p+1}(x) = \mu_p\}.$$

If $x \in K_{p+1}$, then $|\varphi_{p+1}(x) - f_{p+1}(x)| < \frac{1}{3} \varepsilon_{p+1} M_p$ and

$$f_{p+1}(x) > \sup \{\varphi_{p+1}(\xi): \xi \in K_p\} - \tfrac{1}{3} \varepsilon_{p+1} M_p > \left(1 - \tfrac{2}{3} \varepsilon_{p+1}\right) M_p.$$

Since $\varphi_{p+1}(x) > f_{p+1}(x) - \frac{1}{3} \varepsilon_{p+1} M_p$ and $\|x\| \leqq M_p$ if $x \in K_p$, we have

$$\varphi_{p+1}(x) \geqq (1 - \varepsilon_{p+1}) \|x\| \quad \text{if} \quad x \in K_{p+1}. \tag{6}$$

Let $\{w_n\}$ be any sequence for which $w_n \in K_n$ for each n. It follows from (6) that $\{w_n / \|w_n\|\}$ is asymptotically normed by Φ, so $\{w_n\}$ is convergent. Since $\{w_n\}$ was arbitrary except that $w_n \in K_n$ for each n, $\bigcap_1^\infty K_n$ is a single point. This point is an extreme point of K_0.

For the following lemma, let a *branching set* be any set $\{x^{n,i}: 1 \leqq i \leqq N(n), n \geqq 1\}$ that satisfies (a) of the introduction and is in the unit ball of a Banach space, but does not necessarily have a separation constant and therefore need not be a bush. A *branch* of $\{x^{n,i}\}$ is a sequence $\{w_n\}$ for which each w_n is $x^{n,i}$ for some value of i, and $w_{n+1} = x^{n+1,j}$ for some j in S_i^{n+1} if $w_n = x^{n,i}$. We say that $(\lambda, \varkappa) \succ (m, i)$ if $(\lambda, \varkappa) = (m, i)$ or $x^{\lambda,\varkappa}$ follows $x^{m,i}$ on some branch. For the next lemma and theorem, we also will use the special notation:

$$S(m, i; f) = \sup \{f(x^{\lambda,\varkappa}): (\lambda, \varkappa) \succ (m, i)\}. \tag{7}$$

Lemma 1.7. *Suppose α and Δ are positive numbers with $0 < \Delta < 1$, and that τ and ε are positive numbers for which*

$$\tau = \tfrac{1}{3} \alpha \quad \text{and} \quad \varepsilon \leqq \tfrac{1}{24} \Delta \alpha.$$

Let $\{x^{n,i}\}$ be a branching set in a Banach space X. Let f be in the unit ball of X^*, and assume that $(p, k) \succ (m, i)$ with

$$f(x^{p,k}) > S(m, i; f) - \tfrac{1}{4} \Delta \varepsilon. \tag{8}$$

If $\|g\| \leqq 1$ and $g(x^{p,k}) > \Delta$, then there exists (μ, ι) such that $(\mu, \iota) \succ (p, k)$ and

(μ, ι) *has the property that, if* $(n,j)\succ(\mu,\iota)$ *and*

$$(f+\tau g)(x^{n,j}) > S(\mu,\iota;\, f+\tau g)-\varepsilon, \tag{9}$$

then

$$f(x^{n,j}) > S(m,i;\, f)-\alpha \quad \text{and} \quad g(x^{n,j}) > A-\Delta. \tag{10}$$

Proof. Suppose the various hypotheses are satisfied. Choose (μ,ι) so that $(\mu,\iota)\succ(p,k)$ and

$$(f+\tau g)(x^{\mu,\iota}) > S(p,k;\, f+\tau g)-\varepsilon. \tag{11}$$

We will complete the proof by showing that (10) is satisfied by any (n,j) such that $(n,j)\succ(\mu,\iota)$ and (9) is satisfied by (n,j). Since $\|g\|\leqq 1$, it follows from (9), (11), and (8) that

$$f(x^{n,j}) > (f+\tau g)(x^{\mu,\iota})-\varepsilon-\tau > S(p,k;\, f+\tau g)-2\varepsilon-\tau$$
$$> f(x^{p,k})-2\varepsilon-2\tau > S(m,i;\, f)-2\varepsilon-2\tau-\tfrac{1}{4}\Delta\alpha.$$

Since $2\varepsilon+2\tau+\frac{1}{4}\Delta\alpha\leqq\frac{1}{12}\Delta\alpha+\frac{2}{3}\alpha+\frac{1}{4}\Delta\alpha<\alpha$, it follows that the first inequality of (10) is satisfied by $x^{n,j}$. Because of (8),

$$f(x^{n,j}) < f(x^{p,k})+\tfrac{1}{4}\Delta\alpha.$$

Now we use this to substitute for part of the left member of (9), and then use (11) to obtain

$$f(x^{p,k})+\tau g(x^{n,j})+\tfrac{1}{4}\Delta\alpha > (f+\tau g)(x^{\mu,\iota})-\varepsilon$$
$$> (f+\tau g)(x^{p,k})-2\varepsilon.$$

Then $\tau g(x^{n,j})>\tau g(x^{p,k})-2\varepsilon-\frac{1}{4}\Delta\alpha$. Since $g(x^{p,k})>A$ and $2\varepsilon+\frac{1}{4}\Delta\alpha\leqq\frac{1}{3}\Delta\alpha=\Delta\tau$, we have

$$g(x^{n,j}) > A-\Delta.$$

Theorem 1.8. *If a separable Banach space* X *has* ANP, *then* X *does not contain a bush.*

Proof. Suppose X has ANP. Let $\{\varphi_n\}$ be a sequence of norming functionals giving ANP-I. Suppose $\{x^{n,i}\}$ is a bush in X with separation constant ε. Let $\{\Delta_n\}$ be a null sequence of positive numbers less than $\frac{1}{8}\varepsilon$. To obtain a contradiction, we will first construct a "tree" of sets

$$\{E^{\lambda,\varkappa}\colon\ 1\leqq\lambda<\infty,\ 1\leqq\varkappa\leqq 2^{\lambda-1}\}$$

as follows. Let $E^{1,1}$ be the union of all the bush elements. Suppose $E^{\lambda,\varkappa}$ has been defined and that there are linear functionals $f_1,\dots,f_\omega$ in the unit ball of X^*; positive numbers ε_ω and $\tau_1,\dots,\tau_\omega$; and an (m,i) for which $E^{\lambda,\varkappa}$ is the set of all $x^{n,j}$ for which $(n,j)\succ(m,i)$ and (n,j) also satisfies

$$f(x^{n,j}) > S(m,i;\, f)-\varepsilon_\omega, \quad \text{where} \quad f=\textstyle\sum_{i=1}^{\omega}\tau_i f_i. \tag{12}$$

Also, suppose there are numbers $A_1, \ldots, A_\omega$ such that, if $x^{n,j} \in E^{\lambda,\varkappa}$, then

$$f_k(x^{n,j}) > A_k - \Delta_k \quad \text{if } 1 \leqq k \leqq \omega.$$

Let us now define the two successors of $E^{\lambda,\varkappa}$. This will be done in two steps, first choosing a subset of $E^{\lambda,\varkappa}$ and then choosing two subsets of this set to be the successors of $E^{\lambda,\varkappa}$. As above, let $f = \sum_{i=1}^{\omega} \tau_i f_i$. Then let

$$I(\sigma) = \{(n, j) \colon (n, j) \succ (m, i) \text{ and } f(x^{n,j}) > S(m, i; f) - \sigma\}.$$

Choose α so that $\alpha \leqq \varepsilon_\omega$ and

$$\sup \{\|x^{n,j}\| \colon (n, j) \in I(\alpha)\} < \lim_{\sigma \to 0} \sup \{\|x^{n,j}\| \colon (n, j) \in I(\sigma)\} + \Delta_\omega.$$

Then choose (p, k) in $I(\frac{1}{24} \Delta_{\omega+1} \alpha)$ so that

$$\|x^{p,k}\| + \Delta_\omega > \sup \{\|x^{n,j}\| \colon (n, j) \in I(\alpha)\}. \tag{13}$$

Choose a norming functional g from $\{\varphi_n\}$ such that

$$g(x^{p,k}) > \|x^{p,k}\| - \Delta_{\omega+1}, \tag{14}$$

and let $A_{\omega+1} = \|x^{p,k}\| - \Delta_{\omega+1}$. Now choose (μ, ι) to follow (p, k) and to have the properties described in Lemma 1.7, where Δ, τ, ε, and A in that lemma are $\Delta_{\omega+1}$, $\tau_{\omega+1}$, $\varepsilon_{\omega+1}$, and $A_{\omega+1}$, respectively, and $f = \sum_{i=1}^{\omega} \tau_i f_i$. Let $F^{\lambda,\varkappa}$ be the set of all $x^{n,j}$ for which $(n, j) \succ (\mu, \iota)$ and (9) is satisfied. If $x^{n,j} \in F^{\lambda,\varkappa}$, then it follows from (13) that

$$\|x^{n,j}\| < \|x^{p,k}\| + \Delta_\omega.$$

It follows from (10) and (14) that we also have

$$\|x^{n,j}\| \geqq g(x^{n,j}) > A_{\omega+1} - \Delta_{\omega+1} = \|x^{p,k}\| - 2\Delta_{\omega+1}.$$

Thus all members of $F^{\lambda,\varkappa}$ have approximately the same norm and are approximately normed by g.

We will now describe how to choose the two desired subsets of $F^{\lambda,\varkappa}$. Let $g = f_{\omega+1}$. For any (n, j) and any $p > n$, $x^{n,j}$ is the average of all $x^{p,k}$ for which $(p, k) \succ (n, j)$. Thus, for any positive δ, if $(n, j) \succ (\mu, \iota)$ and the value of $\sum_{i=1}^{\omega+1} \tau_i f_i(x^{n,j})$ is sufficiently near $S(\mu, \iota; f)$, where $f = \sum_{i=1}^{\omega+1} \tau_i f_i$, then in any later column an arbitrary large part of the followers of (μ, ι) give values of f that are greater than $S(\mu, \iota; f) - \delta$. Because of ε-separation of the bush, it follows that if δ is small enough, then at least two such elements are separated by at least $\frac{3}{4}\varepsilon$. Thus there exist followers (q, r) and (q, s) of (μ, ι) and an h in X^* for which $\|h\| \leqq 1$ and

$$h(x^{q,r}) - h(x^{q,s}) > \tfrac{3}{4}\varepsilon,$$

$$f(x^{q,r}) > S(\mu, \iota; f) - \tfrac{1}{4}\Delta_{\omega+2}\varepsilon_{\omega+1},$$

$$f(x^{q,s}) > S(\mu, \iota; f) - \tfrac{1}{4}\Delta_{\omega+2}\varepsilon_{\omega+1}.$$

Now use Lemma 1.7 to determine a subset $E^{\lambda+1,2\varkappa-1}$ of $F^{\lambda,\varkappa}$ with properties similar to those of $E^{\lambda,\varkappa}$ and such that, if $f_{\omega+2}=h$ and $A_{\omega+2}=h(x^{q,r})$, then for each $x^{n,j}$ in $E^{\lambda+1,2\varkappa-1}$ we have

$$f_k(x^{n,j}) > A_k - \Delta_k \quad \text{if} \quad k \leqq \omega+2.$$

Similarly, we can define $E^{\lambda+1,2\varkappa}$ by letting $f_{\omega+3}=-h$ and $A_{\omega+3}=-h(x^{q,s})$ and have

$$f_k(x^{n,j}) > A_k - \Delta_k \quad \text{if} \quad k \leqq \omega+3.$$

The distance between $E^{\lambda+1,2\varkappa-1}$ and $E^{\lambda+1,2\varkappa}$ is at least as great as $\frac{3}{4}\varepsilon-\Delta_{\omega+2}-\Delta_{\omega+3}$, which is greater than $\frac{1}{2}\varepsilon$.

This defines sets $\{E^{\lambda,\varkappa}\}$ such that, if $\{w_i\}$ is a sequence chosen from successive sets along a branch, then $\lim_{i\to\infty} w_i$ exists. However, two such limits for sequences chosen from different branches are separated by at least $\frac{1}{2}\varepsilon$. This contradicts the fact that the closed linear span of all the bush elements is separable

Theorem 1.9. *If a Banach space X has* ANP-III, *then X has* RNP.

Proof. If X has ANP-III, then it follows from Theorem 1.8 that each separable subspace of X has RNP. This implies X has RNP.

2. ANP and Separable Duals

We now know that any Banach space isomorphic to a subspace of a separable dual has ANP, and that each ANP implies RNP. To show that ANP is more general than "isomorphism with a subspace of a separable dual", we need the next theorem.

Theorem 2.1. *There is a separable Banach space that has* ANP *and is not isomorphic to any subspace of a separable dual.*

Proof. Let $P=c_0\times\prod_{n=1}^{\infty} l_\infty^{(n)}$, where the l_1-norm is used in defining the product. Let $\{t^{k,j}\}$ be the tree in c_0 whose elements are $t^{1,1}=e_1$, $t^{2,1}=e_1+e_2$, $t^{2,2}=e_1-e_2$, $t^{3,1}=e_1+e_2+e_3$, $t^{3,2}=e_1+e_2-e_3$, $t^{3,3}=e_1-e_2+e_3$, $t^{3,4}=e_1-e_2-e_3$, etc. Suppose $\frac{3}{4}<\alpha<1$. For each n, and each (k,j), let $x_n^{k,j}$ be the element of P which has exactly two nonzero components: its c_0 component is $\alpha\cdot t^{k,j}$ and its $l_\infty^{(n)}$ component is $(1-\alpha)t_n^{k,j}$, where $t_n^{k,j}$ is $t^{k,j}$ truncated so as to get an element of $l_\infty^{(n)}$. Let Y_α be the space denoted by Y in Example 9 of [11]. That is,

$$Y_\alpha = \mathrm{cl}\,[\mathrm{lin}\,\{x_n^{k,j}\colon\ n\geqq 1,\ k\geqq 1,\ 1\leqq j\leqq 2^{k-1}\}].$$

Let $\{\varphi^n\}$ be a sequence that norms P, where each φ^n is a finite sum,

$$\varphi^n = \sum_{i=1}^{r(n)} f_i^n,$$

with each f_i^n of unit norm when restricted to the ith component of P and identically zero on all other components. We will use $\{\varphi^n\}$ to show that Y_α has ANP-II. Let $\{w^i\}$ be a sequence in Y_α for which $\|w^i\|=1$ for each i and $\{w^i\}$ is asymptotically normed by $\{\varphi^i\}$. That is, for each positive ε there are integers p and N for which

$$\varphi^p(w^i) > 1-\varepsilon \quad \text{if} \quad i > N. \tag{15}$$

It follows from (15) that if $\varphi^p = \sum_{i=1}^r f_i^p$, then each w^i with $i>N$ is concentrated in its first r components, in the following sense. If w_r^i is obtained from w^i by replacing the first r components of w^i by zeros, then

$$\|w_r^i\| \leqq \varepsilon \quad \text{if} \quad i > N.$$

Since each factor of P other than the first factor is finite-dimensional, there is a subsequence $\{u^n\}$ of $\{w^i \colon i>N\}$ for which

$$\sum_{i=2}^r \|u_i^n - u_i^m\| < \varepsilon \quad \text{for all } n \text{ and } m.$$

Then $\sum_{i=2}^\infty \|u_i^n - u_i^m\| < 3\varepsilon$. Now it follows from (1) of [11] that

$$\|u^n - u^m\| < 3\varepsilon(1-\alpha)^{-1}.$$

Since ε was arbitrary, this process can be repeated and the diagonal method used to obtain a convergent subsequence of $\{w^i\}$. As was done in [12], we now let $X = \prod_{n=1}^\infty Y_{\alpha(n)}$, where the l_1-norm is used in defining the product and the sequence $\{\alpha(n)\}$ is chosen so that X is not isomorphic to a subspace of a separable dual. It follows from the next theorem that X has ANP-II.

It is known that l_p-products of spaces with RNP have RNP if $1 \leqq p < \infty$ [7, page 219]. The analogous theorem for ANP-II also is true. The stronger Theorem 2.2 will be proved, to illustrate what properties of an l_p-product seem to be needed. It is easily seen that the hypotheses of Theorem 2.2 are satisfied if $X = \prod_{\gamma\in\Gamma} X_\gamma$, where each X_γ has ANP-II for a norming set Φ_γ and $\|\{x_\gamma\}\| = [\sum \|x_\gamma\|^p]^{1/p}$ with $1 \leqq p < \infty$. In fact, the norming set Φ can then be the set of all linear functionals

$$\varphi = \{a_\gamma \varphi_\gamma\}, \quad \varphi_\gamma \in \Phi_\gamma \quad \text{for each } \gamma,$$

for which only *finitely* many a_γ's are nonzero and the l_q-norm of $\{a_\gamma\}$ is 1. For a particular $\theta = \{b_\gamma \theta_\gamma\}$, we then let $B = \{\gamma \colon b_\gamma \neq 0\}$ and

$$Y_\theta = \{\{x_\gamma\} \colon x_\gamma = 0 \text{ if } \gamma \notin B\}, \quad Z_\theta = \{\{x_\gamma\} \colon x_\gamma = 0 \text{ if } \gamma \in B\}.$$

Then Y_θ has ANP-II, since it follows from B being finite that any asymptotically normed sequence in Y_θ has a subsequence for which each component either converges to 0 or (when normalized) is asymptotically normed. The other hypotheses of Theorem 2.2 also are satisfied. For example, (ii) with $\tau = \sigma$ follows from the fact that, if $x=y+z$ with $y\in Y_\theta$, $z\in Z_\theta$, and $\|x\|=1$, and if $\varphi(x)>1-\sigma$, then

$$1-\sigma<\varphi(y)+\varphi(z)\leqq\|\varphi\|_1\|y\|+\|\varphi\|_2\|z\|$$
$$\leqq[\|\varphi\|_1^q+\|\varphi\|_2^q]^{1/q}[\|y\|^p+\|z\|^p]^{1/p}=1,$$

where $\|\varphi\|_1$ and $\|\varphi\|_2$ are the norms of φ when restricted to Y_θ and Z_θ, respectively.

Theorem 2.2. *A Banach space X has* ANP-II *if X has a norming set Φ which has the property that, for each $\varepsilon>0$, there is a $\Delta>0$ such that, for each θ in Φ, there are subspaces Y_θ and Z_θ of X such that X is the linear span of Y_θ and Z_θ and:*

(i) $\theta(z)=0$ *if* $z\in Z_\theta$;

(ii) *for each $\sigma>0$ there is a $\tau>0$ such that, if $x=y+z$ with $y\in Y_\theta$, $z\in Z_\theta$, and $\|x\|=1$, and if $\varphi\in\Phi$ with $\varphi(x)>1-\tau$, then*

$$\varphi(y)>\|\varphi\|_1\|y\|-\sigma,$$

where $\|\varphi\|_1$ is the norm of φ when restricted to Y_θ;

(iii) Y_θ *has* ANP-II *with respect to $\{\varphi_1\}$, where*

$$\varphi_1(y)=\frac{\varphi(y)}{\|\varphi\|_1}\quad \textit{if}\quad y\in Y_\theta\quad \textit{and}\quad \varphi\in\Phi\quad \textit{with}\quad \|\varphi\|_1\neq 0;$$

(iv) *if $y\in Y_\theta$, $z\in Z_\theta$, and $\|z\|>\varepsilon\|y\|$, then*

$$\|y+z\|>\|y\|+\Delta\|z\|.$$

Proof. Suppose $\{w^i\}$ is a sequence in X such that $\|w^i\|=1$ for each i and $\{w^i\}$ is asymptotically normed by Φ. For an arbitrary positive $\varepsilon<\frac{1}{2}$, choose the corresponding Δ and then choose $\theta\in\Phi$ and N so

$$|\theta(w^i)|>\frac{1}{1+\Delta\varepsilon}\quad \text{if}\quad i>N. \tag{16}$$

Let $w^i=y^i+z^i$ for each i, with $y_i\in Y_\theta$ and $z_i\in Z_\theta$. Since $\theta(z^i)=0$, (16) implies $\|y^i\|>1/(1+\Delta\varepsilon)$ if $i>N$. It then follows from (iv) that $\|z^i\|\leqq\varepsilon\|y^i\|$ if $i>N$, since otherwise

$$\|w^i\|>\|y^i\|+\Delta\|z^i\|>\frac{1}{1+\Delta\varepsilon}+\frac{\Delta\varepsilon}{1+\Delta\varepsilon}=1.$$

Thus $\|y^i\|\leqq\|w^i\|+\|z^i\|\leqq 1+\varepsilon\|y^i\|$, and

$$\|y^i\|\leqq\frac{1}{1-\varepsilon}\quad \text{if}\quad i>N. \tag{17}$$

For an arbitrary $\sigma>0$, choose $\tau<(\frac{1}{2}-\varepsilon)/(1-\varepsilon)$ so that (ii) is satisfied. Then choose $\varphi\in\Phi$ and $N'\geqq N$ such that $\varphi(w^i)>1-\tau$ if $i>N'$. Since $\varphi(y^i)=\varphi(w^i)-\varphi(z^i)$

and $\|z^i\| \leqq \varepsilon \|y^i\| \leqq \varepsilon/(1-\varepsilon)$, we have

$$\varphi(y^i) > 1-\tau-\frac{\varepsilon}{1-\varepsilon} = \frac{1-2\varepsilon}{1-\varepsilon}-\tau > \frac{\frac{1}{2}-\varepsilon}{1-\varepsilon}.$$

This and (17) imply $\|\varphi\|_1 > \frac{1}{2}-\varepsilon$. From (ii), we have $\varphi(y^i) > \|\varphi\|_1 \|y^i\| - \sigma$. Division by $\|\varphi\|_1$ gives

$$\varphi_1(y^i) > \|y^i\| - \frac{\sigma}{\|\varphi\|_1} \quad \text{if} \quad i > N'. \tag{18}$$

Since σ was arbitrary, (18) implies there is a subsequence of $\{y^i\colon i > N'\}$ whose norms converge and which is asymptotically normed by $\{\varphi_1\colon \varphi \in \Phi \text{ and } \|\varphi\|_1 \neq 0\}$. This and (iii) imply $\{y^i\colon i > N'\}$ has a convergent subsequence. Since $\|z^i\| \leqq \varepsilon \|y^i\|$ if $i > N$, there is a subsequence $\{w^{n(i)}\}$ of $\{w^i\}$ and a $\bar{y}$ such that $\|\bar{y} - w^{n(i)}\| < 2\varepsilon \|\bar{y}\| \leqq 2\varepsilon/(1-\varepsilon)$. Since ε was arbitrary, this implies $\{w^i\}$ has a Cauchy subsequence.

If ANP-II is replaced by RNP in Theorem 2.2, we obtain a valid theorem that is a generalization of the known fact that l_p-products $(1 \leqq p < \infty)$ of spaces with RNP have RNP [7, page 219]. The next theorem includes this as a special case. This theorem was suggested by Corollary 3.11 of [9], but the hypotheses of this corollary have been weakened significantly, and the conclusion strengthened from KMP to RNP. An attempt has been made to have Theorems 2.2 and 2.3 as similar as practical, since the differences might suggest the possibility that RNP does not imply ANP.

Theorem 2.3. *A Banach space X has* RNP *if there are positive numbers $\alpha < 1$, $\beta < (1-\alpha)^2$, and Δ such that, for each x of X, there are subspaces Y and Z of X such that X is the linear span of Y and Z and:*

(i) *$x \in Y$ and Y has* RNP;

(ii) $\|y+z\| > (1-\alpha\Delta)\|y\|$ *if* $y \in Y$ and $z \in Z$;

(iii) *if $y \in Y$, $z \in Z$, and $\|z\| > \beta\|y\|$, then*

$$\|y+z\| > \|y\| + \Delta\|z\|.$$

Proof. Suppose μ is a nonnegative finite measure on a σ-algebra Σ of subsets of a set S. Let m be an X-valued, μ-continuous measure on Σ with $\|m\|$ finite. Since $\Delta \leqq 1$, we have

$$\frac{\beta}{1-\alpha\Delta}+\frac{1-(1-\alpha\Delta)}{\Delta} \leqq \frac{\beta}{1-\alpha}+\alpha < 1,$$

and there are positive numbers σ and τ such that

$$\frac{\beta}{1-\alpha\Delta}+\frac{1-(1-\alpha\Delta)(1-\sigma)}{\Delta}+\tau = \theta < 1. \tag{19}$$

Suppose first that E is any member of Σ which has the property that, if m^E is defined by letting $m^E(F)=m(E\cap F)$ if $F\in\Sigma$, then

$$(1-\sigma)\|m^E\| < \|m(E)\|. \tag{20}$$

Let $x=m(E)$ and choose subspaces Y and Z of X such that (i)—(iii) are satisfied. Define λ and ν for each $F\in\Sigma$ by letting

$$m^E(F) = \lambda(F)+\nu(F), \quad \text{where} \quad \lambda(F)\in Y \quad \text{and} \quad \nu(F)\in Z.$$

Now partition E into disjoint sets $\{F_j\colon 1\leqq j\leqq K\}$ such that

$$\|\nu\| < \sum_{i=1}^{K}\|\nu(F_j)\|+\tau\|m^E\|, \tag{21}$$

For each j, it follows from (iii) that at least one of the following is true:

$$\|\nu(F_j)\| \leqq \beta\|\lambda(F_j)\|; \tag{22}$$

$$\|m^E(F_j)\| > \|\lambda(F_j)\|+\Delta\|\nu(F_j)\|. \tag{23}$$

Let A be the set of all j for which (22) is true, and let B be the set of all j for which (22) is false and (23) is true. Then

$$\begin{aligned}\|m^E\| &\geqq \sum_{j\in A}\|m^E(F_j)\|+\sum_{j\in B}\|m^E(F_j)\|\\ &\geqq \sum_{j\in A}(1-\alpha\Delta)\|\lambda(F_j)\|+\sum_{j\in B}[\|\lambda(F_j)\|+\Delta\|\nu(F_j)\|]\\ &\geqq \sum_{j=1}^{K}(1-\alpha\Delta)\|\lambda(F_j)\|+\Delta\sum_{j\in B}\|\nu(F_j)\|\\ &\geqq (1-\alpha\Delta)\|\lambda(E)\|+\Delta\sum_{j\in B}\|\nu(F_j)\|.\end{aligned}$$

Since $\lambda(E)=m(E)=m^E(E)$, it follows from this and (20) that

$$\|m^E\| > (1-\alpha\Delta)(1-\sigma)\|m^E\|+\Delta\sum_{j\in B}\|\nu(F_j)\|,$$

so we have

$$\sum_{j\in B}\|\nu(F_j)\| < \frac{1-(1-\alpha\Delta)(1-\sigma)}{\Delta}\|m^E\|. \tag{24}$$

Now we can use (21), (22), (24), $(1-\alpha\Delta)\|\lambda(F_j)\|\leqq\|m^E(F_j)\|$, and (19), to obtain

$$\begin{aligned}\|\nu\| &< \sum_{j\in A}\|\nu(F_j)\|+\sum_{j\in B}\|\nu(F_j)\|+\tau\|m^E\|\\ &< \beta\sum_{j\in A}\|\lambda(F_j)\|+\frac{1-(1-\alpha\Delta)(1-\sigma)}{\Delta}\|m^E\|+\tau\|m^E\|\\ &\leqq \frac{\beta}{1-\alpha\Delta}\sum_{j\in A}\|m^E(F_j)\|+\left(\frac{1-(1-\alpha\Delta)(1-\sigma)}{\Delta}+\tau\right)\|m^E\|\\ &< \theta\|m^E\|.\end{aligned}$$

We have shown that $m^E=\lambda+\nu$ and $\|\nu\|<\theta\|m^E\|$, where λ is a Y-valued measure and Y has RNP.

For the original m, partition S into disjoint sets $\{E_i\colon 1\leqq i\leqq N\}$ such that

$$\|m\| < \sum_{i=1}^{N}\|m(E_i)\|+\tfrac{1}{2}\sigma(1-\theta)\|m\|, \tag{25}$$

and let m_i be defined for each i by letting $m_i(F)$ be $m(E_i\cap F)$ if $F\in\Sigma$. Let

$$C=\{i\colon (1-\sigma)\|m_i\|<\|m(E_i)\|\},\quad D=\{i\colon (1-\sigma)\|m_i\|\geqq\|m(E_i)\|\}.$$

It follows from the definition of D that $\sigma\|m_i\|\leqq\|m_i\|-\|m(E_i)\|$ if $i\in D$. Since $\|m\|\geqq\sum_{i=1}^{N}\|m_i\|$, this and (25) imply

$$\sum_{i\in D}\|m_i\|\leqq\frac{1}{\sigma}\Big(\|m\|-\sum_{i=1}^{N}\|m(E_i)\|\Big)<\tfrac{1}{2}(1-\theta)\|m\|.$$

For each $i\in C$, we can have $m_i=\lambda_i+\nu_i$ with $\|\nu_i\|<\theta\|m_i\|$, where for each i all values of λ_i are in a space with RNP. Then

$$m=\sum_{i\in C}\lambda_i+\Big(\sum_{i\in C}\nu_i+\sum_{i\in D}m_i\Big)=\lambda^1+\nu^1,$$

where $\|\nu^1\|\leqq\theta\sum_{i\in C}\|m_i\|+\tfrac{1}{2}(1-\theta)\|m\|\leqq(\tfrac{1}{2}+\tfrac{1}{2}\theta)\|m\|$. Also, λ^1 has the μ-derivative f^1, where f^1 is the sum of the μ-derivatives of λ_i for $i\in C$.

Similarly, $\nu^1=\lambda^2+\nu^2$, where $\|\nu^2\|\leqq(\tfrac{1}{2}+\tfrac{1}{2}\theta)\|\nu^1\|$, and therefore $\|\nu^2\|\leqq(\tfrac{1}{2}+\tfrac{1}{2}\theta)^2\|m\|$, etc. Therefore, $m=\sum_1^\infty\lambda^i$, where each λ^i has a μ-derivative f^i. Then m has the μ-derivative $\sum_{i=1}^\infty f^i$, since

$$\|\lambda^i\|\leqq\|\nu^{i-1}\|+\|\nu^i\|\leqq(\tfrac{1}{2}+\tfrac{1}{2}\theta)^{i-1}(\tfrac{3}{2}+\tfrac{1}{2}\theta)\|m\|,$$

and therefore $\sum_{i=1}^\infty\|\lambda^i\|<\infty$.

3. ANP and Kadeć—Klee-Type Problems

If X^* is separable and also is a dual of a Banach space X, then X^* has ANP (Theorem 1.5) and also has RNP.

We have seen (Theorem 1.9) that a nonseparable Banach space has RNP if it has ANP-III, since in this case no separable subspace contains a bush and therefore all separable subspaces have RNP. It is not known whether RNP implies ANP-III, even for Banach spaces that are duals.

A Banach space is said to have the *Kadeć—Klee property* if $\{w_i\}$ converges strongly to w whenever $\{w_i\}$ converges weakly to w and $\|w\|=\|w_i\|$ for each i. It is easy to see that ANP-II implies the Kadeć—Klee property. For suppose $\{w_i\}$ converges weakly to w with $\|w\|=\|w_i\|=1$ for each i, and that Φ is a norming set that verifies ANP-II. For an arbitrary positive ε, if $\varphi \in \Phi$ and $\varphi(w) > (1-\varepsilon)\|w\|$, then there is an N such that

$$\varphi(w_i) > 1-\varepsilon \quad \text{if} \quad i > N.$$

Thus $\{w_i\}$ is asymptotically normed by Φ and each subsequence of $\{w_i\}$ has a strongly convergent subsequence, whose limit can only be w. Therefore, $\{w_i\}$ converges strongly to w.

The Kadeć—Klee property does not imply ANP-II, since any separable Banach space has an equivalent norm with respect to which it has the Kadeć—Klee property, but not all separable Banach spaces have RNP. The basic differences are that ANP-II puts much weaker restrictions on the sequences $\{w_i\}$ that are required to have convergent subsequences, and also that ANP-II provides no candidate for the limit of $\{w_i\}$, as is given by the Kadeć—Klee property. In order to obtain a candidate for the limit, it is often useful to embed the space in a dual.

Let us say that a subspace W of the dual of a Banach space has the *w^*-Kadeć—Klee property* (w^*-KK property) if $\{w_i\}$ in W converges strongly to w whenever $w \in W$, $\|w\| = \|w_i\|$ for each i, and w is the w^*-limit of $\{w_i\}$. It is known that if W is a separable subspace of the dual X^* of a Banach space X, then X can be given an equivalent norm for which W then has the w^*-KK property [5]. In the following, let $\hat{X}$ be the natural image of X in X^{**}.

Theorem 3.1. *Let X^* be the dual of a Banach space X. Then X^* has the w^*-KK property if it has ANP-II, with respect to its given norm and with the unit sphere of $\hat{X}$ as the corresponding norming set. The converse of this is true if X is separable.*

Proof. First assume X^* has ANP-II with respect to its given norm and with the unit sphere of $\hat{X}$ as the corresponding norming set. Let $\{w_i\}$ be any sequence in X^* which has a w^*-limit w with $\|w\| = \|w_i\| = 1$ for each i. For $\varepsilon > 0$, choose x such that $\|x\| = 1$ and $w(x) > 1-\varepsilon$. Since $w_i(x) \to w(x)$, there is an N such that

$$w_i(x) > 1-\varepsilon \quad \text{if} \quad i > N. \tag{26}$$

That is, $\{w_i\}$ is asymptotically normed by the unit sphere of $\hat{X}$. It follows from ANP-II that each subsequence of $\{w_i\}$ has a strongly convergent subsequence, whose limit must necessarily be w. Therefore, $\{w_i\}$ converges strongly to w.

Now suppose X^* has the w^*-KK property and X is separable. Let $\{w_i\}$ be a sequence in X^* for which $\|w_i\| = 1$ for each i and $\{w_i\}$ is asymptotically normed by the unit sphere of $\hat{X}$. That is, for any $\varepsilon > 0$, there is an x in X and an N for which $\|x\| = 1$ and (26) is satisfied. Let w be a w^*-limit of a subsequence of $\{w_i\}$. Since ε in (26) was arbitrary, we have $\|w\| = 1$. Now the w^*-KK property implies the subsequence converges strongly to w.

The w^*-KK property can be weakened and still yield ANP-III. Let us say that X has the *convex-w^*-Kadeč—Klee property* (cw^*-KK property) if $w \in \bigcap_{n=1}^{\infty} K_n$ whenever $\|w\| = \|w_i\|$ for each i and w is the w^*-limit of $\{w_i\}$, where

$$K_n = \mathrm{cl}\,[\mathrm{co}\,\{w_i\colon\ i \geqq n\}].$$

The proof of the next theorem is so similar to that of Theorem 3.1 that the proof will be omitted.

Theorem 3.2. *Let X^* be the dual of a Banach space X. Then X^* has the cw^*-KK property if it has* ANP-III, *with respect to its given norm and with the unit sphere of $\hat{X}$ as the corresponding norming set. The converse of this is true if X is separable.*

If a separable Banach space W is a w^*-closed subspace of a dual, then it is itself a dual and therefore has ANP and RNP. However, suppose W is a separable subspace of the dual of a separable Banach space and that $w \in W$ if w is the w^*-limit of a sequence $\{w_i\}$ for which $\|w\| = \|w_i\|$ and $w_i \in W$ for each i. Then W does not contain a bush and therefore W has RNP. To see this, one need only observe that all linear functionals used in the proof of Theorem 1.8 could in this case be w^*-functionals, and that instead of using ANP in the last paragraph of the proof of Theorem 1.8 to obtain a limit of $\{w_i\}$, one could use a w^*-limit w of a subsequence of $\{w_i\}$. Because $\{w_i\}$ is asymptotically normed by the w^*-functionals, we have $\|w\| = \lim_{i\to\infty} \|w_i\|$.

We can obtain a necessary and sufficient condition for ANP if we replace "$w \in W$" in the preceding paragraph by a stronger condition, such as (a) or (b) of the next theorem. Note that (a) is not the cw^*-KK property and (b) is not the w^*-KK property, since it is not assumed in either case that $w \in W$.

Theorem 3.3. *Each of the following is a necessary and sufficient condition for a separable Banach space W to have* ANP.

(a) *W can be embedded in a dual of a separable Banach space so that, whenever w is the w^*-limit of a sequence $\{w_i\}$ for which $\|w\| = \|w_i\|$ for each i, we have*

$$w \in \bigcap_{n=1}^{\infty} K_n, \quad \text{where} \quad K_n = \mathrm{cl}\,[\mathrm{co}\,\{w_i\colon\ i \geqq n\}].$$

(b) *W can be embedded in a dual of a separable Banach space so that, whenever w is the w^*-limit of a sequence $\{w_i\}$ for which $\|w\| = \|w_i\|$ for each i, we have*

$$w = \lim_{i\to\infty} w_i.$$

Proof. Suppose W is separable and W is embedded in the dual X^* of a separable Banach space X as described by either (a) or (b). Let W be given the norm $|||\ \ |||$ as described by (2) in the proof of Theorem 1.2, where $\{\varphi_n\}$ is a dense countable set in the unit sphere of $\hat{X}$. Suppose $\{w_i\}$ is asymptotically normed by Ψ, where Ψ is described by (3). Then as in the proof of Theorem 1.2, it follows that $\{w_i\}$ is asymptotically normed by $\{\varphi_n\}$. Therefore,

$$\lim_{i\to\infty} \|w_i\| = \|w\|$$

and it follows from (a), or from (b), that the w^*-limit of any subsequence $\{u_i\}$ of $\{w_i\}$ belongs to $\bigcap_{n=1}^{\infty} K_n$, where $K_n = \mathrm{cl}\,[\mathrm{co}\,\{u_i\colon i \geqq n\}]$. As in the proof of Theorem 1.2, it then follows that $\lim_{i\to\infty} u_i = u$. Thus W has ANP-II with respect to $|||\ |||$.

Now suppose W has a norm with respect to which W has ANP-II. Since W is separable, there is no loss of generality if we assume the corresponding norming set is a countable set $\{\varphi_n\}$. Then W can be embedded in a dual of a separable Banach space X for which W has ANP-II, with the norming set being the unit sphere of X. In fact, X can be the completion of lim $\{\varphi_n\}$ with respect to the norm for which

$$\|\varphi\| = \inf\left\{\sum_{i=1}^{n} |\lambda_i|\colon \varphi = \sum_{i=1}^{n} \lambda_i \varphi_{n(i)}\right\}.$$

With this norm, it follows that if $\{w_i\}$ is asymptotically normed by the unit sphere of X, then some subsequence of $\{w_i\}$ is asymptotically normed by $\{\varphi_n\}$. Now if w is the w^*-limit of a sequence $\{w_i\}$ for which $\|w\| = \|w_i\|$ for each i, then $\{w_i\}$ is asymptotically normed by the unit sphere of X. Then ANP-II implies some subsequence of $\{w_i\}$ is convergent, so we must have $w = \lim_{i\to\infty} w_i$.

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Received October 15, 1979

Robert C. James
Claremont Graduate School
Claremont, Calif. 91 711

Aggie Ho
Dept. of Math.
Iowa State University
Ames, Iowa 50 010