# On the Hardy space $H^1$ of a $C^1$ domain

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## Introduction

In this work we will extend the notion of the classical Hardy spaces $H^p$, $p \geqq 1$ of the unit disc, or of the half space, $\mathbf{R}^n_+ = \mathbf{R}^{n-1} \times (0, \infty)$, to the case of a bounded $C^1$ domain $D$ of $\mathbf{R}^n$. We first recall the definitions of these spaces (see [18]): In the case of $\mathbf{R}^n_+$, $H^p(\mathbf{R}^n_+) = \{\vec{u} = (u_1, \ldots, u_n)$, $\vec{u}$ satisfies the generalized Cauchy—Riemann equations, i.e.

$$\frac{\partial u_j}{\partial x_i} = \frac{\partial u_i}{\partial x_j}, \quad \sum_{j=1}^n \frac{\partial u_j}{\partial x_j} = 0,$$

and $(\vec{u})^*$, the non-tangential maximal function of $\vec{u}$, is in $L^p(\mathbf{R}^{n-1})\}$. To such a system we associate the function $f = u_n|_{\mathbf{R}^{n-1}}$, which is in $L^p(\mathbf{R}^{n-1})$. In every simply connected domain, the conditions on $\vec{u}$ are equivalent with $\vec{u} = \nabla U$, $\Delta U = 0$ on $\mathbf{R}^n_+$, and $f$ then becomes $\dfrac{\partial U}{\partial n}$. It is well known that, for $p > 1$, the mapping $\vec{u} \to f$ induces an isomorphism onto $L^p(\mathbf{R}^{n-1})$, and $\vec{u}$ may be recovered from $f$ as the vector formed by the Poisson integral of the Riesz transforms of $f$ and the Poisson integral of $f$. Another way of describing this procedure is the following: for $X \in \mathbf{R}^n_+$, let $Sf(X) = c_n \int_{\mathbf{R}^{n-1}} \frac{1}{|X-y|^{n-2}} f(y)\, dy$, where for definiteness we have taken $n \geqq 3$. Then, $\vec{u}(X) = \nabla Sf(X)$. In the case $p = 1$, we no longer get all of $L^1(\mathbf{R}^{n-1})$, but only a subspace, called $H^1(\mathbf{R}^{n-1})$. This space has been extensively studied (see [18], [9], [10]). It was proved in [18] that $f \in H^1(\mathbf{R}^{n-1})$ iff $f$ and its Riesz transforms $R_i f$ belong to $L^1(\mathbf{R}^{n-1})$. Under these conditions, $\vec{u}$ is recovered from $f$ as before. It was shown in [9] that $H^1(\mathbf{R}^{n-1})^* = \mathrm{BMO}$, the space of functions of bounded mean oscillation introduced by John and Nirenberg. Also, C. Fefferman observed that as a consequence of this duality, $f \in H^1(\mathbf{R}^{n-1})$ iff $f = \sum \lambda_j a_j$, where $\sum |\lambda_j| < \infty$, supp $a_j \subset B_j$, $B_j$ a ball, $\|a_j\|_{L^\infty} \leqq \dfrac{1}{|B_j|}$, $\int a_j = 0$. This was later on extended to $p < 1$ by R. R. Coifman ([2]) in the case of $H^p$ of the real line, and for the $n$-dimensional case by R. Latter ([16]).

It is this type of definition and properties that we wish to extend to the case of bounded $C^1$ domains. We consider only the case $1 \leqq p < \infty$. Our definition of $H^p(D)$ is the following: $H^p(D) = \{\vec{u} = \nabla U,\ \Delta U = 0,\ \text{such that } (\vec{u})^* \in L^p(\partial D)\}$. When $D$ is simply connected, this corresponds precisely to solutions, $\vec{u}$, of the generalized Cauchy—Riemann system. To any such $\vec{u} = \nabla U$, we associate the function $f = \dfrac{\partial U}{\partial N_Q}$ on $\partial D$. $f$ is in $L_0^p(\partial D) = \{f \in L^p(\partial D),\ \int_{\partial D} f = 0\}$. In the case $1 < p < \infty$, it follows from the results in [7] that this induces an isomorphism onto $L_0^p(\partial D)$, and that $\vec{u}$ can be recovered from $f$ by $\vec{u} = \nabla S(Tf)$ $(*)$, where $T$ is an operator bounded and invertible on $L_0^p(\partial D)$. We study the case $p = 1$ in this paper, obtaining results analogous to the ones described for the flat case of $\mathbf{R}_+^n$: we give a characterization in terms of Riesz transforms, an atomic decomposition, and a duality pairing with BMO $(\partial D)$. Moreover, we show that if $f \in H^1(\partial D)$, then we can recover $\vec{u}$ from it by $(*)$.

In the case when $D = \{z \in \mathbf{C},\ |z| < 1\}$, our spaces also essentially coincide with the classical Hardy spaces. To be specific, classically (see [3]), $\operatorname{Re} H^p(\partial D) = \{f \in L^p(\partial D)$ such that $f = u|_{\partial D}$, where $u + iv = F$ is analytic in $D$, $F^* \in L^p(\partial D)\}$. It is not difficult to see then that using our definition of $H^p(\partial D)$, $H^p(\partial D) = \operatorname{Re} H^p(\partial D) \cap \{\text{mean value } 0\}$.

This remark also generalizes to the unit ball $B_n$ in $\mathbf{R}^n$. For this particular example the case $p > 1$ of our results was established by Koranyi—Vági ([15]), and the case $p = 1$ has recently been studied by Ricci and Weiss ([17]), who obtained the atomic decomposition and the singular integral characterization. Of course, these authors relied on specific formulas and properties available for the case of $B_n$ but unavailable in the general case. To substitute these we use the theorem of A. P. Calderón ([1]) together with the results and techniques of [7], extended to the end point cases of $p = 1$ and BMO, and an extension of a result of Varopoulos ([20]).

Before beginning the major part of this work we need to introduce some of the basic notations and definitions we will use.

Capital letters, $X$, $Y$, $Z$, will denote points of a fixed domain $D \subset \mathbf{R}^n$. Lower case letters $x$, $y$, $z$ are reserved for points in $\mathbf{R}^{n-1}$. The notation $\langle X, Z \rangle$ denotes the inner product in $\mathbf{R}^n$ whereas $x \cdot z$ will be used for the inner product in $\mathbf{R}^{n-1}$. Points on the boundary of $D$, $\partial D$, will usually be denoted by $Q$ and sometimes by $P$. Also letters $t$, $s$ will be reserved for real numbers.

*Definition.* A domain $D \subset \mathbf{R}^n$ is called a $C^1$ domain if corresponding to each point $Q \in \partial D$ there exists a ball, $B$, with center $Q$ and a coordinate system of $\mathbf{R}^n$

with $Q$ as the origin such that with respect to these new coordinates

$$B\cap D = B\cap\{(x,t)\colon x\in\mathbf{R}^{n-1},\ t>\Phi(x),\ \Phi\in C_0^1(\mathbf{R}^{n-1}),$$

$$\Phi(0)=0=\frac{\partial\Phi}{\partial x_i}(0),\ i=1,\dots,n)\}$$

and

$$B\cap\partial D = B\cap\{(x,\Phi(x))\colon x\in\mathbf{R}^{n-1}\}.$$

We will assume throughout this paper that both $D$ and $\mathbf{R}^n\setminus\bar{D}$ are connected.

If $D$ is a bounded $C^1$ domain we will let $N_Q$ denote the unit inner normal to $\partial D$ at $Q$. Given $0<\alpha<1$ we set

$$\Gamma_{\alpha,\delta}(Q)=\{X\in D\colon |X-Q|<\delta,\ \langle X-Q, N_Q\rangle>\alpha|X-Q|\}$$

and

$$\tilde{\Gamma}_{\alpha,\delta}(Q)=\{x\in\mathbf{R}^n\setminus\bar{D}\colon |X-Q|<\delta,\ \langle X-Q, N_Q\rangle<-\alpha|X-Q|\}.$$

In general when the numbers $\alpha$ and $\delta$ are understood we will drop them as subscripts and write $\Gamma(Q)$ and $\tilde{\Gamma}(Q)$ respectively for the interior and exterior cone with vertex $Q$.

By a surface ball with center $Q\in\partial D$ and radius $R>0$ we mean the intersection of $\partial D$ with a ball in $\mathbf{R}^n$ of radius $r$ and center $Q$. We will use the notation $S_r(Q)$ for such a surface ball. Since our domain $D$ is bounded, it is obvious that there exists a constant $A$ such that, for any $Q\in\partial D$, and $r>A$, $S_{r/2}(Q)=\partial D$. Hence, we will restrict our attention to surface balls of diameter less than or equal to $A$. For these balls, there exist constants $c_1$ and $c_2$, depending only on $D$ such that $c_1r^{(n-1)}\leqq\sigma(S_{2r}(Q))\leqq c_2r^{(n-1)}$. Here, $\sigma(E)$, $E\subset\partial D$ denotes the surface area of the set $E$.

If we consider the distance on $\partial D$ inherited from $\mathbf{R}^n$, the balls for this distance coincide with our surface balls, and the triple $(\partial D, d, \sigma)$ becomes a space of homogeneous type (see [3]).

We now introduce the spaces we will be mainly concerned with:

*Definition.* BMO $(\partial D)=\{f\in L^2(\partial D)$ for which there is a constant $c$ such that for all surface balls $S_r$

$$(*)\qquad \left(\frac{1}{\sigma(S_r)}\int_{S_r}|f(Q)-f_{S_r}|^2\,dQ\right)^{1/2}\leqq c,$$

where $f_{S_r}=\dfrac{1}{\sigma(S_r)}\displaystyle\int_{S_r}f(Q)\,dQ$.

We let $\|f\|_{\mathrm{BMO}(\partial D)}=\inf\{c\colon (*)$ holds for $c\}$. Hence, if we identify two functions differing by a constant, BMO $(\partial D)$ becomes a Banach space. It is well-known ([3] page 593) that an equivalent norm is obtained on BMO $(\partial D)$ if we replace the $L^2$-means in $(*)$ by $L^p$-means, $1\leqq q<\infty$.

If $f \in \mathrm{BMO}(\partial D)$, we define $\|f\|_{\mathrm{bmo}(\partial D)} = \|f\|_{\mathrm{BMO}(\partial D)} + \|f\|_{L^2(\partial D)}$. With this definition, bmo $(\partial D)$ is a Banach space.

*Definition.* A real valued function a defined on $\partial D$ is called an atom if there exists a surface ball $S_r$ such that $\operatorname{supp} a \subset S_r$, $\int_{\partial D} a(Q)\,dQ = 0$, and $\|a\|_{L^\infty(\partial D)} \leqq \dfrac{1}{\sigma(S_r)}$.

*Definition.* $h^1(\partial D) = \{f \in L^1(\partial D)$, such that $(\ast)\, f = \sum_{i=1}^\infty \lambda_i a_i$, where the $a_i$'s are atoms, and $\sum_{i=1}^\infty |\lambda_i| < \infty\}$. If $f \in h^1(\partial D)$, $\|f\|_{h^1(\partial D)} = \inf \sum_{i=1}^\infty |\lambda_i|$, such that $(\ast)$ holds.

It is well known (see [3]) that with this norm, $h^1(\partial D)$ becomes a Banach space, and $(h^1(\partial D))^* = \mathrm{BMO}(\partial D)$. Here, the duality pairing is given by $\int_{\partial D} f \cdot a\,dQ$, where $f \in \mathrm{BMO}(\partial D)$ and $a$ is an atom.

In the first section we study the continuity and compactness on BMO $(\partial D)$ of the integral operator obtained by restricting to the $\partial D$ the classical double layer potential.

Precisely we consider the principal value operator

$$Kf(P) = \frac{1}{\omega_n} \text{ p.v. } \int_{\partial D} \frac{\langle P-Q, N_Q \rangle}{|P-Q|^n} f(Q)\,dQ, \quad (n \geqq 2)$$

where

$$\omega_n = \text{area of } \{X \in \mathbf{R}^n,\ |X| = 1\}.$$

We then apply these results to the study of the Dirichlet problem with boundary data in BMO $(\partial D)$ and the Neumann problem with boundary data in $h^1(\partial D)$.

In the second section the results on the Neumann problem with boundary data in $h^1(\partial D)$ are used to study the Hardy space $H^1(\partial D)$. The main result of this paper, Theorem 2.6, is proved in this section.

We remark that the Dirichlet problem with boundary data in BMO $(\partial D)$, on a bounded starshaped Lipschitz domain has been previously considered in [8]. What is important to us here is that for $C^1$ domains the solution can be expressed in terms of the double layer potential.

## Section 1.

**Theorem 1.1.** *$K$: BMO $(\partial D) \to$ BMO $(\partial D)$, and is in fact compact on this space.*

*Proof.* Our first remark is that $K$ is well defined on BMO $(\partial D)$ since $K(c) = \frac{1}{2} c$ for any constant $c$. (See [7], page 170.)

Fix $f \in \mathrm{BMO}(\partial D)$ and a surface ball $S_r (\equiv S_r(P_0))$ of radius $r$ and center

$P_0 \in \partial D$. Now,

$$Kf(P) = \frac{1}{\omega_n}\int_{S_{2r}} \frac{\langle P-Q, N_Q\rangle}{|P-Q|^n}(f(Q)-f_{S_{2r}})\,dQ$$

$$+\frac{1}{\omega_n}\int_{\partial D\setminus S_{2r}}\left(\frac{\langle P-Q, N_Q\rangle}{|P-Q|^n}-\frac{\langle P_0-Q, N_Q\rangle}{|P_0-Q|^n}\right)(f(Q)-f_{S_{2r}})\,dQ+A(r, P_0)$$

where $A(r, P_0)=\frac{1}{\omega_n}\int_{\partial D\setminus S_{2r}}\frac{\langle P_0-Q, N_Q\rangle}{|P_0-Q|^n}(f(Q)-f_{S_{2r}})\,dQ+\frac{1}{2}f_{S_{2r}}$. Using the fact that $K$ is continuous on $L^2(\partial D)$, it is easy to see that

$$\left(\frac{1}{\sigma(S_r)}\int_{S_r}|Kf(P)-A(r, P_0)|^2\,dP\right)^{1/2} \leqq c\left(\frac{1}{\sigma(S_r)}\int_{S_{2r}}|f(Q)-f_{S_{2r}}|^2\right)^{1/2}$$

$$+r^{-\frac{n-1}{2}}\int_{\partial D\setminus S_{2r}}\left(\int_{S_r}\left|\frac{\langle P-Q, N_Q\rangle}{|P-Q|^n}-\frac{\langle P_0-Q, N_Q\rangle}{|P_0-Q|^n}\right|^2 dP\right)^{1/2}|f(Q)-f_{S_{2r}}|\,dQ$$

$$\leqq c\left[\|f\|_{\mathrm{BMO}(\partial D)}+\int_{\partial D\setminus S_{2r}}\frac{r}{|P_0-Q|^n}|f(Q)-f_{S_{2r}}|\,dQ\right] \leqq c\|f\|_{\mathrm{BMO}(\partial D)}.$$

To show the compactness of $K$ on BMO it will suffice to show compactness on bmo. By the use of a finite partition of unity we may further reduce the problem to the compactness on bmo of the operator, $f\to K(\psi f)$, where $\psi\in C_0^\infty(B_\delta)$ and $B_\delta$ is a ball of radius $\delta>0$ and center on $\partial D$ such that

$$B_\delta\cap\partial D=\{(x, \Phi(x)):\ x\in\mathbf{R}^{n-1},\ \Phi\in C_0^1(\{|x|<1\}),\ |\nabla\Phi|\leqq m_0\}$$

(see [7]). Choose now $\theta\in C_0^\infty(B_\delta)$ such that $\theta\equiv 1$ on a neighborhood of the support of $\psi$. It is easily seen that $(1-\theta)K\psi$ is compact on bmo. Our problem is reduced to the study of the compactness of $\theta K\psi$.

We now pick a sequence $\{\Phi_j\}\in C_0^\infty(\{|x|<1\})$ such that $\Phi_j\to\Phi$ and $\nabla\Phi_j\to\nabla\Phi$ uniformly on $\mathbf{R}^{n-1}$ and for $f\in$ bmo we define

$$K_j f(P)=\theta(P)\int_{\partial D}k_j(P, Q)\psi(Q)f(Q)\,dQ$$

where $k_j$ is defined on $B_{4\delta}\cap\partial D\times B_{4\delta}\cap\partial D$ as

$$k_j(P, Q)=\frac{\Phi_j(x)-\Phi_j(z)-\nabla\Phi_j(z)\cdot(x-z)}{[|x-z|^2+(\Phi_j(x)-\Phi_j(z))^2]^{n/2}}\frac{1}{\sqrt{1+|\nabla\Phi(z)|^2}}.$$

$(P=(x, \Phi(x))$ and $Q=(z, \Phi(z)).)$

The operator $K_j$ is compact on bmo.

We now show that $K_j\to\theta K\psi$ on bmo $(\partial D)$. The first observation we need is that $K_j\to\theta K\psi$ on $L^p(\partial D)$, for all $1<p<\infty$.

Now let $g(P)=\psi(P)\cdot f(P)$. Then $\|g\|_{\mathrm{bmo}(\partial D)}\leqq c\|f\|_{\mathrm{bmo}(\partial D)}$, and $g$ is supported on $\partial D\cap B_\delta$. Define now a function $\tilde g$ on $\mathbf{R}^{n-1}$, by $\tilde g(x)=g(x, \Phi(x))$. Then

$\tilde{g}\in\mathrm{BMO}(\mathbf{R}^{(n-1)})$, $\|\tilde{g}\|_{\mathrm{BMO}(\mathbf{R}^{n-1})}\leqq c\|g\|_{\mathrm{bmo}(\partial D)}$, and $\operatorname{supp}\tilde{g}\subset\{|x|<1\}$. We now introduce a sequence of operators on $\mathbf{R}^{n-1}$:

$$K_\Phi\tilde{g}(x)=\frac{1}{\omega_n}\text{ p.v. }\int\frac{\Phi(x)-\Phi(x)-\nabla\Phi(z)\cdot(x-z)}{(|x-z|^2+|\Phi(x)-\Phi(z)|^2)^{n/2}}\tilde{g}(z)\,dz,$$

and

$$K_{\Phi_j}\tilde{g}(x)=\frac{1}{\omega_n}\text{ p.v. }\int\frac{\Phi_j(x)+\Phi_j(z)-\nabla\Phi_j(z)\cdot(x-z)}{(|x-z|^2+|\Phi_j(x)-\Phi_j(z)|^2)^{n/2}}\tilde{g}(z)\,dz.$$

It is easy to verify that for any constant $c$, $K_\Phi(c)=K_{\Phi_j}(c)=0$. Moreover, if $P\in B_{4\delta}\cap\partial D$, $P=(x,\Phi(x))$; $\theta(P)K(\psi f)(P)=\theta(P)\cdot K_\Phi(\tilde{g})(x)$, and $K_jf(P)=\theta(P)\cdot K_{\Phi_j}(\tilde{g})(x)$. As $K_{\Phi_j}\to K_\Phi$ in $L^p(\mathbf{R}^{n-1})$ $1<p<\infty$, $\Phi_j\to\Phi$ and $\nabla\Phi_j\to\nabla\Phi$ uniformly, using the proof of the first part of the theorem, we see that $\|(K_{\Phi_j}-K_\Phi)\tilde{g}\|_{\mathrm{BMO}(\mathbf{R}^{n-1})}\leqq\varepsilon_j\|\tilde{g}\|_{\mathrm{BMO}(\mathbf{R}^{n-1})}$, where $\varepsilon_j\to_{j\to\infty}0$. Using these facts once more, it follows that if $S_r(P_0)$ is a surface ball on $\partial D$, such that $S_r(P_0)\subset B_{4\delta}$, then

$$\frac{1}{\sigma(S_r(P_0))}\int_{S_r(P_0)}|\theta(P)(K\psi f)(P)-K_jf(P)-[\theta K(\psi f)-K_jf]_{S_r}|\,dP$$
$$\leqq\varepsilon_j(\|\tilde{g}\|_{\mathrm{BMO}(\mathbf{R}^{n-1})}+\|\tilde{g}\|_{L^{n-1}(\mathbf{R}^{n-1})})\leqq c\varepsilon_j\|f\|_{\mathrm{bmo}(\partial D)}.$$

Now, as $K_j\to\theta K\psi$ in $L^p(\partial D)$ for $1<p<\infty$ and from the above estimate in BMO $(\partial D)$, we conclude that $K_j\to\theta K\psi$ in bmo $(\partial D)$.

**Theorem 1.2.** *$\frac{1}{2}I+K$ is invertible on* BMO $(\partial D)$, *and $\frac{1}{2}I-K^*$ is invertible on $h^1(\partial D)$. ($K^*$ denotes the adjoint of $K$.)*

*Proof.* It was shown in [7] that $\frac{1}{2}I+K$ is invertible on $L^2(\partial D)$. (It was actually shown in [7] that $\frac{1}{2}I+K$ is invertible on $L^2(\partial D)$ for $n\geqq 3$. The result however remains true in dimension $n=2$ and the proof given in [7] is valid also for this case. Referring the reader to the proof of Theorem 2.1 in [7] one only needs to observe that if $f$ satisfies $(\frac{1}{2}I+K^*)f=0$ then $\int_{\partial D}f\,dQ=0$ and, therefore

$$\int_{\partial D}\log|X-Q|f(Q)\,dQ=O(|X|^{-1})\quad\text{as}\quad(|X|\to\infty).$$

From the invertibility on $L^2(\partial D)$ it is immediate that $\frac{1}{2}I+K$ is $1-1$ on BMO $(\partial D)$, and, hence, by Theorem 1.1 it is invertible on BMO $(\partial D)$.

It was also shown in [7] that $\frac{1}{2}I-K^*$ is invertible on the space

$$L_0^2(\partial D)=\left\{f\in L^2(\partial D)\colon\int_{\partial D}f=0\right\}.$$

(Again the invertibility of $\frac{1}{2}I-K^*$ on $L_0^2(\partial D)$ was stated in [7] only for $n\geqq 3$. The proof given there (Theorem 2.5 in [7]) is also valid for $n=2$ since, as pointed out above, $\int_{\partial D}\log|X-Q|f(Q)dQ\to 0$ as $|X|\to\infty$ provided $\int_{\partial D}f=0$.) The invertibility of $\frac{1}{2}I-K^*$ on $L_0^2(\partial D)$ implies the same for the adjoint operator, $\frac{1}{2}I-K$, on the quotient space, $L^2(\partial D)$/constants. In particular $\frac{1}{2}I-K$ is $1-1$ on BMO $(\partial D)$, and so, by Theorem 1.1, it is invertible.

Since the dual space of $h^1(\partial D)$ is BMO (see [3]) the invertibility of $\frac{1}{2}I-K^*$ on $h^1(\partial D)$ will follow from the invertibility of $\frac{1}{2}I-K$ on BMO provided we know that $\frac{1}{2}I-K^*$ is indeed continuous on $h^1(\partial D)$. The validity of the continuity on $h^1(\partial D)$ is in fact a consequence of the continuity on $h^1$ of a space of homogeneous type of a general class of operators discussed by Coifman and Weiss on pages 598—600 in [3].

The invertibility of $\frac{1}{2}I-K^*$ on $h^1(\partial D)$ immediately suggests the solvability of the Neumann problem for the Laplace operator in the form of a single layer potential. In the next few results we want to formalize the notion of solvability.

**Theorem 1.3.** *Suppose $a$ is an atom. Let*

$$S_a(X) = c_n \int_{\partial D} \frac{a(Q)}{|X-Q|^{n-2}}\,dQ \quad \left(c_n = -\frac{1}{(n-2)\omega_n}\right) \quad \textit{for} \quad n \geqq 3$$

*and*

$$S_a(X) = \int_{\partial D} \log|X-Q|\,a(Q)\,dQ \quad \textit{for} \quad n = 2.$$

*Given $0<\alpha<1$, there exists $\delta_{\alpha,D}$ such that*

$$(\nabla S_a)^*(Q) = \sup_{\Gamma_Q} |\nabla_X S_Q(X)|, \quad Q\in\partial D$$

*belongs to $L^1(\partial D)$ and*

$$\int_{\partial D} (\nabla S_a)^*\,dQ \leqq c, \quad \textit{independent of } a.$$

*(Recall $\Gamma_Q=\{X\in D\colon |X-Q|<\delta_\alpha$ and $\langle X-Q, N_Q\rangle>\alpha|X-Q|\}$.)*

*Proof.* Assume $a$ is supported in the surface ball, $S_r$, of radius $r>0$. Now $\int a\,dQ=0$ and $\|a\|_{L^\infty(\partial D)}\leqq cr^{1-n}$ with $c$ depending only on $\partial D$.

$$\int_{S_{5r}} (\nabla S_a)^*\,dQ \leqq cr^{n-1/2}\Big(\int_{S_{5r}} (\nabla S_a)^{*2}\,dQ\Big)^{1/2} \leqq cr^{n-1/2}\|a\|_{L^2(\partial D)} \leqq c.$$

Let $\tilde{Q}$ denote the center of $S_r$.

$$\nabla_X S_a(X) = \int_{\partial D} (k(X,Q)-k(X,\tilde{Q}))\,a(Q)\,dQ$$

where $k(X,Q) = -\dfrac{1}{\omega_n}\dfrac{X-Q}{|X-Q|^n}$.

Set $\Gamma_P=\{X\in D\colon |X-P|<\delta,\ \langle X-P, N_P\rangle>\alpha|X-P|\}$. Clearly for $\delta$ small enough, depending only on $\alpha$ and $D$, $\bar{\Gamma}_P\cap\partial D=\{P\}$ $\forall P\in\partial D$. For

$$X\in\Gamma_P,\ |\nabla_X S_a(X)| \leqq c_\alpha |X-\tilde{Q}|^{-n}\int_{S_r} |Q-\tilde{Q}|\,|a(Q)|\,dQ \leqq c_\alpha r|X-\tilde{Q}|^{-n}.$$

Using this last estimate we have

$$\int_{\partial D\setminus S_{\delta r}} (\nabla S_a)^*(P)\,dP \leqq c_\alpha.$$

As an immediate consequence we have the

**Corollary.** *Given* $0<\alpha<1$, *there exist* $\delta>0$ *and* $c>0$, *such that for any* $f\in h^1(\partial D)$

$$\|(\nabla S_f)^*\|_{L^1(\partial D)} \leqq C\|f\|_{h^1(\partial D)}.$$

*Here, as in Theorem* 1.3,

$$S_f(X) = c_n \int_{\partial D} \frac{1}{|X-Q|^{n-2}} f(Q)\,dQ \quad for \quad n \geqq 3,$$

$$S_f(X) = \int_{\partial D} \log(X-Q) f(Q)\,dQ \quad for \quad n=2$$

*and*

$$(\nabla S_f)^*(Q) = \operatorname*{Sup}_{\Gamma_Q} |\nabla_X S_f(X)|.$$

We will now show that the Neumann problem is solvable in the form of a single layer potential with a density in $h^1(\partial D)$ provided of course the data also belongs to $h^1(\partial D)$.

**Theorem 1.4.** *Given* $g\in h^1(\partial D)$ *there exists a unique (modulo constants) harmonic function,* $u(X)$, *such that*

i) *for any* $0<\alpha<1$ *the function*

$$(\nabla u)^*(Q) = \sup_{\Gamma_Q} |\nabla_X u(X)|$$

*belongs to* $L^1(\partial D)$ *and* $\|(\nabla u)^*\|_{L^1(\partial D)} \leqq C\|g\|_{L^1}$ *with* $C$ *independent of* $g$, *and*

ii) $\langle \nabla_X u(X), N_Q\rangle \to g(Q)$ *pointwise for almost every* $Q\in\partial D$ *as* $X\in\Gamma_Q$ *tends to* $Q$.

*Moreover* $u$ *may be written as the single layer potential of* $\left(\frac{1}{2}I-K^*\right)^{-1}g$ *i.e.*

$$u(X) = S_{\left(\frac{1}{2}I-K^*\right)^{-1}g}(X).$$

*Proof.* The invertibility of $\frac{1}{2}I-K^*$ on $h^1(\partial D)$ (Theorem 1.2) implies it is sufficient for a proof of Theorem 1.4 to show the nontangential convergence almost everywhere of the normal derivative of

$$u(X) = S_f(X)$$

to the value $\left(\frac{1}{2}I-K^*\right)f$ when $f\in h^1(\partial D)$. More precisely we would like to show that $\langle \nabla_X S_f(X), N_Q\rangle \to \left(\frac{1}{2}I-K^*\right)f(Q)$ for almost every $Q\in\partial D$.

We write $f=\sum_{i=1}^\infty \lambda_i a_i$ where $a_i$ is an atom and $|\lambda_i|<\infty$. From the Corollary of Theorem 1.3

$$(\nabla S_{\sum_N^\infty \lambda_i a_i})^*$$

has arbitrarily small $L^1$ norm over $\partial D$ for $N$ sufficiently large. Hence the existence almost everywhere of the above limit and its equality with $(\frac{1}{2}I-K^*)f$ will follow provided for almost every $Q$

$$\langle \nabla_X S_a(X), N_Q \rangle \to \left(\tfrac{1}{2}I-K^*\right)a(Q)$$

as $X\to Q$, $X\in\Gamma_Q$, when $a$ is an atom. Since atoms in particular belong to $L^2(\partial D)$ this last fact was already shown in [7].

For the uniqueness part of the theorem, we need two lemmas. The first one, although not explicitly stated there in the form we need it, was proved in [7]. We first need some notation.

Since $D$ is a $C^1$ domain, there exist $\delta>0$, and a finite covering of the set $\{X,\ \operatorname{dist}(X,\partial D)\leqq\delta\}$ by balls $B_j=B(P_j,r_j)$ with center $P_j\in\partial D$ such that $B(P_j,4r_j)\cap D=B(P_j,4r_j)\cap\{(x,y);\ y>\Phi_j(X)\}$. (See the Introduction.)

Now, let $\{\psi_j\}$ be a finite partition of unity for the set $\{X,\ \operatorname{dist}(X,\partial D)\leqq\delta\}$, subordinate to the cover $\{B_j\}$. We assume each $\psi_j\in C_0^\infty$.

For each $t>0$ and sufficiently small, we set

$$D_{t,j} = B(P_j,4r_j)\cap\{(x,y);\ x\in\mathbf{R}^{n-1},\ y>\Phi_j(x)+t\},$$

and

$$\Gamma_{t,j} = B(P_j,4r_j)\cap\{(x,\Phi_j(x)+t);\ x\in\mathbf{R}^{n-1}\}.$$

**Lemma 1.5.** *Suppose that $\Delta u=0$ in $D$, and that for some $p$, $1<p<\infty$,*

$$\Big(\int_{\Gamma_{t,j}} |\psi_j(Q_t)\cdot u(Q_t)|^p\,d(Q_t)\Big)^{1/p}\leqq C,\quad (Q_t\in\Gamma_{t,j})$$

*where $C$ is independent of $t$ and $j$, for sufficiently small $t$. Then, if $u(X)\to 0$ as $X\to Q\in\partial D$ nontangentially for a.e. $Q$, $u\equiv 0$ in $D$.*

The proof is given in Theorem 2.3 of [7].

**Lemma 1.6.** *Suppose $\Delta u=0$ in $D$, and for some $0<\alpha<1$, $(\nabla u)^*(Q)\equiv \sup_{\Gamma_Q}|\nabla u(X)|\in L^1(\partial D)$. If $\dfrac{\partial u}{\partial N_Q}\equiv 0$ on $\partial D$, then $u$ is constant in $D$.*

*Proof.* Using the fundamental theorem of calculus to express $u$ in terms of $\nabla u$, and the convexity of cones, it is not hard to show that $u^*\in L^1(\partial D)$. Thus, $u$ has a nontangential limit a.e. on $\partial D$. Now, pick a $j$, and a sufficiently small $t$. For $x\in\mathbf{R}^{n-1}$, let $f_{j,t}(x)=\psi_j(x,t+\Phi_j(x))\cdot u(x,t+\Phi_j(x))$. Then, $f_{j,t}$ is in $L^1(\mathbf{R}^{n-1})$ independently of $t$, and $\nabla f_{j,t}(x)$ is in $L^1(\mathbf{R}^{n-1})$ independently of $t$. Hence, by the Sobolev embedding theorem, $f_{j,t}\in L^{n-1/n-2}(\mathbf{R}^{n-1})$. Thus, the nontangential limit of $u$ on $\partial D$ belongs to $L^{n-1/n-2}(\partial D)$, and $u$ satisfies the hypothesis of Lemma 1.5, with $p=n-1/n-2$. Hence, if we show that $u|_{\partial D}=c$, Lemma 1.6 will follow from Lemma 1.5.

We are going to show that $\int_{\partial D} u(Q)\cdot\Phi(Q)\,dQ=0$ for every $\Phi\in C(\partial D)$ such that $\int_{\partial D}\Phi\,dQ=0$. This will certainly imply that $u|_{\partial D}=c$. Pick such a $\Phi$. Let $B$ be the solution to the Neumann problem $\Delta B=0$, in $D$, $\dfrac{\partial B}{\partial N_Q}(Q)=\Phi(Q)$ on $\partial D$ (see [7]). As $B$ is the single layer potential of a function in $L^p(\partial D)$ $\forall\, 1<p<\infty$, $B$ is bounded on $\bar D$. Moreover, it was shown in [7] that $(\nabla B)^*(Q)\in L^p(\partial D)$ $\forall\, 1<p<\infty$.

Now, let

$$u|_{\partial D}=g\cdot\int_{\partial D} g(Q)\cdot\Phi(Q)\,dQ=\sum_j\int_{\partial D}\psi_j(Q)\cdot g(Q)\cdot\Phi(Q)\,dQ.$$

$$\int_{\partial D}\psi_j g\cdot\Phi\,dQ=\lim_{t\to 0^+}\int_{\Gamma_{j,t}}\psi_j(Q_t)u(Q_t)\langle\nabla B(Q_t),N_{Q_t}\rangle\,dQ_t$$

$$=\lim_{t\to 0^+}\int_{D_{t,j}}\langle\nabla(\psi_j u)(X),\nabla B(X)\rangle\,dX.$$

$$\int_{D_{t,j}}\langle\nabla(\psi_j u),\nabla B\rangle\,dX=\int_{\Gamma_{t,j}}\langle\nabla u(Q_t),N_{Q_t}\rangle\psi_j(Q_t)B(Q_t)\,dQ_t$$

$$+\int_{\Gamma_{t,j}}\langle\nabla\psi_j(Q_t),N_{Q_t}\rangle u(Q_t)B(Q_t)\,dQ_t-\int_{D_{t,j}}(\Delta\psi_j)(X)\cdot u(X)B(X)\,dX$$

$$-2\int_{D_{t,j}}\langle\nabla\psi_j(X),\nabla u(X)\rangle B(X)\,dX.$$

Letting $t\to 0^+$, we have

$$\int_{\partial D}\psi_j\cdot g\cdot\Phi\cdot dQ=\int_{\partial D}\frac{\partial\psi_j}{\partial N_Q}(Q)\cdot g(Q)\cdot\Phi(Q)\,dQ$$

$$-\int_D(\Delta\psi_j)(X)\cdot u(X)\cdot B(X)\,dX-2\int_D\langle\nabla\psi_j(X),\nabla u(X)\rangle\cdot B(X)\,dX.$$

Adding in $j$, we get

$$\int_{\partial D} g\cdot\Phi\cdot dQ=-\int_D\Delta\Big(\sum_j\psi_j\Big)\cdot u(X)\cdot B(X)\,dX$$

$$-2\int_D\Big\langle\nabla\sum_j\psi_j(X),\nabla u(X)\Big\rangle B(X)\,dX=-\int_D\Delta\Big[\Big(\sum_j\psi_j\Big)\cdot u\Big]\cdot B(X)\,dX.$$

Let now $\psi=1-\sum_j\psi_j$, then, $-\Delta\big[(\sum_j\psi_j)\cdot u\big]=\Delta(\psi\cdot u)$. Moreover, if $v=\psi u$, $v$ vanishes on a neighborhood of $\partial D$, and so, an integration by parts shows that $\int_D\Delta(v)\cdot B\,dX=0$. Hence, $\int_{\partial D} g\cdot\Phi\cdot dQ=0$, and the lemma and also the uniqueness part of Theorem 1.4 are established.

We now turn to the Dirichlet problem with BMO data. As mentioned in the introduction, this problem has already been treated in the more general case of Lipschitz domains in [8]. We will sketch an alternative approach for the case of $C^1$ domains.

*Definition 1.7.* A measure $\mu$ on $D$ is called a Carleson measure if $\mu\{X\in D, |X-Q|<r\}\leqq Mr^{n-1}$ for all $Q\in\partial D$. The least such $M$ is called the Carleson norm of $\mu$.

**Theorem 1.8.** *Given* $g\in\mathrm{BMO}(\partial D)$, *there exists a harmonic function,* $u(X)$ *such that*

i) *The measure* $d\mu=d(X)\cdot|\nabla u(X)|^2dX$ *is a Carleson measure on D, with Carleson norm bounded by a constant times the square of the* BMO $(\partial D)$ *norm og g; here* $d(X)=\mathrm{dist}\,(X,\partial D)$.

ii) $u(X)\to g(Q)$ *for almost every* $Q\in\partial D$ *as* $X\in\Gamma_Q$ *tends to* $Q$. *Moreover, u maybe written as*

$$(*)\qquad u(X)=c_n\int_{\partial D}\frac{\langle X-Q,N_Q\rangle}{|X-Q|^n}\left(\tfrac{1}{2}I+K\right)^{-1}(g)(Q)\,dQ.$$

iii) *Conversely, if* $d\mu=d(X)\cdot|\nabla u(X)|^2dX$ *is a Carleson measure, then there exists a function* $g\in\mathrm{BMO}(\partial D)$ *such that* $u\to g$ *nontangentially and such that* $(*)$ *holds.*

*Proof.* For the proof of iii), we refer to [8]. Since $g\in L^2(\partial D)$, using the uniqueness results of [7], and Theorem 1.2, we see that all we have to show is that if $u(X)=c_n\int_{\partial D}\frac{\langle X-Q,N_Q\rangle}{|X-Q|^n}f(Q)dQ$, where $f\in\mathrm{BMO}(\partial D)$, then $\mu$ as defined in (i) is a Carleson measure.

Fix a $Q_0\in\partial D$, and an $r>0$. Let $S_r=S_r(Q_0)$. Then $f=(f-f_{S_{2r}})\chi_{S_{2r}}+(f-f_{S_{2r}})\chi_{S_{2r}^c}+f_{S_{2r}}=f_1+f_2+f_{S_{2r}}$, and so $u(X)=u_1(X)+u_2(X)+f_{S_{2r}}$, and hence, $\nabla u(X)=\nabla u_1(X)+\nabla u_2(X)$. We note that

$$\int|f_1|^2\leqq\|f\|^2_{\mathrm{BMO}(\partial D)}\sigma(S_{2r}),\quad\text{and so,}\quad\int|Tf_1|^2\leqq C\|f\|^2_{\mathrm{BMO}(\partial D)}\sigma(S_{2r})$$

where $T=\left(\frac{1}{2}I+K\right)$. Hence, if we use the fact that if $u$ is harmonic, then

$$\int_D d(X)\cdot|\nabla u(X)|^2\,dx\cong\int_{\partial D}|u(Q)|^2\,dQ,$$

(see [6]); the term corresponding to $u_1$ is taken care of.

By the formula we have for $u_2$, it is easy to see that

$$|\nabla u_2(X)|\leqq c\int_{S_{2r}^c}\frac{1}{|Q_0-Q|^n}|f(Q)-f_{S_{2r}}|\,dQ\leqq c\int_{S_{2r}^c}\frac{1}{|Q_0-Q|^n}|f(Q)-f_{S_{2r}}|\,dQ$$

$$\leqq\frac{c}{r}\|f\|_{\mathrm{BMO}(\partial D)}.$$

From this, the desired estimate for $u_2$ easily follows.

## Section 2.

In this section we will study Hardy spaces on $D$ and we will identify them with spaces of functions on $\partial D$. We start with three technical lemmas. The first one is by now classical:

**Lemma 2.1.** *Suppose $U$ is harmonic in $D\subset\mathbf{R}^n$. Let $\vec{u}=\nabla U$. Then for $p\geqq\frac{(n-2)}{(n-1)}$, $|\vec{u}|^p$ is subharmonic in $D$.*

For a proof of this theorem, see [19], page 234.

**Lemma 2.2.** *Suppose $v(X)$ is continuous, nonnegative and subharmonic in $D$. In addition, assume there is a $p$, $1<p\leqq\infty$, such that for each $\alpha$, $0<\alpha<1$, $v^*\in L^p(\partial D)$, and assume $v(X)\to g(Q)$ as $X\to Q$ nontangentially for almost every $Q\in\partial D$.*

*Let $P(g)(X)$ denote the Poisson integral of $g$ (i.e., $u(X)\equiv P(g)(X)$ satisfies $\Delta u=0$ in $D$, $u^*\in L^p(\partial D)$ for each $0<\alpha<1$, and $u\to g$ non-tangentially almost everywhere). Then $v(X)\leqq P(g)(X)$ in $D$.*

*Proof.* There exists a sequence, $\{V_j(X)\}$, of nonnegative subharmonic functions with each $V_j\in C^2(D)$ and such that $V_j\to V$ uniformly on each compact subdomain of $D$. In particular $\Delta V_j\geqq 0$ in $D$.

Take now a function $\Phi_\varepsilon(X)\in C_0^\infty(D)$ with $\Phi_\varepsilon\geqq 0$ in $D$, $\Phi_\varepsilon\equiv 1$ on

$$\{X\in D:\ \mathrm{dist}\,(X,\partial D)\geqq\varepsilon\}$$

and $\Phi_\varepsilon\equiv 0$ on

$$\{X\in D:\ \mathrm{dist}\,(X,\partial D)\leqq\varepsilon/2\}.$$

If $G(X, Y)$ denotes the Green's function for the domain $D$ (see [7, page 183]) we have

$$\Phi_\varepsilon(X)V_j(X)=-\int_D G(X,Y)\Delta(V_j\Phi_\varepsilon)(Y)\,dY$$

$$\leqq 2\int_D\langle\nabla_Y G(X,Y),\nabla_Y\Phi(Y)\rangle V_j(Y)\,dY+\int_D G(X,Y)\Delta\Phi_\varepsilon(Y)V_j(Y)\,dY.$$

Letting $j\to\infty$ we obtain the same inequality for $V$. It easily follows that the same inequality holds for the subharmonic function

$$W(X)=V(X)-P(g)(X).$$

Since $W^*\in L^p(\partial D)$ (for each $0<\alpha<1$) and $W(X)\to 0$ nontangentially at almost every point of the boundary, the argument given in [7, page 184] shows that

$$\lim_{\varepsilon\to 0}\int_D\langle\nabla_Y G(X,Y),\nabla_Y\Phi_\varepsilon(Y)\rangle W(Y)\,dY=0,$$

and

$$\lim_{\varepsilon\to 0}\int_D G(X,Y)\Delta\Phi_\varepsilon(Y)W(Y)\,dY=0.$$

This of course implies $W\leqq 0$ in $D$.

We also need the following extension of a result of N. Varopoulos ([20]), for the half-space, to the case of a bounded $C^1$ domain.

**Lemma 2.3.** *Assume* $f\in \mathrm{BMO}(\partial D)$. *Then, there exists* $F(X)\in C^1(D)$ *such that for almost every* $Q\in\partial D$, $F(X)\to f(Q)$ *as* $X\to Q$ *nontangentially, and* $|\nabla F(X)|$ *is density for a Carleson measure, with Carleson norm bounded by the* BMO *norm of* $f$. *Moreover, if we use the notation introduced before Lemma* 1.5, $F=\sum_j F_j$, *where each* $F_j$ *is compactly supported in* $B_j$, $|\nabla F_j|$ *is the density for a Carleson measure, and there exists* $g_j\in L^p(\partial D)$ *for all* $1\leqq p<\infty$ *such that for* $t$ *sufficiently small and*

$$X = (x, \Phi_j(x)+t)\in\Gamma_{t,j}, \quad |F_j(X)| \leqq g_j(x, \Phi_j(x)).$$

*Proof.* We may assume $\int_{\partial D} f=0$, and hence $\|f\|_{\mathrm{bmo}(\partial D)}\leqq C\|f\|_{\mathrm{BMO}(\partial D)}$.

Let $\psi_j$ be as in Lemma 1.5. The function $f_{\psi_j}(x)=\psi_j(x, \Phi_j(x))\cdot f(x, \Phi_j(x))$ belongs to $\mathrm{BMO}(\mathbf{R}^{n-1})$. Fix now a non-negative function $K\in C_0^\infty(|x|<1)$ with integral 1. By Theorem 4.3 of [12], there exists a Carleson measure $\mu_{f_{\psi_j}}$ on $\mathbf{R}^n_+$ such that $f_{\psi_j}(x)=\iint_{\mathbf{R}^n_+} K_y(x-z)\cdot d\mu_{f_{\psi_j}}(z, y)$, where $K_y(z)=y^{-(n-1)}K\left(\frac{z}{y}\right)$. Moreover, the Carleson norm for $\mu_{f_{\psi_j}}$ is bounded by the BMO norm of $f_{\psi_j}$. Fix now a $C^\infty$ function $b$ on $[0, \infty)$ such that $b(t)\equiv 1$ for $0\leqq t\leqq 1/2$, $b(t)=0$ for $t\geqq 1$. Let now $F_{\psi_j}(x, t)=\iint_{\mathbf{R}^n_+} K_j(x-z)\, b\left(\frac{t}{y}\right) d\mu_{f_{\psi_j}}(z, y)$. Then, arguing as in the second proof of Theorem 1.1 in [20], we see that $|\nabla F_{\psi_j}(x, t)|$ is a Carleson measure in $\mathbf{R}^n_+$, with Carleson constant bounded by the BMO norm of $f_{\psi_j}$. Moreover, $F_{\psi_j}(x, t)\to_{t\to 0} f_{\psi_j}(x)$, and $|F_{\psi_j}(x, t)|\leqq\iint_{\mathbf{R}^n_+} K_y(x-z)|d\mu_{f_{\psi_j}}(z, y)|\equiv g_j(x)$. $g_j(x)$ is in $\mathrm{BMO}(\mathbf{R}^{n-1})$ and, hence, locally in every $L^p$ space, $1\leqq p<\infty$.

Pick now $\theta_j\in C_0^\infty(B_j)$, $\theta_j\equiv 1$ in a neighborhood of the support of $\psi_j$. Set now, for $X\in B_j\cap D$, $X=(x, y)$, $y>\Phi_j(x)$, $F_j(X)=\theta_j(X)\cdot F_{\psi_j}(x, y-\Phi_j(x))$, and $F_j\equiv 0$ outside $B_j\cap D$. Let now $F(X)=\sum_j F_j(X)$.

Certainly, $F(X)\in C^1(D)$, and for almost every $Q\in\partial D$, $F(X)\to f(Q)$ nontangentially. Moreover, if $X\in B_j\cap D$, $X=(x, \Phi_j(x)+t)\in\Gamma_{t,j}$, then $|F_j(X)|\leqq |\theta_j(x, t+\Phi_j(x))|\cdot|F_{\psi_j}(x, t)|\leqq|\theta_j(x, t+\theta_j(x))|\,|g_j(x)|\leqq c\cdot\chi_{B_j\cap\partial D}(x, \Phi_j(x))\cdot g_j(x)$.

All that remains to show then, is that $\nabla F_j(X)$ is a Carleson measure on $D$ with Carleson norm bounded by the BMO norm of $f$. It is enough to consider balls $B_r=B(Q_0)$ with $0\leqq 2r\leqq A$, where $A$ is as in the introduction, $Q_0\in\partial D$.

As we can assume that if $X\in B_j\cap D$, $X=(x, \Phi_j(x)+y)$, then $\operatorname{dist}(X, \partial D)\cong y$, we see that if $B_r\cap B_j\cap D\neq\emptyset$, and $X\in B_r\cap B_j\cap D$, then there exists a constant $c$ depending only on $D$ such that $B_r\cap B_j\cap D\subset B_{cr}((x, \Phi_j(x)))\cap D$. Thus, it is enough to check that for the function $|\nabla_{x,t}(\theta_j(x, t+\Phi_j(x))\cdot F_{\psi_j}(x, t))|$ and for any ball

$A_\delta$ contained in $\mathbf{R}^{n-1}$, with radius $\delta \leqq cA$, the

$$\int_0^\delta \int_{A_\delta} \left|\nabla_{x,t}\big(\theta_j(x, t+\Phi_j(x)\big)\cdot F_{\psi_j}(x, t)\right| dx\, dt \leqq c\cdot \delta^{(n-1)} \|f\|_{\mathrm{BMO}(\partial D)}.$$

Now,

$$\int_0^\delta \int_{A_\delta} |F_{\psi_j}(x, t)|\, dx\, dt \leqq \int_0^\delta \int_{A_\delta} |F_{\psi_j}(x, t)-f_{\psi_j}(x)|\, dx\, dt+\delta \int_{A_\delta} |f_{\psi_j}(x)|\, dx.$$

$$\delta \int_{A_\delta} |f_{\psi_j}(x)|\, dx \leqq \delta^{n-1} \|f_{\psi_j}\|_{L^{(n-1)}(\mathbf{R}^{n-1})} \leqq c\delta^{n-1}\|f\|_{\mathrm{bmo}(\partial D)} \leqq c\delta^{n-1}\|f\|_{\mathrm{BMO}(\partial D)}.$$

Also,

$$\int_0^\delta \int_{A_\delta} |F_{\psi_j}(x, t)-f_{\psi_j}(x)|\, dx\, dt \leqq \int_0^\delta \int_{A_\delta} \Big(\int_0^t |\nabla F_{\psi_j}(x, s)|\, ds\Big)\, dx\, dt$$

$$\leqq c\cdot\delta^n\cdot \|f_{\psi_j}\|_{\mathrm{BMO}(\mathbf{R}^{n-1})} \leqq c\cdot A\cdot \delta^{n-1}\|f\|_{\mathrm{BMO}(\partial D)}.$$

As we already know the required estimate for $\nabla_{x,t} F_{\psi_j}(x, t)$, the proof of the lemma is completed.

*Definition 2.4.* (a) $H^p(D)=\{\vec{u}=(u_1, \dots, u_n)$; such that $\vec{u}=\nabla U$, $U$ harmonic in $D$, and for some $0<\alpha<1$ the function $|\vec{u}|^*(Q)=\sup_{\Gamma_Q} |\vec{u}(x)|$ belongs to $L^p(\partial D)\}$. Here we will consider only the case $1\leqq p<\infty$.

*Remark.* If $D$ is simply connected, the vectors $\vec{u}=\nabla U$, $\Delta U=0$ on $D$ coincide with the set of vectors $\vec{u}=(u, \dots, u_n)$ such that $\Delta\vec{u}=0$, $\operatorname{div}\vec{u}=0$, and $\dfrac{\partial u_i}{\partial x_j}=\dfrac{\partial u_j}{\partial x_k}$, i.e. $\vec{u}$ satisfies the generalized Cauchy—Riemann equations, and our definition coincides with the one used by Stein and Weiss in [18].

For a vector $\vec{u}$ in $H^p(D)$, we set $\|\vec{u}\|_{H^p(D)}=\|\,|\vec{u}|^*\|_{L^p(\partial D)}$.

From the results of Hunt—Wheeden [11], and Dahlberg [4], we know the existence of non-tangential limits for each $\vec{u}\in H^p(D)$ at almost every (surface measure) point $Q\in\partial D$.

In particular, if $\vec{u}=\nabla U$, then

$$\lim_{\substack{X\to Q\\ \text{non-tang.}}} \langle N_Q, \nabla U(X)\rangle = \frac{\partial U}{\partial N_Q}(Q)$$

exists for almost every $Q$. Moreover, $\dfrac{\partial U}{\partial N_Q}\in L^p(\partial D)$, and a localization procedure as the one used in Lemma 1.6 shows that $\int_{\partial D} \dfrac{\partial U}{\partial N_Q}=0$.

*Definition 2.4.* (b) $H^p(\partial D)=\{f\in L^p(\partial D);\ f(Q)=\langle N_Q, \vec{u}(Q)\rangle,$ for $\vec{u}\in H^p(D)\}$. We also set $\|f\|_{H^p(\partial D)}=\|\,|\vec{u}|^*\|_{L^p(\partial D)}$.

*Remark.* If $1<p<\infty$, the results in [7] show that $H^p(\partial D)$ with the norm defined above is a Banach space which coincides as a set with $L_0^p(\partial D)=\{f\in L^p(\partial D);\ \int_{\partial D} f=0\}$ moreover the $H^p$ norm is equivalent to the $L^p$ norm.

Hence, we will restrict ourselves from now on to the case $p=1$. Lemma 1.6 shows that $H^1(\partial D)$ is a Banach space, which can be identified with $H^1(D)$.

**Lemma 2.5.** *Assume $\vec{u}\in H^1(D)$. Then $\|\vec{u}\|_{H^1(D)}\cong|||\vec{u}|||_{L^1(\partial D)}$. Hence, if $f\in H^1(\partial D)$, $f(Q)=\langle N_Q, \vec{u}(Q)\rangle$, $\|f\|_{H^1(\partial D)}\cong|||\vec{u}|||_{L^1(\partial D)}$.*

*Proof.* Use Lemma 2.1 to pick $r$, $0<r<1$ such that $v=|\vec{u}|^r$ is subharmonic in $D$. By Lemma 2.2, with $p=1/r$, we see that $v\leqq P(|\vec{u}|^r)$. As $|v^*|^{1/r}\leqq[P(|\vec{u}|^r)^*]^{1/r}$, we see that

$$\int_{\partial D}|\vec{u}|^*\,dQ \leqq \int_{\partial D}[P(|\vec{u}|^r)^*]^{1/r}\,dQ \leqq c\int_{\partial D}|\vec{u}|\,dQ,$$

and hence the Lemma is established.

*Remark.* This Lemma also shows that different $\alpha$'s yield comparable norms in $H^1(D)$ and in $H^1(\partial D)$.

We are now ready to state our main theorem, which gives several equivalent characterizations of $H^1(\partial D)$.

**Theorem 2.6.** *Assume that $f\in L^1(\partial D)$, $\int_{\partial D} f=0$. Then, the following conditions are equivalent:*

i) $f\in H^1(\partial D)$, *i.e.* $f(Q)=\langle N_Q, \vec{u}(Q)\rangle$, $\vec{u}=\nabla U$, $\Delta U=0$, *and* $(\vec{u})^*\in L^1(\partial D)$.

ii) $f\in h^1(\partial D)$, *i.e.* $f=\sum_{i=1}^{\infty}\lambda_i a_i$, *where the* $a_i$ *are atoms, and* $\sum|\lambda_i|<+\infty$.

iii) $(\nabla S_f)^*\in L^1(\partial D)$.

iv) $R_j f(P)=c_n$ p.v. $\displaystyle\int_{\partial D}\frac{(P_j-Q_j)}{|P-Q|^n}f(Q)\,dQ$, $1\leqq j\leqq n$, *belong to* $L^1(\partial D)$.

*Moreover, all the corresponding norms are equivalent, and if $f$ satisfies any of the equivalent conditions, $(\frac{1}{2}I-K^*)^{-1}f$ is defined, and $\vec{u}=\nabla S_{((1/2)I-K^*)^{-1}f}$.*

As a consequence of this theorem, and its proof, and of Theorem 1.8, we obtain the following result for BMO $(\partial D)$:

**Theorem 2.7.** *The following conditions are equivalent for a function $f$ in $L^1(\partial D)$:*

i) *$f$ is in* BMO $(\partial D)$.

ii) *The measure $d(X)|\nabla u(X)|^2dX$ on $D$ is a Carleson measure, here $d(X)=$ dist $(X,\partial D)$, and $u$ is the solution to the Dirichlet problem with $f$ as boundary data.*

iii) *$f$ is the boundary value of an $F\in C^1(D)$, such that $|\nabla F|$ is a Carleson measure, and $F$ satisfies the conditions in Lemma 2.3.*

iv) $f=f_0+\sum_{j=1}^n R_j f_j$, *where* $f_0, f_1, \dots, f_n \in L^\infty(\partial D)$, *and* $R_j$ *are the operators defined in Theorem* 2.6.

We now turn to the proof of Theorem 2.6. This will be accomplished in several stages. We will first show that i)$\leftrightarrow$ii).

Theorem 1.4 shows ii)$\to$i). The proof that i)$\to$ii) is accomplished in two steps. In the first one, we show the equivalence for all starshaped $C^1$ domains, and in the second step we pass to the general case. We now study the starshaped case.

**Lemma 2.8.** *If $D$ is a starshaped $C^1$ domain, the continuous functions with mean value zero are dense in $H^1(\partial D)$. In particular, for starshaped $C^1$ domains, $h^1(\partial D)$ is dense in $H^1(\partial D)$.*

*Proof.* We may assume that $D$ is starshaped with respect to the origin. For a given $\vec{u}\in H^1(D)$, set $\vec{u}_t(X)=\vec{u}(tX)$, $0<t<1$. The function $\langle N_Q, \vec{u}(tQ)\rangle = f_t(Q)$ is continuous, with mean value zero, hence, it belongs to $h^1(\partial D)$. Moreover, by Lemma 2.5,

$$\|\langle N_Q, \vec{u}(tQ)-\vec{u}(Q)\rangle\|_{H^1(\partial D)} = \int_{\partial D} |\vec{u}(tQ)-\vec{u}(Q)|\, dQ.$$

This last integral converges to zero as $t$ converges to 1, since $|\vec{u}|^* \in L^1(\partial D)$.

**Theorem 2.9.** *Given $g\in \mathrm{BMO}(\partial D)$, define for each $f\in H^1(\partial D)$*

$$\lg(f) = \int_D \langle \nabla G(x), \nabla U(x)\rangle\, dX,$$

*where $G$ is the function constructed in Lemma* 2.3, *and $\vec{u}=\nabla U\in H^1(\partial D)$ satisfies $\dfrac{\partial U}{\partial N_Q}(Q)=f(Q)$. Then*

$$|\lg(f)| \leqq c\|g\|_{\mathrm{BMO}(\partial D)} \cdot \|f\|_{H^1(\partial D)},$$

*where $c$ depends only on $D$. Moreover, if for $p>1, f\in H^1(\partial D)\cap L^p(\partial D) = L_0^p(\partial D)$, then* $\lg(f) = \int_{\partial D} g\cdot f dQ$.

*Furthermore, if $D$ is starshaped, and $l\in H^1(\partial D)^*$, then there exists a unique $g\in \mathrm{BMO}(\partial D)$ such that $l=lg$.*

*Proof.* By Lemma 2.1, we can find a number $r$, $0<r<1$, such that $|\nabla U(X)|^r$ is subharmonic in $D$. By Lemma 2.2, $|\nabla U(X)|\leqq P[|\nabla U|^r]^{1/r}(X)$ in $D$. Hence,

$$\int_D |\nabla G(X)|\,|\nabla U(X)|\, dX \leqq \int_D |\nabla G(X)|\, P(|\nabla U|^r)^{1/r}(X)\, dX,$$

and since $|\nabla G|$ is the density for a Carleson measure, we have the last integral bounded by $c\cdot\|g\|_{\mathrm{BMO}(\partial D)} \int_{\partial D} |\nabla U(Q)| dQ \leqq c\cdot \|g\|_{\mathrm{BMO}(\partial D)} \cdot \|f\|_{H^1(\partial D)}$, (see [5]).

*Remark.* The above proof can also be obtained using a purely geometric argument, without using the subharmonicity property of the gradient, or the results in [5].

Assume now that $f \in L_0^p(D)$, $1 < p < \infty$. Then, by [7], $(\nabla U)^* \in L^p(\partial D)$.

$$\lg(f) = \int_D \langle \nabla G, \nabla U \rangle \, dx = \sum_j \int_D \langle \nabla G_j, \nabla U \rangle \, dx;$$

here we are using the notation in Lemma 2.3. Moreover,

$$\int_D \langle \nabla G_j, \nabla U \rangle \, dx = \int_{B_j \cap D} \langle \nabla G_j, \nabla U \rangle \, dx = \lim_{t \to 0^+} \int_{D_{t,j}} \langle \nabla G_j, \nabla U \rangle \, dx,$$

since the integral on the left hand side is absolutely convergent by the first part of the proof. Now, using Green's theorem, we see that

$$\int_{D_{t,j}} \langle \nabla G_j, \nabla U \rangle \, dx = \int_{\Gamma_{t,j}} G_j(Q_t) \cdot \langle \nabla U(Q_t), N_{Q_t} \rangle \, dQ_t.$$

Moreover, since $|G_j(Q_t)| \leq g_j(Q)$, $g_j \in L^{p'}(\partial D)$, and $(\nabla U)^* \in L^p(\partial D)$, the dominated convergence theorem shows that the last integral converges to $\int_{\partial D} G_j(Q) \cdot f(Q) dQ$ as $t \to 0$. Adding now in $j$, we see that $\lg(f) = \int_{\partial D} g \cdot f \cdot dQ$.

Assume now that $D$ is starshaped, and $l \in H^1(\partial D)^*$. Since $h^1(\partial D) \subset H^1(\partial D)$, there exists $g \in \mathrm{BMO}(\partial D)$ such that for any atom, $a$, $l(a) = \int_{\partial D} a(Q) g(Q) dQ = \int_D \langle \nabla U_a, \nabla G(X) \rangle dX$, where $U_a(X)$ is harmonic in $D$, and satisfies $\dfrac{\partial U_a}{\partial N_Q} = a$, and $G(X)$ is the function constructed in Lemma 2.3. Using the first part of the present theorem, and Lemma 2.8, we have $l(f) = \int_D \langle \nabla U, \nabla G \rangle dx$ for all $f \in H^1(\partial D)$, with $\Delta U = 0$ on $D$, and $\dfrac{\partial U}{\partial N_Q} = f$. The uniqueness of $g$ is obvious.

*Remark.* Once we establish the equivalence of i) and ii) for general $C^1$ domains the condition that $D$ be starshaped in the last part of Theorem 2.9 can be dropped. Also, Theorem 2.7 is then seen to follow from Theorem 2.6 and Theorem 2.9, using the argument given on page 145 of [10].

**Corollary 2.10.** *If $D$ is a starshaped $C^1$ domain, then* $h^1(\partial D) = H^1(\partial D)$.

*Proof.* By Lemma 2.8, $h^1(\partial D)$ is dense in $H^1(\partial D)$. By Theorem 2.9, it is closed in $H^1(\partial D)$. Hence, both spaces are equal. We now show that i)$\leftrightarrow$ii) for arbitrary $C^1$-domains.

**Theorem 2.11.** *For $D$ a bounded $C^1$-domain,* $h^1(\partial D) = H^1(\partial D)$.

*Proof.* We just have to show $H^1(\partial D)\subset h^1(\partial D)$. Let $f\in H^1(\partial D)$; let $\vec{u}\in H^1(D)$ be such that $\vec{u}=\nabla U$, $f=\dfrac{\partial U}{\partial N_Q}$ on $\partial D$.

Using a regularized version (to make it of class $C^1$) of the construction given in Section 2 of [21], we see that for each point $Q\in\partial D$, we can find a ball $B=B_r$, with center $Q$ and radius $r$, and we can construct a $C^1$-starshaped domain $D_B$ such that $B_{4r}\cap D\supset D_B$, $\partial D_B\supset B_{2r}\cap\partial D$, and with the additional property that $\vec{u}\in H^1(D)\subset H^1(D_B)$.

We then cover $\partial D$ by a finite number, $\{B_j\}$, of the above balls, and we let $\psi_j\in C^1(\partial D)$ be a finite partition of unity of $\partial D$ subordinate to the above cover.

From Corollary 2.10, $f_j=\langle\vec{u}, N_{D_{B_j}}\rangle$ belongs to $h^1(\partial D_{B_j})$. Since we may assume atoms are supported in surface balls of diameter smaller than any preassigned positive number, we can write $\psi_j f=\psi_j g_j$, where $g_j\in h^1(\partial D)$. We now observe that if $h\in h^1(\partial D)$, and $\psi\in C^1(\partial D)$, then $\psi\cdot h-m(\psi\cdot h)\in h^1(\partial D)$, here $m(\psi h)=\dfrac{1}{\sigma(\partial D)}\int_{\partial D}\psi\cdot h$. Since $\int_{\partial D}f=0$, $\sum_j m(\psi_j f)=0$, and so $\sum_j m(\psi_j g_j)=0$. Moreover, $f=\sum_j\psi_j f=\sum_j\psi_j g_j=\sum_j(\psi_j g_j-m(\psi_j g_j))=\sum_j h_j$, where $h_j\in h^1(\partial D)$, and thus $f\in h^1(\partial D)$.

*Remark.* Theorem 2.11 shows that as sets $h^1(\partial D)=H^1(\partial D)$. However, since $h^1$ is continuously embedded in $H^1$, the open mapping theorem gives the equivalence of the two norms.

We proceed with our proof of Theorem 2.6.

The corollary to Theorem 1.3, shows that ii)$\to$iii). For the converse implication, we need the following uniqueness result:

**Lemma 2.12.** *Assume* $f\in L^1(\partial D)$; $\int_{\partial D}f=0$. *Assume that* $\|\nabla_X S_f(X)\|_{H^1(D)}=0$. *Then,* $f=0$.

*Proof.* Lemma 1.6 implies that $S_f(X)=c$, a constant for all $X\in D$. In $\mathbf{R}^n\setminus\bar{D}$, $S_f$ is harmonic, and since $\int_{\partial D}f=0$, $S_f(X)=O(|X|^{1-n})$ as $|X|\to\infty$. Also it is easily seen that the function $\tilde{S}_f^*(Q)=\sup_{\tilde{\Gamma}_Q}|S_f(X)|$ ($\tilde{\Gamma}_Q$ is the truncated exterior cone defined in the introduction.) belongs to $L^p(\partial D)$ for some $p>1$, and moreover, $\lim_{X\to Q,\, X\in\tilde{\Gamma}_Q}S_f(X)$ exists for almost every $Q\in\partial D$. This exterior nontangential limit is the same as the interior one, which is, of course, the constant $c$.

Let us assume that $c\neq 0$, and for simplicity we take $c>0$. Choose a ball $B$ so large that $\bar{D}\subset B$ and $|S_f|<\varepsilon$ on $\partial B$, with $\varepsilon$ an arbitrary but fixed positive number.

From Lemma 2.2, we have $|S_f(X)|<c+\varepsilon$ $\forall X\in B\setminus\bar{D}$, and from this we conclude $|S_f(X)|\leqq c$ in all of $\mathbf{R}^n\setminus\bar{D}$. Hence, either $S_f(X)=\pm c$ in all of $\mathbf{R}^n\setminus\bar{D}$

or $|S_f(X)|<c$ in all of $\mathbf{R}^n\setminus\bar{D}$. Hence, at almost every point $Q\in\partial D$,

$$0 \leqq \lim_{\substack{X\to Q\\ X\in\Gamma_Q}} \langle N_Q, \nabla S_f(X)\rangle = (\tfrac{1}{2}I+K^*)f(Q).$$

Moreover, from the interior normal derivative, we have $\frac{1}{2}f-K^*f=0$ on $\partial D$, and hence $K^*f$ has mean value 0 on $\partial D$. We then have $(\frac{1}{2}I+K^*)f=0$, and so $f=0$ on $\partial D$.

If the constant $c$ equals 0 we easily deduce that $S_f(X)$ is zero also in $\mathbf{R}^n\setminus\bar{D}$, and once again, the jump relations on the normal derivative give $f=0$ on $\partial D$.

We now prove iii)$\to$ii). Assume $f$ satisfies iii). By the equivalence of i) and ii), we have that $(\frac{1}{2}I-K^*)f=\dfrac{\partial}{\partial N_Q}S_f\in h^1(\partial D)$. By Theorem 1.2, there exists $\tilde{f}\in h^1(\partial D)$ such that

$$(\tfrac{1}{2}I-K^*)\tilde{f}=(\tfrac{1}{2}I-K^*)f.$$

The function $S_{(f-\tilde{f})}(X)$ satisfies the hypothesis of Lemma 1.6, and hence it is identically constant in $D$. By Lemma 2.12, $f=\tilde{f}$, and thus $f\in h^1(\partial D)$.

We now show that iii)$\leftrightarrow$iv) in Theorem 2.6.

For $f\in L^1(\partial D)$, $D_{X_i}S_f(X)$ has a nontangential limit at almost every point $Q\in\partial D$, and this limit equals $\frac{1}{2}N_p^i\cdot f(P)+R_if(P)$. It now follows that if iii) is satisfied, and $f\in L^1(\partial D)$, then iv) holds.

Conversely, now assume iv). From what we just noted in the above paragraph, $|\nabla S_f|\in L^1(\partial D)$, and $\||\nabla S_f|\|_{L^1(\partial D)}\leqq c\{\|f\|_{L^1(\partial D)}+\sum_{i=1}^n\|R_if\|_{L^1(\partial D)}\}$.

For any $f\in L^1(\partial D)$, it was remarked in [7], that $(\nabla S_f)^*$ belongs to weak $L^1$ of $\partial D$. In particular, for each $r$, $0<r<1$, $(\nabla S_f^*)^r$ belongs to $L^{1/r+\varepsilon}(\partial D)$, $\varepsilon>0$. However, by Lemma 2.1, there exists $r$, $0<r<1$, such that $|\nabla S_f(X)|^r$ is subharmonic in $D$. If we choose $\varepsilon>0$ such that $1-\varepsilon>r$, and let $p=\dfrac{1}{r+\varepsilon}$, then $p>1$, and so we can apply Lemma 2.2, to show that $|\nabla S_f(X)|^r\leqq P(|\nabla S_f|^r)(X)$, and so $(\nabla S_f)^*(Q)\leqq P(|\nabla S_f|^r)^*(Q)^{1/r}$. Hence we conclude that

$$\int_{\partial D}(\nabla S_f)^*\,dQ \leqq c\int_{\partial D}|\nabla S_f(Q)|\,dQ,$$

and so iv)$\to$iii). Thus the proof of Theorem 2.6 is finished.

We give an application of Theorem 2.6 to a special two-dimensional situation.

**Theorem 2.13.** *Let $\Gamma$ be a simple closed $C^1$ Jordan curve in the complex plane* $\mathbf{C}$. *Set*

$$Cf(z)=\frac{1}{2\pi i}\int_\Gamma\frac{f(\zeta)}{z-\zeta}\,d|\zeta| \equiv \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{|z-\zeta|<\varepsilon}\frac{f(\zeta)}{z-\zeta}\,d|\zeta|$$

*where $d|\zeta|=$arc length and $z\in\Gamma$, and let $H^1(\Gamma)=\{f\in L^1(\Gamma)$: $f$ real valued, $\int_\Gamma f\,d|\zeta|=0$,* *and* $Cf\in L^1(\Gamma)\}$ *with*

$$\|f\|_{H^1(\Gamma)} = \|f\|_{L^1(\Gamma)}+\|Cf\|_{L^1(\Gamma)}.$$

*Then* $H^1(\Gamma)^* = \mathrm{BMO}\,(\Gamma)$.

*Proof.* We need only observe that $Cf\in L^1(\Gamma)$ if and only if the principal value operators

$$\frac{1}{2\pi}\int_\Gamma f(\zeta)\frac{(\mathrm{Re}\,(z)-\mathrm{Re}\,(\zeta))}{|z-\zeta|^2}\,d|\zeta| \quad \text{and} \quad \frac{1}{2\pi}\int_\Gamma f(\zeta)\frac{(\mathrm{Im}\,(z)-\mathrm{Im}\,(\zeta))}{|z-\zeta|^2}\,d|\zeta|$$

belong to $L^1(\Gamma)$. These two operators are exactly the Riesz operators defined in Theorem 2.6 for the domain $D$=inside of $\Gamma$. The conclusion of Theorem 2.13 is now immediate from Theorems 2.6 and 2.9.

This completes, in the case of bounded $C^1$ domains in the complex plane, a remark made in [14]. We can conclude, using the notation of [14], that if $f/\omega$ and $C(f/\omega)$ are in $L^1(\Lambda, d|\zeta|)$, then $f$ is the real part of the boundary values of an analytic function $F\in H^1(0, d\omega)$. The opposite implication was shown in [14].

*Some concluding remarks:* Let $D$ be a bounded $C^1$ domain in $\mathbf{R}^n$. Let $X^*$ be a fixed point in $D$, and let $d\omega = d\omega^{X^*}$ be harmonic measure in $D$, with pole at $X^*$, and $\omega=\frac{d\omega}{d\sigma}$ be its density with respect to surface measure. If $n=2$, the equivalence between i) and ii) in Theorem 2.6 follows combining the results in [13] and [14], even in the case of Lipschitz domains. Moreover, if we let $H^1_{\mathscr{D}}(\partial D, d\omega)=\{f,\, f=u|_{\partial D},\, \Delta u=0$ in $D$ and $u^*\in L^1(\partial D, d\omega)\}$ (here $\mathscr{D}$ stands for 'Dirichlet data'), then, for $n=2$ and arbitrary Lipschitz domains, $f\in H^1_{\mathscr{D}}(\partial D, d\omega)$ iff $g=f\cdot\omega\in H^1(\partial D)$, where this is the space defined in this paper. This also follows from [13] & [14].

Also, if again $n=2$, and $D$ is $C^1$ $\left(\text{to insure that } \frac{d\sigma}{\omega}\in A_\infty\right)$, and we let $H^1\left(D, \frac{d\sigma}{\omega}\right)=\left\{\vec{u},\, \vec{u}=\nabla U,\, \nabla U=0, \text{ and } (\vec{u})^*\in L^1\left(\partial D, \frac{d\sigma}{\omega}\right)\right\}$,

$$H^1\left(\partial D, \frac{d\sigma}{\omega}\right) = \left\{f=\frac{\partial U}{\partial N_Q},\ \vec{u}=\nabla U,\ \vec{u}\in H^1\left(D, \frac{d\sigma}{\omega}\right)\right\},$$

and $H^1_{\mathscr{D}}(\partial D, d\sigma)=\{f,\, f=u|_{\partial D},\, \Delta u=0$ in $D\}$, and $u^*\in L^1(\partial D, d\sigma)$, then using [13] and [14] we can see that $f\in H^1_{\mathscr{D}}(\partial D, d\sigma)$ iff $g=f\cdot\omega\in H^1\left(\partial D, \frac{d\sigma}{\omega}\right)$. Moreover, $g\in H^1\left(\partial D, \frac{d\sigma}{\omega}\right)$ iff $g=\sum\lambda_j a_j$, $\sum\lambda_j a_j$, $\sum|\lambda_j|<+\infty$, $\operatorname{supp} a_j\subset B_j$, $\|a_j\|_\infty\leqq \left(\int_{B_j}\frac{d\sigma}{\omega}\right)^{-1}$, $\int a_j\,d\sigma=0$. Also, $f\in H^1_{\mathscr{D}}(\partial D, d\sigma)$ iff $f=\sum\lambda_j b_j$, $\sum|\lambda_j|<+\infty$, $\operatorname{supp} b_j\subset B_j$ $\|b_j\|_\infty\leqq\frac{1}{\sigma(B_j)}$, $\int b_j\,d\omega=0$.

We conjecture that these results also hold for $n>2$, and already have some partial positive results in this direction.

Finally, we remark that the analysis of $H^p(\partial D)$, $p<1$, along the lines of this paper, on a bounded $C^1$ domain remains open. Moreover, the study of the situation considered here in the setting of an arbitrary Lipschitz domain in $\mathbf{R}^n$, $n>2$, is also open, even in the case $p>1$.

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