Function theory and M-ideals

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1. Introduction

This paper deals with some questions that lie at the interface between functional analysis and complex function theory. Let H^∞ denote the algebra of bounded holomorphic functions on the open unit disk. We may identify a function with its radial boundary values, and thus view H^∞ as a subalgebra of the algebra L^∞ of bounded measurable functions on the unit circle. Since H^∞ is a weak-star closed subspace of L^∞, it is easy to see that for every function f in L^∞, there is a function g in H^∞ such that $\|f-g\|=\text{distance}\,(f;\, H^\infty)$; i.e., g is a *best approximant* to f from H^∞. Sarason [10] asked whether functions in L^∞ always have best approximants from the space $H^\infty+C$ spanned by H^∞ and the algebra C of continuous functions on the circle.

The space $H^\infty+C$ plays a special role in function theory, since Sarason [9] has shown that it is in fact a closed algebra and is contained in every closed subalgebra of L^∞ that properly contains H^∞. The space $H^\infty+C$ also plays a special role in operator theory, in the following way. Let L^2 denote the Hilbert space of square integrable functions on the circle, let H^2 denote the closed subspace spanned by the non-negative powers of the function z, and let $(H^2)^\perp$ be the closed subspace spanned by the negative powers of z (which is just the orthogonal complement of H^2). We write Q for the orthogonal projection of L^2 onto $(H^2)^\perp$. For each $f\in L^\infty$, we define a *Hankel operator* $H_f\colon H^2\to(H^2)^\perp$ by $H_f(g)=Q(gf)$. If we use the usual bases for H^2 and $(H^2)^\perp$, we may characterize Hankel operators as those whose matrices have constant cross diagonals. It turns out that the distance (in the operator norm) from the Hankel operator H_f to the space of all compact operators from H^2 to $(H^2)^\perp$ is the same as the distance from H_f to the space of compact Hankel operators, and the latter coincides with the distance (in the L^∞-norm) from the

* Research supported in part by grants from the National Science Foundation.

function f to the space $H^\infty+C$. (In particular, H_f is a compact operator if and only if $f\in H^\infty+C$.) Sarason's question about best approximants from $H^\infty+C$ therefore has a purely operator-theoretic formulation: Does every Hankel operator have a best approximant from the space of compact Hankel operators?

In its operator-theoretic formulation, Sarason's question was answered affirmatively by Axler, Berg, Jewell and Shields [2]. Subsequently, a simple function-theoretic proof was found by Luecking [8]. Luecking used the F. and M. Riesz Theorem to show that the quotient space $(H^\infty+C)/H^\infty$ is an M-ideal in L^∞/H^∞. (The definition of M-ideal is given at the end of this introduction.) By work of Alfsen and Effros [1], this implies that elements of L^∞/H^∞ have best approximants from $(H^\infty+C)/H^\infty$. Since H^∞ is weak-star closed in L^∞, it is simple to conclude that functions in L^∞ have best approximants in $H^\infty+C$.

In this paper, we consider Luecking's result in a more general context. Let A be a uniform algebra on a compact space X, with maximal ideal space M_A. Fix $\varphi\in M_A$, and let σ be a representing measure on X for φ. We denote by $H^\infty(\sigma)$ the weak-star closure of A in $L^\infty(\sigma)$, and by $[H^\infty(\sigma)+C(X)]$ the closed linear span of $H^\infty(\sigma)$ and $C(X)$. We ask when $[H^\infty(\sigma)+C(X)]/H^\infty(\sigma)$ is an M-ideal in $L^\infty(\sigma)/H^\infty(\sigma)$. In Section 3 we obtain information in a general setting, which is sufficient to answer the question completely for certain algebras which arise in rational approximation theory.

Let, for instance, D be a bounded domain in the complex plane, and let $A(D)$ be the algebra of continuous functions on $\bar{D}$ which are analytic on D. The results apply to $A(D)$, with σ a representing measure on ∂D for some point $p\in D$, and X the closed support of σ. In this case, $[H^\infty(\sigma)+C(X)]/H^\infty(\sigma)$ is an M-ideal in $L^\infty(\sigma)/H^\infty(\sigma)$ if and only if the representing measures on X for p form a weakly compact set of measures. This latter condition is satisfied when D is the open unit disc and σ is normalized arc-length measure on ∂D, since in this case $d\theta/2\pi$ is a unique representing measure on ∂D for the evaluation functional at $0\in D$.

The results in Section 3 depend upon a careful analysis of M-ideals in quotient spaces of $C(X)$. This is accomplished in Section 2 via bands of measures (cf. Theorem 2). Section 2 also contains some applications to the tight uniform algebras introduced by Cole and Gamelin [3]. In Section 4 we return to the theme of best approximants to give an elementary proof of the Alfsen—Effros result that M-ideals have the best approximation property.

We conclude this introduction by recalling some facts about L-projections, M-ideals and bands of measures. An *L-projection* on a Banach space E is a linear operator $P\colon E\to E$ such that $P^2=P$ and

$$\|Px\|+\|x-Px\| = \|x\|$$

for each $x\in E$. A subspace F_0 of the Banach space F is an *M-ideal* if its annihilator

$F_0^\perp$ in the dual space F^* is the range of an L-projection P on F^*. If in addition there is a subspace F_1 of F such that $F_0 \oplus F_1 = F$ and $F_1^\perp$ is the range of $I-P$, we say that F_0 is an *M-summand*. These notions were introduced by Alfsen and Effros [1], and we refer the reader to those papers for further information. (A word of caution: Alfsen and Effros consider only real Banach spaces, while we are concerned with complex Banach spaces. However, we shall have no occasion to use those results of Alfsen and Effros which are particular to the case of real scalars. For other applications of M-ideals to operator theory, see for example the paper of Holmes, Scranton and Ward [7].

For X a compact Hausdorff space, we let $C(X)$ be the algebra of continuous, complex-valued functions on X and let $\mathfrak{M}(X)$ be its dual space, the space of regular complex Borel measures on X. A *band* of measures on X is a (norm) closed subspace $\mathfrak{B}$ of $\mathfrak{M}(X)$ with the property that if $\mu \in \mathfrak{B}$ and σ is absolutely continuous with respect to μ, then σ also belongs to $\mathfrak{B}$. The *complementary band* to $\mathfrak{B}$ is the band $\mathfrak{B}'$ of all measures singular with respect to every measure in $\mathfrak{B}$. The Lebesgue decomposition theorem shows that $\mathfrak{M}(X) = \mathfrak{B} \oplus \mathfrak{B}'$; the associated projection $P_{\mathfrak{B}}$ of $\mathfrak{M}(X)$ onto $\mathfrak{B}$ will be called the *band projection* associated with $\mathfrak{B}$. We say that the band $\mathfrak{B}$ is *reducing* for a subspace A of $C(X)$ if $P_{\mathfrak{B}}(A^\perp) \subset A^\perp$. The reducing band $\mathfrak{B}$ is *minimal* if it is not the zero subspace and contains no reducing bands other than the zero subspace and $\mathfrak{B}$ itself. Minimal reducing bands which consist of multiples of a point mass are called *trivial minimal reducing bands;* all others are *non-trivial minimal reducing bands*. For further information concerning bands of measures, we refer to [3].

2. L-projections, M-ideals and tight algebras

The Lebesgue Decomposition Theorem gives rise to simple examples of L-projections in the following way. Fix a (complex, regular Borel) measure μ on the compact Hausdorff space X. For each measure σ on X we may then write $\sigma = \sigma_a + \sigma_s$, where σ_a is absolutely continuous with respect to μ and σ_s is singular with respect to μ. The operator P: $\mathfrak{M}(X) \to \mathfrak{M}(X)$ given by $P\sigma = \sigma_a$ is then an L-projection on $\mathfrak{M}(X)$. Our first result shows that if we replace the single measure μ by an appropriate band of measures we can obtain all L-projections on all subspaces of $\mathfrak{M}(X)$.

Theorem 1. *Let X be a compact Hausdorff space, let $\mathfrak{E}$ be a (norm) closed subspace of $\mathfrak{M}(X)$ and let P be an L-projection on $\mathfrak{E}$. Let $\mathfrak{B}$ be the band of measures generated by $P\mathfrak{E}$ and $\mathfrak{B}'$ the complementary band. Then*

(i) $\mathfrak{E} = (\mathfrak{E} \cap \mathfrak{B}) \oplus (\mathfrak{E} \cap \mathfrak{B}')$;
(ii) *P is the projection of $\mathfrak{E}$ onto $\mathfrak{E} \cap \mathfrak{B}$;*
(iii) *for each measure $\mu \in \mathfrak{E}$, there is a Borel set $E \subset X$ such that $P\mu = \chi_E \mu$.*

Proof. Let $\mu\in\mathfrak{E}$ and use the Lebesgue Decomposition Theorem and the Radon—Nikodym Theorem to write $P\mu=h\mu+\eta$, where $h\in L^1(\mu)$ and η is singular with respect to μ. We claim that $\eta=0$ and that $0\leqq h\leqq 1$. To see this, note that, because P is an L-projection and η is singular, we have

$$\|\mu\| = \|P\mu\|+\|\mu-P\mu\| = \|h\mu+\eta\|+\|(1-h)\mu-\eta\|$$
$$= \|h\mu\|+\|\eta\|+\|(1-h)\mu\|+\|\eta\| \geqq \|\mu\|+2\|\eta\|.$$

This immediately yields $\eta=0$. Hence

$$\int d|\mu| = \int |h|\, d|\mu| + \int |1-h|\, d|\mu|.$$

Since $|h|+|1-h|\geqq 1$, with equality only when $0\leqq h\leqq 1$, we have our claim.

Now suppose that $\sigma\in\mathfrak{E}$ is absolutely continuous with respect to μ, so that $\sigma=g\mu$ for some $g\in L^1(\mu)$; we want to see that $P\sigma=gP\mu$. To see this, we use the above to write

$$P\sigma = f\sigma = fg\mu,$$
$$P\mu = h\mu,$$
$$P(\sigma+\mu) = k(\sigma+\mu),$$

with f, h and k between 0 and 1. Since P is linear, $fg+h=k(g+1)$ (a.e. with respect to μ). Thus $(f-k)g=k-h$. On the set where the imaginary part of g is not zero, we must have $f-k=0$ and $k-h=0$ since f, h, k are all real. Thus $f=h$ wherever the imaginary part of g is not zero. If we replace σ by $i\sigma=ig\mu$ then $P\sigma=ifg\mu$. Arguing in the same way as for σ, we see that $f=h$ wherever the real part of g is not zero. Thus $P\sigma=fg\mu=hg\mu=gP\sigma$, as desired.

Combining the two above analyses, we see that for each $\mu\in\mathfrak{E}$, there is a function h such that $P\mu=h\mu$ and $P(h\mu)=h^2\mu$. Since $P^2=P$, we conclude that h is the characteristic function of some Borel set E, which is (iii).

We next wish to see that $P\mathfrak{E}=\mathfrak{E}\cap\mathfrak{B}$. Since $\mathfrak{B}$ is the band generated by $P\mathfrak{E}$, the inclusion $P\mathfrak{E}\subset\mathfrak{E}\cap\mathfrak{B}$ is clear. To obtain the reverse inclusion, let $\sigma\in\mathfrak{E}\cap\mathfrak{B}$. We can then find measures $\sigma_1, \sigma_2, \ldots$ in $P\mathfrak{E}$ and functions $f_1, f_2, \ldots$ such that $\sigma=\sum_{j=1}^{\infty} f_j\sigma_j$ (norm convergent). (Note that the measures $f_j\sigma_j$ need not belong to $\mathfrak{E}$.) We may write $\sigma=g_1\sigma_1+\sigma'$, where σ' is singular with respect to σ_1; we may then write $\sigma'=g_2\sigma_2+\sigma''$ where σ'' is singular with respect to σ_1 and σ_2. Continuing in this way, we obtain a representation $\sigma=\sum_{j=1}^{\infty} g_j\sigma_j$, where $g_j\sigma_j$ is singular with respect to σ_k whenever $j\neq k$. There is a Borel set E such that $P\sigma=\chi_E\sigma$; if we show that $\chi_E g_j\sigma_j=g_j\sigma_j$ for each j, we will have that $P\sigma=\sigma$, or equivalently that $\sigma\in P\mathfrak{E}$, which is what we desire.

To this end, we fix an index j and write $\beta=\sum_{k\neq j} g_k\sigma_k$, $\alpha=\sigma_j$ and $g=g_j$ so that $\sigma=g\alpha+\beta$, and β is singular with respect to α. Since $P\sigma=\chi_E\sigma$, $P\alpha=\alpha$ and P

is an L-projection, computing $P(\alpha+\sigma)$ yields

$$\|(1+g)\alpha+\beta\| = \|\alpha+\chi_E g\alpha+\chi_E\beta\|+\|(1-\chi_E)(g\alpha+\beta)\|.$$

Since β is singular with respect to α, we may rewrite both sides of this equation to obtain

$$\|(1+g)\alpha\|+\|\beta\| = \|\alpha+\chi_E g\alpha\|+\|\chi_E\beta\|+\|(1-\chi_E)g\alpha\|+\|(1-\chi_E)\beta\|.$$

Now the terms involving β obligingly cancel out. Writing the norms as integrals we then obtain

$$\int |1+g|\,d|\alpha| = \int |1+\chi_E g|\,d|\alpha|+\int |(1-\chi_E)g|\,d|\alpha|.$$

If we break the integrals up into integrals over E and over $X-E$, we see that the integrals over E cancel out and we are left with

$$\int_{X-E} |1+g|\,d|\alpha| = \int_{X-E} d|\alpha|+\int_{X-E} |g|\,d|\alpha|.$$

Hence $g\geqq 0$ almost everywhere on $X-E$ (with respect to $|\alpha|$). On the other hand, if we replace $\alpha+\sigma$ by $\alpha-\sigma$ and perform the same calculation, we find $g\leqq 0$ almost everywhere on $X-E$ (with respect to $|\alpha|$). In other words, $(1-\chi_E)g\alpha=0$ so $\chi_E g\alpha= g\alpha$. Since $g\alpha=g_j\sigma_j$ and j was arbitrary, we find that $P\sigma=\chi_E\sigma=\sigma$, as desired, so that $P\mathfrak{C}=\mathfrak{C}\cap\mathfrak{B}$.

To complete the proof, we need only show that the null space of P, which is the range of $I-P$, coincides with $\mathfrak{C}\cap\mathfrak{B}'$. To see this, note first that if $\sigma\in(\mathfrak{C}\cap\mathfrak{B}')$ then $P\sigma\in\mathfrak{C}\cap\mathfrak{B}=P\mathfrak{C}$, so $P\sigma$ is singular to σ. On the other hand, $P\sigma$ is always absolutely continuous with respect to σ, so $P\sigma=0$ and the null space of P includes $\mathfrak{C}\cap\mathfrak{B}'$. To obtain the reverse inclusion we must show that if $\tau\in\mathfrak{C}$ and $P\tau=0$ then τ is singular with respect to every measure in $\mathfrak{B}$. Since $P\mathfrak{C}$ generates $\mathfrak{B}$, it suffices to show that τ is singular with respect to every measure $\sigma\in P\mathfrak{C}$. We know there is a set E such that $P(\sigma+\tau)=\chi_E(\sigma+\tau)$. Since $P(\sigma+\tau)=\sigma$, this yields $\sigma=\chi_E(\sigma+\tau)$ and $(1-\chi_E)\sigma=\chi_E\tau$, which forces both $(1-\chi_E)\sigma$ and $\chi_E\tau$ to be zero. Thus σ is carried by E, τ is carried by $X-E$, and σ and τ are indeed singular. This completes the proof. □

Since M-ideals are defined in terms of L-projections in the dual space, we can use Theorem 1 to obtain information about M-ideals. We are interested in pairs A, B of closed subspaces of $C(X)$ with $A\subset B$; we want to know when B/A is an M-ideal in $C(X)/A$. By the definition of M-ideal and the usual identification of dual spaces, this is equivalent to the existence of an L-projection from the annihilator $A^\perp$ in $\mathfrak{M}(X)=C(X)^*$ onto $B^\perp$. In view of Theorem 1, this implies the decomposition $A^\perp=B^\perp\oplus Q$, where every measure in Q is singular to every measure in $B^\perp$, and Q may be identified with the dual space $(B/A)^*$. We can say a bit more.

Theorem 2. *Let A, B be closed subspaces of $C(X)$ with $A \subset B$. Let $\mathfrak{B}$ be the band generated by $B^\perp$. Then B/A is an M-ideal in $C(X)/A$ if and only if*

(i) *$\mathfrak{B}$ is reducing for A, and*
(ii) *$\mathfrak{B} \cap A^\perp = B^\perp$.*

Proof. If (i) and (ii) are true, let P be the restriction to $A^\perp$ of the band projection of $\mathfrak{M}(X) = \mathfrak{B} \oplus \mathfrak{B}'$ onto $\mathfrak{B}$. Since $\mathfrak{B}$ is reducing for A, the operator P maps $A^\perp$ into itself; since $\mathfrak{B} \cap A^\perp = B^\perp$, P is an L-projection of $A^\perp$ onto $B^\perp$, so B/A is an M-ideal in $C(X)/A$.

Conversely, if P: $A^\perp \to A^\perp$ is an L-projection onto $B^\perp$, then Theorem 1 assures us that $A^\perp = (A^\perp \cap \mathfrak{B}) \oplus (A^\perp \cap \mathfrak{B}')$ and that P is just projection onto the first summand, which coincides with $B^\perp$. This yields both (i) and (ii). □

If we start with a given band of measure $\mathfrak{B}$, we may ask when the projection onto $\mathfrak{B}$ is the L-projection associated with some subspace B of A. We leave it to the reader to check that this will be so if and only if $\mathfrak{B}$ is reducing for A and $\mathfrak{B} \cap A^\perp$ is weak-star closed in $\mathfrak{M}(X)$. Note that in this case, $\mathfrak{B}$ need not be the band generated by $B^\perp$, but rather will be the direct sum of the band generated by $B^\perp$ and a band singular to $A^\perp$.

Finally, we consider the case in which the subspace A of $C(X)$ is actually a uniform algebra. For each $g \in C(X)$, we consider the Hankel operator S_g: $A \to C(X)/A$ given by $S_g(f) = gf + A$. Following Cole and Gamelin [3], we shall say that A is a *tight* uniform algebra on X if each of the Hankel operators S_g is weakly compact. Tightness may be thought of as an abstract version of the solvability of a certain $\bar{\partial}$-problem with a gain in smoothness. Among familiar uniform algebras, both $R(K)$ (the algebra of uniform limits on K of rational functions) and $A(K)$ (the algebra of functions continuous on K and holomorphic on the interior of K) are tight for every compact set $K \subset C$. In higher dimensions, $A(\bar{D})$ is tight if D is a strictly pseudoconvex domain in C^n but is not tight if D is a polydisk (and $n \geqq 2$). For tight uniform algebras we can sharpen Corollary 2 and obtain a complete description of M-ideals in $C(X)/A$ in terms of reducing bands for A.

Theorem 3. *Let A be a tight uniform algebra on X, and let $\{\mathfrak{B}_\alpha\}_{\alpha \in I}$ be the family of non-trivial minimal reducing bands for A. Then there is a one-to-one correspondence between subspaces B of $C(X)$ which contain A and for which B/A is an M-ideal in $C(X)/A$, and subsets J of the index set I, determined by*

$$B^\perp = A^\perp \cap \Big(\bigoplus_{\alpha \in J} \mathfrak{B}_\alpha\Big).$$

Furthermore, each such B is a subalgebra of $C(X)$, and B/A is an M-summand of $C(X)/A$.

Proof. For J a subset of I, it follows from Theorem 10.3 of [3] that $A^{\perp} \cap (\bigoplus_{\alpha \in J} \mathfrak{B}_{\alpha})$ is a weak-star closed subspace of $\mathfrak{M}(X)$. It follows from Theorem 11.4 of [3] that $B_J = \bigcap_{\alpha \in J} (H^{\infty}(\mathfrak{B}_{\alpha}) \cap C(X))$ is a closed subalgebra of $C(X)$, that B_J contains A, and that $B_J^{\perp} = A^{\perp} \cap (\bigoplus_{\alpha \in J} \mathfrak{B}_{\alpha})$. Moreover, the band projection onto $\bigoplus_{\alpha \in J} \mathfrak{B}_{\alpha}$ provides an L-projection of $A^{\perp}$ onto $B_J^{\perp}$, so that B_J/A is an M-ideal in $C(X)/A$.

Conversely, if B/A is an M-ideal in $C(X)/A$, let $\mathfrak{B}$ be the band of measures generated by $A^{\perp}$. By Theorem 10.5 of [3],

$$A^{\perp} = \bigoplus_{\alpha \in I} (A^{\perp} \cap \mathfrak{B}_{\alpha}).$$

By Theorem 2, $\mathfrak{B}$ is reducing for A, so the minimality of each $\mathfrak{B}_{\alpha}$ yields

$$B^{\perp} = \bigoplus_{\alpha \in J} (\mathfrak{B}_{\alpha} \cap A_{aa})$$

for some subset J of I.

If B is the algebra corresponding to $J \subset I$ and B' is the algebra corresponding to $I - J$, then Theorem 11.3 of [3] yields that $B/A \oplus B'/A = C(X)/A$; i.e., every M-ideal in $C(X)/A$ is an M-summand. This completes the proof. □

We mention in passing that there seems to be some similarity between the final assertion of the theorem, that the M-ideals of $C(X)/A$ correspond to subalgebras B of $C(X)$ containing A, and results of Smith and Ward. They show in [11] that M-ideals in a Banach algebra with identity are subalgebras, and that they are ideals if the algebra is commutative.

As a very simple corollary to Theorem 3, note that if A is a tight uniform algebra with only one nontrivial minimal reducing band, then there are no proper M-ideals in $C(X)/A$. This applies for instance to the algebra $A(D)$ on $X = \partial D$ in the case that D is a bounded domain in the complex plane. It also applies to $A(D)$ in the case that D is a bounded, strictly pseudoconvex domain in C^n. (See [3], particularly Theorem 12.3.)

3. The space $H^{\infty}(\sigma) + C(X)$

Now fix a uniform algebra A on a compact space X, and a representing measure σ on X for some $\varphi \in M_A$. Let $\Sigma(\sigma)$ denote the spectrum of $L^{\infty}(\sigma)$. Then $\Sigma(\sigma)$ is a compact space, and

$$L^{\infty}(\sigma) \cong C(\Sigma(\sigma)).$$

The measure σ lifts to a measure $\hat{\sigma}$ on $\Sigma(\sigma)$. Recall that $H^{\infty}(\sigma)$ is the weak-star closure of A in $L^{\infty}(\sigma)$. If $H^{\infty}(\sigma)$ is regarded as a closed subspace of $C(\Sigma(\sigma))$, then $\hat{\sigma}$ is a representing measure for the weak-star continuous extension $\hat{\varphi}$ of φ to $H^{\infty}(\sigma)$. We will frequently make use of the following result.

Dual Version of the Hoffman—Rossi Theorem [4, Theorem IV.2.3]: The set of representing measures on $\Sigma(\sigma)$ for $\hat{\varphi}$ is the weak-star closure of the set of representing measures for $\hat{\varphi}$ which are absolutely continuous with respect to $\hat{\sigma}$.

We are particularly interested in the case in which $\hat{\sigma}$ is a dominant representing measure, that is, in which every representing measure on $\Sigma(\sigma)$ for $\hat{\varphi}$ is absolutely continuous with respect to $\hat{\sigma}$. From the dual version of the Hoffman—Rossi theorem, it follows easily (cf. [4, Corollary IV.2.4]) that $\hat{\sigma}$ is dominant if and only if the set of representing measures for φ which are absolutely continuous with respect to σ is a weakly compact set of measures, i.e., the Radon—Nikodym derivatives form a uniformly integrable subset of $L^1(\sigma)$.

In this section, we are concerned with the problem of determining when $[H^\infty(\sigma)+C(X)]/H^\infty(\sigma)$ is an M-ideal in $L^\infty(\sigma)/H^\infty(\sigma)$. By Theorem 2, this occurs if and only if there is a band projection from $H^\infty(\sigma)^\perp$ to $[H^\infty(\sigma)+C(X)]^\perp$, where these are both subspaces of $\mathfrak{M}(\Sigma(\sigma))$. Our main necessary condition for the existence of such a band projection is the following.

Theorem 4. *Let A be a uniform algebra on a compact metrizable space X. Let σ be a representing measure for some point $\varphi\in M_A$, and suppose that $[H^\infty(\sigma)+C(X)]/H^\infty(\sigma)$ is an M-ideal in $L^\infty(\sigma)/H^\infty(\sigma)$. Then $\hat{\sigma}$ is dominant; equivalently, the set of representing measures for φ which are absolutely continuous with respect to σ is weakly compact.*

Proof. Let $\mathfrak{B}$ be the band of measures on $\Sigma(\sigma)$ generated by $[H^\infty(\sigma)+C(X)]^\perp$. By Theorem 2, $\mathfrak{B}$ is a reducing band for $H^\infty(\sigma)$, and

$$(*)\qquad \mathfrak{B}\cap H^\infty(\sigma)^\perp=[H^\infty(\sigma)+C(X)]^\perp.$$

Let $\mathfrak{B}_0$ be the band of measures on $\Sigma(\sigma)$ generated by the representing measures for $\hat{\varphi}$. By [3, Lemma 20.6], $\mathfrak{B}_0$ is a minimal reducing band. Since $\mathfrak{B}\cap\mathfrak{B}_0$ is a reducing band, then either $\mathfrak{B}=\mathfrak{B}_0$, or else $\mathfrak{B}\cap\mathfrak{B}_0=\{0\}$.

We may assume that σ is not the point mass at φ, since the conclusions of the theorem are trivial in that case. Then there is a function $f\in A$ such that $f(\varphi)=0$, while f is not identically zero on the closed support of σ. Thus $f\sigma$ is a nonzero measure in $A^\perp$. Hence $f\hat{\sigma}\perp H^\infty(\sigma)$, but $f\hat{\sigma}$ is not orthogonal to $C(X)$. Thus $f\hat{\sigma}\notin[H^\infty(\mu)+C(X)]^\perp$, so that from (*) above we obtain $f\hat{\sigma}\notin\mathfrak{B}$. On the other hand, $f\hat{\sigma}\in\mathfrak{B}_0$. We conclude that $\mathfrak{B}\neq\mathfrak{B}_0$. By our earlier remarks, $\mathfrak{B}\cap\mathfrak{B}_0=\{0\}$, and $\mathfrak{B}$ is singular to $\mathfrak{B}_0$.

We next consider the space S of positive measures on X which represent φ and which are absolutely continuous with respect to σ. We want to see that S is weak-star closed. If this is not so, we let η be a measure in the weak-star closure of S but not in S. Since $C(X)$ is separable and S is a subset of the unit ball of $\mathfrak{M}(X)$, we can choose a sequence $\{f_j\sigma\}$ in S which converges weak-star to η. We are going to extract two subsequences from $\{f_j\sigma\}$.

By assumption, η is not absolutely continuous with respect to σ, so there is a compact subset E of X for which $\sigma(E)=0$ and $\eta(E)>0$. Let ε be a small positive number (to be chosen later), and choose open sets U_0, V_0 in X such that $E\subset U_0\subset \bar{U}_0\subset V_0$ and $\eta(V_0)<\eta(E)+\varepsilon$. Let φ, Ψ be continuous functions on $C(X)$ such that $0\leqq\Psi\leqq 1$, $\varphi\equiv 1$ on E, $\varphi\equiv 0$ off U_0, $\Psi\equiv 1$ on $\bar{U}_0$, $\Psi\equiv 0$ off V_0. Then

$$\eta(E)-\varepsilon<\eta(E)\leqq\int\varphi\, d\eta,$$

$$\int\Psi\, d\eta\leqq\eta(V_0)<\eta(E)+\varepsilon.$$

Since $f_j\sigma\to\eta$ weak-star, we can find an index j_0 such that

$$\eta(E)-\varepsilon<\int\varphi\, d(f_{j_0}\sigma)$$

$$\int\Psi\, d(f_{j_0}\sigma)<\eta(E)+\varepsilon.$$

Combining these two inequalities yields

$$\eta(E)-\varepsilon<f_{j_0}\sigma(\bar{U}_0)<\eta(E)+\varepsilon.$$

We can now repeat this process, and continue inductively to obtain a sequence $\{U_k\}$ of open sets and a subsequence $\{f_{j_k}\sigma\}$ of the sequence $\{f_j\sigma\}$ such that:

(i) $$E\subset U_{k+1}\subset\bar{U}_{k+1}\subset U_k,$$

(ii) $$j_{k+1}>j_k,$$

(iii) $$f_{j_k}\sigma(\bar{U})<\eta(E)+\varepsilon,$$

(iv) $$f_{j_k}\sigma(\bar{U}_k)>\eta(E)-\varepsilon.$$

Since each of the measures $f_j\sigma$ is absolutely continuous with respect to σ, and $\sigma(E)=0$, we can also arrange that:

(v) $$f_{j_k}\sigma(\bar{U}_{k+1})<\varepsilon.$$

Henceforth, we will write g_k for f_{j_k}. Set

$$W=\bigcup_{k=0}^{\infty}(\bar{U}_{2k}-\bar{U}_{2k+1});$$

W is a Borel set, and we write χ_W for its characteristic function. Then

$$\int\chi_W g_{2k}\, d\sigma=g_{2k}\sigma(W)$$
$$\geqq g_{2k}\sigma(\bar{U}_{2k})-g_{2k}\sigma(\bar{U}_{2k+1})\geqq\eta(E)-2\varepsilon.$$

On the other hand,

$$\int\chi_W g_{2k+1}\, d\sigma=g_{2k+1}\sigma(W)$$
$$\leqq g_{2k+1}\sigma(U_0-U_{2k+1})+g_{2k+1}\sigma(U_{2k+2})\leqq\big(\eta(E)+\varepsilon\big)-\big(\eta(E)-\varepsilon\big)+\varepsilon.$$

Since the measures $g_k\sigma$ are absolutely continuous with respect to σ and represent φ on the algebra A, they have lifts to $\Sigma(\sigma)$, which we denote by $\hat{g}_k\hat{\sigma}$, which are absolutely continuous with respect to $\hat{\sigma}$ and represent $\hat{\varphi}$ on the algebra $H^\infty(\sigma)$. Let λ_0 be any weak-star cluster point of $\{\hat{g}_{2k}\hat{\sigma}\}$ and let λ_1 be any weak-star cluster point of $\{\hat{g}_{2k+1}\hat{\sigma}\}$. If we regard χ_W as a continuous function on $\Sigma(\sigma)$, the estimates obtained above imply that

$$\int \chi_W \, d\lambda_0 \geqq \eta(E)-2\varepsilon,$$

$$\int \chi_W d\lambda_1 \leqq 3\varepsilon.$$

In particular, $\lambda_0 \neq \lambda_1$ if $\varepsilon < \eta(E)/5$.

On the other hand, the measure $\lambda_0-\lambda_1$ belongs to the band $\mathfrak{B}_0$ generated by the representing measures for $\hat{\varphi}$ and is orthogonal to $H^\infty(\sigma)$, since λ_1 and λ_0 both represent $\hat{\varphi}$. Moreover, if $h \in C(\partial D)$ then $\int h\, d(\hat{g}_k\hat{\sigma}) = \int h\, d(g_k\sigma)$ since $\hat{g}_k\hat{\sigma}$ is the lift of $g_k\sigma$. Since $g_k\sigma \to \eta$ weak-star we conclude that $\int h\, d\lambda_0 = \int h\, d\lambda_1$ for each $h \in C(X)$; i.e., $\lambda_0-\lambda_1$ is orthogonal to $H^\infty(\sigma)+C(X)$. In particular, $\lambda_0-\lambda_1$ belongs to the band $\mathfrak{B}$ as well as to the band $\mathfrak{B}_0$. Since $\mathfrak{B}$ and $\mathfrak{B}_0$ are singular, we conclude that $\lambda_0-\lambda_1=0$. This contradiction shows that the space S is indeed weak-star closed, as desired.

We can now show that $\hat{\sigma}$ is a dominant representing measure. Let ν be any representing measure for $\hat{\varphi}$ on $\Sigma(\sigma)$. By the dual version of the Hoffman—Rossi Theorem there is a net $\{\nu_\beta\}$ of representing measures absolutely continuous with respect to $\hat{\sigma}$ which converges weak-star to ν. Since the measures ν_β are absolutely continuous with respect to $\hat{\sigma}$, each of them is the lift to $\Sigma(\sigma)$ of a measure $f_\beta\sigma$ on X which is absolutely continuous with respect to σ and represents φ for the algebra A. Choose any weak-star cluster point of the net $\{f_\beta\sigma\}$; since the space S is closed, this cluster point is a representing measure for φ of the form $f\sigma$. If we lift $f\sigma$ to $\Sigma(\sigma)$ to obtain the measure $\hat{f}\hat{\sigma}$, we see as before that $\nu-\hat{f}\hat{\sigma}$ belongs to the band $\mathfrak{B}_0$ and to $H^\infty(\sigma)^\perp$, since it is the difference of representing measures. Since ν_β and $f_\beta\sigma$ agree on $C(X)$, while $f\sigma$ and $\hat{f}\hat{\sigma}$ also agree on $C(X)$, it follows from weak-star convergence that $\nu-\hat{f}\hat{\sigma}$ is also orthogonal to $C(X)$, and thus belongs to the band $\mathfrak{B}$. Since $\mathfrak{B}_0$ and $\mathfrak{B}$ are singular, we conclude that $\nu=\hat{f}\hat{\sigma}$; i.e., $\hat{\sigma}$ is a dominant representing measure. This completes the proof. □

The hypothesis that X be metrizable, in Theorem 4, is necessary. A counterexample to the conclusions can be obtained by taking A to be itself an algebra of the form $H^\infty(\mu)$, for μ the harmonic measure on the boundary ∂D of an appropriate domain D in the complex domain, a "road runner" domain. One can choose D so that $H^\infty(\mu)$ separates the points of $\Sigma(\sigma)$, and so that $\hat{\mu}$ is not dominant. Then set $A=H^\infty(\mu)$, $X=\Sigma(\mu)$ and $\sigma=\hat{\mu}$. Since A is already weak-star closed, we obtain

$H^\infty(\sigma) \cong A$, and consequently $\Sigma(\sigma) \cong X$, $L^\infty(\sigma) \cong C(X)$, and $\hat{\sigma} = \hat{\mu}$. Since $\hat{\sigma}$ is not dominant, neither is $\hat{\sigma}$. On the other hand, $[H^\infty(\sigma)+C(X)]^\perp = \{0\}$, so that the zero operator is a band projection of $H^\infty(\sigma)^\perp$ onto $[H^\infty(\sigma)+C(X)]^\perp$, and $[H^\infty(\sigma)+C(X)]/H^\infty(\sigma)$ is an M-ideal in $L^\infty(\sigma)/H^\infty(\sigma)$.

Now we wish to obtain some sufficient conditions for $[H^\infty(\sigma)+C(X)]/H^\infty(\sigma)$ to be an M-ideal. We do not know whether this occurs just as soon as σ is dominant. However, under some rather severe hypotheses, modelled on the algebra $A(D)$, we can obtain a converse to Theorem 4.

Theorem 5. *Let A be a uniform algebra on a compact space X, let $\varphi \in M_A$, and let σ be a representing measure on X for φ. Assume that $\hat{\sigma}$ is a dominant representing measure for $\hat{\varphi}$ on $H^\infty(\sigma)$. Assume furthermore that there is a function $g \in A$ such that gA coincides with the ideal of functions in A vanishing at φ, and such that the linear span of A and the functions $1/g^m$, $m \geqq 1$, is dense in $C(X)$. Let P be the band projection of $\mathfrak{M}(\Sigma(\sigma))$ onto the band of measures singular to $\hat{\sigma}$. Then P maps $H^\infty(\sigma)^\perp$ onto $[H^\infty(\sigma)+C(X)]^\perp$. In particular, $[H^\infty(\sigma)+C(X)]/H^\infty(\sigma)$ is an M-ideal in $L^\infty(\sigma)/H^\infty(\sigma)$.*

Proof. Since $\hat{\sigma}$ is dominant, the F. and M. Riesz theorem shows that the measures singular to $\hat{\sigma}$ form a reducing band. Consequently P maps $H^\infty(\sigma)^\perp$ into $H^\infty(\sigma)^\perp$.

Denote the kernel of $\hat{\varphi}$ on $H^\infty(\sigma)$ by $H_0^\infty(\sigma)$. Since gA coincides with the kernel of φ, we see upon passing to weak-star closures that $gH^\infty(\sigma) = H_0^\infty(\sigma)$. Thus if $\nu \perp H^\infty(\sigma)$, then $\frac{1}{g}\nu \perp H_0^\infty(\sigma)$.

Suppose $\nu \in H^\infty(\sigma)^\perp$ is singular to $\hat{\sigma}$. Then $\frac{1}{g}\nu \in H_0^\infty(\sigma)^\perp$ is also singular to $\hat{\sigma}$. By a corollary to the F. and M. Riesz Theorem [4, Corollary II.7.9], $\frac{1}{g}\nu \perp H_0^\infty(\sigma)$. Proceeding by induction, we obtain

$$\frac{\nu}{g^m} \perp H^\infty(\sigma), \quad m \geqq 1.$$

In particular, ν is orthogonal to the powers $1/g^m$ for $m \geqq 1$. Since these together with A span a dense subset of $C(X)$, $\nu \perp [H^\infty(\sigma)+C(X)]$. Thus

$$P(H^\infty(\sigma)^\perp) \subseteqq [H^\infty(\sigma)+C(X)]^\perp.$$

To prove the reverse inclusion, suppose that $\nu \perp H^\infty(\sigma)+C(X)$. Write $\nu = h\hat{\sigma} + P(\nu)$. By the F. and M. Riesz theorem, $P(\nu) \perp H^\infty(\sigma)$. According to what we have proved already, $P(\nu) \perp H^\infty(\sigma)+C(X)$. Hence $h\hat{\sigma} = \nu - P(\nu) \perp C(X)$. It follows that $h=0$ a.e. $(d\hat{\sigma})$, and $\nu = P(\nu)$ lies in the range of P. This completes the proof. □

Before turning to the algebra $A(D)$, we wish to consider the problem of capturing a version of the F. and M. Riesz theorem when $\hat{\sigma}$ is dominant. When is every measure in $A^{\perp}$ absolutely continuous with respect to σ? The type of problem we might run into is illustrated by the following example (which however does not satisfy the hypotheses of Theorem 5).

Let $\Delta=\{|z|<1\}$ be the open unit disc in the complex plane. Let E be a compact perfect subset of the unit circle $\partial\Delta$ of zero length, let B be some nontrivial uniform algebra on E, and let $x\in E$ be a peak point for B. Let A consist of all functions $f\in A(\Delta)$ such that $f|_E\in B$, and $f(x)=f(1/2)$. Let φ be the evaluation homomorphism at $0\in\Delta$, and let σ be normalized arc-length measure on $\partial\Delta$. By considering the difference of the point mass at x and the Poisson measure, one sees that there is no hope of obtaining an F. and M. Riesz theorem for measures in $B^{\perp}$ with respect to σ. Yet since $H^{\infty}(\sigma)\cap C(\partial\Delta)$ is the usual disc algebra $A(\Delta)$, we do have an F. and M. Riesz theorem for measures orthogonal to $H^{\infty}(\sigma)\cap C(\partial\Delta)$.

In the case at hand, we prove the following F. and M. Riesz theorem for the "localized" version $H^{\infty}(\sigma)\cap C(X)$ of the algebra A.

Theorem 6. *Suppose that the hypotheses of Theorem 5 are met, and that furthermore $H^{\infty}(\sigma)+C(X)$ is a closed subspace of $L^{\infty}(\sigma)$. Then every measure on X orthogonal to $H^{\infty}(\sigma)\cap C(X)$ is absolutely continuous with respect to σ. Consequently, every representing measure on X for $\hat{\varphi}$ with respect to the algebra $H^{\infty}(\sigma)\cap C(X)$ is absolutely continuous with respect to σ.*

Proof. Suppose $\nu\perp H^{\infty}(\sigma)\cap C(X)$. Then there is a well-defined functional L on $H^{\infty}(\sigma)+C(X)$, such that

$$L(f+h)=\int h\,d\nu,\quad f\in H^{\infty}(\sigma),\quad h\in C(X).$$

Now the natural projection

$$H^{\infty}(\sigma)+C(X)\to C(X)/[H^{\infty}(\sigma)\cap C(X)]$$

is a closed operator, hence continuous. Since the functional $g+(H^{\infty}(\sigma)\cap C(X))\to\int g\,d\nu$ is evidently well-defined and continuous on $C(X)/(H^{\infty}(\sigma)\cap C(X))$, and since L is obtained by composing this functional with the projection above, L is continuous. Hence there is a measure α on $\Sigma(\sigma)$ such that

$$L(F)=\int F\,d\alpha,\quad F\in H^{\infty}(\sigma)+C(X).$$

Since L annihilates $H^{\infty}(\sigma)$, α is orthogonal to $H^{\infty}(\sigma)$. Let $\alpha=\alpha_s+\hat{h}\hat{\sigma}$ be the Lebesgue decomposition of α with respect to $\hat{\sigma}$, where $h\in L^1(\sigma)$. By Theorem 5, α_s is orthogonal to $H^{\infty}(\sigma)+C(X)$. Hence the measure $\hat{h}\hat{\sigma}$ also represents L. If $G\in C(X)$ we then have

$$\int G\,d\nu=L(G)=\int G\hat{h}\,d\hat{\sigma}=\int Gh\,d\sigma.$$

It follows that $\nu=h\sigma$, and in particular ν is absolutely continuous with respect to σ. $\square$

Now we specialize our results to algebras of analytic functions.

Theorem 7. *Let D be a bounded domain in the complex plane, and let μ be harmonic measure on ∂D for some point $p\in D$. Suppose that the closed support of μ coincides with ∂D. Then the following are equivalent:*

(i) $[H^\infty(\mu)+C(\partial D)]/H^\infty(\mu)$ *is an M-ideal in* $L^\infty(\mu)/H^\infty(\mu)$,

(ii) *there is a band projection mapping $H^\infty(\mu)^\perp$ onto* $[H^\infty(\mu)+C(\partial D)]^\perp$,

(iii) *$\hat{\mu}$ is a dominant representing measure for the evaluation homomorphism at p of the algebra $H^\infty(\mu)$,*

(iv) *the set of representing measures on ∂D for p is a weakly compact set of measures.*

Proof. The equivalence of (i) and (ii) follows from Theorem 2. That (i) implies (iii) follows immediately from Theorem 4.

Suppose that (iii) is true. It is a very useful fact from rational approximation theory that the linear span of $A(D)$ and the powers $1/(z-p)^m$, $m\geqq 1$, is dense in $C(\partial D)$. (To see this, observe first from Runge's theorem that this linear span is dense in $R(\partial D)$. Since each point of ∂D lies in the closure of a single component of the complement of ∂D, each point is a peak point for $R(\partial D)$, and hence by Bishop's theorem, $R(\partial D)=C(\partial D)$.) Furthermore, $(z-p)A(D)$ coincides with the ideal of functions in $A(D)$ which vanish at p. Hence the hypotheses of Theorem 5 are verified, with $g(z)=z-p$. We conclude from Theorem 5 that (i) and (ii) are true.

If (iv) is true, then in particular the representing measures which are absolutely continuous with respect to μ form a weakly compact set. By the dual version of the Hoffman—Rossi theorem, (iii) is valid. To complete the proof, it suffices then to show that the equivalent conditions (i), (ii), and (iii) imply (iv). This can be done in several ways.

One method, following [5], is to invoke a theorem of Gamelin and Garnett [6, Corollary 7.4 and Theorem 8.1], to the effect that the representing measures for p which are absolutely continuous with respect to μ are weak-star dense in the set of all representing measures for p which are carried by the closed support of μ. Suppose (iii) is true. By the dual version of the Hoffman—Rossi theorem, the representing measures absolutely continuous with respect to μ form a weakly compact set, hence a weak-star closed set. By the theorem just cited, all representing measures for p are absolutely continuous with respect to μ. Hence the set of all representing measures for p is weakly compact, and (iv) is true.

Another way to show that (iii) implies (iv) is to appeal to Theorem 6 and the distance estimate

$$\text{distance}\,(h, A(D)) = \text{distance}\,(h, H^\infty(\mu)), \quad h \in C(\partial D),$$

[6, Theorems 6.3 and 8.1]. This distance estimate shows that $C(\partial D) \cap H^\infty(\mu)$ coincides with the restriction of $A(D)$ to ∂D. It also shows that $H^\infty(\mu) + C(\partial D)$ is closed. Thus, assuming (iii), the hypotheses of Theorem 6 are met, and we obtain that every representing measure for p on ∂D is absolutely continuous with respect to μ. This fact, combined with the Hoffman—Rossi theorem, establishes (iv). □

One consequence of our analysis is that, under the equivalent hypotheses of Theorem 7, all representing measures for p are absolutely continuous with respect to harmonic measure μ. It is proved in [5] that if the representing measures for p form a weakly compact set, then all representing measures for p are *mutually* absolutely continuous.

We refer the reader also to [5] for a discussion of which domains D have weakly compact sets of representing measures. The situation for road runner domains is easiest to describe. These are domains obtained from the punctured unit disc $\Delta \setminus \{0\}$ by excising a sequence of closed, pairwise-disjoint subdiscs which cluster only at 0. In this case the set of representing measures on ∂D for $p \in D$ is weakly compact if and only if 0 is a peak point for $A(D)$. The situation for champagne bubble domains is much more complicated. See [5] for various constructions.

4. Best approximation

Having obtained necessary and sufficient conditions that $(H^\infty(\mu) + C(\partial D))/H^\infty(\mu)$ be an M-ideal in $L^\infty(\mu)/H^\infty(\mu)$, we want to close this circle of ideas by obtaining best approximation results. The fact that best approximation is possible from M-ideals was proved by Alfsen and Effros [1, Corollary 5.6]; since the proof they give is rather complicated (it is embedded in a much larger and more powerful argument), it seems worthwhile to give an independent, elementary proof. (In fact, we get the full strength of their Corollary 5.6.)

Recall that Alfsen and Effros define a subspace F to have the *n-ball property* in a Banach space E if whenever n open balls, each of which meets F, have a nonempty intersection, then that intersection meets F. The paper [1] is devoted to proving that for each $n \geqq 3$, the n-ball property characterizes M-ideals in E. We aim to show that a weak form of the two-ball property already guarantees the existence of best approximants. Then we will show by a short direct proof that M-ideals have the property, hence they have best approximants.

We say that a subspace F of a Banach space E has the *weak two-ball property in E* if there is a positive, continuous, increasing function $\varphi(\varrho)$, $\varrho>0$, for which

$$\int_0^1 \frac{\varphi(\varrho)}{\varrho}\, d\varrho < \infty,$$

and such that for each $0<\varrho<1$ and each $x\in E$ satisfying

$$\|x\| < 1+\varrho, \text{ distance}\,(x, F) < 1,$$

there is a $y\in F$ satisfying

$$\|y\| < \varphi(\varrho), \quad \|x-y\| < 1+\varrho/2.$$

The purpose of the integrability condition on φ is to assure that

$$\sum_{j=1}^{\infty} \varphi(x^j \varrho) < \infty$$

for any $0<\varrho<1$ and $0<c<1$.

The two-ball property trivially implies the weak two-ball property, with $\varphi(\varrho)=\varrho/2$. Indeed, suppose $x\in E$ satisfies $\|x\|<1+\varrho$ and distance $(x, F)<1$. Then the open balls $B(0;\ \varrho/2)$ and $B(x;\ 1+\varrho/2)$ meet each other, and each of them meets F. By the two-ball property, there exists $y\in F\cap B(0;\ \varrho/2)\cap B(x;\ 1+\varrho/2)$, and this y does the trick, with $\varphi(\varrho)=\varrho/2$.

It is fairly straightforward to check that any closed subspace of a Hilbert space has the weak two-ball property, with $\varphi(\varrho)=2\sqrt{\varrho}$. Furthermore, no larger power of ϱ will serve for Hilbert space.

An iteration argument, using the two-ball property, allows us to obtain in short order the existence of best approximants.

Theorem 8. *If F has the weak two-ball property in E, then every element of E has a best approximant in F.*

Proof. Let $z\in E$, and fix $0<\varrho<1$. We may assume without loss of generality that distance $(z, F)=1$, and furthermore that $\|z\|<1+\varrho$. By the two-ball property, applied to $x=(1+\varrho/2)z/(1+3\varrho/4)$, there exists $y\in F$ with $\|y\|<\varphi(\varrho)$ and $\|x-y\|<1+\varrho/2$. Then $w_1=(1+3\varrho/4)y/(1+\varrho/2)$ satisfies

$$\|w_1\| \leqq 2\|y\| < 2\varphi(\varrho),$$

$$\|z-w_1\| < 1+3\varrho/4.$$

Repeating this procedure, with ϱ replaced by $3\varrho/4$ and z replaced by $z-w_1$, we obtain $w_2\in F$ such that

$$\|w_2\| < 2\varphi(3\varrho/4),$$

$$\|z-w_1-w_2\| < 1+\left(\frac{3}{4}\right)^2 \varrho.$$

Proceeding in this fashion, we obtain a sequence $\{w_j\}$ in F such that

$$\|w_m\| < 2\varphi\left(\left(\frac{3}{4}\right)^{m-1} \varrho\right),$$

$$\|z-w_1-\ldots-w_m\| < 1+\left(\frac{3}{4}\right)^m \varrho.$$

The estimate for $\|w_m\|$ shows that the series Σw_j converges to $w\in F$, and evidently $\|z-w\|=1=\text{distance}\,(z, F)$. □

We remark that the above proof shows that there is continuity of best approximants, in the sense that if $z\in E$ satisfies distance $(z, F)=1$ and $\|z\|<1+\varrho$, there exists $w\in F$ such that $\|z-w\|=1$ and $\|w\|<\Psi(\varrho)$, where $\Psi(\varrho)\to 0$ as $\varrho\to 0$. In fact, we may take

$$\Psi(\varrho) = 8\int_0^{\varrho} \frac{\varphi(t)}{t}\,dt.$$

Using the separation theorem for convex sets, we may now show that M-ideals have the weak two-ball property and hence the best approximation property.

Proposition 9. *If F is an M-ideal in E then it has the weak two-ball property, and hence every element of E has a best approximant in F.*

Proof. We are going to verify the weak two-ball property with the function $\varphi(\varrho)\equiv\varrho$. Let $x\in E$, with $\|x\|<1+\varrho<2$, $x\notin F$, and distance $(x, F)<1$. Let B be the open unit ball of E. We claim that there is an element z in the intersection of the two convex sets $x+\varrho(B\cap F)$ and $(1+\varrho/2)B$. If this is so, we may set $y=x-z$, so that $y\in F$ and $\|y\|<\varrho$ (because z is in $x+\varrho(B\cap F)$) while $\|x-y\|=\|z\|<1+\varrho/2$ (because z is in $(1+\varrho/2)B$), which is what we desire.

It remains to prove the claim. If it were not so, the separation theorem would enable us to find a nonzero continuous linear functional Ψ in E^*, the dual space of E, such that

$$\sup_{u\in(1+\varrho/2)B} \operatorname{Re}\Psi(u) \leqq \inf_{v\in B\cap F} \operatorname{Re}\Psi(x+\varrho v)$$

thus

$$(1+\varrho/2)\|\Psi\| \leqq |\Psi(x)|-\varrho \sup_{v\in B\cap F} \operatorname{Re}\Psi(v)$$
$$= |\Psi(x)|-\varrho\|\Psi\|_{F^*}.$$

Since F is an M-ideal in E, there is an L-projection P of E^* onto $F^\perp$. From the Hahn—Banach theorem and the defining property of L-projections, we have

$$\|\Psi\|_{F^*} = \inf\{\|\Psi-\theta\|:\ \theta\in F^\perp\}$$
$$= \inf\{\|P\Psi-\theta\|+\|\Psi-P\Psi\|:\ \theta\in F^\perp\} = \|\Psi-P\Psi\|.$$

Thus using $P\Psi\perp F$ below, we obtain

$$(1+\varrho/2)\|\Psi\| \leqq |\Psi(x)|-\varrho\|\Psi-P\Psi\|$$
$$\leqq |\Psi(x)-P\Psi(x)|+|P\Psi(x)|-\varrho\|\Psi-P\Psi\|$$
$$\leqq \|\Psi-P\Psi\|\,\|x\|+\|P\Psi\|\operatorname{distance}(x,F)-\varrho\|\Psi-P\Psi\|$$
$$\leqq \|\Psi-P\Psi\|(1+\varrho)+\|P\Psi\|-\varrho\|\Psi-P\Psi\| = \|\Psi\|.$$

This contradiction establishes the proposition.

As another application of the weak two-ball property, we mention the following, which is related to the Hilbert space example mentioned earlier.

Proposition 10. *Suppose H is a closed self-adjoint subalgebra of $C(X)$, where X is compact Hausdorff. Then H has the weak two-ball property, with $\varphi(\varrho)=2\sqrt{\varrho}$, and hence every element of $C(X)$ has a best approximant in H.*

Sketch of proof. Fix $0<\varrho<1$, and suppose $f\in C(X)$ satisfies $\|f\|<1+\varrho$, and $\|f-g\|<1$ for some $g\in H$. Define

$$h(x) = \begin{cases} g(x), & |g(x)| \leqq \sqrt{3\varrho}, \\ \sqrt{3\varrho}\,g(x)/|g(x)|, & |g(x)| > \sqrt{3\varrho}. \end{cases}$$

Then $h\in H$ satisfies $\|h\|<\varphi(\varrho)$. Some elementary geometry in the plane shows that if $|a|\leqq 1+\varrho$ and if $|b|\geqq\sqrt{3\varrho}$, $|b-a|<1$, then the radial projection of b onto the circle $\{|\zeta|=\sqrt{3\varrho}\}$ lies inside the circle $\{|\zeta-a|<1\}$. Applying this fact to $a=f(x)$ and $b=g(x)$, we obtain $|h(x)-f(x)|<1$, which is a stronger estimate than that required for the weak two-ball property. □

Finally, we combine the above results with weak-star compactness to obtain best approximation from $H^\infty(\mu)+C(\partial D)$.

Corollary 11. *Let D be a bounded domain in the complex plane, and let μ be harmonic measure on ∂D for a point $p\in D$. Suppose that the closed support of μ coincides with ∂D, and that $\hat{\mu}$ is a dominant representing measure. Then every function in $L^\infty(\mu)$ has a best approximant in $H^\infty(\mu)+C(\partial D)$.*

Proof. By Theorem 7, $(H^\infty(\mu)+C(\partial D))/H^\infty(\mu)$ is an M-ideal in $L^\infty(\mu)/H^\infty(\mu)$. Thus, given $f\in L^\infty(\mu)$ we can find (by Theorem 8 and Proposition 9) a function $g\in H^\infty(\mu)+C(\partial D)$ such that distance $(f-g, H^\infty(\mu))=$ distance $(f, H^\infty(\mu)+C(\partial D))$. Since norm-closed balls in $H^\infty(\mu)$ are weak-star compact, there is an $h\in H^\infty(\mu)$ such that

$$\|f-g-h\| = \text{distance } (f-g, H^\infty(\mu))$$

$$= \text{distance } (f, H^\infty(\mu)+C(\partial D));$$

i.e., $g+h$ is a best approximant to f from $H^\infty(\mu)+C(\partial D)$. $\square$

In closing we remark that the question of best approximation from other subalgebras of $L^\infty(\mu)$ is not well-understood, although Sundberg [12] has shown that there are subalgebras between H^∞ and L^∞ which do not have the best approximation property.

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Received March 29, 1984

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