# Phragmén—Lindelöf's and Lindelöf's theorems

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## 1. Introduction

Both Phragmén—Lindelöf's and Lindelöf's theorems consider behavior of a function at a boundary point and, originally, cf. [PL], [L], the proofs for these theorems employ harmonic measure. In local properties the measure theoretic aspect of harmonic measure plays a minor role and in this paper we show that even for non-linear partial differential equations and quasiregular mappings it is possible to prove corresponding results using so called $F$-harmonic measure, which is intimately connected with the corresponding differential equation or variational integral, cf. [GLM2].

We shall study the conformally invariant case, i.e. we consider extremals of the variational integral

$$\int F(x, \nabla u)\, dm,$$

where $F(x, h) \approx |h|^n$ and $n$ is the dimension of the Euclidean space $\mathbf{R}^n$. Thus the plane harmonic case is included but the classical harmonic case in space $\mathbf{R}^n$, $n \geqq 3$, is not. In general, our methods only work in the "borderline" case $F(x, h) \approx |h|^n$.

The proofs for Phragmén—Lindelöf's theorem in domains more general than sectors usually combine the method invented by T. Carleman [C], cf. also [T, Theorem III. 67], with a principle which we call Phragmén—Lindelöf's principle. This is a slight misuse of the name, cf. e.g. [A, p. 40]. The principle, Theorem 3.5, relates in classical terms the growth of a harmonic function with the density of a harmonic measure at $\infty$. The density concept extends to the non-linear case and hence the principle holds in the more general situation even in a sharp form. Carleman's method is based on the study of the Carleman mean

$$\int_{G \cap S^1(t)} u^2\, ds$$

of a suitably chosen harmonic measure $u$. For a good account of the development in the field see [Ha]. Via Wirtinger's inequality [T, p. 112] the Carleman mean can be related to the gradient of $u$ and the estimate of the harmonic density at $\infty$ then makes ingenious use of differential inequalities. Our method is based on a direct study of $\max_{G\cap S^1(t)} u$ which, via a refinement of F. W. Gehring's oscillation lemma [G], is compared with the gradient of $u$. The linear structure of solutions is not employed and no differential inequalities are needed. Instead we make use of an inequality similar to the so called standard estimate, well-known in the theory of non-linear partial differential equations. Thus our approach considerably simplifies Carleman's method even in the plane harmonic situation. However, we do not obtain the best possible constants.

The second part of the paper deals with Lindelöf's theorem which in the classical form states that if a bounded analytic function $f\colon B^2\to\mathbf{C}$ has an asymptotic limit at a boundary point $z_0\in\partial B^2$, then it has the same limit in each Stolz angle at $z_0$. For quasiregular mappings in higher dimensional Euclidean spaces this theorem does not hold as shown by S. Rickman [R]. However, we shall show that the corresponding result in all dimensions can be formulated by means of a principle which we call Lindelöf's principle. This principle again employs the density of the $F$-harmonic measure and it rests on the sub-$F$-extremality of $\log|f|$ for a quasiregular mapping $f$ and for a suitable kernel $F$. The principle can be formulated in any domain without any restrictions on the set along which $f$ has a limit. It is also best possible for plane analytic or quasiregular mappings. As a consequence of this principle we prove Lindelöf's theorem, Theorem 4.27, in all dimensions.

For the proof of Lindelöf's classical theorem see [N, p. 44] or [A, p. 40] and for the theory of quasiregular mappings we refer to [MRV] and [GLM1].

The paper has been organized as follows. Non-linear variational integrals and the $F$-harmonic measure are considered in Chapter 2. Chapter 3 deals with Phragmén—Lindelöf's theorem and Chapter 4 is devoted to Lindelöf's theorem. We have also included basic facts about quasiregular mappings in Chapter 4.

## 2. $F$-harmonic measure

**2.1. Variational integrals and extremals.** Suppose that $G$ is a domain in $\mathbf{R}^n$ and let $F\colon G\times\mathbf{R}^n\to\mathbf{R}$ be a variational kernel satisfying the assumptions:

(a) For each $\varepsilon>0$ there is a closed set $C$ in $G$ such that $m(G\setminus C)<\varepsilon$ and $F|C\times\mathbf{R}^n$ is continuous.

(b) For a.a. $x\in G$ the function $h\mapsto F(x,h)$ is strictly convex and differentiable in $\mathbf{R}^n$.

(c) There are $0<\alpha\leqq\beta<\infty$ such that for a.a. $x\in G$

$$\alpha|h|^n\leqq F(x,h)\leqq\beta|h|^n,\quad h\in\mathbf{R}^n.$$

(d) For a.a. $x\in G$

$$F(x, \lambda h) = |\lambda|^n F(x, h), \quad \lambda\in\mathbf{R}, h\in\mathbf{R}^n.$$

An example of a kernel satisfying (a)—(d) is $F(x, h)=|h|^n$.

Let $W_n^1(G)$ denote the Sobolev-space of functions in $L^n(G)$ whose distributional first partial derivatives belong to $L^n(G)$. The corresponding local space is denoted by $\operatorname{loc} W_n^1(G)$. A function $u\in C(G)\cap \operatorname{loc} W_n^1(G)$ is called an $F$-extremal if for all domains $D\subset\subset G$

$$I_F(u, D) = \inf_{v\in\mathcal{F}_u} I_F(v, D),$$

where

$$I_F(v, D) = \int_D F(x, \nabla v(x))\, dm(x)$$

is the variational integral generated by $F$ and

$$\mathcal{F}_u = \{v\in C(\bar{D})\cap W_n^1(D)\colon\ v = u \text{ in } \partial D\}.$$

A function $u$ is an $F$-extremal if and only if $u\in C(G)\cap \operatorname{loc} W_n^1(G)$ is a solution of the Euler equation

$$\nabla\cdot\nabla_h F(x, \nabla u) = 0 \tag{2.2}$$

in the weak sense, i.e.

$$\int_G \nabla_h F(x, \nabla u)\cdot\nabla\varphi\, dm = 0$$

for all $\varphi\in C_0^\infty(G)$, cf. [GLM1].

For later reference we recall some basic properties of $F$-extremals. The form of $F$ and (d) imply that $u+\lambda$ and $\lambda u$ are $F$-extremals whenever $u$ is an $F$-extremal and $\lambda\in\mathbf{R}$. Each $F$-extremal is locally Hölder-continuous, more precisely,

$$\operatorname{osc}\big(u, B^n(x_0, r)\big) \leqq c(r/R)^\varkappa \operatorname{osc}\big(u, B^n(x_0, R)\big), \tag{2.3}$$

where $0<r\leqq R$, $B^n(x_0, R)\subset G$, $\varkappa$ depends only on $n$ and $\beta/\alpha$, $c$ is an absolute constant and

$$\operatorname{osc}(u, A) = \sup_A u - \inf_A u$$

denotes the oscillation of $u$ on $A$. If $u$ is a non-negative $F$-extremal, then $u$ satisfies Harnack's inequality

$$\sup u \leqq c_0 \inf u \tag{2.4}$$

in $B^n(x_0, r)$ where $c_0$ is of the form

$$c_0 = \exp\big(c'(\log R/r)^{-1}\big),$$

$0<r\leqq R$, $B^n(x_0, R)\subset G$ and $c'$ depends only on $n$ and $\beta/\alpha$. $F$-extremals satisfy Harnack's principle, i.e. if $u_i\colon G\to\mathbf{R}$ is an increasing sequence of $F$-extremals in $G$, then

$$u = \lim u_i$$

is either $=+\infty$ in $G$ or $u$ is an $F$-extremal in $G$. Finally, there are plenty of Dirichlet-regular sets. Especially if $G$ is a bounded regular domain, i.e. $\partial G$ is without point components, then for each $f\in C(\partial G)$ there exists a unique $u\in C(\overline{G})\cap \operatorname{loc} W_n^1(G)$ which is $F$-extremal in $G$ and $u=f$ in $\partial G$. If $f\in C(\overline{G})\cap W_n^1(G)$, then also $u\in C(\overline{G})\cap W_n^1(G)$.

For simple proofs of the above facts see [GLM1].

**2.5. Sub-$F$-extremals.** An upper semi-continuous function $u\colon G\to \mathbf{R}\cup\{-\infty\}$ is called a sub-$F$-extremal if $u$ satisfies the $F$-comparison principle in $G$, i.e. if $D\subset\subset G$ is a domain and $h\in C(\overline{D})$ is an $F$-extremal in $D$, then $h\geqq u$ in $\partial D$ implies $h\geqq u$ in $D$. The PWB-method applies to sub-$F$-extremals, cf. [GLM3]. Let $G$ be bounded and let $f\colon \partial G\to \mathbf{R}\cup\{-\infty,\infty\}$ be any function. The family

$$\mathscr{L}_f=\Big\{u\colon G\to \mathbf{R}\cup\{-\infty\}\colon \varlimsup_{x\to y} u(x)\leqq f(y),\ y\in\partial G,\ u \text{ is a sub-}F\text{-extremal and bounded from above in } G\Big\}$$

is called the lower Perron class associated with $f$. Note that $\mathscr{L}_f\neq\emptyset$, since $u\equiv-\infty$ belongs to $\mathscr{L}_f$. The function $\underline{H}_f=\sup\{u\colon u\in\mathscr{L}_f\}$ satisfies one of the following conditions:

(i) $\underline{H}_f$ is an $F$-extremal in $G$,

(ii) $\underline{H}_f(x)=\infty$ for all $x\in G$,

(iii) $\underline{H}_f(x)=-\infty$ for all $x\in G$.

If $m\leqq f\leqq M$, then also $m\leqq \underline{H}_f\leqq M$ and hence only (i) is possible.

A function $u\colon G\to\mathbf{R}\cup\{\infty\}$, $G$ domain in $\mathbf{R}^n$, is called a super-$F$-extremal if $-u$ is a sub-$F$-extremal. In a similar way we define the upper class $\mathscr{U}_f$ of $f\colon \partial G\to\mathbf{R}\cup\{-\infty,\infty\}$

$$\mathscr{U}_f=\Big\{u\colon G\to \mathbf{R}\cup\{\infty\}\colon \varliminf_{x\to y} u(x)\geqq f(y),\ y\in\partial G,\ u \text{ is a super-}F\text{-extremal and bounded from below in } G\Big\}$$

on a bounded domain $G$ and set $\overline{H}_f=\inf\{u\colon u\in\mathscr{U}_f\}$. The function $\overline{H}_f$ also satisfies one of the conditions (i)—(iii).

The use of sub-$F$-extremals and super-$F$-extremals is based on the following $F$-comparison principle, cf. [GLM3, Lemma 2]. Suppose that $G$ is a bounded domain and that $u$ is a sub-$F$-extremal and $v$ a super-$F$-extremal in $G$. If

$$\varlimsup_{x\to y} u(x)\leqq \varliminf_{x\to y} v(x) \tag{2.6}$$

for all $y\in\partial G$ and if the left and right hand sides of (2.5) are neither $\infty$ nor $-\infty$ at the same time, then $u\leqq v$ in $G$.

Finally we note that the above concepts can be used in an arbitrary open set $G$ of $\mathbf{R}^n$ or in a bounded open set $G$ of $\mathbf{R}^n$, if necessary. The corresponding properties then hold in each component of $G$.

**2.7.** ***F*-harmonic measure.** Suppose that $G$ is a bounded open set and that $C$ is a subset of $\partial G$. Let $f$ be the characteristic function of $C$. The $F$-extremal $\bar{H}_f$ in $G$ is called the $F$-harmonic measure of $C$ with respect to $G$ and denoted by $\omega(C, G; F)$. The upper Perron class $\mathcal{U}_f$ associated with $f$ is written as $\mathcal{U}(C, G; F)$. Clearly

$$0 \leqq \omega(C, G; F) \leqq 1$$

and if $C' \subset C$, then $\omega(C', G; F) \leqq \omega(C, G; F)$. For $n=2$ and $F(x, h) = |h|^2$, $\omega(C, G; F)$ is the classical outer harmonic measure of $C$ with respect to $G$.

In order to form the $F$-harmonic measure $\omega(C, G; F)$ the set $C$ should be a subset of $\partial G$. However, in Chapter 4 the following extension of the definition will turn useful. Suppose that $G$ is a bounded open set. A set $C$ in $\mathbf{R}^n$ is called *G-admissible*, if $G \cap C$ is closed in $G$. If a kernel $F$ is defined on $G$, then the $F$-harmonic measure $\omega(C \cap \partial(G \setminus C), G \setminus C; F)$ is defined for all $G$-admissible sets $C$ in the open set $G \setminus C$ and we denote it simply by $\omega(C, G \setminus C; F)$, although $C$ need not be a set on $\partial(G \setminus C)$. The same notation $\mathcal{U}(C, G \setminus C; F)$ is also used for the corresponding upper class. Note that any set $C$ in $\mathbf{R}^n \setminus G$ is $G$-admissible and that any closed set $C$ in $\mathbf{R}^n$ is $G$-admissible for all bounded open sets $G$.

The following basic principles will be employed in Chapters 3 and 4.

**2.8. Lemma.** *(Carleman's principle) Suppose that a set $C$ is both $G_1$- and $G_2$-admissible, that $G_1 \subset G_2$ and that the kernel $F$ is defined on $G_2$. Then*

$$\omega(C, G_1 \setminus C; F) \leqq \omega(C, G_2 \setminus C; F)$$

*in $G_1 \setminus C$.*

*Proof.* If $\varphi$ belongs to $\mathcal{U}(C, G_2 \setminus C; F)$, then $\varphi | G_1 \setminus C$ belongs to $\mathcal{U}(C, G_1 \setminus C; F)$ and the lemma follows.

**2.9. Lemma.** *Suppose that the sets $C_1$ and $C_2$ are both G-admissible with $C_1 \subset C_2$. Then*

$$\omega(C_1, G \setminus C_1; F) \leqq \omega(C_2, G \setminus C_2; F)$$

*in $G \setminus C_2$.*

*Proof.* Let $\varphi$ belong to the upper class $\mathcal{U}(C_2, G \setminus C_2; F)$ and let $\varepsilon > 0$. The function $v = \min(\varphi + \varepsilon, 1)$ in $G \setminus C_2$ and $v = 1$ in $G \cap (C_2 \setminus C_1)$ belongs to $\mathcal{U}(C_1, G \setminus C_1; F)$. Hence

$$\omega(C_1, G \setminus C_1; F) \leqq v \leqq \varphi + \varepsilon$$

in $G \setminus C_2$ which proves the desired inequality.

If $C$ is a closed subset of the boundary $\partial G$ of a bounded open set $G$, then it is possible to define the $F$-harmonic measure $\omega(C, G; F)$ in a more practical way. Let $\varphi_i \in C(\partial G)$, $i=1, 2, \ldots$, be a sequence of non-negative functions such that $\varphi_i | C \geqq 1$, $\varphi_1 \geqq \varphi_2 \geqq \ldots$ and

$$\lim_{i\to\infty} \varphi_i(y) = \begin{cases} 1, & y\in C, \\ 0, & y\in \partial G \setminus C. \end{cases}$$

In [GLM3, Chapter 5] the following lemma was proved under the hypothesis that $G$ is a regular domain.

**2.10. Lemma.** $\lim_{i\to\infty} \overline{H}_{\varphi_i} = \omega(C, G; F)$.

*Proof.* For $i=1, 2, \ldots$ write $h_i = \overline{H}_{\varphi_i}$. Now $\varphi_i \geqq \varphi_{i+1}$ yields $h_i \geqq h_{i+1}$ and since $\varphi_i | C \geqq 1$, $h_i \geqq \overline{H}_f = \omega(C, G; F)$, where $f$ is the characteristic function of $C$. Hence

$$\lim_{i\to\infty} h_i \geqq \omega(C, G; F).$$

To prove the converse inequality let $\varepsilon > 0$. Pick $v \in \mathcal{U}_f$ and set

$$v^*(y) = \varliminf_{x\to y} v(x), \quad y\in\partial G.$$

Then $v^*\colon \partial G \to \mathbf{R} \cup \{\infty\}$ is lower semincontinuous and since $\partial G$ is compact and $\varphi_1 \geqq \varphi_2 \geqq \ldots$, there is an $i_\varepsilon$ such that for $i \geqq i_\varepsilon$

$$v^* + \varepsilon > \varphi_i$$

in $\partial G$. On the other hand $v + \varepsilon \in \mathcal{U}_{v^*+\varepsilon}$ and thus $v+\varepsilon \geqq \overline{H}_{v^*+\varepsilon}$ in $G$. Hence for $i \geqq i_\varepsilon$ we obtain

$$v+\varepsilon \geqq \overline{H}_{v^*+\varepsilon} \geqq h_i. \tag{2.11}$$

If we let $i\to\infty$ and then $\varepsilon\to 0$, (2.11) yields $v \geqq \lim_{i\to\infty} h_i$. Thus $\omega(C, G; F) \geqq \lim_{i\to\infty} h_i$ as desired.

If $G$ is a bounded regular open set, i.e. each component of $G$ is a bounded regular domain, then it is possible to give a variational interpretation of the sequence $\overline{H}_{\varphi_i}$ in Lemma 2.10. Let $\varphi_i$ be a $(C, G)$-boundary sequence, i.e. $\varphi_i \in C(\overline{G}) \cap W_n^1(G)$, $1 \geqq \varphi_1 \geqq \varphi_2 \geqq \ldots \geqq 0$, $\varphi_i = 1$ on $C$ and

$$\bigcap_i \operatorname{spt} \varphi_i = C,$$

cf. [GLM2, Chapter 2]. For $i=1, 2, \ldots$ let $u_i \in C(\overline{G}) \cap W_n^1(G)$ be the unique $F$-extremal in $G$ with boundary values $\varphi_i$. The sequence $u_i$ is called a generating sequence for $\omega(C, G; F)$. Now $\overline{H}_{\varphi_i} = u_i$ in $G$ and by Harnack's principle and Lemma 2.10, the sequence $u_i$ converges uniformly on compact subsets of $G$ to $\omega(C, G; F)$. Actually, a little more is true in this case. For the next lemma observe that a regular open set $G$

may have point components in its boundary. Especially, a boundary point $x_0 \in \partial G$ need not be a boundary point of any component of $G$.

**2.12. Lemma.** *Suppose that $G$ is a bounded regular open set in $\mathbf{R}^n$ and that $C \subset \partial G$ is closed. Then the sequence $u_i$ converges uniformly on compact subsets of $\bar{G} \setminus C$.*

*Proof.* It suffices to show that the family $\{u_i\}$ is equicontinuous at $x_0 \in \partial G \setminus C$. Choose a ball $B = B^n(x_0, r_0)$ such that $\bar{B} \cap C = \emptyset$ and $S^{n-1}(x_0, t) \cap \partial G \neq \emptyset$ for $0 < t \leqq r_0$. Set $u_i(x) = 0$ for $x \in B \setminus \bar{G}$. Then $u_i \in C(B) \cap W_n^1(B)$ for large $i$, say $i \geqq i_0$, and

$$\int_B |\nabla u_i|^n \, dm \leqq M < \infty$$

where $M$ is independent of $i$. This follows from the proof of [M, Lemma 2.8] since $0 \leqq u_i \leqq 1$ and $u_i = 0$ in $B \setminus G$.

Fix $i \geqq i_0$. Since $S^{n-1}(x_0, t) \cap \partial G \neq \emptyset$ for $0 < t \leqq r_0$ and $u_i$ is monotone in $G$, cf. [GLM1, 2.8],

$$\operatorname{osc}\left(u_i, S^{n-1}(x_0, t)\right) = \operatorname{osc}\left(u_i, B^n(x_0, t)\right), \quad 0 < t \leqq r_0.$$

Hence for $0 < r \leqq t \leqq r_0$

$$\operatorname{osc}\left(u_i, S^{n-1}(x_0, t)\right) \geqq \operatorname{osc}\left(u_i, S^{n-1}(x_0, r)\right)$$

and F. W. Gehring's oscillation lemma, see [GLM1, Lemma 2.7] or Lemma 3.2 below, yields for each $r$, $0 < r < r_0$,

$$\operatorname{osc}\left(u_i, B^n(x_0, r)\right)^n \log \frac{r_0}{r} = \operatorname{osc}\left(u_i, S^{n-1}(x_0, r)\right)^n \log \frac{r_0}{r}$$

$$\leqq \int_r^{r_0} \frac{\operatorname{osc}\left(u_i, S^{n-1}(x_0, t)\right)^n}{t} \, dt \leqq A_n \int_B |\nabla u_i|^n \, dm \leqq A_n M.$$

Thus

$$\operatorname{osc}\left(u_i, B^n(x_0, r)\right) \leqq (A_n M)^{1/n} \left(\log \frac{r_0}{r}\right)^{-1/n}$$

and since the right hand side is independent of $i$ and approaches 0 as $r \to 0$, the equicontinuity of $\{u_i\}$ at $x_0$ has been proved.

**2.13. Corollary.** *Let $G$ and $C$ be as in Lemma* 2.12. *Then* $\lim_{x \to y} \omega(C, G; F)(x) = 0$ *for all* $y \in \partial G \setminus C$.

**2.14. Remark.** If we set $\omega(C, G; F)(x) = 1$, $x \in C$, and $\omega(C, G; F)(x) = 0$, $x \in \partial G \setminus C$, then a slight modification of the above proof shows that the sequence $u_i$ converges uniformly in $\bar{G}$ to $\omega(C, G; F)$ if $\operatorname{dist}(\partial G \setminus C, C) > 0$ and if $C$ is a non-degenerate continuum. In fact, to prove the equicontinuity of $\{u_i\}$ at $x_0 \in C$, the proof of Lemma 2.12 can be applied to $1-u$. The above proof also implies that $u_i$ converges uniformly on compact subsets of $\bar{G}\setminus\partial_{\partial G}C$ to $\omega(C, G; F)$ provided that $C$ is a non-degenerate continuum. Here $\partial_{\partial G}C$ means the boundary of $C$ with respect to $\partial G$.

**2.15. Lemma.** *Suppose that $C$ is a closed and connected set in $\partial G$ and that $C$ contains at least two points. Then*

$$\lim_{x\to y}\omega(C, G; F)(x) = 1 \tag{2.16}$$

*for all interior points $y$ of $C$ with respect to $\partial G$.*

*Proof.* Let $y$ be an interior point of $C$ with respect to $\partial G$. Choose $r>0$ such that $\partial G\cap \bar{B}^n(y, r)\subset C$ and let $C'$ be the component of $\complement G$ which contains $C$. Write $G'=B^n(y, r)\setminus C'$. Then $G'\subset G$ is a regular open set and if $u_i$ is a generating sequence for $\omega(C, G'; F)$, then Remark 2.14 yields

$$\lim_{x\to y}\omega(C, G'; F)(x) = 1. \tag{2.17}$$

By Lemma 2.8

$$1 \geqq \omega(C, G; F) \geqq \omega(C, G'; F)$$

in $G'$ and (2.16) follows from (2.17).

### 3. Phragmén—Lindelöf's theorem

The classical version of Phragmén—Lindelöf's theorem [PL] considers a subharmonic function $u$ in the plane sector $|\arg z|<\theta/2\leqq\pi$. The theorem states that if $\overline{\lim}\, u\leqq 0$ on the boundary, then either $u\leqq 0$ in the whole sector or an asymptotic growth condition

$$M(r) = \max_{\varphi} u(re^{i\varphi}) \gtrsim r^{\pi/\theta}$$

holds as $r\to\infty$.

In an arbitrary unbounded plane domain $G$ the method of *T.* Carleman [C], see also [T, p. 112], can be used to prove an asymptotic growth condition

$$M(r) \gtrsim \exp\left(\pi\int_{1^*}^{r}\frac{dt}{t\theta(t)}\right),$$

where $\theta(t)$, $0\leqq\theta(t)\leqq 2\pi$, is the angle measure of $G$ on the sphere $S^1(t)$ and the star * indicates that the integration is extended only over those radii $t$ for which $\partial G\cap S^1(t)$ is non-empty.

Let $G$ be an unbounded domain in $\mathbf{R}^n$ and let $\theta(t)$ denote the angle measure of $G\cap S^{n-1}(t)$, i.e. $t^{n-1}\theta(t)$ is the $(n-1)$-area of $G\cap S^{n-1}(t)$. In this chapter we shall derive an asymptotic growth condition

$$M(r) \gtrsim \exp\left(c_n \sqrt[n]{\frac{\alpha}{\beta}} \int_{1^*}^{r} \frac{dt}{t\theta(t)^{1/(n-1)}}\right)$$

for a sub-$F$-extremal $u$ in $G$ with $\overline{\lim}\, u \leqq 0$ on the boundary. Here the constant $c_n > 0$ depends only on $n$. The proof considerably simplifies that of $M$. Tsuji. On the other hand our constant $c_n$ for $n=2$ is less than $\pi$.

**3.1. An oscillation lemma.** We start with a refinement of F. W. Gehring's lemma. If $A$ is a set in $\mathbf{R}^n$ and $u\colon A \to \mathbf{R} \cup \{-\infty, \infty\}$ is a function, we recall that $\operatorname{osc}(u, A)$ is the oscillation of $u$ on $A$. With minor modifications the proof of Lemma 3.2 follows from the proofs of [G, Lemma 1] or [Mo, Lemma 4.1]. A direct proof based on a slightly different reasoning is given in Yu. Rešetnjak's new book [Re, pp. 57—59].

**3.2. Lemma.** *Suppose that $K_r$ is an $(n-1)$-dimensional spherical cap on $S^{n-1}(r)$ with the $(n-1)$-area $r^{n-1}\theta$. Let $u$ be continuously differentiable on $K_r$. Then*

$$\operatorname{osc}(u, K_r)^n \leqq A_n r\theta^{1/(n-1)} \int_{K_r} |\nabla u|^n \, dS, \tag{3.3}$$

*where the constant $A_n < \infty$ depends only on $n$ and $S$ is the $(n-1)$-measure on $S^{n-1}(r)$.*

**3.4. Phragmén—Lindelöf's principle.** Suppose that $G$ is an unbounded domain in $\mathbf{R}^n$. Then each component of $G_r = G \cap B^n(r)$, $r > 0$, is open and we let $\omega(x; r)$ denote the value of the $F$-harmonic measure $\omega(S^{n-1}(r), G_r; F)$ at the point $x \in G$, $|x| < r$, see 2.7.

The following general principle easily follows from the construction of the $F$-harmonic measure, see [GLM2, Theorem 3.10]. We assume that the kernel $F$ satisfies (a)—(d) of 2.1 in $G$.

**3.5. Theorem.** *Suppose that $u\colon G \to R \cup \{-\infty\}$ is a sub-$F$-extremal with*

$$\overline{\lim_{x \to \xi}}\, u(x) \leqq 0 \quad \textit{for all} \quad \xi \in \partial G.$$

*Then, either $u \leqq 0$ in $G$ or*

$$M(r) = \sup_{\substack{x \in G \\ |x| = r}} u(x)$$

*grows so fast that*

$$\lim_{r \to \infty} M(r)\omega(x; r) > 0 \tag{3.6}$$

*for each $x \in G$.*

*Proof.* Suppose that $u(x_0) > 0$ at some point $x_0 \in G$. By the maximum principle, see 2.5,

$$M(r) = \sup_{x \in G_r} u(x) > 0$$

at least for $r \geqq |x_0|$. Let $v$ be any super-$F$-extremal in the upper Perron class $\mathcal{U}(S^{n-1}(r), G_r; F)$.

By (2.6)

$$\frac{u}{M(r)} \leqq v$$

in $G_r$ and hence

$$u(x_0) \leqq M(r)\,\omega(x_0; r)$$

for $r > |x_0|$. We have shown that (3.6) holds for $x = x_0$.

If $x, y \in G$, then Harnack's inequality (2.4) and the form of $c_0$ give a constant $L < \infty$ such that

$$\omega(x; r) \leqq L\omega(y; r)$$

for all sufficiently large $r$ and $L$ is independent of $r$. Hence if (3.6) holds at some point $x_0 \in G$, it holds for each point $x \in G$ as desired.

The following simple examples of plane harmonic functions illustrate Phragmén—Lindelöf's principle and also show that it is the best possible:

(1) Suppose that $G$ is the infinite annulus $1 < |z| < \infty$ in the plane and that $F(x, h) = |h|^2$. Then $\omega(z; r) = \log|z|/\log r$ for $1 < |z| < r$ and (3.6) takes the form

$$\varliminf_{r\to\infty} \frac{M(r)}{\log r} > 0.$$

The functions $\lambda \log|z|$, $\lambda > 0$, show that it is not possible to replace (3.6) by

$$\varliminf_{r\to\infty} M(r)\,\omega(x; r) \geqq \gamma$$

for any $\gamma > 0$. The same example can be used in all dimensions $n$ for the kernel $F(x, h) = |h|^n$.

(2) Suppose that $G$ is the upper half plane $\operatorname{Im} z > 0$ and $F$ as above. Now

$$\omega(z; r) = 2\left(1 - \frac{1}{\pi}\arg\frac{z-r}{z+r}\right)$$

and the condition (3.6) takes the form

$$\varliminf_{r\to\infty} \frac{M(r)}{r} > 0.$$

This is again the best possible as shown by the functions $\lambda \operatorname{Im} z$, $\lambda > 0$.

**3.7. A standard estimate near the boundary.** Suppose that $\partial G$ contains no point components. Then $G_t$ is a regular open set and the $F$-harmonic measure $\omega(x; r) = \omega(S^{n-1}(r), G_r; F)$ is in $C(\bar{G}_t)$, $0 < t < r$, when extended as $= 0$ to $\partial G \cap B^n(r)$, see Corollary 2.13. The proof of the next lemma shows that $\omega$ is actually in $W_n^1(G_t)$, $0<t<r$, and, moreover, it gives a useful estimate for

$$\int_{G_t} |\nabla\omega|^n \, dm.$$

**3.8. Lemma.** *Let* $R>0$ *and let*

$$m(t) = \max_{|x|=t,\, x\in G} \omega(x; R).$$

*Then for* $0<r<r_2<R$

$$\int_{G_r} |\nabla\omega|^n \, dm \leqq n^n \frac{\beta}{\alpha} \left[\int_r^{r_2} \frac{dt}{t m(t)^{n/(n-1)} \theta(t)^{1/(n-1)}}\right]^{1-n}, \tag{3.9}$$

*where* $\omega(x)=\omega(x; R)$.

*Proof.* Choose a generating sequence $u_1 \geqq u_2 \geqq \ldots$ of $F$-extremals in $C(\bar{G}_R)\cap W_n^1(G_R)$. By Lemma 2.12 the sequence $u_i$ converges uniformly on compact subsets of $\bar{G}_R \setminus \overline{G\cap S^{n-1}(R)}$ to $\omega$. We recall that $\omega(x)=0$ for $x\in\partial G_R\setminus\overline{G\cap S^{n-1}(R)}$.

Consider a test-function $\zeta$ for the condenser $(B^n(r_2), \bar{B}^n(r))$, see [GLM1, 2.3]. Especially, $\zeta\,|B^n(r)=1$, $0\leqq\zeta\leqq 1$, and $\zeta\,|R^n\setminus B^n(r_2)=0$. The proper choice of a radial $\zeta$ will be specified later.

The function

$$v_i = (1-\zeta^n)u_i$$

has the distributional gradient

$$\nabla v_i = (1-\zeta^n)\nabla u_i - n\zeta^{n-1}u_i\nabla\zeta.$$

Now the functions $v_i$ have the same boundary values as $u_i$ in $G_{r_2}$ for large $i$ and hence by the extremality of $u_i$,

$$I_F(u_i, G_{r_2}) \leqq I_F(v_i, G_{r_2}). \tag{3.10}$$

The convexity of $F$ and (c) yield

$$F(x, \nabla v_i) \leqq (1-\zeta^n)F(x, \nabla u_i) + \beta n^n |u_i \nabla\zeta|^n$$

for a.e. $x\in G_{r_2}$. Integrating this inequality over $G_{r_2}$ and using (3.10) we have the estimate

$$\int_{G_{r_2}} \zeta^n F(x, \nabla u_i)\, dm \leqq \beta n^n \int_{G_{r_2}} |u_i|^n |\nabla\zeta|^n \, dm$$

for $i=1, 2, \ldots$. Thus

$$\alpha \int_{G_r} |\nabla\omega|^n \, dm \leqq \beta n^n \int_{G_{r_2}} |\omega|^n |\nabla\zeta|^n \, dm \tag{3.11}$$

by the well-known lower semicontinuity, cf. e.g. [GLM1, Theorem 3.10],

$$\int_{G_r} |\nabla \omega|^n \, dm \leqq \varliminf_{i\to\infty} \int_{G_r} |\nabla u_i|^n \, dm$$

and by (c) of 2.1.

Next we choose

$$1-\zeta(x) = \left[\int_r^{|x|} \frac{dt}{tm(t)^{n/(n-1)}\theta(t)^{1/(n-1)}}\right]\left[\int_r^{r_2} \frac{dt}{tm(t)^{n/(n-1)}\theta(t)^{1/(n-1)}}\right]^{-1},$$

$r<|x|<r_2$, and $\zeta(x)=1$, $|x|\leqq r$, $\zeta(x)=0$, $|x|\geqq r_2$. Then $\zeta$ is a radial test function; observe that if the integral in the denominator is zero, there is nothing to prove. Using the definition of $m(t)$ we can now write (3.11) in the form

$$\alpha \int_{G_r} |\nabla \omega|^n \, dm \leqq \beta n^n \int_r^{r_2} m(t)^n |\zeta'(t)|^n \theta(t) t^{n-1} \, dt, \tag{3.12}$$

where we have used the same symbol for the radial function $\zeta(t)=\zeta(|x|)$ as for the function $\zeta$. The inequality of the lemma follows from (3.12).

In the next lemma the star is used to indicate that the left hand side integral in (3.14) is taken only over those radii $t$ for which $\partial G \cap S^{n-1}(t) \neq \emptyset$.

**3.13. Lemma.** *For* $0<r_1<r<R$

$$\int_{r_1}^{r}{}^{*} \frac{m(t)^n \, dt}{t\theta(t)^{1/(n-1)}} \leqq A_n \int_{G_r} |\nabla \omega|^n \, dm, \tag{3.14}$$

*where the constant $A_n$ is the same as in Lemma 3.2.*

*Proof.* Consider the radii $t$ meeting the boundary of $G$, i.e. $\partial G \cap S^{n-1}(t) \neq \emptyset$. For almost all such radii $t$, $0<t<R$, the inequality (3.3) via an obvious approximation argument, cf. [GLM1, Lemma 2.7], yields

$$m(t)^n = \operatorname{osc}^n(\omega, K_t) \leqq A_n t (\theta_{K_t})^{1/(n-1)} \int_{K_t} |\nabla \omega|^n \, dS,$$

where $K_t$ is an open cap in $G \cap S^{n-1}(t)$ chosen so that at the midpoint $x_0$ of $K_t$

$$\omega(x_0) = \max_{x \in G \cap S^{n-1}(t)} \omega(x)$$

and $\bar{K}_t$ meets $\partial G$. Moreover, $t^{n-1}\theta_{K_t}$ is the $(n-1)$-area of $K_t$. Since $\theta_{K_t} \leqq \theta(t)$, the above inequality yields

$$\frac{m(t)^n}{t\theta(t)^{1/(n-1)}} \leqq A_n \int_{K_t} |\nabla \omega|^n \, dS \leqq A_n \int_{G \cap S^{n-1}(t)} |\nabla \omega|^n \, dS.$$

An integration now completes the proof.

**3.15. Carleman's theorem.** The following theorem gives a counterpart of T. Carleman's method to the non-linear case.

**3.16. Theorem.** *Suppose that $G$ is an unbounded domain in $\mathbf{R}^n$ and that the kernel $F$ satisfies* (a)—(d) *in $G$. Let $u\colon G\to\mathbf{R}\cup\{-\infty\}$ be a sub-$F$-extremal in $G$ such that $\overline{\lim}\, u\leqq 0$ at each boundary point of $G$. Then, either $u\leqq 0$ or the quantity*

$$M(r)=\max_{|x|=r,\, x\in G} u(x)$$

*grows so fast that*

$$\lim_{r\to\infty} M(r)\exp\left(-c_n\sqrt[n]{\frac{\alpha}{\beta}}\int_{1\,*}^{r}\frac{dt}{t\theta(t)^{1/(n-1)}}\right)>0. \tag{3.17}$$

*Here $c_n>0$ depends only on $n$ and the set of integration is $[1,r]\cap\{t\colon\partial G\cap S^{n-1}(t)\neq\emptyset\}$.*

*Proof.* First observe that it suffices to prove the theorem for regular domains. In fact, given $\varepsilon>0$ the function $u_\varepsilon=\max(u,\varepsilon)-\varepsilon$ is a sub-$F$-extremal in $G$ and $\overline{\lim}\, u_\varepsilon\leqq 0$ on the boundary of some regular domain $G_\varepsilon$ contained in $G$. Moreover, $G_\varepsilon$ approximates $G$ from inside. If the growth condition (3.17) holds for $u_\varepsilon$ and $G_\varepsilon$ for all $\varepsilon>0$, then clearly it holds for $u$ in $G$.

Next assume that $G$ is a regular domain. Now (3.17) follows from the next lemma and Phragmén—Lindelöf's principle, Theorem 3.5.

**3.18. Lemma.** *Suppose that $G$ is a regular domain. Then*

$$\omega(x)\leqq 4\exp\left[-c_n\left(\frac{\alpha}{\beta}\right)^{1/n}\int_{|x|\,*}^{R}\frac{dt}{t\theta(t)^{1/(n-1)}}\right] \tag{3.19}$$

*for $x\in G_R$. Here $\omega(x)$ denotes the value of the $F$-harmonic measure $\omega(S^{n-1}(R), G_R; F)$ taken at the point $x$.*

*Proof.* The proof is a technical interpretation of Lemmas 3.8 and 3.13. First, by the maximum principle

$$m(t)=\max\{\omega(x)\colon x\in G_t\},\quad 0<t<R,$$

and thus

$$m(t_1)\geqq m(t_2),\quad t_1\geqq t_2.$$

Hence (3.9) and (3.14) yield

$$A_n n^n\frac{\beta}{\alpha}\geqq\left[\frac{m(r_1)}{m(r_2)}\right]^n\left[\int_{r_1\,*}^{r}\frac{dt}{t\theta(t)^{1/(n-1)}}\right]\left[\int_{r}^{r_2}\frac{dt}{t\theta(t)^{1/(n-1)}}\right]^{n-1}$$

$$\geqq\left(\frac{m(r_1)}{m(r_2)}\right)^n\int_{r_1\,*}^{r}\Big(\int_{r\,*}^{r_2}\Big)^{n-1}$$

for $0<r_1<r<r_2<R$ with an obvious abbreviation on the last line. Now choose $r, r_1<r<r_2$, such that

$$\int_r^{r_2}{}_* = (n-1) \int_{r_1}^{r}{}_* .$$

Then

$$m(r_1) \int_{r_1}^{r_2}{}_* \leqq Km(r_2), \tag{3.20}$$

where $K=A_n^{1/n} n^2 (n-1)^{(1-n)/n} (\beta/\alpha)^{1/n}$.

Let $x \in G_R$ and set $r=|x|$. It suffices to prove (3.19) for $m(r)$. We may assume that

$$\int_r^R{}_* > 0$$

since $m(r) \leqq 1$ and the inequality follows in the opposite case trivially.

To this end we iterate (3.20). Choose radii

$$0 < r = r_0 < r_1 < \ldots < r_k = R$$

such that

$$\int_{r_0}^{r_1}{}_* = \int_{r_1}^{r_2}{}_* = \ldots = \int_{r_{k-1}}^{r_k}{}_* = \frac{1}{k} \int_r^R{}_* .$$

Applying (3.20) successively to each pair of consequent radii we obtain the upper bound

$$m(r) \left( \frac{1}{kK} \int_r^R{}_* \right)^k \leqq m(R) \leqq 1. \tag{3.21}$$

Finally we choose a positive integer $k$ so that

$$e(k-1) \leqq \frac{1}{K} \int_r^R{}_* < ek, \tag{3.22}$$

where $e$ is Neper's number. Write

$$\frac{1}{K} \int_r^R{}_* = \delta > 0.$$

If $k=1$, then $\delta < e$ and hence

$$4e^{-\delta/e} > 4e^{-1} > 1 \geqq m(r).$$

Thus the inequality (3.19) follows with

$$c_n = K^{-1} e^{-1} = A_n^{-(1/n)} n^{-2} e^{-1} (n-1)^{(n-1)/n}.$$

If $k \geqq 2$, then

$$\left(1-\frac{1}{k}\right)^{-k} \leqq 4$$

and hence by (3.21) and (3.22)

$$m(r) \leqq \left(\frac{k}{\delta}\right)^k \leqq \left(\frac{k}{(k-1)e}\right)^k = \left(1-\frac{1}{k}\right)^{-k} e^{-k} \leqq 4e^{-k} < 4e^{-(\delta/e)}$$

and (3.19) again follows with the same $c_n$ as before. The proof is complete.

**3.23. Remark.** If $n=2$, then $A_2=1$, and hence $c_2=1/4$.

**3.24. Remark.** Phragmén—Lindelöf's theorem for subharmonic functions can be used to study the growth of analytic functions, see e.g. [N, p. 43]. In view of 4.1 and Theorem 3.16 these results can be extended to quasiregular mappings in a natural way. We leave these quite straightforward applications to the reader.

## 4. Lindelöf's theorem

Suppose that $f\colon B^2 \to \mathbf{R}^2$ is a bounded analytic function in the unit disk $B^2$ and that $f$ has an asymptotic limit at $x_0 \in \partial B$, i.e. there exists a path $\gamma\colon [0, b] \to \bar{B}^2$ such that $\gamma[0, b) \subset B^2$, $\gamma(b)=x_0$ and

$$\lim_{t\to b} f(\gamma(t)) = w_0.$$

Lindelöf's classical theorem [L] states that $f$ has the same limit $w_0$ in every Stolz angle with vertex at $x_0$. There are various generalizations of this result, see e.g. [He].

This chapter is devoted to study this problem for quasiregular mappings in any Euclidean space $\mathbf{R}^n$, $n \geqq 2$. In the classical formulation the theorem is not true for quasiregular mappings of $B^3$, see [R] for an example. However, our approach is based on the density concept of the $F$-harmonic measure and the method works in the same way in all dimensions. In particular, it can be used to prove the aforementioned Lindelöf's theorem in the plane. Our main result, Theorem 4.21, seems to be new even for analytic functions. Theorem 4.27 exhibits, together with Rickman's example, some basic differences in the boundary behavior of quasiregular mappings in $\mathbf{R}^2$ and in $\mathbf{R}^n$, $n \geqq 3$.

**4.1. Quasiregular mappings.** A mapping $f\colon G \to \mathbf{R}^n$ is called quasiregular (qr) if the coordinate functions of $f$ belong to $C(G) \cap \operatorname{loc} W_n^1(G)$ and for some $K \geqq 1$

$$|f'(x)|^n \leqq KJ(x,f) \tag{4.2}$$

a.e. in $G$. Here $|A|$ is the supremum norm of a linear map $A\colon \mathbf{R}^n \to \mathbf{R}^n$ and $J(x,f)$ is the Jacobian determinant of $f$ at $x$. For the basic theory of qr mappings we refer to [MRV]. We remind the reader that if $n=2$ and if (4.2) holds with $K=1$, then $f$ is analytic.

We shall employ the following property of qr mappings $f\colon G \to \mathbf{R}^n$. Suppose that $G'$ is a domain in $\mathbf{R}^n$ and $G' \supset f(G)$. Let $F\colon G' \times \mathbf{R}^n \to \mathbf{R}$ be a variational kernel satisfying (a)—(d) in $G'$. Define $f^{\#}F\colon G \times \mathbf{R}^n \to \mathbf{R}$ as

$$f^{\#}F(x,h) = \begin{cases} F(f(x), J(x,f)^{1/n} f'(x)^{-1*} h), & \text{if} \quad J(x,f) \neq 0, \\ |h|^n, \quad \text{if} \quad J(x,f) = 0 \quad \text{or} \quad J(x,f) \quad \text{does not exist}. \end{cases}$$

Here $A^*$ means the adjoint of a linear map $A\colon \mathbf{R}^n \to \mathbf{R}^n$. It follows from [GLM1, Lemma 6.4] that $f^{\#}F$ satisfies the same assumptions (a)—(d) in $G$ possibly with different $\alpha$ and $\beta$. Note that for $n=2$, $f\colon G \to \mathbf{R}^2$ analytic and $F(x,h)=|h|^2$ the kernel $f^{\#}F(x,h)$ will again be the classical Dirichlet kernel $|h|^2$ in $G$. In fact, we shall only use the kernel $F(x,h)=|h|^n$ and the induced kernel $f^{\#}F$ of a qr mapping $f\colon G \to \mathbf{R}^n$.

Suppose that $u\colon \mathbf{R}^n \to \mathbf{R} \cup \{\infty\}$ is a super-$F$-extremal in $\mathbf{R}^n$ and that $f\colon G \to \mathbf{R}^n$ is qr. Then [GLM1, Theorem 7.10] implies that $u \circ f\colon G \to \mathbf{R} \cup \{\infty\}$ is a super-$f^{\#}F$-extremal in $G$. It is easy to check that the function

$$u(x) = -\log |x|$$

is a super-$F$-extremal in $\mathbf{R}^n$ for the kernel $F(x,h)=|h|^n$. For this one merely calculates that $u$ satisfies (2.2) in $\mathbf{R}^n \setminus \{0\}$. Hence

$$u \circ f(x) = -\ln |f(x)| \tag{4.3}$$

is a super-$f^{\#}F$-extremal in $G$. This is the only property of quasiregular mappings employed in the Lindelöf-type results.

**4.4. Cones and $F$-harmonic densities.** Let $G$ be a bounded domain in $\mathbf{R}^n$. For $x_0 \in \partial G$ and $A \subset \overline{G}$ we use the abbreviation

$$A_r = A \cap \overline{B}^n(x_0, r) \setminus \{x_0\}, \quad r > 0.$$

If $E$ is an open, non-empty subset of $G$, we let

$$K(E, x_0) = \{x = t x_0 + (1-t)x\colon\ 0 \leq t < 1,\ x \in E\}$$

be the open cone generated by $E$ with vertex at $x_0$. The cone $K(E, x_0)$ is a *Stolz-cone* in $G$ if for some $\delta > 0$

$$K(E + B^n(\delta), x_0) \subset G,$$

where $E + B^n(\delta)$ is the $\delta$-inflation of $E$, i.e.

$$E + B^n(\delta) = \{x \in \mathbf{R}^n\colon\ d(x, E) < \delta\}.$$

Suppose that the kernel $F$ satisfies the assumptions (a)—(d) in $G$. Let $x_0 \in \partial G$ and let $C \subset \bar{G}$ be a $G$-admissible set. For $r > 0$ the $F$-harmonic measure

$$u_r = \omega(C_r, G \setminus C_r; F)$$

is defined in the open set $G \setminus C_r$, cf. 2.7. If $A$ is a subset of $G$ with $x_0 \in \bar{A}$, then the double limit

$$D(x_0, C, A; F) = \lim_{r \to 0} \varliminf_{\substack{x \to x_0 \\ x \in A \setminus C}} u_r(x) \tag{4.5}$$

is called the *lower $F$-harmonic density of $C$ along $A$*. If $x_0 \notin \overline{A \setminus C}$, we set $D(x_0, C, A; F) = 1$ and call this situation the trivial case. Observe that for $0 < r \leqq r'$ Lemma 2.9 implies $u_r \leqq u_{r'}$ in $G \setminus C$ and hence the first limit in (4.5) exists.

Let $y \in G$ and suppose that the segment

$$L(y, x_0) = \{ty + (1-t)x_0 \colon 0 < t \leqq 1\}$$

lies in $G$. The number $D(x_0, C, L(y, x_0); F)$ is called the *lower $F$-harmonic radial density of $C$ along $L(y, x_0)$*. Together with this concept we shall mainly use the lower $F$-harmonic density of $C$ along a Stolz-cone in $G$.

**4.6. Density in the plane.** Since the definition for $D(x_0, C, A; F)$ is complicated, we compare it in the classical plane case with more familiar concepts. Let $B^2$ be the unit disk in the plane and let $F(x, h) = |h|^2$ be the classical Dirichlet kernel. Suppose that $x_0 \in \partial B^2$ and that $A \subset B^2$ with $x_0 \in \bar{A}$.

**4.7. Theorem.** *If $C$ is a subset of $\partial B^2$, then*

$$D(x_0, C, A; F) = \varliminf_{\substack{x \to x_0 \\ x \in A}} u(x),$$

*where $u = \omega(C, B^2; F)$ is the classical outer harmonic measure of $C$ with respect to $B^2$.*

**4.8. Remark.** If $C$ is Lebesgue measurable in $\partial B^2$, then the function $u$ in Theorem 4.7 has an ordinary Poisson representation in terms of the characteristic function of $C$. For a general kernel $F$ the authors do not know if the theorem holds in $B^n$, $n \geqq 2$, with $u = \omega(C, B^n; F)$ even for a closed set $C$ in $\partial B^n$.

*Proof.* First note that points on $\partial B^2$ have zero $F$-harmonic measure and the classical outer harmonic measure is sub-additive. Hence $u \leqq u_r + \tilde{u}_r$ in $B^2$ where

$$\tilde{u}_r = \omega(\bar{C} \setminus B^2(x_0, r),\ B^2;\ F)$$

and since $\lim \tilde{u}_r(x)=0$ as $x$ tends to $x_0$ along $A$, cf. Corollary 2.13,

$$\varliminf_{\substack{x\to x_0\\ x\in A}} u(x) \leqq \varliminf_{\substack{x\to x_0\\ x\in A}} u_r(x) + \lim_{\substack{x\to x_0\\ x\in A}} \tilde{u}_r(x) = \varliminf_{\substack{x\to x_0\\ x\in A}} u_r(x).$$

Now $r\to 0$ yields the non-trivial part of the desired result.

**4.9. Radial density and the density in a Stolz-cone.** In this section the lower $F$-harmonic radial density and the lower $F$-harmonic density along a Stolz-cone are compared.

**4.10. Lemma.** *Suppose that $G$ is a bounded domain in $\mathbf{R}^n$ and that $C$ is $G$-admissible. Let $E$ be an open connected subset of $G$, $y\in E$ and $x_0\in\partial G$. If $K(E, x_0)$ is a Stolz-cone in $G$ and if $D(x_0, C, L(y, x_0); F)>0$ then $D(x_0, C, K(E, x_0); F)>0$.*

For the proof technical preparations are needed. First we present a useful estimate for $F$-extremals.

Let $D$ be a domain in $\mathbf{R}^n$, $z_0\in\partial D$, $r_0>0$ and $y_0\in D$. Suppose that there is a connected set $K$ in $\complement D$ such that $B^n(z_0, r_0)\cap\partial D\subset K$ and $K\cap S^{n-1}(z_0, r_0)\neq\emptyset$. Suppose, furthermore, that there exists a rectifiable curve $\gamma\colon [0, l]\to\bar{D}$, arc length as parameter, with $\gamma(l)=y_0$, $\gamma(0)=z_0$ and

$$d(\gamma(t), \partial D\setminus K) \geqq at, \quad a>0, \tag{4.11}$$

for all $t\in[0, l]$. In this situation we prove

**4.12. Lemma.** *Suppose that $u\colon D\to[0, 1]$ is an $F$-extremal in $D$ with*

$$\varliminf_{x\to y} u(x) \geqq c > 0$$

*for all $y\in K\cap\partial D$. Then*

$$u(y_0) \geqq c' > 0$$

*where $c'$ depends only on $n$, $a$, $c$, $\beta/\alpha$ and* $\max(l/r_0, 1)$.

*Proof.* Since $z_0\in\partial D$, $a\leqq 1$. Set $a'=a/2$ and let

$$t_0 = \sup\{t\in[0, l]\colon d(\gamma(t), K) \leqq a't\}.$$

Pick $z_0'\in\bar{K}$ such that

$$|\gamma(t_0)-z_0'| = d(\gamma(t_0), K).$$

Then

$$|\gamma(t_0)-z_0'| \leqq a't_0.$$

Let $\gamma_1$ be the straight line segment from $z_0'$ to $\gamma(t_0)$. The curve $\gamma_2=\gamma|[l-t_0, l]+\gamma_1$: $[0, l_2]\to\bar{D}$ joins $z_0'$ to $y_0$ and we parametrize $\gamma_2$ by arc length starting from $z_0'$. Observe that

$$l_2 = l-t_0+l_1$$

where $l_1$ is the length of $\gamma_1$. We shall prove that

$$d(\gamma_2(t), \partial D) \geqq a' t \tag{4.13}$$

for all $t \in [0, l_2]$.

To this end note that (4.13) clearly holds for $t \in [0, l_1]$ since $a' \leqq 1/2$. Next suppose that $l_1 < t \leqq l_2$. Then

$$\begin{aligned} d(\gamma_2(t), \partial D) &= d(\gamma(t-l_1+t_0), \partial D) \\ &\geqq a'(t-l_1+t_0) \geqq a' t \end{aligned}$$

since

$$l_1 \leqq t_0 \tag{4.14}$$

for $l_1 > t_0$ contradicts the definition of $t_0$.

Set $r_0' = a' r_0/4$. We shall show that $S^{n-1}(z_0', t)$ meets $\partial D$ for all $0 < t < r_0'$ and that

$$\partial D \cap B^n(z_0', r_0') \subset K. \tag{4.15}$$

If $z_0' \in B^n(z_0, r_0/2)$, then this follows from the assumptions made. If $z_0' \notin B^n(z_0, r_0/2)$, then $t_0 \geqq r_0/4$, for $t_0 < r_0/4$ would imply

$$\begin{aligned} |z_0 - z_0'| &\leqq |z_0 - \gamma(t_0)| + |\gamma(t_0) - z_0'| \\ &\leqq t_0 + a' t_0 < r_0/4 + r_0/8 < r_0/2. \end{aligned}$$

Hence

$$\begin{aligned} d(z_0', \partial D \setminus K) &\geqq d(\gamma(t_0), \partial D \setminus K) - l_1 \geqq a t_0 - a' t_0 \\ &= a' t_0 \geqq a r_0/4 = r_0'. \end{aligned} \tag{4.16}$$

On the other hand $\bar{K}$ is a continuum which contains $z_0$ and $z_0'$, thus $S^{n-1}(z_0', t) \cap \partial D \neq \emptyset$ for $0 < t < r_0'$ and (4.16) clearly implies (4.15).

After these preliminaries we shall complete the proof. Letting $v = \frac{1}{c} \max(0, c-u)$ we obtain a sub-$F$-extremal $v \colon D \to [0, 1]$ which is monotone in $D$. The proof for Lemma 4.4 in [GLM2] can be used to conclude that there is $\varkappa \in (0, 1)$ and $c_1 < 1$ both depending only on $n$ and $\beta/\alpha$ such that

$$v(x) \leqq c_1 < 1, \quad x \in D \cap B^n(z_0', \varkappa r_0').$$

Hence $u$ satisfies

$$u(x) \geqq c(1-c_1) = c_2 > 0 \tag{4.17}$$

for all $x \in D \cap B^n(z_0', \varkappa r_0')$ and $c_2$ depends only on $n$ and $\beta/\alpha$. Next choose points $t_0 = \varkappa r_0'$, $t_j = (1+a/4)^j t_0$, $j = 1, 2, \ldots$, and let $k$ be the first $j$ such that $t_k > l_2$. Write $\tilde{z}_j = \gamma_2(t_j)$, $r_j = \frac{a}{4} t_j$ and $B_j = B^n(\tilde{z}_j, r_j)$ for $j = 0, \ldots, k-1$. Then $\tilde{z}_j \in \bar{B}_{j-1}$, $j = 1, \ldots$ $\ldots, k-1$, and by (4.13)

$$B^n(\tilde{z}_j, 2r_j) \subset D$$

for $j=0, \ldots, k-1$. Harnack's inequality implies

$$\sup_{B_j} u \leqq c_3 \inf_{B_j} u$$

where $c_3 \in (1, \infty)$ depends only on $n$ and $\beta/\alpha$. Hence

$$\text{(4.18)} \qquad u(y_0) \geqq \inf_{B_{k-1}} u \geqq \frac{1}{c_3} \sup_{B_{k-1}} u \geqq \frac{1}{c_3} \inf_{B_{k-2}} u \geqq \frac{1}{c_3^2} \sup_{B_{k-2}} u$$

$$\geqq \ldots \geqq \frac{1}{c_3^k} \sup_{B_0} u \geqq \frac{1}{c_3^k} c_2$$

where (4.17) has been used in the last step. On the other hand, $t_{k-1} \leqq l_2$ or in other words

$$(1+a/4)^{k-1} \varkappa a' r_0/4 \leqq l_2 \leqq l$$

by (4.14). Hence $k$ has an upper bound in terms of $\max(l/r_0, 1)$, $a, n$ and $\beta/\alpha$ and (4.18) gives the desired result.

*Proof of Lemma 4.10.* We may assume $x_0=0$. Since $K(E, x_0)$ is a Stolz-cone in $G$ and $G$ is bounded, $\bar{E}$ is compact in $G$. Pick $r'>0$ so small that $\bar{B}^n(r') \cap \bar{E} = \emptyset$. Let $0<r<r'$ and set

$$v_r = \omega(L(y, x_0)_r, G \setminus L(y, x_0)_r; F).$$

We shall show that there exists $c>0$ independent of $r$ such that

$$\text{(4.19)} \qquad v_r(y_0) \geqq c, \quad y_0 \in V(r) = K(E, x_0)_{r/2} \setminus L(y, x_0).$$

Fix $y_0 \in V(r)$. The sets $K(s) = S^{n-1}(s) \cap K(E, x_0)$, $0<s<r'$, are similar domains in $S^{n-1}(s)$ and their $\delta s$-inflation for some $\delta>0$ is contained in $G$. Hence it is easy to see that the point $z_0 = |y_0| y/|y|$ can be joined to $y_0$ with a rectifiable curve $\gamma\colon [0, l] \to S^{n-1}(|y_0|)$ satisfying

$$d(\gamma(t), \partial G) \geqq at, \quad t \in [0, l], \quad l \leqq M|y_0|,$$

where $a>0$ and $M<\infty$ are independent of $r$. We use Lemma 4.12 in the domain $D = G \setminus L(y, x_0)$ with $K = L(y, x_0)_r$. The number $r_0$ in Lemma 4.12 can be chosen equal to $|y_0|\delta$. By Lemma 2.15 $\lim_{z \to z'} v_r(z) = 1$ for all $z' \in L(y, x_0)_{r/2}$, hence Lemma 4.12 yields (4.19).

To complete the proof we first consider the non-trivial case, i.e. $x_0 \in \overline{L(y, x_0) \setminus C}$. Fix $r>0$ such that $r<r'$ and

$$\varliminf_{\substack{x \to x_0 \\ x \in L(y, x_0) \setminus C}} u_r(x) > m = D(x_0, C, L(y, x_0); F)/2 > 0,$$

where $u_r = \omega(C_r, G \setminus C_r; F)$. Let $\varphi$ belong to the upper class $\mathscr{U}(C_r, G \setminus C_r; F)$. Fix $0<s<r$ with $u_r(x) \geqq m$ for $x \in L(y, x_0)_s \setminus C$. Now the function $\psi = \varphi/m$

belongs to the upper class $\mathscr{U}(C_r \cup L(y, x_0)_s, G\setminus(C_r \cup L(y, x_0)_s); F)$, hence by Lemma 2.9, $v_s \leqq \psi$ in $G\setminus(C \cup L(y, x_0))$. Thus (4.19) yields $\varphi \geqq cm$ in $V(s)\setminus C$ and, consequently, $u_r \geqq cm$ in $V(s)\setminus C$. By continuity

$$\varliminf_{\substack{x \to x_0 \\ x \in K(E, x_0)\setminus C}} u_r \geqq cm,$$

which is the desired inequality since $cm > 0$ is independent of $r$.

In the trivial case $x_0 \notin \overline{L(y, x_0)\setminus C}$, $u_r \geqq v_r$ in $G\setminus(C \cup L(y, x_0))$ for small $r > 0$ and the result follows from (4.19). This completes the proof.

**4.20. Lindelöf's theorems.** We begin with a theorem which may be called Lindelöf's principle. Let $G$ be a bounded domain in $\mathbf{R}^n$ and let $F$: $\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ be the kernel $F(x, h) = |h|^n$.

**4.21. Theorem.** *Suppose that $A \subset G$ and that $x_0 \in \bar{A} \cap \partial G$. Let $C \subset \bar{G}$ be a $G$-admissible set and let $f$: $G \cup C\setminus\{x_0\} \to \mathbf{R}^n$ be a continuous bounded mapping which is quasiregular in $G$. If $D(x_0, C, A; f^\# F) > 0$ and if*

$$\lim_{\substack{x \to x_0 \\ x \in C}} f(x) = w_0,$$

*then $f$ has the same limit $w_0$ at $x_0$ along the set $A$.*

*Proof.* If $x_0 \notin \overline{A\setminus C}$, then there is nothing to prove and we may assume that the trivial case is excluded. We may also assume that $x_0 = 0$, that $|f(x)| < 1$ for all $x \in G$ and that $w_0 = 0$. Write $\delta = D(x_0, C, A; f^\# F)$ and for $r > 0$ let $u_r = \omega(C_r, G\setminus C_r; f^\# F)$, see 4.4.

Set $v(x) = \ln(1/|f(x)|)$. Then $v$ is a super-$f^\# F$-extremal in $G$, see 4.1. Let $M > 0$ and choose $r' > 0$ such that

$$v(y) \geqq M \quad \text{for all} \quad y \in C_{r'}. \tag{4.22}$$

Fix $r$, $0 < r < r'$. Since $v \geqq 0$, (4.22) implies that $v/M$ belongs to the upper class $\mathscr{U}(C_r, G\setminus C_r; f^\# F)$. Hence $v/M \geqq u_r$ in $G\setminus C$ and thus

$$\varliminf_{\substack{x \to x_0 \\ x \in A\setminus C}} v(x) \geqq M \varliminf_{\substack{x \to x_0 \\ x \in A\setminus C}} u_r(x)$$

and letting $r \to 0$ we obtain

$$\varliminf_{\substack{x \to x_0 \\ x \in A\setminus C}} v(x) \geqq M\delta > 0.$$

Hence

$$\varlimsup_{\substack{x \to x_0 \\ x \in A\setminus C}} |f(x)| \leqq e^{-M\delta}$$

and thus $M\to\infty$ yields $f(x)\to 0=w_0$ as $x\to x_0$ along $A\setminus C$ and since the limit is $w_0$ along $C$, the theorem follows.

**4.23. Remarks.** (a) Let $\varphi\colon [0,\infty)\to[0,\infty)$ be an increasing function with $\lim_{t\to 0}\varphi(t)=0$. For $x\in G$ let $D(x,t)$ be the ball centered at $x$ of radius $t>0$ in the quasihyperbolic metric of $G$, cf. [V3, 2.8]. Write $C'=C\cap G$ where $C$ is as in Theorem 4.21. If a bounded quasiregular mapping has a limit as $x\to x_0$ along $C'$, then it has the same limit in the set

$$C'_\varphi = \bigcup_{x\in C'} D(x, \varphi(|x-x_0|)),$$

cf. [V1, Lemma 4.5]. Hence the set $C$ in Theorem 4.21 can always be replaced by a larger set $(C\cap\partial G)\cup C'_\varphi$.

(b) If $n=2$ and $f$ is analytic in Theorem 4.21, then $f^\# F=F$ is the classical Dirichlet kernel and $D(x_0, C, A; f^\# F)$ reduces to the usual lower harmonic density of $C$ at $x_0$ along the set $A$.

Theorem 4.21 implies several results on the boundary behavior of quasiregular mappings. These results are well-known for plane analytic functions.

**4.24. Corollary.** *Let $f$, $C$ and $x_0$ be as in Theorem 4.21. Suppose that $K(E,x_0)$ is a Stolz-cone in $G$ and that $E$ is connected with $y\in E$. If $D(x_0, C, L(y,x_0); f^\# F)>0$ and if*

$$\lim_{\substack{x\to x_0\\ x\in C}} f(x) = w_0,$$

*then $f$ has the same limit $w_0$ as $x\to x_0$ in $K(E,x_0)$.*

*Proof.* By Lemma 4.10, $D(x_0, C, K(E,x_0); f^\# F)>0$ and the corollary follows from Theorem 4.21.

**4.25. Corollary.** *Suppose that $f\colon B^n\to\mathbf{R}^n$ is a bounded quasiregular mapping. If $f$ has a radial limit at $x_0\in\partial B^n$, then $f$ has the same limit in each Stolz-cone $K(E,x_0)\subset B^n$ at $x_0$.*

*Proof.* We may assume that $E$ is connected, since otherwise we could replace $E$ by a larger connected open set. Now $D(x_0, L(0,x_0), L(0,x_0); f^\# F)=1$, the trivial case, and the result follows from Corollary 4.24.

**4.26. Remarks.** (a) In [MR] Corollary 4.25 was proved using a normal family argument. In [V3] another proof based on Remark 4.23 (a) has been given. Note that Lindelöf's principle, Theorem 4.21, covers the tangential and non-tangential approach at the same time.

(b) The sharpness of the condition $D(x_0, C, L(y,x_0); f^\# F)>0$ in Corollary 4.24 can be easily proved in the classical analytic case for a closed set $C$ on the boundary $\partial B^2$ of the unit disk $B^2$ such that the symmetric derivative of the measure $x_C d\theta$ exists at $x_0 \in C$. Let $L$ be the radius $\{tx_0 : 0 \leqq t < 1\}$. In view of Theorem 4.7 and Remark 4.23 (b) the condition $D(x_0, C, L; f^\# F) > 0$ takes the form

$$\lim_{r \to 0} \frac{m_1(C \cap B^2(x_0, r))}{2r} > 0.$$

Now this condition is also necessary for a radial limit of a bounded analytic function at $x_0$. Indeed, if the above limit is $=0$, then there is a function $u \in C^\infty(\partial B^2 \setminus \{x_0\})$ such that $u$ belongs to $L^1$ on $\partial B^2$, $u \geqq 0$, $u(x) \to \infty$ as $x \to x_0$ along the set $C$ and the upper symmetric derivative of the measure $u d\theta$ at $x_0$ is finite. Let $u$ be the Poisson-representation of $u$ in $B^2$, $v$ its conjugate function and define $f = \exp(-u - iv)$. Then $f$ is continuous on $\bar{B}^2 \setminus \{x_0\}$, $|f| \leqq 1$ and $f$ does not have the radial limit 0 at $x_0$ although $f(x) \to 0$ as $x \to x_0$ along $C$.

For quasiregular mappings $f: B^2 \to \mathbf{R}^2$ the condition $D(x_0, C, L; f^\# F) > 0$ is sharp as well since $f = g \circ h$ where $g: B^2 \to \mathbf{R}^2$ is analytic and $h: \bar{B}^2 \to \bar{B}^2$ is a homeomorphism quasiconformal in $B^2$ with $h(x_0) = x_0$ provided that $h(C)$ satisfies the above condition. Now

$$\begin{aligned} D(x_0, C, L; f^\# F) &= D(x_0, h(C), h(L); h^{-1} f^\# F) \\ &= D(x_0, h(C), h(L); F), \end{aligned}$$

cf. [GLM2, Theorem 5.4], and since the quasiconformality of $h$ shows that $h(L)$ remains in a Stolz cone with vertex at $x_0$, the condition $D(x_0, C, L; f^\# F) > 0$ takes the form

$$\lim_{r \to 0} \frac{m_1(h(C) \cap B^2(x_0, r))}{2r} > 0.$$

Thus the above construction can be repeated in the plane quasiregular case. It remains an open question in which sense the condition $D(x_0, C, L; f^\# F) > 0$ of Corollary 4.24 is necessary for the radial limit of a quasiregular mapping $f: B^n \to \mathbf{R}^n$, $n \geqq 3$.

Next we apply Theorem 4.21 to prove the classical Lindelöf's theorem for plane analytic functions and its counterpart for quasiregular mappings in $\mathbf{R}^n$, $n \geqq 2$. Let $G$ be the domain

$$G = B^2 \cap \{x \in \mathbf{R}^2 : x_2 > 0\}, \quad n = 2,$$

$$G = B^n \setminus \{x \in \mathbf{R}^n : x_2 = x_3 = \ldots = x_n = 0\}, \quad n \geqq 3.$$

The essential difference of the domain $G$ for $n=2$ and for $n \geqq 3$ is that the line $\{x \in \mathbf{R}^n : x_2 = x_3 = \ldots = x_n = 0\}$ cuts for $n=2$ the ball $B^n$ into two parts. Let $\gamma: [a, b) \to G$ be a path such that $\gamma(t) \to 0$ as $t \to b$.

**4.27. Theorem.** *(Lindelöf's theorem) Suppose that $f\colon G\to\mathbf{R}^n$ is a bounded quasiregular mapping and that*

$$\lim_{t\to b} f(\gamma(t)) = w_0.$$

*Then f has the same limit at $x_0=0$ in each Stolz-cone $K(E, 0)$ in $G$.*

*Proof.* For each $i=1, 2, \dots, n$ we let $L_i$ denote the $x_i$-axis and $L_i^+$ its positive half. We write $T^+$ for the two dimensional positive $x_1x_n$-half plane $x_n>0$ in $\mathbf{R}^n$. For $A\subset G$ the set $S(A)$ is the rotation of $A$ about $L_1$ in $G$, i.e.

$$S(A) = \{x\in G\colon d(x, L_1) = d(y, L_1),\ x_1 = y_1 \quad \text{for some } y\in A\}.$$

Observe that for $n=2$, $S(A)=A$. We also recall that for a set $A\subset\mathbf{R}^n$ and $r>0$, $A_r=\bar{B}^n(r)\cap A\setminus\{0\}$.

We shall show that

$$D(0, \gamma, L_n^+\cap B^n; f^\# F) > 0, \tag{4.28}$$

where $F(x, h)=|h|^n$. Since $\frac{1}{2}e_n\in E$ represents no restriction, Corollary 4.24 will then complete the proof. To this end choose $r>0$ such that $S^{n-1}(r)$ meets $\gamma$ and let $u_r=\omega(\gamma_r, G\setminus\gamma_r; f^\# F)$. By Lemmas 2.9 and 2.15, $\lim_{x\to y} u_r(x)=1$ for all $y\in \big(\gamma\cap\partial(G\setminus\gamma)\big)\cap B^n(r)$, hence we may extend $u_r$ as a continuous function to $B^n(r)\cap\gamma$ by setting

$$u_r(x) = 1, \quad x\in B^n(r)\cap\gamma. \tag{4.29}$$

Next we shall prove that there is $c'>0$ independent of $r$ such that

$$u_r(y_0) \geqq c' \tag{4.30}$$

for all $y_0\in S(\gamma_{r/2})$. The inequality (4.30) is the crucial step in the proof. Note that for $n=2$, $S(\gamma_{r/2})=\gamma_{r/2}$ and (4.30) follows from (4.29).

To prove (4.30) we fix $y_0\in S(\gamma_{r/2})\setminus\gamma$ and employ Lemma 4.12 in the $y_0$-component $D$ of $G\setminus\gamma$. Now $y_0\in S(z_0)$ for some $z_0\in\gamma_{r/2}\cap\partial D$ and for a curve $\gamma$ in Lemma 4.12 we choose a circular arc from $z_0$ to $y_0$ in the set $S(z_0)$. Lemma 4.12 together with (4.29) then implies (4.30).

To complete the proof of (4.28) we again use Lemma 4.12. It suffices to show that

$$u_r(y_0) \geqq c, \quad y_0\in(L_n^+)_{r/4}\setminus S(\gamma), \tag{4.31}$$

where $c>0$ is independent of $r$, since (4.30) takes care of the points in $S(\gamma_{r/2})$. Fix $y_0\in(L_n^+)_{r/4}\setminus S(\gamma)$. Now a component $K$ of $S(\gamma)_r$ separates $y_0$ either from $L_1^+$ or from the negative half of $L_1$ in $(T^+)_r$. Let $D$ be the $y_0$-component of $B^n(r/2)\cap G\setminus K$ and write $u$ for the $f^\# F$-harmonic measure $\omega(K, D, f^\# F)$. By (4.30), $u\leqq u_r/c'$ in $D$ and it suffices to prove (4.31) for $u$. To this end let $\gamma_1\colon [0, l]\to D\cup\{z_0\}$ be a circular arc in the half plane $T^+$ with $|\gamma_1(t)|=|y_0|$, $t\in[0, l]$, joining a point $z_0\in K$ to $y_0$.

The curve $\gamma_1$ has been parametrized by arc length from $z_0$. Now $l \leqq \pi|y_0|/2$ and the number $r_0$ in Lemma 4.12 can be chosen equal to $|y_0|/2$. Since $d(\gamma_1(t), \partial D \setminus K) \geqq t/4$ for all $t \in [0, l]$ and since $\lim_{x \to y} u(x) = 1$ for all $y \in \partial D \cap K \cap B^n(r/2)$ by Lemma 2.15, Lemma 4.12 finally gives the estimate (4.31) for $u$. The proof is complete.

**4.32. Remark.** Rickman [R] showed that a bounded quasiregular mapping $f$ of the upper half space $H^+$ into $\mathbf{R}^n$ has a limit in each Stolz-cone $K$ at 0 provided that the limit

$$\lim_{\substack{x \to 0 \\ x \in T}} f(x)$$

exists, where $T$ is a smooth $(n-1)$-dimensional relatively thick tangential surface ending at 0. It is not difficult to see that $D(0, T, K; f^{\#} F) > 0$ and hence Rickman's result follows from Theorem 4.21.

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