Wiener's theorem, the Radon—Nikodym theorem, and $m_0(\mathbf{T})$

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1. Introduction

Let $M(\mathbf{T})$ denote the class of complex Borel measures on the circle $\mathbf{T}=\mathbf{R}/\mathbf{Z}$ and $M_0(\mathbf{T})$ the subclass $\{\mu\colon \lim_{n\to\infty}\hat{\mu}(n)=0\}$. It was recently proved [5, 6] that $M_0(\mathbf{T})$ is characterized by its class of common null sets. To make this more precise, we use the following notation. For any subclass $\mathscr{C}\subset M(\mathbf{T})$, we let

$$\mathscr{C}^{\perp} = \{E\subset T\colon E \text{ is a Borel set and } \forall\mu\in\mathscr{C}\ |\mu|(E) = 0\}$$

be the class of common null sets of $\mathscr{C}$. Likewise, if $\mathscr{E}$ is a class of Borel subsets of $\mathbf{T}$, we write

$$\mathscr{E}^{\perp} = \{\mu\in M(\mathbf{T})\colon \forall E\in\mathscr{E}\,|\mu|(E) = 0\}$$

for the class of measures annihilating $\mathscr{E}$. Then by definition, the class of sets of uniqueness in the wide sense, U_0, is equal to $M_0(\mathbf{T})^{\perp}$ and [6] shows that $U_0^{\perp}=M_0(\mathbf{T})$. That is, $M_0(\mathbf{T})^{\perp\perp}=M_0(\mathbf{T})$.

Now notice that we can write $M_0(\mathbf{T})$ in another way. Let PM be the pseudo-measure topology on $M(\mathbf{T})$: $\|\mu\|_{PM}\equiv\sup_{n\in\mathbf{Z}}|\hat{\mu}(n)|$. If $\mathscr{P}$ denotes the trigonometric polynomials and λ Lebesgue measure on $\mathbf{T}$, then $M_0(\mathbf{T})$ is the PM-closure of $\mathscr{P}.\lambda$.

If M denotes the usual norm topology on $M(\mathbf{T})$, then the M-closure of $\mathscr{P}.\sigma$, for any $\sigma\in M(\mathbf{T})$, is $L^1(\sigma)=\{f.\sigma\colon \int|f|\,d|\sigma|<\infty\}$. It is clear that $L^1(\sigma)^{\perp}=\{E\colon |\sigma|(E)=0\}$, whence the Radon—Nikodym theorem is equivalent to the assertion $L^1(\sigma)^{\perp\perp}=L^1(\sigma)$. This leads us to ask if the analogous theorem holds for PM. In other words, if $L^{PM}(\sigma)$ denotes the PM-closure of $\mathscr{P}.\sigma$, is $L^{PM}(\sigma)^{\perp\perp}=L^{PM}(\sigma)$?

Consider now Wiener's theorem [3, p. 42], which says that for all $\mu\in M(\mathbf{T})$,

$$V(\mu) \equiv \lim_{N\to\infty}\left(\frac{1}{2N+1}\sum_{|n|\leq N}|\hat{\mu}(n)|^2\right)^{1/2} \tag{1}$$

exists and equals

$$V(\mu) = (\sum_{\tau\in\mathbf{T}} |\mu(\{\tau\})|^2)^{1/2}. \tag{2}$$

In particular, $V(\mu)=0$ if and only if μ is a continuous measure: $\mu\in M_c(\mathbf{T})$. Let us introduce the "Wiener norm"

$$\|\mu\|_{WN} \equiv \sup_{N\geq 0}\left(\frac{1}{2N+1}\sum_{|n|\leq N}|\hat{\mu}(n)|^2\right)^{1/2}.$$

Then $V(\mu)=0$ if and only if μ belongs to the WN-closure of $\mathcal{P}.\lambda$, which we denote $L^{WN}(\lambda)$. In other words, $L^{WN}(\lambda)=M_c(\mathbf{T})$, from which it immediately follows that $L^{WN}(\lambda)^{\perp\perp}=L^{WN}(\lambda)$. Again, we ask if this holds with λ replaced by any $\sigma\in M(\mathbf{T})$.

2. Statements of results

The problem appears quite difficult for the PM topology. In view of the following theorem, $L^{PM}(\sigma)^{\perp\perp}=L^{PM}(\sigma)$ for discrete σ $\big(\sigma\in M_d(\mathbf{T})\big)$ and the general problem is reduced to the case of continuous σ:

Theorem 1. *If σ_c and σ_d are the continuous and discrete parts of any $\sigma\in M(\mathbf{T})$, then*

$$L^{PM}(\sigma) = L^{PM}(\sigma_c)+L^1(\sigma_d)$$

and $L^{PM}(\sigma_c)\subset M_c(\mathbf{T})$.

On the other hand, the Wiener norm is fully tractable. Let supp σ denote the support of σ and let $M_c(E)$ be the class of continuous measures supported in E. Then the fact that $L^{WN}(\sigma)^{\perp\perp}=L^{WN}(\sigma)$ follows from

Theorem 2. *For all $\sigma\in M(\mathbf{T})$,*

$$L^{WN}(\sigma) = M_c(\text{supp }\sigma)+L^1(\sigma_d).$$

The proof of Theorem 2 is based on a reduction to the weak* topology. For it will be easy to show that the weak*-closure $L^{w^*}(\sigma)$ of $\mathcal{P}.\sigma$ is given by

Proposition 3. *For all $\sigma\in M(\mathbf{T})$,*

$$L^{W^*}(\sigma) = M(\text{supp }\sigma).$$

Of course, it follows that $L^{w^*}(\sigma)^{\perp\perp}=L^{w^*}(\sigma)$. The reduction to this topology will be effected by means of a surprising

Lemma 4. *If $\{\mu_m\}$ is a sequence of positive measures converging weak* to a continuous measure ν, then $\|\mu_m-\nu\|_{WN}\to 0$.*

In words, this says that pointwise convergence $\hat{\mu}_m(n)\to\hat{\nu}(n)$ implies uniform Cesaro convergence! This lemma, interesting in its own right, has the following extension.

Proposition 5. *Let $\{\mu_m\}$ be a sequence of positive measures converging weak* to ν. Let $E=\{\tau\in\mathbf{T}\colon \nu(\{\tau\})\neq 0\}$. Then the following are equivalent:*

i) $$\|\mu_m-\nu\|_{WN}\to 0;$$

ii) $$\lim_{m\to\infty}\sup_{\tau\in\mathbf{T}}|\mu_m(\{\tau\})-\nu(\{\tau\})| = 0;$$

iii) $$\lim_{m\to\infty}\sup_{\tau\in E}|\mu_m(\{\tau\})-\nu(\{\tau\})| = 0.$$

Easy examples show that the hypothesis $\mu_m\geqq 0$ is indispensable.

The reader has surely wondered whether a general result holds for all "reasonable" topologies: if $\mathscr{C}$ is a "reasonable" topology on $M(\mathbf{T})$ and $L^{\mathscr{C}}(\sigma)$ denotes the $\mathscr{C}$-closure of $\mathscr{P}.\sigma$, is $L^{\mathscr{C}}(\sigma)^{\perp\perp}=L^{\mathscr{C}}(\sigma)$? If σ is a discrete measure with finite support, the answer is trivially "yes" because of the well-known fact that finite-dimensional vector spaces have a unique topology, which is hence complete. Therefore $L^{\mathscr{C}}(\sigma)=L^1(\sigma)$. In general, however, even for discrete measures or Lebesgue measure and even for norm topologies, the answer is "no".

Theorem 6. *Define*

$$\|\mu\| = \sup\left(\left\{\frac{|\hat{\mu}(n)|}{|n|+1}\colon n\in\mathbf{Z}\right\}\cup\{|\hat{\mu}_{sc}(n)|\colon n\in\mathbf{Z}\}\right),$$

where μ_{sc} is the continuous part of μ singular to λ. Then

$$L^{\|\,\|}(\lambda) = M_d(\mathbf{T})+L^1(\lambda)$$

and for discrete σ,

$$M_d(E)\subset L^{\|\,\|}(\sigma)\subset M_d(E)+L^1(\lambda|_E),$$

where $E=\operatorname{supp}\sigma$.

It follows that $L^{\|\,\|}(\lambda)^{\perp\perp}=M(\mathbf{T})\neq L^{\|\,\|}(\lambda)$ and that $L^{\|\,\|}(\sigma)^{\perp\perp}=M(E)\neq L^{\|\,\|}(\sigma)$ for $\sigma\in M_d(\mathbf{T})$.

3. Proofs

We note first the following trivial facts. For any topology $\mathscr{C}$, $L^{\mathscr{C}}(\sigma)\subset L^{\mathscr{C}}(\sigma)^{\perp\perp}$. If $\mathscr{C}_1\subset\mathscr{C}_2$, then $L^{\mathscr{C}_1}(\sigma)\supset L^{\mathscr{C}_2}(\sigma)$. If $\mathscr{C}$ is weaker than the M-topology, as all our topologies are, then $L^{\mathscr{C}}(\sigma)$ is the $\mathscr{C}$-closure of $L^1(\sigma)$. We denote the dual of $M(\mathbf{T})$ when equipped with the topology $\mathscr{C}$ by $(M(\mathbf{T}),\mathscr{C})'$. For $c\subset M(\mathbf{T})$, let $\operatorname{ann}_{\mathscr{C}} c$ be the annihilator of c in $(M(\mathbf{T}),\mathscr{C})'$. For $\mathscr{U}\subset(M(\mathbf{T}),\mathscr{C})'$, let $\ker\mathscr{U}$ be the kernel of $\mathscr{U}$ in $M(\mathbf{T})$. Then a well-known consequence of the Hahn—Banach theorem says that for any locally convex $\mathscr{C}$ and any subspace $c\subset M(\mathbf{T})$, the $\mathscr{C}$-closure of c is equal to $\ker(\operatorname{ann}_{\mathscr{C}} c)$. In particular,

(3) $$L^{\mathscr{C}}(\sigma) = \ker(\operatorname{ann}_{\mathscr{C}} L^1(\sigma)).$$

Proposition 3 follows immediately from this. For we have $(M(\mathbf{T}), w^*)'=c(\mathbf{T})$, so that

$$L^{w^*}(\sigma) = \ker(\operatorname{ann}_{w^*} L^1(\sigma))$$
$$= \ker\{f\in(\mathbf{T}) : f = 0 \quad \text{on supp } \sigma\} = M(\operatorname{supp} \sigma).$$

The next lemma is useful in proving Theorems 1 and 2.

Lemma 7. *If* $\mu\in L^{WN}(\sigma)$, *then* $\mu_d\in L^1(\sigma_d)$.

Proof. With $V(\mu)$ as in (1), we see by (2) that for all τ, $|\mu(\{\tau\})|\leqq V(\mu)\leqq\|\mu\|_{WN}$, so that $\mu\mapsto\mu(\{\tau\})$ is WN-continuous. Thus, if $\sigma(\{\tau\})=0$, also $\mu(\{\tau\})=0$ for all $\mu\in L^{WN}(\sigma)$.

From the well-known fact that $\|\sigma_d\|_{PM}\leqq\|\sigma\|_{PM}$ for any σ (see [2, p. 110]), we deduce

Lemma 8. $\sigma\mapsto\sigma_d$ *and* $\sigma\mapsto\sigma_c$ *are PM-continuous.* $M_c(\mathbf{T})$ *and* $M_d(\mathbf{T})$ *are PM-closed.*

We may now proceed to the

Proof of Theorem 1. By Lemma 8,

$$L^{PM}(\sigma) = L^{PM}(\sigma_c)+L^{PM}(\sigma_d)$$

and $L^{PM}(\sigma_c)\subset M_c(\mathbf{T})$, $L^{PM}(\sigma_d)\subset M_d(\mathbf{T})$. Also, by Lemma 7, $L^{WN}(\sigma_d)\cap M_d(\mathbf{T})= L^1(\sigma_d)$. Since $\|\mu\|_{WN}\leqq\|\mu\|_{PM}\leqq\|\mu\|_M$, we have

$$L^1(\sigma_d)\subset L^{PM}(\sigma_d)\subset L^{WN}(\sigma_d)\cap M_d(\mathbf{T}) = L^1(\sigma_d),$$

from which the theorem follows.

Proof of Lemma 4. Let
$\Omega_\mu(h)=\sup\{|\mu I| : I$ is a closed arc of $\mathbf{T}$ of length $h\}$. Then Wiener showed (see [1, Chap. II, § 2]) that for all μ,

$$\frac{1}{2N+1}\sum_{|n|\leqq N}|\hat{\mu}(n)|^2 \leqq \frac{\pi^2}{4}\|\mu\|_M\Omega_\mu\left(\frac{1}{2N}\right).$$

Hence if $\Delta_m=\sup_h \Omega_{\mu_m-\nu}(h)$, we have

$$\|\mu_m-\nu\|_{WN}^2 \leqq \frac{\pi^2 C}{4}\Delta_m,$$

where $C=\sup_m\|\mu_m-\nu\|_M<\infty$. But $\Delta_m\to 0$ as $m\to\infty$ (see [7, p. 317] or [4, Chap. 2, Theorem 1.1, p. 89] for the case $\nu=\lambda$; the proof is the same for all $\nu\in M_c$).

Theorem 2 now follows from Lemma 7, Lemma 4, and the following two propositions.

Proposition 8. *If* $0\leqq\nu\in M(\operatorname{supp}\sigma)$, *then there exist positive* $\mu_m\in L^1(\sigma)$ *converging weak* to* ν.

Proof. That the result holds when ν is concentrated on a point τ is trivial: $\|\nu\|_M/|\sigma|(I_n)\ |\sigma|\big|I_n \xrightarrow{w^*} \nu$, where $I_n=(\tau-1/n)\cdot\tau+1/n$. Hence the result holds when ν is discrete. But it is well-known that we can use positive discrete measures to approximate any positive measure. ∎

Proposition 9. *If* $\mu \ll \nu \in L^{PM}(\sigma)$, *then* $\mu \in L^{PM}(\sigma)$.

Proof. It is clear that if $\nu \in L^{PM}(\sigma)$, then $\mathscr{P}.\nu \subset L^{PM}(\sigma)$. Therefore $L^{PM}(\sigma)$ contains the PM-closure of $\mathscr{P}.\nu$, which in turn contains the M-closure, namely, $L^1(\nu)$.

We now show how Proposition 5 follows from Lemma 4.

Proof of Proposition 5. That (i)⇒(ii) follows from (2), and (ii)⇒(iii) is trivial. Assume (iii). Write $E^C=\mathbf{T}\setminus E$, $\sigma_m=\mu_m|_{E^c}$, and $\varrho_m=(\mu_{m|E})-\nu_d$. Then $\sigma_m+\varrho_m=\mu_m-\nu_d \xrightarrow{w^*} \nu_c$. Splitting $\varrho_m=\varrho_m^+-\varrho_m^-$ into its positive and negative parts, we claim it suffices to show that $\|\varrho_m^-\|_M\to 0$. For then we would have $\sigma_m+\varrho_m^+ \xrightarrow{w^*} \nu_c$. But $\sigma_m+\varrho_m^+\geqq 0$, so that Lemma 4 implies $\sigma_m\varrho_m^+ \xrightarrow{WN} \nu_c$. Since $\varrho_m^- \xrightarrow{wN} 0$, we conclude that $\sigma_m+\varrho_m \xrightarrow{WN} \nu_c$, whence $\mu_m-\nu_d \xrightarrow{WN} \nu_c$, or (i).

To show that $\|\varrho_m^-\|_M\to 0$, pick $\varepsilon>0$. Choose a finite set $F\subset E$ such that $\sum_{\tau\notin F}\nu(\{\tau\})<\varepsilon$. Let m_0 be such that $\sup_{\tau\in E}|\mu_m(\{\tau\})-\nu(\{\tau\})|<\varepsilon/|F|$ for $m\geqq m_0$. Write $E_m^-=\{\tau\colon \mu_m(\{\tau\})<\nu(\{\tau\})\}$. Then we have

$$\|\varrho_m^-\|_m = \sum\nolimits_{\tau\in E_m^-}|\mu_m(\{\tau\})-\nu(\{\tau\})| \leqq \sum\nolimits_{\tau\in F}+\sum\nolimits_{\tau\in E_m^-/F}$$

$$\leqq |F|\frac{\varepsilon}{|F|}+\sum\nolimits_{\tau\notin F}\nu(\{\tau\}) < 2e$$

for $m\geqq m_0$.

Our last task is the

Proof of Theorem 6. Let $\Lambda_n(\mu)=\hat{\mu}_{sc}(n)$. Then $\Lambda_n\in(M(\mathbf{T}),\|\ \|)'$, whence by (3),

$$L^{\|\,\|}(\lambda)\subset\ker\{\Lambda_n\}_{-\infty}^{\infty} = M_d(\mathbf{T})+L^1(\lambda).$$

Since $\|\mu\|\leqq\|\mu\|_M$, we have $L^1(\lambda)\subset L^{\|\,\|}(\lambda)$. It remains to show that $M_d(\mathbf{T})\subset L^{\|\,\|}(\lambda)$. Now if $\mu\in M_d(\mathbf{T})$ and $D_N(t)=\sum_{|n|\leqq N}e^{2\pi int}$ is the Dirichlet kernel, we have

$$\|D_N*\mu-\mu\| = \sup_n\frac{|(D_N*\mu)^\wedge(n)-\hat{\mu}(n)|}{|n|+1}$$

$$= \sup_{|n|>N}\frac{|\hat{\mu}(n)|}{|n|+1}\leqq\frac{\|\mu\|_M}{N+2}.$$

Hence $D_N*\mu \xrightarrow{\|\,\|} \mu$. Since $D_N*\mu\in L^1(\lambda)$, it follows that $\mu\in L^{\|\,\|}(\lambda)$.

The argument above also shows that for any discrete σ,

$$L^{\|\,\|}(\sigma) \subset M_d(\mathbf{T}) + L^1(\lambda).$$

But it is clear that every C^∞ function belongs to $(M(\mathbf{T}), \|\ \|)'$, whence by (3), $L^{\|\,\|}(\sigma) \subset M(E)$. Combining these two inclusions gives

$$L^{\|\,\|}(\sigma) \subset M_d(E) + L^1(\lambda_{|E}).$$

Finally, in order to prove that $M_d(E) \subset L^{\|\,\|}(\sigma)$, it suffices to prove that $\delta_x \in L^{\|\,\|}(\sigma)$ for every $x \in E$, where δ_x is the Dirac measure at x. But for every $\varepsilon > 0$, there exists y with $|x-y| < \varepsilon$ and $\delta_y \in L^1(\sigma)$. Since

$$\|\delta_x - \delta_y\| = \sup_n \frac{|\hat{\delta}_x(n) - \hat{\delta}_y(n)|}{|n|+1} = \sup \frac{|e^{-2\pi inx} - e^{-2\pi iny}|}{|n|+1}$$

$$\leqq \sup \frac{|2\pi nx - 2\pi ny|}{|n|+1} \leqq 2\pi|x-y| < 2\pi\varepsilon,$$

the result follows.

References

1. Nina K. Bari, *A Treatise on Trigonometric Series*. Translated from the Russian by M. F. Mullins. Vol. I: New York: Macmillan, 1964.
2. Colin C. Graham and O. Carruth McGehee, *Essays in Commutative Harmonic Analysis*. New York: Springer-Verlag, 1979.
3. Yitzhak Katznelson, *An Introduction to Harmonic Analysis*. Second corrected edition. New York: Dover, 1976.
4. L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*. New York: Wiley, 1974.
5. Russell Lyons, Characterizations of measures whose Fourier—Stieltjes transforms vanish at infinity, *Bull. Amer. Math. Soc.* **10** (1984), 93—96.
6. Russell Lyons, A characterization of measures whose Fourier—Stieltjes transforms vanish at infinity (to appear).
7. Hermann Weyl, Über die Gleichverteilung von Zahlen mod. Eins, *Math. Ann.* **77** (1916), 313—352.

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