Positive solutions of elliptic equations in nondivergence form and their adjoints

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Introduction

We consider uniformly elliptic operators of the form

$$(*) \qquad L = \sum_{i,j=1}^{n} a_{ij}(X) \cdot D^2_{X_i X_j} + \sum_{i=1}^{n} b_i(X) \cdot D_{X_i}$$

with real-valued, bounded measurable coefficients defined in $\mathbf{R}^n$ (for $n \geqq 2$). The functions, a_{ij}, are assumed to be uniformly continuous in $\mathbf{R}^n$ (with no restriction on the modulus of continuity) and satisfy $a_{ij} = a_{ji}$. Operators of this type correspond to diffusion processes in $\mathbf{R}^n$ (see [16]) and hence will be called diffusion operators.

Our main objective is to prove a comparison theorem (Theorem 2.1) for positive solutions of $Lu=0$ in a bounded Lipschitz domain, D, in $\mathbf{R}^n$. The theorem asserts that any two positive solutions of $Lu=0$ in D which vanish on a portion of the boundary must vanish at the same rate. More precisely, if $Q \in \partial D$, $B(8r, Q)$ is a ball of radius $8r$ centered at Q, and u_1 and u_2 are positive solutions of $Lu=0$ in $B(8r, Q) \cap D$ which vanish continuously on ∂D, then

$$\frac{1}{c} \cdot \frac{u_1(X)}{u_1(A_r)} \leqq \frac{u_2(X)}{u_2(A_r)} \leqq c \cdot \frac{u_1(X)}{u_2(A_r)}$$

for all X in $B(r, Q) \cap D$. Here, A_r is a point in $B(r, Q) \cap D$ whose distance from ∂D is proportional to r. The constant, c, is independent of Q, r, u_1, and u_2.

The comparison theorem was proved for harmonic functions in 1968 by Hunt and Wheeden ([9]). It was extended to solutions of $Lu=0$ for operators with Hölder continuous coefficients by A. Ancona in 1978 ([2]). A consequence of the comparison theorem is that the representation theorem and Fatou-type results for positive harmonic functions in D (see [9]) extend to positive solutions of $Lu=0$.

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In addition, Hunt and Wheeden's estimates of harmonic measure in D can be extended to the "L-harmonic measure" corresponding to the diffusion operator, L, in D. These results will appear in a separate publication. (See also [3].)

Much of our work in this paper concerns the behavior of the Green's function and nonnegative solutions of the adjoint equation, $L^*v=0$. This is because the comparison theorem is shown to be equivalent to the inequality (Theorem 2.6):

$$(**) \qquad \int_{\psi(r)} G_r(X,Y)\,dY \leqq c\cdot\int_{\Phi(r)} G_r(X,Y)\,dY$$

for all $X\in B(r,Q)\cap D$, where $Q\in\partial D$, $G_r(X,Y)$ is the Green's function for L in $B(4r,Q)\cap D$, $\psi(r)=[B(3r,Q)\setminus B(2r,Q)]\cap D$, and $\Phi(r)$ is a cube in $B(4r,Q)\cap D$ whose distance from ∂D is proportional to r. The constant, c, is independent of Q, r and X.

The main difficulty in proving the above inequality is that the Green's function for L in D, which we denote by $G(X,Y)$, need not behave like the Green's function for the Laplacian. In particular, the examples of Gilbarg and Serrin ([7]) show that $G(X,Y)$ need not be proportional to $|X-Y|^{2-n}$ (for $n\geqq 3$) when X is near Y in D. Moreover, we have constructed an example in which $G(X,\cdot)\notin L^\infty_{\mathrm{loc}}(D\setminus\{X\})$. (See [4].)

We prove inequality $(**)$ (and hence the comparison theorem) as a consequence of our *a priori* estimates of nonnegative solutions of $L^*v=0$. A solution of $L^*v=0$ in D is defined to be a function, v, in $L^1_{\mathrm{loc}}(D)$ such that

$$\int_D v(Y)\cdot L\varphi(Y)\,dY = 0$$

for all $\varphi\in C_0^\infty(D)$. The functions, $G(X,\cdot)$ and $G_r(X,\cdot)$, above satisfy $L^*v=0$ in subdomains of $D\setminus\{X\}$ and $[B(4r,Q)\cap D]\setminus\{X\}$, respectively.

Our results on nonnegative solutions of $L^*v=0$ include an interior A_∞-estimate. (See Corollary 3.4.) In addition, assuming the coefficients of L are smooth functions we prove classical estimates on suitably "normalized" adjoint solutions which are independent of the smoothness of the coefficients. (See Section 4.) For example, we prove a Harnack inequality (Theorem 4.4) for functions of the form

$$\tilde v(Y) = v(Y)/G(X,Y)$$

in subdomains of $D\setminus\{X\}$, where v is a nonnegative solution of $L^*v=0$ and $G(X,Y)$ is the Green's function for L in D. The constants in these estimates depend only on the ellipticity, bounds, and modulus of continuity of the coefficients.

It follows from the above results that the Green's function for L in D is the product of an A_∞-weight (as defined by Muckenhoupt) and a positive, continuous function of X and Y (for $X \neq Y$) which satisfies a Harnack inequality in each variable and vanishes continuously on ∂D. (See Section 5.) This is the main idea of the proof of inequality $(**)$ and may be of independent interest.

1. Notation

We assume throughout this paper that L is a diffusion operator defined in $\mathbf{R}^n$ (for $n \geqq 2$) as described in $(*)$. We let $w: \mathbf{R}^+ \to \mathbf{R}^+$ denote a nondecreasing function such that $w(t) \to 0$ as $t \to 0$ and

$$\sup \{|a_{ij}(X) - a_{ij}(Y)|: |X-Y| \leqq t, \ 1 \leqq i, j \leqq n\} \leqq w(t).$$

We denote by λ a positive number such that

$$\|a_{ij}\|_{L^\infty(\mathbf{R}^n)} + \|b_i\|_{L^\infty(\mathbf{R}^n)} \leqq 1/\lambda \quad \text{for} \quad 1 \leqq i, j \leqq n$$

and $\lambda |\xi|^2 \leqq \sum_{i,j=1}^n a_{ij}(X) \cdot \xi_i \xi_j$ for all $(X, \xi) \in \mathbf{R}^n \times \mathbf{R}^n$.

We assume that D is a bounded Lipschitz domain in $\mathbf{R}^n$ and let $m \geqq 1$ and $r_0 \leqq 1$ denote positive numbers such that the following holds: to each $Q \in \partial D$ there corresponds a coordinate system (x, y) of $\mathbf{R}^{n-1} \times \mathbf{R}$ (obtained by a translation and rotation of the standard coordinate system) and a function $\varphi: \mathbf{R}^{n-1} \to \mathbf{R}$ such that

$$8|\varphi(x) - \varphi(x')| \leqq m|x-x'|$$

and

$$B(10mr_0, Q) \cap \{(x, y): y > \varphi(x)\} = B(10mr_0, Q) \cap D.$$

Here, $B(s, Q)$ denotes a ball of radius s centered at Q. If $r > 0$ and $Q = (x_0, y_0)$ with respect to this coordinate system, we define

$$\Omega(r, Q) = \{(x, y): |x - x_0| < r, \ |y - y_0| < mr\}.$$

We denote by $A_r(Q)$ the point in $\Omega(r, Q)$ with coordinates $(x_0, y_0 + mr/2)$. The set $\Omega(r, Q) \cap \partial D$ is denoted by $\Delta(r, Q)$.

If $p \in (1, \infty)$, we let

$$W^{2,p}(D) = \{u: D^\alpha u \in L^p(D), \ |\alpha| \leqq 2\}$$

and

$$W^{2,p}_{\text{loc}}(D) = \{u: D^\alpha u \in L^p_{\text{loc}}(D), \ |\alpha| \leqq 2\}.$$

We define $\mathring{W}^{2,p}(D)$ to be the closure of $\{u \in C^2(\bar{D}): u = 0 \text{ on } \partial D\}$ in $W^{2,p}(D)$ with respect to the norm

$$\|u\|_{W^{2,p}(D)} = \sum_{|\alpha| \leqq 2} \|D^\alpha u\|_{L^p(D)}.$$

By a solution of $Lu=0$ in D we mean a continuous function, u, in $W^{2,p}_{\mathrm{loc}}(D)$ for some $p>1$ such that $Lu=0$ pointwise almost everywhere in D. The L^p-Schauder estimates ([1]) and the Sobolev inequalities imply that solutions as defined above are in $W^{2,p}_{\mathrm{loc}}(D)$ for every $p>1$.

We denote by $\Phi(r, Q)$ the cube of side length r centered at Q in $\mathbf{R}^n$. If N is a unit vector in $\mathbf{R}^n$ and $0<\theta<\pi$, we define the truncated cone

$$\Gamma(r, \theta, N, Q) = B(r, Q) \cap \{X \in \mathbf{R}^n \setminus \{0\} : \cos^{-1}(N \cdot (X-Q)/|X-Q|) < \theta\}.$$

The Lebesgue measure of a measurable set, E, in $\mathbf{R}^n$ is denoted by $|E|$.

2. The comparison theorem: Motivation for a study of adjoint solutions

In this section we *state* the comparison theorem and show that it follows from an integral inequality on the Green's function (Theorem 2.6). We shall prove this inequality in Section 5 as a consequence of our estimates on nonnegative solutions of $L^*v=0$.

Theorem 2.1 (Comparison Theorem). *Suppose $Q \in \partial D$ and $0<r<r_0$. If u and v are positive solutions of $Lu=0$ in $\Omega(8r, Q) \cap D$ which vanish continuously on $\Delta(8r, Q)$, then*

$$\frac{1}{c} \cdot \frac{u(X)}{u(A_r(Q))} \leqq \frac{v(X)}{v(A_r(Q))} \leqq c \cdot \frac{u(X)}{u(A_r(Q))}$$

for all $X \in \Omega(r, Q) \cap D$. The constant, c, above depends only on λ, n, w, r_0, and m.

To prove that the comparison theorem follows from an inequality on the Green's function, we will need the following results (Lemmas 2.2—2.5) which are standard consequences of the maximum principle ([5]), Harnack principle ([11] or [17]), and existence of uniform barriers on cones ([12]).

Lemma 2.2. *Suppose $Q \in \partial D$, $0<r<r_0$, and u is a positive solution of $Lu=0$ in $\Omega(2r, Q) \cap D$. There exists $c>0$ depending only on λ, n, and m so that if h is any natural number, we have*

$$u(X) \leqq c^h \cdot u(Y)$$

for all X and $Y \in \Omega(3r/2, Q) \cap D$ such that $\operatorname{dist}(X, \partial D) > r/2^h$ *and* $\operatorname{dist}(Y, \partial D) > r/2^h$.

Proof. From the Lipschitz structure of D it follows that there is a natural number M depending only on m so that for X and Y as above, there is a chain of balls, $B_0, \ldots, B_j$ with $j \leqq Mh$ such that $X \in B_0$ and $Y \in B_j$; $B_i \cap B_{i+1} \neq \emptyset$ for $i=0, \ldots, j$; and $2B_i \subset \Omega(2r, Q) \cap D$ for $i=0, \ldots, j$. (Here, $2B_i$ denotes the open ball concentric with B_i with radius twice that of B_i.) The Harnack principle implies that

$$\sup_{B_i} u \leqq c_0 \cdot \inf_{B_i} u$$

for $i=0, \ldots, j-1$ where c_0 depends only on λ and n. Hence

$$u(X) \leqq c_0^{2j} \cdot u(Y) \leqq c_0^{2Mh} \cdot u(Y).$$

Lemma 2.3. *Suppose $Q \in \partial D$, $0 < r < r_0$, and u is a positive solution of $Lu=0$ in $\Omega(r, Q) \cap D$ which vanishes continuously on $\Delta(r, Q)$. There exists c and $\alpha > 0$ depending only on $\lambda, n,$ and m such that*

$$u(X) \leqq c \cdot (|X-Q|/r)^{\alpha} \cdot M(u)$$

for all $X \in \Omega(r, Q) \cap D$, where $M(u) = \sup \{u(X) \colon X \in \Omega(r, Q) \cap D\}$.

Proof. This is an immediate consequence of the maximum principle and K. Miller's construction of Hölder continuous barriers defined on cones in $\mathbf{R}^n$.

The following result is often called a Carleson estimate.

Lemma 2.4. *Suppose $Q \in \partial D$, $0 < r < r_0$, and u is a positive solution of $Lu=0$ in $\Omega(2r, Q) \cap D$ which vanishes continuously on $\Delta(2r, Q)$. There exists $c > 0$ depending only on $\lambda, n,$ and m such that*

$$u(X) \leqq c \cdot u(A_r(Q))$$

for all $X \in \Omega(r, Q) \cap D$.

Proof. By Lemma 2.3 there exists $c_1 \geqq 2$ depending on $\lambda, n,$ and m so that if $P \in \Delta(2r, Q)$ and $\Omega(s, P) \subset \Omega(2r, Q)$ (where $\Omega(s, P)$ is defined with respect to the coordinate system in $\Omega(2r, Q)$), then

(1) $$\sup \{u(X) \colon X \in \Omega(s/c_1, P) \cap D\} \leqq (1/2) \cdot \sup \{u(X) \colon X \in \Omega(s, P) \cap D\}.$$

Normalize u so that $u(A_r(Q)) = 1$. By Lemma 2.2, there exists $c_2 > 1$ depending on $\lambda, n, m,$ and c_1 so that if $Y \in \Omega(3r/2, Q) \cap D$ and $u(Y) > c_2^h$, then

(2) $$\operatorname{dist}(Y, \partial D) < c_1^{-h} r.$$

Choose $M \geqq 1$ so that $2^M > c_2$. Let $N = M+5$ and define $c = c_2^N$. Suppose there exists $Y_0 = (x_0, y_0)$ in $\Omega(r, Q) \cap D$ with $u(Y_0) > c \cdot u(A_r(Q)) = c = c_2^N$. Then $\operatorname{dist}(Y_0, \partial D) < c_1^{-N} r$. If $Q_0 = (p_0, q_0)$ is a point of ∂D nearest to Y_0 and if $Q = (p, q)$, we have

$$\begin{aligned} |p_0 - p| &\leqq |p_0 - x_0| + |x_0 - p| \\ &< c_1^{-N} r + r \leqq (2^{-5} + 1) r, \end{aligned}$$

and

$$\begin{aligned} |q_0 - q| &\leqq |q_0 - y_0| + |y_0 - q| \\ &\leqq c_1^{-N} r + mr \leqq (2^{-5} + 1) \cdot mr. \end{aligned}$$

Thus $Q_0 \in \Omega(33r/32, Q)$ and we conclude that $\Omega(c_1^{-5}r, Q_0) \subset \Omega(3r/2, Q)$. Applying (1), we have:

$$\begin{aligned}
&\sup\{u(X)\colon X \in \Omega(c_1^{-5}r, Q_0) \cap D\} \\
&= \sup\{u(X)\colon X \in \Omega(c_1^{-N+M}r, Q_0) \cap D\} \\
&\geqq 2^M \cdot \sup\{u(X)\colon X \in \Omega(c_1^{-N}r, Q_0) \cap D\} \\
&\geqq 2^M \cdot u(Y_0) > c_2^{N+1}.
\end{aligned}$$

Hence we can choose $Y_1 = (x_1, y_1) \in \Omega(c_1^{-5}r, Q_0) \cap D$ such that $u(Y_1) > c_2^{N+1}$. By (2), $\operatorname{dist}(Y_1, \partial D) < c_1^{-N-1}r$. Let $Q_1 = (p_1, q_1)$ be a point of ∂D closest to Y_1. Continuing in this manner, we obtain two sequences, $\{Y_k\}$ and $\{Q_k\}$, with $Y_k = (x_k, y_k)$ and $Q_k = (p_k, q_k) \in \partial D$ such that:

(i) $$Y_k \in \Omega(c_1^{M-N-(k-1)}r, Q_{k-1}) \cap D = \Omega(c_1^{-5-(k-1)}r, Q_{k-1}) \cap D$$

(ii) $$\operatorname{dist}(Y_k, \partial D) = |Y_k - Q_k| < c_1^{-N-k}r$$

(iii) $$\Omega(c_1^{-5-(k-1)}r, Q_{k-1}) \subset \Omega(3r/2, Q)$$

(iv) $$u(Y_k) > c_2^{N+k}.$$

The fact that $\operatorname{dist}(Y_k, \partial D) \to 0$ and $u(Y_k) \to +\infty$ contradicts the hypothesis that u vanishes continuously on $\Delta(2r, Q)$.

Lemma 2.5. *Suppose $Q = (x_0, y_0) \in \partial D$, $0 < r < r_0$, and u and v are positive solutions of $Lu = 0$ in $\Omega(4r, \Omega) \cap D \cap \{(x, y)\colon y < y_0 + 2mr\}$ which vanish continuously on the bottom and sides,*

$$\Delta(4r, Q) \cup \left[\partial\Omega(4r, Q) \cap \{(x, y)\colon |x - x_0| = 4r,\ \varphi(x) \leqq y < y_0 + 2mr\}\right].$$

There exists $c > 0$ depending only on λ, n, and m such that

$$\frac{u(X)}{u(A_r(Q))} \leqq c \cdot \frac{v(X)}{v(A_r(Q))}$$

for all $X \in \Omega(4r, Q) \cap D \cap \{(x, y)\colon y < y_0 + mr\}$.

Proof. By the maximum principle it is sufficient to prove the above inequality for all $X \in \overline{\Omega(4r, Q)} \cap \{(x, y)\colon y = y_0 + mr\} \equiv \Sigma$. For simplicity, we shall denote all positive constants depending only on λ, n, and m by c.

Fix $P \in \partial\Omega(4r, Q) \cap \Sigma$ and let N denote the inward unit normal at P in $\Omega(4r, Q)$. Let $P_0 = P + rN/4$ and define

$$h(X) = \frac{\left(e^{-64n|X-P_0|^2/\lambda^2 r^2} - e^{-4n/\lambda^2}\right)}{\left(e^{-n/\lambda^2} - e^{-4n/\lambda^2}\right)}.$$

An elementary calculation implies that

$$Lh(X) \geqq [128ne^{-64n|X-P_0|^2/\lambda^2 r^2}/\lambda^2 r^2(e^{-n/\lambda^2}-e^{-4n/\lambda^2})]$$
$$\cdot[(128n|X-P_0|^2/\lambda r^2)-2n/\lambda] \geqq 0$$

for all $X \in B(r/4, P_0) \cap B(r/8, P)$ and

$$\inf\left\{\frac{\partial h}{\partial N}(P+tN)\colon 0 \leqq t \leqq r/16\right\} \geqq c/r.$$

By the Harnack principle, we have

$$u(X) \geqq c \cdot u(A_r(Q)) \geqq c \cdot u(A_r(Q)) \cdot h(X)$$

for all $X \in B(r/4, P_0) \cap \partial B(r/8, P)$. Since h is zero on $\partial B(r/4, P_0) \cap B(r/8, P)$, the above inequality extends (by the maximum principle) to all $X \in B(r/4, P_0) \cap B(r/8, P)$ and hence

(1) $$u(P+tN) \geqq u(A_r(Q)) \cdot ct/r$$

whenever $0 \leqq t \leqq r/16$.

We proceed now to prove a similar estimate on u from above. Let $P_1 = P - rN/16$, with P and N as before, and define

$$f(X) = 1-[r/(16|X-P_1|)]^{2n/\lambda^2}$$

in $\Omega(4r, Q)$. An elementary calculation implies that

$$Lf(X) \leqq (r/16)^{2n/\lambda^2} \cdot |X-P_1|^{-2-2n/\lambda^2} \cdot (-4n/\lambda) \leqq 0$$

for all $X \in \Omega(4r, Q) \cap B(r/8, P_1)$ and

$$\sup\left\{\frac{\partial f}{\partial N}(P+tN)\colon 0 \leqq t \leqq r/16\right\} \leqq c/r.$$

By Lemma 2.4, we have

$$u(X) \leqq c \cdot u(A_r(Q))$$

for all $X \in B(r/8, P_1) \cap \Omega(4r, Q)$. Since $f \geqq 0$ in $\Omega(4r, Q)$ and $f \geqq c$ on $\partial B(r/8, P_1)$, we deduce by the maximum principle that

$$u(X) \leqq c \cdot u(A_r(Q)) \cdot f(X)$$

for all $X \in B(r/8, P_1) \cap \Omega(4r, Q)$. Since $f(P)=0$, we obtain

(2) $$u(P+tN) \leqq u(A_r(Q)) \cdot ct/r$$

whenever $0 \leqq t \leqq r/16$.

Inequalities (1) and (2) hold for both u and v. Thus

$$u(X)/u(A_r(Q)) \leqq c \cdot v(X)/v(A_r(Q))$$

whenever $X \in \Sigma$ and $\operatorname{dist}(X, \partial\Sigma) \leqq r/16$. By the Harnack principle we may redefine c so that the above inequality holds for all X in Σ.

For completeness we define what we mean by the Green's function for L in a bounded Lipschitz domain, D. Our definition is based on the fact that for each f in $L^p(D)$ with $p>n$, there is a unique function, u, in $W^{2,p}_{\text{loc}}(D) \cap C(\bar{D})$ such that $Lu=-f$ in D and $u=0$ on ∂D. (This follows by approximating f and the coefficients of L with smooth functions and using the L^p-Schauder estimates ([1]) and barriers constructed in [12] to show that the corresponding solutions converge uniformly in $\bar{D}$.) Pucci's estimate ([14]) says that

$$\|u\|_{L^\infty(D)} \leqq c \cdot \|f\|_{L^p(D)}.$$

Hence for each fixed X in D, the mapping $f \to u(X)$ is a continuous, positive linear functional on $L^p(D)$. The Riesz Representation Theorem implies the existence of a nonnegative function, $G(X, \cdot)$, in $L^{p/(p-1)}(D)$ such that $u(X)= \int_D G(X, Y) f(Y)\, dY$. The function, $G(X, Y)$, is called the Green's function for L in D.

The following result (which we *state* as Theorem 2.6) is proved in Section 5.

Theorem 2.6. *Suppose* $Q=(x_0, y_0) \in \partial D$ *and* $0<r<r_0$. *Let* $A_r'(Q)= (x_0, y_0+3mr)$ *and* $\psi(r, Q)=D \cap [\Omega(3r, Q) \setminus \Omega(2r, Q)]$. *There exists* $c>0$ *depending only on* λ, n, w, r_0, *and* m *such that*

$$\int_{\psi(r,Q)} G_r(X, Y)\, dY \leqq c \cdot \int_{\Phi(r/\sqrt{n},\, A_r'(Q))} G_r(X, Y)\, dY$$

for all $X \in \Omega(r, Q) \cap D$, *where* $G_r(X, Y)$ *is the Green's function for* L *and the domain* $\Omega(4r, Q) \cap D$.

We now *claim* that the above theorem implies the comparison theorem (Theorem 2.1). The proof is the following:

Proof of the Claim. Suppose Theorem 2.6 holds. Fix $Q=(x_0, y_0) \in \partial D$ and $r<r_0$, and suppose u and v are positive solutions of $Lu=0$ in $\Omega(8r, Q) \cap D$ which vanish continuously on $\Delta(8r, Q)$. Let ω_r^X be the L-harmonic measure at X for the domain $\Omega(4r, Q) \cap D$. (That is, for each $X \in D$, ω_r^X is the measure corresponding to the linear functional: $\varphi \to u(X)$, where φ is a continuous function defined on the boundary of $\Omega(4r, Q) \cap D$ and u is the solution of $Lu=0$ in $\Omega(4r, Q) \cap D$ with boundary values, φ.) Let $\alpha_r = \bar{D} \cap \partial\Omega(4r, Q)$ and $\beta_r = \partial\Omega(4r, Q) \cap \{(x, y)\colon y=y_0+4mr\}$. By Lemma 2.4 there exists $c_1>0$ depending on λ, n, and m such that

$$u(X) \leqq c_1 \cdot u(A_r(Q)) \cdot \omega_r^X(\alpha_r)$$

for all $X \in \Omega(4r, Q) \cap D$. By the Harnack principle, we have

$$\omega_r^X(\beta_r) \cdot v(A_r(Q)) \leqq c_2 \cdot v(X)$$

for all $X\in\beta_r$ and hence for all $X\in\Omega(4r, Q)\cap D$ by the maximum principle. Thus we obtain the conclusion of Theorem 2.1 if we show that

(1) $$\omega_r^X(\alpha_r) \leqq c\cdot\omega_r^X(\beta_r)$$

for all $X\in\Omega(r, Q)\cap D$, where c depends on λ, n, w, r_0, and m.

Choose $h\in C^\infty(\mathbf{R}^n)$ such that $0\leqq h\leqq 1$ in $\mathbf{R}^n$, $h=0$ in $\Omega(2r, Q)$, $h=1$ in $\mathbf{R}^n\setminus\Omega(3r, Q)$, and

$$\|h\|_{C^2(\mathbf{R}^n)} \leqq \eta/r^2,$$

where η depends only on n and m. If $X\in\Omega(r, Q)\cap D$, we have

$$\omega_r^X(\alpha_r) \leqq \int_{\partial[\Omega(4r, Q)\cap D]} h(P)\,d\omega_r^X(P) = \int_{\partial[\Omega(4r, Q)\cap D]} h(P)\,d\omega_r^X(P) - h(X)$$

$$= \int_{\Omega(4r, Q)\cap D} G_r(X, Y)\,Lh(Y)\,dY \leqq \frac{c_3}{r^2}\cdot\int_{\psi(r, Q)} G_r(X, Y)\,dY,$$

where c_3 depends only on η, λ, and n. The above inequality and Theorem 2.6 imply that

(2) $$\omega_r^X(\alpha_r) \leqq \frac{c_3\cdot c}{r^2}\cdot\int_{\Phi(r/\sqrt{n}, A_r'(Q))} G_r(X, Y)\,dY \equiv c_3 c\cdot f(X)$$

for all $X\in\Omega(r, Q)\cap D$. The functions, $f(X)$ and $\omega_r^X(\beta_r)$, satisfy the hypotheses of Lemma 2.5 in $\Omega(4r, Q)\cap D$. In addition, it is easily seen (by a dilation argument) that $f(A_r(Q))$ and $\omega^{A_r(Q)}(\beta_r)$ are bounded above and below by positive constants depending only on λ, n, and m. Hence (by Lemma 2.5)

(3) $$f(X) \leqq c_4\cdot\omega_r^X(\beta_r)$$

for all $X\in\Omega(r, Q)\cap D$, where c_4 depends only on λ, n, and m. We obtain inequality (1) by combining (2) and (3).

3. Interior A_∞-estimates of nonnegative adjoint solutions

We have reduced the proof of the comparison theorem to an integral inequality on the Green's function. In Sections 3 and 4 we prove *a priori* estimates of nonnegative solutions of $L^*v=0$ which enable us to prove this inequality.

We defined solutions of $L^*v=0$ in the introduction. We shall say that $L^*v\geqq 0$ in D (or $\leqq 0$) if $v\in L^1_{\mathrm{loc}}(D)$ and

$$\int_D v(Y)\,L\varphi(Y)\,dY \geqq 0 \quad (\text{or} \leqq 0)$$

for all nonnegative $\varphi\in C_0^\infty(D)$.

Throughout this section we denote by $\Phi(r)$ an arbitrary open cube in $\mathbf{R}^n$ of side length r. The concentric open cube of side length σr will be denoted by $\Phi(\sigma r)$.

Theorem 3.1. *Suppose* $0<r\leqq 1$, $1<p<n/(n-1)$, *and* $0<\sigma<\gamma<1$. *There exists* $c>0$ *such that if* $v\in L^p(\Phi(r))$, $v\geqq 0$ *in* $\Phi(r)$, *and* $L^*v\geqq 0$ *in* $\Phi(r)$, *then*

$$\left(\frac{1}{|\Phi(\sigma r)|}\cdot\int_{\Phi(\sigma r)} v(Y)^p\,dY\right)^{1/p}\leqq\frac{c}{|\Phi(\gamma r)|}\cdot\int_{\Phi(\gamma r)} v(Y)\,dY.$$

The constant, c, depends only on $\lambda, n, w, p, \sigma,$ *and* γ.

Proof. It is sufficient to prove this result in the case $r=1$; the theorem then follows by a change of variables.

Fix a smooth domain, Ω, in $\Phi(1)$ such that $\overline{\Phi(\gamma)}\subset\Omega$. Choose $h\in C_0^\infty(\Phi(\gamma))$ such that $h\geqq 0$ in $\Phi(\gamma)$ and $h=1$ in $\Phi(\sigma)$. Suppose $v\in L^p(\Phi(1))$, $v\geqq 0$ in $\Phi(1)$, and $L^*v\geqq 0$ in $\Phi(1)$. Define $f\in L^{p'}(\Omega)$, where $1/p+1/p'=1$, by setting $f=v^{p-1}$ in $\Phi(\sigma)$ and $f=0$ in $\Omega\setminus\Phi(\sigma)$. Let u be the unique function in $\mathring{W}^{2,p'}(\Omega)\cap C(\overline{\Omega})$ which satisfies $Lu=f$ in Ω. From the maximum principle, we have $u\leqq 0$ in Ω. Hence

$$\begin{aligned}\int_{\Phi(\sigma)} v^p\,dY&=\int_{\Phi(\sigma)} vf\,dY\leqq\int_{\Phi(\gamma)} hv\cdot Lu\,dY\\&=\int_{\Phi(\gamma)}[L(uh)-u\cdot Lh-2a_{ij}\cdot D_{Y_i}u\cdot D_{Y_j}h]v\,dY\\&\leqq\int_{\Phi(\gamma)}[-u\cdot Lh-2a_{ij}\cdot D_{Y_i}u\cdot D_{Y_j}h]v\,dY.\end{aligned}$$

Since $p'>n$, we deduce from the Sobolev inequalities, L^p-Schauder estimates, and Pucci's estimate ([14]) that

$$\begin{aligned}\int_{\Phi(\sigma)} v^p\,dY&\leqq c\cdot\|u\|_{W^{2,p'}(\Omega)}\cdot\int_{\Phi(\gamma)} v\,dY\\&\leqq c\cdot(\|Lu\|_{L^{p'}(\Omega)}+\|u\|_{L^{p'}(\Omega)})\cdot\int_{\Phi(\gamma)} v\,dY\\&\leqq c\cdot\|Lu\|_{L^{p'}(\Omega)}\cdot\int_{\Phi(\gamma)} v\,dY=c\cdot\left(\int_{\Phi(\sigma)} v^p\,dY\right)^{1/p'}\cdot\int_{\Phi(\gamma)} v\,dY,\end{aligned}$$

where c depends on $\lambda, n, w, p, h,$ and Ω. This proves the theorem.

Theorem 3.2. *Suppose* $r>0$ *and* $0<\sigma<\gamma<1$. *There exists* $c>0$ *depending only on* $\lambda, n, \sigma,$ *and* γ *such that if* $v\geqq 0$ *in* $\Phi(r)$ *and* $L^*v\leqq 0$ *in* $\Phi(r)$, *then* $\int_{\Phi(\gamma r)} v\,dY\leqq c\cdot\int_{\Phi(\sigma r)} v\,dY$.

Proof. The theorem follows if we show that there exists $\theta\in(0, 1)$ depending only on λ and n such that if $\theta<\gamma<1$ and $r>0$, then

$$\int_{\Phi(\gamma r)} v\,dY\leqq c\cdot\int_{\Phi(\theta r)} v\,dY \tag{1}$$

whenever $v \geqq 0$ and $L^*v \leqq 0$ in $\Phi(r)$, where c depends only on λ, n, θ, and γ. Iterations of the above result would imply that

$$\int_{\Phi(\gamma r)} v\, dY \leqq c^k \cdot \int_{\Phi(\theta^k r/\gamma^{k-1})} v\, dY$$

for any natural number k; by choosing k so that $\theta^k < \gamma^{k-1}\sigma$, we obtain the result of the theorem.

To prove (1) we may assume $r=1$ and $\Phi(1)$ is centered at the origin; as before, the general result follows by a change of variables. Consider the function in $C_0^\infty(\mathbf{R}^n)$ defined by

$$h(X) = \begin{cases} e^{\sum_{i=1}^n \frac{-1}{1-4x_i^2}}, & \text{if } X \in \Phi(1) \\ 0, & \text{if } X \in \mathbf{R}^n \setminus \Phi(1), \end{cases}$$

where $X=(x_1, \dots, x_n)$. If $X \in \Phi(1)$ and we set $s=s(X)=\max\{2|x_i| : 1 \leqq i \leqq n\}$, we have

$$Lh(X) = 8e^{\sum_{i=1}^n \frac{-1}{1-4x_i^2}}$$

$$\cdot \left\{ \sum_{i,j=1}^n \frac{a_{ij}(X) \cdot 8x_i x_j}{(1-4x_i^2)^2 \cdot (1-4x_j^2)^2} - \sum_{i=1}^n \left[\frac{a_{ii}(X)(1+12x_i^2)}{(1-4x_i^2)^3} + \frac{b_i(X) \cdot x_i}{(1-4x_i^2)^2} \right] \right\}$$

$$\geqq \frac{8e^{-n/(1-s^2)}}{(1-s^2)^2} \cdot \left[\frac{(2\lambda s^2)}{(1-s^2)^2} - \frac{4n}{\lambda(1-s^2)} - \frac{n}{\lambda} \right].$$

Hence if we choose $\theta \in (0, 1)$ so that

$$\inf_{\theta \leqq s \leqq 1} \left\{ \frac{2\lambda s^2}{(1-s^2)^2} - \frac{4n}{\lambda(1-s^2)} \right\} \geqq \frac{2n}{\lambda}$$

we have $Lh \geqq 0$ in $\Phi(1) \setminus \Phi(\theta)$ and $Lh \geqq c_1 > 0$ in $\Phi(\gamma) \setminus \Phi(\theta)$ for each $\gamma \in (\theta, 1)$, where c_1 depends only on λ, n, θ, and γ.

Now suppose $v \geqq 0$ and $L^*v \leqq 0$ in $\Phi(1)$. Choose c_2 depending on λ and n such that $|Lh| \leqq c_2$ in $\Phi(1)$. If $\gamma \in (\theta, 1)$ and h is the function defined above, then

$$c_1 \int_{\Phi(\gamma)\setminus\Phi(\theta)} v\, dY \leqq \int_{\Phi(\gamma)\setminus\Phi(\theta)} v \cdot Lh\, dY \leqq \int_{\Phi(1)\setminus\Phi(\theta)} v \cdot Lh\, dY$$
$$\leqq -\int_{\Phi(\theta)} v \cdot Lh\, dY \leqq c_2 \cdot \int_{\Phi(\theta)} v\, dY$$

and we conclude that

$$c_1 \cdot \int_{\Phi(\gamma)} v\, dY \leqq (c_1 + c_2) \cdot \int_{\Phi(\theta)} v\, dY.$$

This proves (1) when $r=1$.

The two previous theorems imply the following "reverse-Hölder" inequality.

Theorem 3.3. *Suppose* $0 < r \leqq 1$, $1 < p < n/(n-1)$, *and* $0 < \sigma < 1$. *There exists* $c > 0$ *depending only on* λ, n, w, p, *and* σ *such that if* $v \in L^p(\Phi(r))$, $v \geqq 0$ *in* $\Phi(r)$, *and* $L^*v=0$ *in* $\Phi(r)$, *then*

$$\left(\frac{1}{|\Phi(\sigma r)|}\cdot\int_{\Phi(\sigma r)} v^p\,dY\right)^{1/p} \leqq \frac{c}{|\Phi(\sigma r)|}\cdot\int_{\Phi(\sigma r)} v\,dY.$$

R. Coifman and C. Fefferman ([6]) showed that Theorems 3.2 and 3.3 imply the following estimates (Corollaries 3.4 and 3.5):

Corollary 3.4. *Suppose* $0<r\leqq 1$, $1<p<n/(n-1)$, *and* $0<\sigma<1$. *There exist positive constants* $q>1$, c_1, c_2, *and* δ *depending only on* λ, n, w, p *and* σ *so that if* $v\in L^p(\Phi(r))$, $v\geqq 0$ *in* $\Phi(r)$, *and* $L^*v=0$ *in* $\Phi(r)$, *then:*

(i) A_∞*-estimate: For any measurable subset,* E, *of* $\Phi(\sigma r)$, *we have*

$$\frac{v(E)}{v(\Phi(\sigma r))} \leqq c_1\left(\frac{|E|}{|\Phi(\sigma r)|}\right)^{\delta},$$

where $v(E)=\int_E v\,dY$.

(ii) A_q*-estimate:*

$$\left(\frac{1}{|\Phi(\sigma r)|}\cdot\int_{\Phi(\sigma r)} v\,dY\right)\cdot\left(\frac{1}{|\Phi(\sigma r)|}\cdot\int_{\Phi(\sigma r)} v^{-1/(q-1)}\,dY\right)^{q-1} \leqq c_2.$$

Corollary 3.5. *Suppose* $0<r\leqq 1$, $1<p<n/(n-1)$, *and* $0<\sigma<1$. *Given* $\varepsilon>0$, *there exists* $\beta>0$ *depending only on* ε, c_2, *and* q *(the constants in the* A_q*-estimate) so that if* $v\in L^p(\Phi(r))$, $v\geqq 0$ *in* $\Phi(r)$, *and* $L^*v=0$ *in* $\Phi(r)$, *then*

$$(1-\varepsilon)\cdot|\Phi(\sigma r)|$$
$$\leqq \left|\left\{Y\in\Phi(\sigma r)\colon \frac{\beta}{|\Phi(\sigma r)|}\cdot\int_{\Phi(\sigma r)} v(Z)\,dZ \leqq v(Y) \leqq \frac{1}{\beta\,|\Phi(\sigma r)|}\cdot\int_{\Phi(\sigma r)} v(Z)\,dZ\right\}\right|.$$

The A_∞-estimate of Corollary 3.4 may be interpreted as a generalized Harnack principle; its relationship to the classical Harnack inequality is shown in Corollary 3.5. It is known that if the coefficients of L are Hölder continuous, nonnegative solutions of $L^*v=0$ have continuous representatives ([15]) and satisfy a classical Harnack inequality ([2]). These properties do not hold without such restrictions, as shown by the example in which the Green's function, $G(X,\cdot)$, is not in $L^\infty_{\mathrm{loc}}(D\setminus\{X\})$. (See [4].)

4. Normalized adjoint solutions

Throughout this section we assume (in addition to our previous hypotheses) that the coefficients of L are *smooth* functions. We assume $\mathbf{B}$ is a fixed open ball in $\mathbf{R}^n$ and A is a fixed point in $\mathbf{B}$. We let $G_{\mathbf{B}}(X, Y)$ denote the representative of the Green's function for L and $\mathbf{B}$ which is continuous at all points, (X, Y) in $\mathbf{B}\times\mathbf{B}$ such that $X\neq Y$.

If Ω is an open subset of $\mathbf{B}\setminus\{A\}$, we say that $\tilde{v}$ is a normalized adjoint solution (with respect to L and $G_{\mathbf{B}}(A,\cdot)$) in Ω if $\tilde{v}\in C(\Omega)$ and

$$\tilde{v}(Y)=v(Y)/G_{\mathbf{B}}(A,Y)$$

for all Y in Ω, where $L^*v=0$ in Ω.

Our objective in this section is to establish uniform pointwise estimates on normalized adjoint solutions which will be used to prove Theorem 2.6. In particular, we prove a maximum principle and a Harnack principle and we construct uniform barriers on truncated cones. Based on these properties we obtain a Carleson-type estimate for positive normalized adjoint solutions which vanish on a portion of the boundary of D. Although we consider only those operators with smooth coefficients, our estimates depend only on λ, n, and w.

The maximum principle and Harnack principle will be proved as consequences of the following proposition.

Proposition 4.1. *Suppose Ω is a smooth domain such that $\bar{\Omega}\subset\mathbf{B}\setminus\{A\}$. Given $\varphi\in C(\partial\Omega)$, there is a unique function $v\in C(\bar{\Omega})$, such that $L^*v=0$ in Ω and $v=\varphi$ on $\partial\Omega$. Moreover,*

$$v(Y)=\int_{\partial\Omega}\Phi(Q)\cdot\frac{\partial G}{\partial\nu_Q}(Q,Y)\,ds(Q)$$

for all $Y\in\Omega$. Here, $G(X,Y)$ is the Green's function for L in Ω; $\frac{\partial G}{\partial\nu_Q}(Q,Y)$ is the inward conormal derivative of $G(\cdot,Y)$ at $Q\in\partial\Omega$ (i.e.,

$$\frac{\partial G}{\partial\nu_Q}(Q,Y)=\sum_{i,j=1}^{n}a_{ij}(Q)\cdot N_j(Q)\cdot\frac{\partial G}{\partial x_i}(Q,Y),$$

where $N(Q)=(N_1,\dots,N_n)$, the inward unit normal at Q); and ds denotes surface measure on $\partial\Omega$.

Proof. See [13], p. 77. (The result also holds when L has Hölder continuous coefficients; see [15].)

Theorem 4.2 (Maximum Principle). *Suppose Ω is a Lipschitz domain and $\bar{\Omega}\subset\mathbf{B}\setminus\{A\}$. If $\tilde{v}$ is a normalized adjoint solution in Ω and $\tilde{v}\in C(\bar{\Omega})$, then*

$$\|\tilde{v}\|_{L^\infty(\Omega)}\leqq\|\tilde{v}\|_{L^\infty(\partial\Omega)}.$$

Proof. If Ω is a smooth domain, Proposition 4.1 implies that

$$|\tilde{v}(Y)|=\left|\int_{\partial\Omega}\tilde{v}(Q)\cdot\frac{G_{\mathbf{B}}(A,Q)}{G_{\mathbf{B}}(A,Y)}\cdot\frac{\partial G_\Omega}{\partial\nu_Q}(Q,Y)\,ds(Q)\right|$$

$$\leqq\|\tilde{v}\|_{L^\infty(\partial\Omega)}\cdot\int_{\partial\Omega}\frac{G_{\mathbf{B}}(A,Q)}{G_{\mathbf{B}}(A,Y)}\cdot\frac{\partial G_\Omega}{\partial\nu_Q}(Q,Y)\,ds(Q)=\|\tilde{v}\|_{L^\infty(\partial\Omega)}.$$

The theorem now follows by applying the above result to smooth subdomains of Ω.

The following result will be used in our proof of the Harnack principle. We omit the proof, since it is essentially the same as the proof of Lemma 2.5.

Lemma 4.3. *Suppose $r>0$ and $P\in\mathbf{R}^n$. There exist positive constants, c_1 and c_2, depending only on λ and n such that if u is a positive solution of $Lu=0$ in $B(3r, P)\setminus\overline{B(r, P)}$ which vanishes continuously on $\partial B(3r, P)$, then*

$$\frac{c_1}{r}\cdot u(Q+rN_Q) \leqq \frac{\partial u}{\partial \nu_Q}(Q) \leqq \frac{c_2}{r}\cdot u(Q+rN_Q)$$

for all $Q\in\partial B(3r, P)$. Here, $N_Q=(P-Q)/|P-Q|$ and $\partial u/\partial \nu_Q$ is the inward conormal derivative of u at Q.

Theorem 4.4 (Harnack Principle). *Suppose $r>0$ and $\overline{B(4r, P)}\subset\mathbf{B}\setminus\{A\}$. There exists $c>0$ depending only on λ and n so that if $\tilde{v}$ is a nonnegative normalized adjoint solution in $B(4r, P)$, then*

$$\sup_{B(r,P)} \tilde{v} \leqq c\cdot \inf_{B(r,P)} \tilde{v}.$$

Proof. Let $G_r(X, Y)$ be the Green's function for L and $B(3r, P)$. Suppose $Y\in B(r, P)$. For each $Q\in\partial B(3r, P)$, let $X_Q=Q+r(P-Q)/|P-Q|$. By Lemma 4.3 we have

$$\frac{c_1}{r}\cdot G_r(X_Q, Y) \leqq \frac{\partial G_r}{\partial \nu_Q}(Q, Y) \leqq \frac{c_2}{r}\cdot G_r(X_Q, Y).$$

Thus

$$\tilde{v}(Y) = \int_{\partial B(3r,P)} \tilde{v}(Q)\cdot\frac{G_{\mathbf{B}}(A, Q)}{G_{\mathbf{B}}(A, Y)}\cdot\frac{\partial G_r}{\partial \nu_Q}(Q, Y)\,ds(Q) \tag{1}$$

$$\leqq \frac{c_2}{r}\cdot\int_{\partial B(3r,P)} \tilde{v}(Q)\cdot\frac{G_{\mathbf{B}}(A, Q)}{G_{\mathbf{B}}(A, Y)}\cdot G_r(X_Q, Y)\,ds(Q).$$

Fix a point $P_0\in\partial B(2r, P)$. From inequality (1) and the Harnack principle for nonnegative solutions of $Lu=0$, we obtain

$$\tilde{v}(Y) \leqq \frac{c_3}{r}\cdot\frac{G_r(P_0, Y)}{G_{\mathbf{B}}(A, Y)}\cdot\int_{\partial B(3r,P)} \tilde{v}(Q)\cdot G_{\mathbf{B}}(A, Q)\,ds(Q). \tag{2}$$

Similarly, we deduce from Proposition 4.1 and Lemma 4.3 that

$$G_{\mathbf{B}}(A, Y) = \int_{\partial B(3r,P)} G_{\mathbf{B}}(A, Q)\cdot\frac{\partial G_r}{\partial \nu_Q}(Q, Y)\,ds(Q)$$

$$\geqq \frac{c_4}{r}\cdot G_r(P_0, Y)\cdot\int_{\partial B(3r,P)} G_{\mathbf{B}}(A, Q)\,ds(Q).$$

Hence

$$\tilde{v}(Y) \leqq c\cdot\int_{\partial B(3r,P)} \tilde{v}(Q)\cdot G_{\mathbf{B}}(A, Q)\, ds(Q)\Big/\int_{\partial B(3r,P)} G_{\mathbf{B}}(A, Q)\, ds(Q),$$

where c is a positive constant depending only on λ and n. A completely analogous argument implies that

$$\tilde{v}(Y) \geqq c^{-1}\cdot\int_{\partial B(3r,P)} \tilde{v}(Q)\cdot G_{\mathbf{B}}(A, Q)\, ds(Q)\Big/\int_{\partial B(3r,P)} G_{\mathbf{B}}(A, Q)\, ds(Q)$$

which proves the theorem.

The Harnack principle and the fact that constants are normalized adjoint solutions imply the following interior Hölder estimate. We omit the standard proof. (See [8], p. 190.)

Theorem 4.5. *Suppose* $r>0$ *and* $\overline{B(2r, P)}\subset\mathbf{B}\setminus\{A\}$. *There exist positive constants* c *and* α *depending only on* λ *and* n *so that if* $0<\sigma<1$ *and* $\tilde{v}$ *is a normalized adjoint solution in* $B(2r, P)$, *then*

$$\operatorname*{osc}_{B(\sigma r,P)} \tilde{v} \leqq c\cdot\sigma^{\alpha}\cdot \operatorname*{osc}_{B(r,P)} \tilde{v}.$$

Here, $\operatorname*{osc}_{E} \tilde{v}\equiv\sup\{|\tilde{v}(Y_1)-\tilde{v}(Y_2)|: Y_1, Y_2\in E\}$.

The following lemma will be used to construct uniform barriers for normalized adjoint solutions on small cones.

Lemma 4.6. *Suppose* $0<\theta<\pi$ *and* $r>0$. *Let* $B_r=B(r, 0)$, $\Gamma_r=B_r\cap\{X\in\mathbf{R}^n\setminus\{0\}: \cos^{-1}(x_n/|X|)<\theta\}$ (*with* $X=(x_1, \dots, x_n)$) *and* $A_r=(0, \dots, 0, 3r/4)$. *For each* $\varepsilon>0$ *there exist positive constants* $R\leqq 1$ *and* $\delta\leqq 1/4$ *depending only on* $\varepsilon, \theta, \lambda, n$, *and* w *so that if* $0<r<R$, *we have*

$$\sup\{G_{\Gamma_r}(A_r, Y)/G_{B_{2r}}(A_r, Y): Y\in\Gamma_r\cap B_{\delta r}\} \leqq \varepsilon.$$

Here $G_{\Gamma_r}(X, Y)$ *and* $G_{B_{2r}}(X, Y)$ *denote the Green's functions for* L *and the domains,* Γ_r *and* B_{2r}, *respectively.*

Proof. Suppose the lemma is false. Then there exist $\varepsilon_0>0$ and sequences $\{r_k\}$, $\{\delta_k\}$, and $\{Y_k\}$ such that $0<r_k\leqq 2^{-k}$, $0<\delta_k\leqq 2^{-k}$, $Y_k\in\Gamma_{\delta_k r_k}$, and

$$G_{\Gamma_{r_k}}(A_{r_k}, Y_k)/G_{B_{2r_k}}(A_{r_k}, Y_k) \geqq \varepsilon_0 \tag{1}$$

for all k. Let $\Gamma=\Gamma_1$, $B=B_2$, and

$$L_k = \sum_{i,j=1}^{n} a_{ij}(r_k X)\cdot D^2_{X_i X_j} + \sum_{i=1}^{n} r_k\cdot b(r_k X)\cdot D_{X_i}.$$

Let $G^k_\Gamma(X, Y)$ and $G^k_B(X, Y)$ denote the Green's functions for L_k in the domains, Γ and B, respectively. Inequality (1) implies that

$$G^k_\Gamma(A_1, Z_k)/G^k_B(A_1, Z_k) \geqq \varepsilon_0$$

for all k, where $Z_k=Y_k/r_k$.

We may assume without loss of generality that $a_{ij}(0)=\delta_{ij}$, so that the coefficients of $\{L_k\}$ converge uniformly on compact sets in $\mathbf{R}^n$ to the coefficients of the Laplace operator. Define

$$u_k(X) = G_\Gamma^k(X, Z_k)/G_B^k(A_1, Z_k)$$

for all $X\in\bar{\Gamma}\setminus\{Z_k\}$. Then $u_k\geqq 0$ and $L_k u_k=0$ in $\Gamma\setminus\{Z_k\}$, $u_k=0$ on $\partial\Gamma$, and $\varepsilon_0\leqq u_k(A_1)\leqq 1$. The L^p-Schauder estimates and Lemma 2.4 imply that a subsequence of $\{u_k\}$, which we still call $\{u_k\}$, converges in $L^\infty_{\text{loc}}(\bar{\Gamma}\setminus\{0\})$ to a continuous, nonnegative function, u, in $\bar{\Gamma}\setminus\{0\}$ such that $\Delta u=0$ in $\Gamma, u=0$ on $\partial\Gamma\setminus\{0\}$, and

(2) $$\varepsilon_0 \leqq u(A_1) \leqq 1.$$

Let

$$g_k(X) = G_B^k(X, 0)/G_B^k(A_1, 0)$$

for all $X\in\bar{B}\setminus\{0\}$. Reasoning as we did above, we see that a subsequence (which we still call $\{g_k\}$) converges in $L^\infty_{\text{loc}}(\bar{B}\setminus\{0\})$ to a continuous, nonnegative function, g, in $\bar{B}\setminus\{0\}$ such that $\Delta g=0$ in $B\setminus\{0\}$ and $g=0$ on ∂B. By Theorem 4.4, there exists $c_0>0$ depending only on λ and n such that

$$u_k(X) \leqq G_B^k(X, Z_k)/G_B^k(A_1, Z_k) \leqq c_0\cdot g_k(X)$$

whenever $X\in\bar{\Gamma}\setminus B_{3r}$ and $Z_k\in B_r$. Since the above inequality holds for arbitrarily small r, we conclude that

(3) $$u(X) \leqq c_0\cdot g(X)$$

for all $X\in\bar{\Gamma}\setminus\{0\}$.

We now claim that $g(X)=G_B(X, 0)/G_B(A_1, 0)$, where $G_B(X, Y)$ is the Green's function for the Laplacian in B. This follows because for each $X\in B\setminus\{0\}$, $\{G_B^k(X, \cdot)\}$ converges weakly to $G_B(X, \cdot)$ in $L^{p/(p-1)}(B)$ for $p>n$. On the other hand, Theorem 4.5 implies that $\{G_B^k(X, \cdot)/G_B^k(A_1, \cdot)\}$ is uniformly bounded and equicontinuous in B_{2r} whenever $B\setminus B_{3r}$ contains X and A_1; hence a subsequence converges uniformly in B_r to $G_B(X, \cdot)/G_B(A_1, \cdot)$. Since $0\in B_r$, we have shown that a subsequence of $\{g_k(X)\}$ converges to $G_B(X, 0)/G_B(A_1, 0)$ and thus

(4) $$g(X) = G_B(X, 0)/G_B(A_1, 0)$$

for each $X\in\bar{B}\setminus\{0\}$. We will show that this leads to a contradiction.

Let $v(X)=g(X)-u(X)/2c_0$ for each $X\in\bar{\Gamma}\setminus\{0\}$. Then $v\in C(\bar{\Gamma}\setminus\{0\})$, $\Delta v=0$ in Γ, $v=g$ on $\partial\Gamma\setminus\{0\}$, and $v\geqq g/2>0$ in Γ. The results of Hunt and Wheeden on nonnegative harmonic functions ([9]) imply that

$$v(X) = \int_{\partial\Gamma} g(Q)\,d\omega^X(Q)+\int_{\partial\Gamma} K(X, Q)\,d\nu(Q)$$

for all $X\in\Gamma$, where ω^X is the harmonic measure at X in Γ, $K(X, Q)=(d\omega^X/d\omega^{A_1})(Q)$, and ν is a finite nonnegative Borel measure on $\partial\Gamma$ such that

$v\perp\omega^{A_1}$. Thus

$$v(X) \geqq \int_{\partial\Gamma} g(Q)\,d\omega^X(Q)$$

for all $X\in\Gamma$. The above inequality and equation (4) imply that $|Q|^{2-n}$ (or $\log|Q|$ if $n=2$) is in $L^1(d\omega^X)$. Hence if we let $P_j=(0,\dots,0,-2^{-j})$ for $j=1,2,\dots,$ we obtain (by the dominated convergence theorem)

$$\begin{aligned} v(X) &\geqq \lim_{j\to\infty}\int_{\partial\Gamma}[G_B(Q,P_j)/G_B(A_1,0)]\,d\omega^X(Q)\\ &= \lim_{j\to\infty} G_B(X,P_j)/G_B(A_1,0) = g(X)\end{aligned}$$

for all $X\in\Gamma$. Since $v=g-u/2c_0$, this implies that $u\leqq 0$ in Γ which contradicts (2).

In the following theorem we construct uniform barriers for normalized adjoint solutions on small cones in $\mathbf{B}\setminus\{A\}$. The theorem is based on the A_∞-estimates in Section 3 and the properties of normalized adjoint solutions established thus far.

Theorem 4.7. *Suppose $0<\theta<\pi$ and N is a unit vector in $\mathbf{R}^n$. Given $\varepsilon>0$, there exist positive constants $R\leqq 1$ and $\delta\leqq 1/4$ so that if $r\leqq R$ and $\overline{B(4r,P)}\subset\mathbf{B}\setminus\{A\}$, there is a normalized adjoint solution, $\tilde h$, in $\Gamma(r,\theta,N,P)$ which satisfies:*

(i) $\tilde h\in C\left(\overline{\Gamma(r,\theta,N,P)}\right)$, $\tilde h>0$ *in* $\Gamma(r,\theta,N,P)$, *and* $\tilde h(P)=0$,
(ii) $\tilde h\geqq 1$ *on* $\partial\Gamma(r,\theta,N,P)\cap\partial B(r,P)$,
(iii) $\sup\{\tilde h(Y)\colon Y\in\Gamma(r,\theta,N,P)\cap B(\delta r,P)\}\leqq\varepsilon$.

The constants, R and δ, above depend only on $\varepsilon,\theta,\lambda,n,$ and w.

Proof. Suppose $0<r\leqq 1$ and $\overline{B(4r,P)}\subset\mathbf{B}\setminus\{A\}$. Let $\Gamma_r=\Gamma(r,\theta,N,P)$, $\Gamma_{2r}=\Gamma(2r,(\theta+\pi)/2,N,P)$, $A_r=P+Nr$, and $A_r'=P+3Nr/2$. Let $B_s=B(s,P)$ for any $s>0$. Choose $\eta>0$ (depending on θ and n) so that $\overline{\Phi(4\eta r,A_r')}\subset\Gamma_{2r}\setminus\Gamma_r$ and $\overline{\Phi(4\eta r,A_r)}\subset\Gamma_{2r}\setminus\{A_r'\}$. Define

$$H(Y)=\frac{G_{\Gamma_{2r}}(A_r',Y)}{G_{\mathbf{B}}(A,Y)}\cdot\frac{1}{r^2}\int_{\Phi(\eta r,A_r')}G_{\mathbf{B}}(A,Z)\,dZ$$

for all $Y\in\bar\Gamma_{2r}\setminus\{A_r'\}$, where $G_{\Gamma_{2r}}(X,Y)$ is the Green's function for L in Γ_{2r}. Then any positive multiple of H satisfies (i) in $\bar\Gamma_r$.

By Theorem 3.2, Corollary 3.5, and Theorem 4.4, there exist positive constants c_1 and c_2 depending only on $\lambda,n,\theta,$ and w such that

$$\begin{aligned}&\inf_{\partial\Gamma_r\cap\partial B_r}\tilde H\geqq c_1\cdot\sup_{\Phi(\eta r,A_r)}\tilde H\\ &\geqq c_1\cdot r^{-2}\cdot\int_{\Phi(\eta r,A_r')}G_{\mathbf{B}}(A,Z)\,dZ\cdot\int_{\Phi(\eta r,A_r)}G_{\Gamma_{2r}}(A_r',Z)\,dZ\Big/\int_{\Phi(\eta r,A_r)}G_{\mathbf{B}}(A,Z)\,dZ\\ &\geqq c_2\cdot r^{-2}\cdot\int_{\Phi(\eta r,A_r)}G_{\Gamma_{2r}}(A_r',Z)\,dZ,\end{aligned}$$

Since $G_{\Gamma_{2r}}(A'_r, \cdot)$ is a nonnegative adjoint solution in $\Gamma_{2r}\setminus\{A'_r\}$ and satisfies $L^*v\leqq 0$ in Γ_{2r}, Theorem 3.2 implies that

$$r^{-2}\cdot\int_{\Phi(\eta r, A_r)} G_{\Gamma_{2r}}(A'_r, Z)\,dZ \geqq c_3\cdot r^{-2}\int_{B(r/4, A'_r)} G_{\Gamma_{2r}}(A'_r, Z)\,dZ.$$

By the maximum principle, we have

$$r^{-2}\cdot\int_{B(r/4, A'_r)} G_{\Gamma_{2r}}(X, Z)\,dZ \geqq c_4(1-16|X-A'_r|^2/r^2)$$

for all $X\in B(r/4, A'_r)$, where c_4 depends only on λ and n. Setting $X=A'_r$ and combining the above inequalities, we conclude that $\tilde{H}\geqq c>0$ on $\partial\Gamma_r\cap\partial B_r$ where c depends only on $\lambda, n, \theta,$ and w. Hence if we define

$$\tilde{h}(Y) = c^{-1}\cdot H(Y)$$

for all $Y\in\bar{\Gamma}_r$, we obtain a function which satisfies (i) and (ii).

Now suppose $\varepsilon>0$. Let $G_{B_{4r}}(X, Y)$ denote the Green's function for L and B_{4r}. Choose R and δ depending on ε, $(\theta+\pi)/2$, $\lambda, n,$ and w as defined in Lemma 4.6. If $0<2r<R$ and $Y\in\Gamma_{2r}\cap B_{\delta r}$, we have

$$\begin{aligned}\tilde{h}(Y) &= \frac{G_{\Gamma_{2r}}(A'_r, Y)}{G_{B_{4r}}(A'_r, Y)}\cdot\frac{G_{B_{4r}}(A'_r, Y)}{G_{\mathbf{B}}(A, Y)}\cdot\frac{1}{cr^2}\int_{\Phi(\eta r, A'_r)} G_{\mathbf{B}}(A, Z)\,dZ \\ &\leqq \varepsilon\cdot c^{-1}\cdot r^{-2}\cdot[G_{B_{4r}}(A'_r, Y)/G_{\mathbf{B}}(A, Y)]\cdot\int_{\Phi(\eta r, A'_r)} G_{\mathbf{B}}(A, Z)\,dZ.\end{aligned}$$

Reasoning as in the proof of (ii), it follows that

$$\tilde{h} \leqq c_5\cdot\varepsilon\cdot r^{-2}\cdot\int_{B_{r/4}} G_{B_{4r}}(A'_r, Z)\,dZ \tag{1}$$

in $\Gamma_{2r}\cap B_{\delta r}$, where c_5 depends on $\theta, \lambda, n,$ and w. By Pucci's estimate (and a change of variables), we have

$$r^{-2}\cdot\int_{B_{r/4}} G_{B_{4r}}(A'_r, Z)\,dZ \leqq c_6, \tag{2}$$

where c_6 depends only on λ and n. Inequalities (1) and (2) imply that $\tilde{h}$ satisfies (iii).

The barrier constructed above enables us to prove a Carleson-type estimate on nonnegative normalized adjoint solutions which vanish on a portion of a bounded Lipschitz domain, D. This estimate, Theorem 4.9, is the main result which we use (in Section 5) to prove Theorem 2.6. We assume in the remainder of this section that $\mathbf{B}$ and A satisfy $\bar{D}\subset\mathbf{B}\setminus\{A\}$ and $\operatorname{dist}(\partial\mathbf{B}, \bar{D})\geqq\operatorname{dist}(A, \bar{D})\geqq 10mr_0$.

Lemma 4.8. *Suppose* $r>0$, $Q\in\partial D$, *and* $\overline{\Omega(4r, Q)}\subset\mathbf{B}\setminus\{A\}$. *There exist positive constants* $r_1\leqq r_0/2$, c *and* α *depending only on* λ, n, w, r_0, *and* m *so that if* $r\leqq r_1$ *and* $\tilde{v}$ *is a nonnegative normalized adjoint solution in* $\Omega(r, Q)\cap D$ *which* *vanishes continuously on* $\Delta(r, Q)$, *then*

$$\tilde{v}(Y) \leqq c\cdot(|Y-Q|/r)^{\alpha}\cdot M(\tilde{v})$$

for all $Y\in\Omega(r, Q)\cap D$, *where* $M(\tilde{v})=\sup\{\tilde{v}(Y)\colon Y\in\Omega(r, Q)\cap D\}$.

Proof. Choose $\theta\in(0,\pi)$ depending only on m so that if $N=(A_r(Q)-Q)/|A_r(Q)-Q|$ and $r<r_0$, then $B(r, Q)\cap D\subset\Gamma(r,\theta,N,Q)$. Set $\varepsilon_0=1/2$ and let R and δ be the constants defined in Theorem 4.7. Set $r_1=\min\{R, r_0/2\}$.

Suppose r and $\tilde{v}$ satisfy the hypotheses of the theorem. Let $\tilde{h}$ be the "ε_0-barrier" as constructed in Theorem 4.7. By the maximum principle for normalized adjoint solutions, $\tilde{v}\leqq M(\tilde{v})\cdot\tilde{h}$ in $B(r/2, Q)\cap D$. Since $\tilde{v}\leqq M(\tilde{v})$ in $\Omega(r, Q)\cap D$, it is sufficient to prove that

$$\tilde{h}(Y) \leqq c(|Y-Q|/r)^{\alpha} \tag{1}$$

for all $Y\in B(r/2, Q)\cap D$, where c and α are positive numbers depending only on δ.

Let $\tilde{h}_1$ be the "ε_0-barrier" in $\Gamma(\delta r/2,\theta,N,Q)$. Since $\tilde{h}\leqq 1/2$ in $\partial B(\delta r/2, Q)\cap D$, the maximum principle implies that $\tilde{h}\leqq\tilde{h}_1/2$ in $\Gamma(\delta r/2,\theta,N,Q)$ and hence $\tilde{h}\leqq 1/4$ in $B(\delta^2 r/2, Q)\cap D$. Iterating this procedure, we deduce that $\tilde{h}\leqq 2^{-j}$ in $B(\delta^j r/2, Q)\cap D$ for $j=1,2,\dots$, which proves (1).

The above lemma is the analog of Lemma 2.3 which was used to prove the Carleson-type estimate of Theorem 2.4. By the same argument, we obtain:

Theorem 4.9. *Suppose* $Q\in\partial D$, *and* $0<r<r_1/2$ *(where* r_1 *is the constant of Lemma* 4.8*). There exists* $c>0$ *depending only on* λ, n, w, *and* m *so that if* $\tilde{v}$ *is a nonnegative normalized adjoint solution in* $\Omega(2r, Q)\cap D$ *which vanishes continuously on* $\Delta(2r, Q)$, *then*

$$\tilde{v}(Y) \leqq c\cdot\tilde{v}(A_r(Q))$$

for all $Y\in\Omega(r, Q)\cap D$.

5. The Green's function

In this section we prove the inequality on the Green's function which we stated as Theorem 2.6. We also describe some properties of the Green's function which we feel are of independent interest.

Proof of Theorem 2.6. We first prove the theorem assuming L has smooth coefficients. (See Section 2 foı the statement of the theorem.) Choose an open ball, $\mathbf{B}$, and a point, A, in $\mathbf{B}$ such that $\bar{D}\subset\mathbf{B}\setminus\{A\}$ and $10mr_0\leqq\operatorname{dist}(A,\bar{D})\leqq\operatorname{dist}(\partial\mathbf{B},\bar{D})$. Let $G_{\mathbf{B}}(X, Y)$ denote the Green's function for L in $\mathbf{B}$ and $\eta=\min\{1/4, r_1/r_0\}$ with r_1 as defined in Lemma 4.8. For simplicity we will denote by c any positive constant which depends only on λ, n, w, r_0, and m.

Suppose $X\in\Omega(r, Q)\cap D$. If $P\in\overline{\psi(r, Q)}\cap\partial D$, then $\Omega(2\eta r, P)\subset\Omega(4r, Q)\setminus\overline{\Omega(r, Q)}$, where $\Omega(2\eta r, P)$ is defined with respect to the coordinate system in $\Omega(r, Q)$. By Theorems 4.4 and 4.9, there exists $c>0$ independent of X and P such that

$$\sup\{G_r(X, Y)/G_{\mathbf{B}}(A, Y)\colon Y\in\Omega(\eta r, P)\cap D\}\leqq c\cdot G_r(X, A_r'(Q))/G_{\mathbf{B}}(A, A_r'(Q))$$

and hence

$$\begin{aligned}&\sup\{G_r(X, Y)/G_{\mathbf{B}}(A, Y)\colon Y\in\psi(r, Q)\}\\ &\quad\leqq c\cdot\inf\{G_r(X, Y)/G_{\mathbf{B}}(A, Y)\colon Y\in\Phi(r/\sqrt{n}, A_r'(Q))\}.\end{aligned}$$

The above inequality and Theorem 3.2 imply that if $X\in\Omega(r, Q)\cap D$, we have

$$\begin{aligned}&\int_{\psi(r,Q)} G_r(X, Y)\,dY\leqq\int_{\psi(r,Q)} G_{\mathbf{B}}(A, Y)\,dY\cdot\sup\{G_r(X, Y)/G_{\mathbf{B}}(A, Y)\colon Y\in\psi(r, Q)\}\\ (1)\quad&\leqq c\cdot\int_{\Phi(r/\sqrt{n},\, A_r'(Q))} G_{\mathbf{B}}(A, Y)\,dY\cdot\inf\{G_r(X, Y)/G_{\mathbf{B}}(A, Y)\colon Y\in\Phi(r/\sqrt{n}, A_r'(Q))\}\\ &\qquad\leqq c\cdot\int_{\Phi(r/\sqrt{n},\, A_r'(Q))} G_r(X, Y)\,dY.\end{aligned}$$

This proves the theorem when L has smooth coefficients. In the general case, we mollify the coefficients of L to obtain a sequence of operators, $\{L_k\}$, with smooth coefficients such that $G_r^k(X, \cdot)\to G_r(X, \cdot)$ in $L^p(\Omega(4r, Q)\cap D)$ for each $X\in\Omega(4r, Q)\cap D$ and $1<p<n/(n-1)$. (Here, $G_r^k(X, Y)$ is the Green's function for L_k and $\Omega(4r, Q)\cap D$. The convergence of the Green's functions follows from Pucci's estimate, Lemma 2.4, and the L^p-Schauder estimates.) We obtain the conclusion of Theorem 2.6 by applying inequality (1) for each L_k and taking the limit.

We conclude this section by making some remarks on the behavior of the Green's function for L and D. Fix an open ball, $\mathbf{B}$, and a point, A, in $\mathbf{B}$ such that $\bar{D}\subset\mathbf{B}\setminus\{A\}$. Let $G_D(X, Y)$ and $G_{\mathbf{B}}(X, Y)$ denote the Green's functions for L in D and $\mathbf{B}$, respectively. By mollifying the coefficients of L, we may choose a sequence of operators, $\{L_k\}$, with smooth coefficients such that $G_D^k(X, \cdot)\to G_D(X, \cdot)$ in $L^p(D)$ and $G_{\mathbf{B}}^k(A, \cdot)\to G_{\mathbf{B}}(A, \cdot)$ in $L^p(\mathbf{B})$ for each $X\in D$ and $1<p<n/(n-1)$ (where $G_D^k(X, Y)$ and $G_{\mathbf{B}}^k(X, Y)$ are the Green's functions for L_k in D and $\mathbf{B}$, respectively). By the Hölder estimates on normalized adjoint solutions (Theorems 4.5 and 4.9), it follows that $\{G_D^k(X, \cdot)/G_{\mathbf{B}}^k(A, \cdot)\}$ converges uniformly in compact subsets of $\bar{D}\setminus\{X\}$ to $G_D(X, \cdot)/G_{\mathbf{B}}(A, \cdot)$. Hence $G_D(X, \cdot)/G_{\mathbf{B}}(A, \cdot)$ has a continuous representative in $\bar{D}\setminus\{X\}$ and the estimates of Section 4 extend to $G_D(X, \cdot)/G_{\mathbf{B}}(A, \cdot)$ in $D\setminus\{X\}$. Moreover, by writing

$$G_D(X, Y)=G_{\mathbf{B}}(A, Y)\cdot(G_D(X, Y)/G_{\mathbf{B}}(A, Y)),$$

we see that $G_D(X, Y)$ is the product of an A_∞-weight (as defined by Muckenhoupt) and a positive, continuous function of X and Y (for $X\neq Y$) which satisfies a Harnack inequality in each variable and vanishes continuously on ∂D.

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