Holomorphic mappings with prescribed Taylor expansions

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1. Introduction

The letters **N** and **C** denote the set of non-negative integers and the set of complex numbers, respectively.

We study the following interpolation problem:

(1) *"Let E and F be two complex, locally convex spaces, Ω an open subset of E and $(z_n)_{n\in\mathbf{N}}$ a sequence of distinct points of Ω. Given any sequence of polynomials $(P_n)_{n\in\mathbf{N}}$ (e.g. continuous polynomials) from E to F, under what conditions on E, F and $(z_n)_{n\in\mathbf{N}}$ does there exist a holomorphic mapping f from Ω to F such that the partial Taylor series of f at z_n up to order $N(n)\geqq\deg P_n$ is equal to P_n for each $n\in\mathbf{N}$?"*

If E and F are one-dimensional, then the answer is that $(z_n)_{n\in\mathbf{N}}$ shall have no accumulation points in Ω. This follows from a combination of Mittag–Leffler's theorem and Weierstrass' theorem.

Weaker versions of (1) have been solved by Y. Hervier [12] and M. Valdivia [24]:

(2) (Y. Hervier [12, Prop. 1, p. 157]). *"Let E and F be two complex Banach spaces and let Ω be a domain of holomorphy in E. Suppose that $(z_n)_{n\in\mathbf{N}}$ is a sequence of distinct elements of Ω such that $\lim_{n\to+\infty}|g(z_n)|=+\infty$ for some holomorphic function g on Ω. Then, given any sequence of elements $(u_n)_{n\in\mathbf{N}}$ of F, there exists a holomorphic mapping f from Ω to F such that $f(z_n)=u_n$ for all $n\in\mathbf{N}$".*

More generally:

(3) (M. Valdivia [24, Thm. 10]). *"Let E be a complex, locally convex space whose topology is given by a family of continuous norms, Ω an open subset of E and F a complex Fréchet space. Suppose that $(z_n)_{n\in\mathbf{N}}$ is a sequence of distinct elements of Ω such that $\lim_{n\to+\infty}|g(z_n)|=+\infty$ for some holomorphic function g on Ω. Then,*

given any sequence $(u_n)_{n\in\mathbf{N}}$ *of elements of* F*, there exists a holomorphic mapping* f *from* Ω *to* F *such that* $f(z_n)=u_n$ *for all* $n\in\mathbf{N}$*."*

Of course, in both (2) and (3), the assumptions made on the sequence $(z_n)_{n\in\mathbf{N}}$ are necessary.

Bearing (2) and (3) in mind, the following question arise:

(4) *"Under what conditions on* E *and* F *does* (1) *have a solution, assuming that there is a holomorphic function* g *on* Ω *such that* $\lim_{n\to+\infty}|g(z_n)|=+\infty$*?"*

We remark here that it is not possible to apply (2) or (3) in order to solve (4) when $\deg P_n>0$. E.g. if the topology on E is given by a family of continuous norms and if the strong dual E'_β of E is a Fréchet space, then (2) gives the existence of a holomorphic mapping G from Ω to E'_β such that $G(z_n)=\zeta_n$ for all $n\in\mathbf{N}$, where $(z_n)_{n\in\mathbf{N}}$ satisfies the assumptions in (3) and $(\zeta_n)_{n\in\mathbf{N}}$ is any given sequence in E'_β. But the proof of the existence of such a mapping G is not constructive and therefore it is not possible to decide if G is equal to the derivative of a holomorphic function on Ω.

In Section 2 we give the necessary definitions used in this paper.

Section 3 contains an answer to (4) and also an answer to the corresponding questions for Silva- and hypo-holomorphic mappings.

In Section 4 we prove that if E is a Fréchet space then, in general, there is a big difference between interpolating with constant polynomials (as in (2) and (3)) and polynomials of higher orders.

In Section 5 we apply the results of Section 3 and prove that any holomorphically convex, open subset of a (*DFC*)-space is the domain of existence of a meromorphic function. This generalizes a result of M. Valdivia [24, Thm. 8].

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2. Definition and notation

In this section E and F denote two complex, locally convex spaces and Ω an open subset of E.

Definition 2.1. *A mapping* $f\colon \Omega\to F$ *is said to be*

(a) *Gâteaux-holomorphic (G-holomorphic for short) if the restriction of* f *to* $L\cap\Omega$ *is holomorphic for each complex line* L *in* E*. We denote the class of G-holomorphic mappings from* Ω *to* F *by* $H_G(\Omega, F)$*;*

(b) *holomorphic, if* $f \in H_G(\Omega, F)$ *and* f *is continuous. We denote this class by* $H_C(\Omega, F)$,

(c) *hypo-holomorphic, if* $f \in H_G(\Omega, F)$ *and the restriction of* f *to each compact subset of* Ω *is continuous. We denote this class by* $H_H(\Omega, F)$;

(d) *Silva-holomorphic, if for each absolutely convex, bounded subset* B *of* E *the restriction of* f *to* $E_B \cap \Omega$ *is an element of* $H_C(\Omega \cap E_B, F)$, *where* $E_B = \bigcup_{n \in \mathbf{N}} nB$ *algebraically and* E_B *is normed by the gauge of* B. *We denote this class by* $H_S(\Omega, F)$;

The corresponding classes of polynomials from E to F (i.e. G-holomorphic, holomorphic, hypo-holomorphic and Silva-holomorphic polynomials) are denoted by $P_G(E, F)$, $P_C(E, F)$, $P_H(E, F)$ and $P_S(E, F)$ respectively.

If $f \in H_A(\Omega, F)$, $A = G, C, H$ or S, then for any $m \in \mathbf{N}$ and any $z \in \Omega$ the m-th derivative of f at z, denoted by $\hat{d}^m f(z)$, is an element of $P_A(E, \hat{F})$. (Cf. S. Dineen [10]. $\hat{F}$ = the completion of F.)

We now equip $H_C(\Omega, F)$ and $H_H(\Omega, F)$ with the topology τ_0 of uniform convergence on compact subsets of Ω and $H_S(\Omega, F)$ with the topology τ_{s0} of uniform convergence on strictly compact subsets of Ω. (A subset K of Ω is said to be strictly compact if K is contained and compact in $\Omega \cap E_B$ for some bounded, absolutely convex subset B of E, with E_B as in Definition 2.1(d)). We have the following algebraic and topological inclusions:

$$(H_C(\Omega, F), \tau_0) \subset (H_H(\Omega, F), \tau_0) \subset (H_S(\Omega, F), \tau_{s0}).$$

Furthermore, $(H_C(\Omega, F), \tau_0) = (H_H(\Omega, F), \tau_0)$ if E is a k-space, e.g. if E is metrizable or if E is a (DFC)-space (the strong dual of a Fréchet space with its compact-open topology. This follows from the Banach—Dieudonné theorem);

$(H_H(\Omega, F), \tau_0) = (H_S(\Omega, F), \tau_{s0})$ if each compact subset of E is strictly compact, e.g. if E is quasi-complete and E' with its compact-open topology is a Schwartz' locally convex space. (This follows from polarity, cf. H. Hogbé-Nlend [13]);

$(H_C(\Omega, F), \tau_0) = (H_S(\Omega, F), \tau_{s0})$ if e.g. E is a (DFS)-space (the strong dual of a Fréchet—Schwartz space. Since then E is a (DFC)-space, quasi-complete and the strong dual of E is a Fréchet—Schwartz space). It is an open question whether or not this last equality is true for (DFM)-space also. (The strong dual of a Fréchet—Montel space, cf. S. Dineen [9].)

Whenever we write $H_A(\Omega, F)$ or $P_A(E, F)$ we mean that $A \in \{C, H, S\}$, if not otherwise is stated. If $F = \mathbf{C}$ then we write $H_A(\Omega)$ and $P_A(E)$ instead of $H_A(\Omega, \mathbf{C})$ and $P_A(E, \mathbf{C})$, respectively.

For more details about different classes of holomorphic mappings we refer to J. F. Colombeau [3, 5], D. Pisanelli [23] and S. Dineen [8, 10].

We shall now introduce an equivalence relation on $H_A(\Omega, F)$:

Definition 2.2. *If* $f, g \in H_A(\Omega, F)$, $N \in \mathbf{N}$ *and* $z \in \Omega$, *then we say that* f *and* g *are equivalent up to order* N *at* z, *denoted by* $(f \sim g)(N, z)$, *if the first* $(N+1)$*-terms in the Taylor series of* $f-g$ *at* z *all are equal to* 0.

Remark. If $f_i \in H_A(\Omega, F)$, $i=1, \ldots, 4$, $N \in \mathbf{N}$, $z \in \Omega$, and $g \in H_A(\Omega)$, then $(f_1 \sim f_2)(N, z)$ and $(f_3 \sim f_4)(N, z)$ implies that $((f_1+f_3) \sim (f_2+f_4))(N, z)$ and $(gf_1 \sim gf_2)(N, z)$.

Next we define the different sequences studied in Sections 3 and 4:

Definition 2.3. *Let* $(z_n)_{n \in \mathbf{N}}$ *be a sequence of distinct elements of* Ω *and let* F *be a class of complex, locally convex spaces such that* $\mathbf{C} \in F$. *We say that* $(z_n)_{n \in \mathbf{N}}$ *is*

(a) *an A-Weierstrass sequence in* Ω, *denoted* $(z_n)_{n \in \mathbf{N}} \in W_A(\Omega)$, *if there exists* $f \in H_A(\Omega)$ *such that* $\lim_{n \to +\infty} |f(z_n)| = +\infty$;

(b) *a strict A-Weierstrass sequence in* Ω, *denoted* $(z_n)_{n \in \mathbf{N}} \in sW_A(\Omega)$, *if there exists* $f \in H_A(\Omega)$ *such that* $(|f(z_n)|)_{n \in \mathbf{N}}$ *is strictly increasing and unbounded;*

(c) *an FA-interpolation sequence of order* $N \in \mathbf{N}$ *in* Ω, *denoted* $(z_n)_{n \in \mathbf{N}} \in FI_A^N(\Omega)$, *if, given any* $F \in F$ *and any sequence* $(P_n)_{n \in \mathbf{N}}$ *in* $P_A(E, \hat{F})$, *where*

$$P_n(z) = c_n^0 + c_n^1(z-z_n) + \ldots + c_n^N(z-z_n)$$

and c_n^i *are homogeneous of degree* i, *there exists* $f \in H_A(\Omega, F)$ *such that* $(f \sim P_n) \cdot (N, z_n)$ *for all* $n \in \mathbf{N}$;

(d) *an FA-interpolation sequence in* Ω, *denoted* $(z_n)_{n \in \mathbf{N}} \in FI_A(\Omega)$, *if, given any* $F \in F$ *and any sequence* $(P_n)_{n \in \mathbf{N}}$ *in* $P_A(E, \hat{F})$, *where*

$$P_n(z) = c_n^0 + c_n^1(z-z_n) + \ldots + c_n^{N(n)}(z-z_n)$$

and c_n^i *are as in* (c) *above, there exists* $f \in H_A(\Omega, F)$ *such that* $(f \sim P_n)(N(n), z_n)$ *for all* $n \in \mathbf{N}$.

Remark. It follows immediately that

$$FI_A(\Omega) \subset \bigcap_{N \in \mathbf{N}} FI_A^N(\Omega) \subset \bigcup_{N \in \mathbf{N}} FI_A^N(\Omega) = FI_A^0(\Omega) \subset sW_A(\Omega) \subset W_A(\Omega)$$

in general.

Question (4), asked in the introduction, can now be formulated as follows: *"Under what conditions on* E *and* F *is* $FI_A(\Omega) = W_A(\Omega)$, $A = C, H, S$?"

The results of Y. Hervier and M. Valdivia ((2) and (3), respectively, in the introduction) get the following formulation:

(2′) *"If* E *is a complex Banach space,* Ω *is a domain of holomorphy and* F = *the class of complex Banach spaces, then* $FI_C^0(\Omega) = W_C(\Omega)$."

(3′) *"If the topology on* E *is given by a family of continuous norms,* Ω *is an open subset of* E *and* F = *the class of complex Fréchet spaces, then* $FI_C^0(\Omega) = W_C(\Omega)$."

And also, [24, Thm. 11] says that $FI_C^0(\Omega)=sW(\Omega)$ for any manifold (Ω, p) spread over a complex, locally convex space E, where Ω is a Hausdorff topological space and p is a local homeomorphism from Ω into E.

We remark also that, using the technique in M. Valdivia [24, § 3], it is not difficult to prove that $FI_C^0(\Omega)=W_C(\Omega)$ if E is a (DFC)-space and F = the class of complex Fréchet spaces.

3. An interpolation theorem

In this section we study (4) in the introduction for a certain class of complex, locally convex spaces containing all (DF)- and (DFC)-spaces. (A locally convex space E is called a (DF)-space if E has a countable, fundamental system of bounded subsets and if for each sequence $(V_n)_{n\in\mathbf{N}}$ of absolutely convex 0-neighbourhoods in E, such that $V=\bigcap_{n\in\mathbf{N}} V_n$ absorbs each bounded set, V is again a 0-neighbourhood. Strong duals of metrizable spaces and countable, strict inductive limits of normed spaces are (DF)-spaces.) We refer to A. Grothendieck [11] for the theory of (DF)-spaces and to R. Hollstein [15] for the theory of (DFC)-spaces. Results concerning holomorphic mappings on (DFC)-spaces can be found in J. Mujica [21] and M. Valdivia [24].

Theorem 3.1. *Let F denote the class of complex Fréchet spaces and let E be a locally convex space with a countable, fundamental system of bounded subsets. If Ω is an open subset of E, then*

$$W_C(\Omega) \subset FI_H(\Omega) = W_H(\Omega) \subset FI_S(\Omega) = W_S(\Omega).$$

Remark. If E satisfies the assumptions in Theorem 3.1 and if E is also a k-space, e.g. if E is a (DFC)-space, then Theorem 3.1 yields that $W_C(\Omega)=FI_C(\Omega)$ for any open subset Ω of E.

Before the proof of Theorem 3.1 we need some lemmas.

Lemma 3.2. *Let E be a locally convex space with a countable, fundamental system $(B_n)_{n\in\mathbf{N}}$ of absolutely convex bounded subsets. Let τ denote the finest locally convex topology on E for which each B_n, $n\in\mathbf{N}$, is bounded. Then (E, τ) is a bornological (DF)-space.*

Proof. We obviously have that $(B_n)_{n\in\mathbf{N}}$ is a fundamental system of absolutely convex bounded subsets of (E, τ). If $(V_n)_{n\in\mathbf{N}}$ is a sequence of absolutely convex 0-neighbourhoods in (E, τ) such that $V=\bigcap_{n\in\mathbf{N}} V_n$ absorbs each bounded subset of (E, τ) then, by the definition of τ, V is a 0-neighbourhood in (E, τ). Furthermore, (E, τ) is bornological since we have taken as 0-neighbourhoods all absolutely convex sets which absorb each B_n, $n\in\mathbf{N}$.

The next lemma is used in order to reduce the proof of Theorem 3.1 to the case of Banach-valued mappings.

Lemma 3.3. (J.-F. Colombeau and J. Mujica [7, Lemma 3.4]). *Let E and F be two (complex) locally convex spaces such that F is metrizable and for each sequence $(V_n)_{n\in\mathbf{N}}$ of absolutely convex 0-neighbourhoods in E, there exists a sequence $(\lambda_n)_{n\in\mathbf{N}}$ of positive, real numbers, such that $\bigcap_{n\in\mathbf{N}}\lambda_n V_n$ is again a 0-neighbourhood in E. Then, given any sequence $(P_n)_{n\in\mathbf{N}}$ of continuous polynomials from E to F, there exists an absolutely convex 0-neighbourhood V in E, an absolutely convex bounded subset B of F and a sequence $(Q_n)_{n\in\mathbf{N}}$ of continuous polynomials from E_V to F_B, such that the following diagram is commutative for each $n\in\mathbf{N}$:*

$$\begin{array}{ccc} E & \xrightarrow{P_n} & F \\ {\scriptstyle s_V}\downarrow & & \uparrow{\scriptstyle i_B} \\ E_V & \xrightarrow[Q_n]{} & F_B \end{array}$$

(Here $E_V=(E/p_V^{-1}(0), p_V)$, p_V = the gauge of V, s_V = the canonical projection, F_B is as before and i_B = the canonical injection.)

The proof of the next lemma can be found in S. Dineen [10, Example 1.24, p. 17].

Lemma 3.4. *Let E be a bornological (DF)-space and F a locally convex space. Then $P_C(E, F)=P_S(E, F)$ algebraically.*

Lemma 3.5. *Let E be a complex, locally convex space and Ω an open subset of E. Let $g\in H_A(\Omega)$ $(A=G, C, H, S)$ and $r\in\mathbf{R}_+$ be given. If $\Omega_r=\{z\in\Omega\colon |g(z)|\leqq r\}$ and $z_0\in\Omega\setminus\Omega_r$, then given any $\varepsilon>0$ and $N\in\mathbf{N}$, there exists $h\in H_A(\Omega)$ such that* $\sup_{z\in\Omega_r}|h(z)|<\varepsilon$ *and* $(h\sim 1)(N, z_0)$.

Proof. Define $H\in H_A(\Omega)$ by $H(z)=(g(z)/g(z_0))^M$, $M\in\mathbf{N}$. (Note that $g(z_0)\neq 0$ since $z_0\notin\Omega_r$.) If $\delta>0$ is given, then $\sup_{z\in\Omega_r}|H(z)|<\delta$ if $M=M(\delta)$ is chosen large enough. Put $h(z)=1-(1-H(z))^{N+1}\in H_A(\Omega)$. We get that $\sup_{z\in\Omega_r}|h(z)|<\varepsilon$ if $\delta>0$ is chosen small enough, i.e. if M is chosen large enough. Furthermore, it follows immediately from the construction that $(h\sim 1)(N, z_0)$.

Proof of Theorem 3.1. The proof is divided into two steps. First we assume that E is a bornological (DF)-space. Then E satisfies the assumptions in Lemma 3.3 and Lemma 3.4, so the proof is (in all cases) reduced to the situation when F = the class of complex Banach spaces. The second step consists of an application of Lemma 3.2. The proof is then again reduced to the case of Banach-valued mappings, with the only reservation that we have changed the topology on E and therefore the classes $H_C(\Omega, F)$ and $H_H(\Omega, F)$ also have changed. (Note that Silva-holomorphicity has nothing to do with the topology on E as long as the bounded subsets remain the same.) However, the construction of the mapping f (satisfying the assertion in Theorem 3.1) in Step 1 will allow us to draw the conclusion that $W_C(\Omega)\subset W_H(\Omega)=FI_H(\Omega)$ with respect to the original topology on E.

Step 1. E is a bornological (DF)-space.

If F is a complex Fréchet space, then we can choose a basis B of absolutely convex, bounded subsets of F such that F_B is a Banach space for each $B\in B$. Hence, applying Lemma 3.3 and Lemma 3.4 we can, without loss of generality, assume that F = the class of complex Banach spaces.

Let now $(z_n)_{n\in \mathbf{N}}\in W_A(\Omega)$, $F\in F$ and a sequence $(P_n)_{n\in\mathbf{N}}$ in $P_A(E, F)$ be given ($A=C$, H or S). Renumbering the sequences $(z_n)_{n\in\mathbf{N}}$ and $(P_n)_{n\in\mathbf{N}}$, if necessary, we can find a function $g\in H_A(\Omega)$ such that $(|g(z_n)|)_{n\in\mathbf{N}}$ is increasing and unbounded. (Renumbering the sequences does not affect the generality as long as we renumber them in the same manner.) Put $\Omega_n=\{z\in\Omega\colon |g(z)|\leqq|g(z_n)|\}$ and choose a sequence $(\mu(n))_{n\in\mathbf{N}}$ in $\mathbf{N}$ such that $z_0, \dots, z_{\mu(n)}\in\Omega_n$, $z_v\notin\Omega_n$ if $v>\mu(n)$. Let $(B_n)_{n\in\mathbf{N}}$ be a fundamental system of bounded subsets of E and put $L_n=B_n\cap\Omega_n$. Then each (strictly) compact subset K of Ω is contained in L_n if n is large enough, since $K\subset B_n$ if n is large enough and g is bounded on K. (Note that if $g\in H_S(\Omega)$, then g is bounded on strictly compact subsets of Ω, but not necessarily on compact subsets of Ω.) Denote by $\|\cdot\|$ the norm defining the topology on F and define $\|f\|_n=\sup_{z\in L_n}\|f(z)\|\leqq+\infty$, if f is a mapping from Ω to F, and $|f|_n=\sup_{z\in L_n}|f(z)|\leqq+\infty$, if f is a complex-valued function on Ω, where $|\cdot|$ is the usual Euclidean norm on $\mathbf{C}$.

We shall now inductively construct a sequence $(f_n)_{n\in\mathbf{N}}$ in $H_A(\Omega, F)$ with the following properties:

$$(H_n)\colon \begin{cases} \|f_n\|_{n-1}\leqq 2^{-n} \\ (f_n\sim 0)(N(i), z_i) \quad \text{if} \quad i\leqq\mu(n-1) \\ \left(f_n\sim \left(P_i-\sum_{k=0}^{n-1} f_k\right)\right)(N(i), z_i) \quad \text{if} \quad \mu(n-1)<i\leqq\mu(n). \end{cases}$$

For $n=0$ only the last of these properties has a meaning and the construction of f_0 is just a simplification of the construction of f_n below. So, we assume that $f_0, \dots, f_{n-1}\in H_A(\Omega, F)$ with properties $(H_0), \dots, (H_{n-1})$, respectively, have been constructed.

Since $(z_i)_{i=0}^{\mu(n)}$ is finite and $z_i\neq z_j$ if $i\neq j$, an application of the Hahn—Banach theorem gives the existence of a linear form $\zeta_n\in E'$ such that $\zeta_n(z_i-z_j)\neq 0$ if $i\neq j$ and $i, j\leqq\mu(n)$. For $i=\mu(n-1)+1, \dots, \mu(n)$ we set

$$v_{i,n}(z)=\Big(\prod_{\substack{1\leqq j\leqq\mu(n)\\ j\neq i}}\zeta_n(z-z_j)\Big)\Big/\Big(\prod_{\substack{1\leqq j\leqq\mu(n)\\ j\neq i}}\zeta_n(z_i-z_j)\Big).$$

Then obviously $v_{i,n}\in P_C(E)$, $v_{i,n}(z_i)=1$ and $v_{i,n}(z_j)=0$ if $i\neq j$ and $1\leqq j\leqq\mu(n)$.

For i as above we let $Q_{i,n}$ be the Taylor expansion of $(P_i - \sum_{k=0}^{n-1} f_k)/v_{i,n}^{M_n}$ at z_i up to order $N(i)$, where $M_n = \sup_{1 \leq j \leq \mu(n)} N(j)+1$. Now, Lemma 3.5 gives, for each i such that $\mu(n-1) < i \leq \mu(n)$, the existence of a function $h_{i,n} \in H_A(\Omega)$ such that $(h_{i,n} \sim 1)(N(i), z_i)$ and

$$\sup_{z \in \Omega_{n-1}} |h_{i,n}(z)| < \{2^n(\mu(n) - \mu(n-1)) \|Q_{i,n}\|_{n-1} |v_{i,n}|_{n-1}^{M_n}\}^{-1}.$$

(If $\|Q_{i,n}\|_{n-1} |v_{i,n}|_{n-1} = 0$, then we just supress the inequality. Note that $\|Q_{i,n}\|_{n-1} < +\infty$ since $Q_{i,n} \in P_A(E, F)$ and L_{n-1} is a bounded subset of E). Put

$$f_n(z) = \sum_{i=\mu(n-1)+1}^{\mu(n)} Q_{i,n}(z) h_{i,n}(z) (v_{i,n}(z))^{M_n}.$$

Then $f_n \in H_A(\Omega, F)$ and the supremum on L_{n-1} of each term in the sum defining f_n is less than $\{2^n(\mu(n) - \mu(n-1))\}^{-1}$ and the number of terms is equal to $\mu(n) - \mu(n-1)$. Hence, we see that $\|f_n\|_{n-1} \leq 2^{-n}$. Furthermore, if $j \leq \mu(n-1)$, then $v_{i,n}(z_j) = 0$. So $(f_n \sim 0)(N(j), z_j)$ if $j \leq \mu(n-1)$. If $\mu(n-1) < i \leq \mu(n)$, then

$$f_n(z_i) = Q_{i,n}(z_i) h_{i,n}(z_i) (v_{i,n}(z_i))^{M_n}, \quad v_{i,n}(z_i) = 1, \quad (h_{i,n} \sim 1)(N(i), z_i)$$

and

$$(Q_{i,n} \sim \{(P_i - \sum_{k=0}^{n-1} f_k)/v_{i,n}^{M_n}\})(N(i), z_i).$$

From the remark following Definition 2.2 it follows that

$$(f_n \sim (P_i - \sum_{k=0}^{n-1} f_k))(N(i), z_i) \quad \text{if} \quad \mu(n-1) < i \leq \mu(n).$$

Hence, f_n satisfies property (H_n) and the construction of the sequence $(f_n)_{n \in \mathbf{N}}$ is done.

Now we define $f(z) = \sum_{n \in \mathbf{N}} f_n(z)$, $z \in \Omega$. Since each (strictly) compact subset K of Ω is contained in L_n for n large enough, it follows that the series defining f converges uniformly on (strictly) compact subsets of Ω. Thus $f \in H_H(\Omega, F)$ (if $(z_n)_{n \in \mathbf{N}} \in W_C(\Omega)$ or $W_H(\Omega)$) and $f \in H_S(\Omega, F)$ (if $(z_n)_{n \in \mathbf{N}} \in W_S(\Omega)$).

It remains to show that $(f \sim P_i)(N(i), z_i)$ for all $i \in \mathbf{N}$. Let $z \in \Omega$ and $x \in E$ be given. We have that

$$\hat{d}f(z)(x) = (2\pi i)^{-1} \int_{|\lambda| = \varrho_x} \sum_{n \in \mathbf{N}} f_n(z + \lambda x) \lambda^{-2} \, d\lambda,$$

where ϱ_x is any positive, real number such that $\{z + \lambda x \colon |\lambda| \leq \varrho_x\} \subset \Omega$. But $\{z + \lambda x \colon |\lambda| \leq \varrho_x\}$ is strictly compact in Ω for fixed z and x and since $\sum_{n \in \mathbf{N}} f_n$ converges uniformly on (strictly) compact subsets of Ω, it follows that

$$\hat{d}f(z)(x) = (2\pi i)^{-1} \sum_{n \in \mathbf{N}} \int_{|\lambda| = \varrho_x} f_n(z + \lambda x) \lambda^{-2} \, d\lambda = \sum_{n \in \mathbf{N}} \hat{d}f_n(x).$$

In the same manner we get that $\hat{d}^m f = \sum_{n \in \mathbf{N}} \hat{d}^m f_n$, $m \geq 2$. These equalities and the properties (H_n) yields that $(f \sim P_i)(N(i), z_i)$ for all $i \in \mathbf{N}$.

Step 2. E is not a bornological (DF)-space.

In this case we equip E with the finest locally convex topology τ as in Lemma 3.2. Denote by Ω_τ the set Ω regarded as an open subset of (E, τ).

First we consider the case of Silva-holomorphic mappings. Since the identity mapping $(E, \tau)\to E$ is continuous and has a bounded inverse, we get that $H_S(\Omega_\tau, F)= H_S(\Omega, F)$ algebraically and topologically.

From Step 1 it follows that

$$W_S(\Omega) = W_S(\Omega_\tau) = FI_S(\Omega_\tau) = FI_S(\Omega).$$

If $(z_n)_{n\in\mathbf{N}}\in W_A(\Omega)$, $A=C$ or H, then obviously $(z_n)_{n\in\mathbf{N}}\in W_A(\Omega_\tau)$. Applying Step 1 we therefore get a sequence $(f_n)_{n\in\mathbf{N}}$ in $H_A(\Omega_\tau, F_B)$ such that $f=\sum_{n\in\mathbf{N}}f_n\in H_H(\Omega_\tau, F_B)$ has the desired properties. (Here F is a complex Fréchet space and $B\subset F$ is bounded and absolutely convex.) But from the construction of the f_n: s in Step 1 it is easily seen that they can be chosen such that $\sum_{n\in\mathbf{N}}f_n$ converges uniformly on compact subsets of Ω (i.e. not only on compact subsets of Ω_τ). From this it follows that

$$i_B\circ f= i_B\circ \sum_{n\in\mathbf{N}}f_n = \sum_{n\in\mathbf{N}} i_B\circ f_n\in H_H(\Omega, F).$$

This ends the proof of Theorem 3.1.

Corresponding results for mappings defined on manifolds spread over certain locally convex spaces are achieved analogously. (For notation and definitions of such mappings we refer to G. Coeuré [2].)

Theorem 3.6. *Let E be a complex, locally convex space with a countable fundamental system of bounded subsets, (Ω, p) a manifold spread over E and F = the class of complex Fréchet spaces. Then*

$$sW_C(\Omega)\subset FI_H(\Omega) = sW_H(\Omega) \quad \text{and} \quad sW_S(\Omega) = FI_S(\Omega).$$

(Here $sW_A(\Omega)$ and $FI_A(\Omega)$, $A=C, H, S$, are defined in complete analogy with Definition 2.3.*)*

Proof. As before, we first assume that E is a bornological (DF)-space. Then we can apply the proof of Theorem 3.1 with the only difference that we do not use the Hahn—Banach theorem in the construction of the sequence $(f_n)_{n\in\mathbf{N}}$. Instead we use the assumption that $(z_n)_{n\in\mathbf{N}}\in sW_A(\Omega)$.

If E is not a bornological (DF)-space, then we apply Lemma 3.2 and equip E with the topology τ. Define Ω_τ by taking inverse images of open subsets of (E, τ) under the projection p. We then get a commutative diagram

$$\begin{array}{ccc} \Omega_\tau & \longrightarrow & \Omega \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ (E, \tau) & \longrightarrow & E \end{array}$$

where the horizontal arrows denote the identities. Proceeding as in Step 2 of the proof of Theorem 3.1 we then get the desired result.

Remark. (1) Since Silva-holomorphicity of a mapping only depends on the bornology on the space, and not on the topology, the proof of Theorem 3.1 for Silva-holomorphic mappings also gives that $W_S(\Omega)=FI_S(\Omega)$ if Ω is a Mackey-open subset of a convex, regular bornological space E with a countable base for the bornology and F = the class of complex Banach spaces. (For the definitions of these concepts concerning bornological vector spaces we refer to H. Hogbé-Nlend [13].)

(2) If we in Theorem 3.2 assume that E has a countable, fundamental system of compact sets, e.g. if E is a (DFC)-space, and that $H_C(\Omega)$ separates points in Ω, then we can replace the assumption that $(z_n)_{n\in\mathbf{N}}\in sW_A(\Omega)$ by $(z_n)_{n\in\mathbf{N}}\in W_A(\Omega)$, $A=C$ or H. The proof of Theorem 3.2 then applies. Note that each component in Ω of the inverse image under p of a compact subset of E is a bounding subset for $H_H(\Omega, F)$.

4. Examples and counterexamples

We begin by proving that if E is a Fréchet space, then E must satisfy some extra assumptions in order that $FI_C^1(E)\neq\emptyset$ where F = any class of complex, locally convex spaces containing $\mathbf{C}$.

First we need a definition.

Definition 4.1. *Let E be a metrizable locally convex space and let $(\|\cdot\|_n)_{n\in\mathbf{N}}$ denote an increasing system of seminorms defining the topology on E. We say that E has property (DN) if there exists a continuous norm $\|\ \|$ on E, such that for each $m\in\mathbf{N}$ there exists $k\geqq m$ and $C>0$ such that $\|z\|_m^2\leqq C\|z\|\cdot\|z\|_k$ for all $z\in E$.*

This property of a metrizable locally convex space was first studied by D. Vogt [26].

Proposition 4.2. *Let E be a complex Fréchet space and F any class of complex, locally convex spaces containing* $\mathbf{C}$. *If $FI_C^1(E)\neq\emptyset$ then E has property (DN).*

Proof. From the proof of D. Vogt [25, 3.5, p. 280] and from D. Vogt [26, 1.4, p. 110] it follows that E has property (DN) if, given any sequence $(\zeta_n)_{n\in\mathbf{N}}$ in E', there exists $h\in H_C(\mathbf{C}, E'_\beta)$ such that $h(n)=\zeta_n$ for all $n\in\mathbf{N}$. (Cf. also R. Meise and D. Voigt [19].)

Let $(z_n)_{n\in\mathbf{N}}\in FI_C^1(E)$ and $(\zeta_n)_{n\in\mathbf{N}}$ in E' be given. Choose $f\in H_C(E)$ such that $(f\sim\zeta_n)(1, z_n)$ for all $n\in\mathbf{N}$. Now, $\hat{d}f$ is an element of $H_C(E, E'_\beta)$ and furthermore, we can choose $g\in H_C(\mathbf{C}, E)$ such that $g(n)=z_n$ for all $n\in\mathbf{N}$. If we define $h=(\hat{d}f)\circ g$, then $h(n)=\zeta_n$ for all $n\in\mathbf{N}$ and thus we get that E has property (DN).

Remark. There are lots of examples of Fréchet spaces that do not have property (DN). Of course, any Fréchet space without continuous norm does not have property (DN). Furthermore, according to D. Vogt [26, 2.4, p. 115], any power series space of finite type does not have property (DN), but nevertheless they all have a continuous norm. So, if E is a power series space of finite type, then Proposition 4.2 implies that $FI^1_C(E)=\emptyset$, but M. Valdivia [24, Thm. 10] implies that $FI^0_C(E)=W_C(E)\neq\emptyset$, where F is any subclass of the class of complex Fréchet spaces and $\mathbf{C}\in F$. Hence, there is, in general, a big difference between the classes $FI^1_C(E)$ and $FI^0_C(E)$.

The following example illuminates, in a perhaps more clear way, the difference between $FI^0_C(E)$ and $FI^1_C(E)$ in some particular cases.

Example 4.3. Let $E=\Lambda_1(\alpha)$ where $\Lambda_1(\alpha)$ is the power series space (of finite type) $\{z\in\mathbf{C}^{\mathbf{N}}: \Pi_r(z)=\sum_{n\in\mathbf{N}}|z_n|r^{\alpha_n}<+\infty,\ 0<r<1\}$ endowed with the Fréchet space topology induced by the norms $\{\Pi_r:\ 0<r<1\}$. (Cf. R. Meise and D. Vogt [18].) R. Meise and D. Vogt [17] have proved that if $\Lambda_1(\alpha)$ is nuclear (which is equivalent with the condition $\lim_{n\to+\infty}\ln(n+1)/\alpha_n=0$), then any $f\in H_C(E)$ factorizes through E_V for some 0-neighbourhood V in E. (Note that V can be chosen such that the gauge of V is a norm.)

Let $(V_n)_{n\in\mathbf{N}}$ be a sequence of strictly decreasing 0-neighbourhoods in E and choose $\zeta_n\in E'$ such that ζ_{n+1} is bounded on V_{n+1} but not on V_n. If follows that there is no $f\in H_C(E)$ such that $(f\sim\zeta_n)(1, z_n)$ for all $n\in\mathbf{N}$ (where $(z_n)_{n\in\mathbf{N}}$ is any given sequence of points in E). We conclude once again that $FI^1_C(E)=\emptyset$.

Remark. (1) If U is a strictly pseudo-convex domain in a k-dimensional Stein manifold then $H_C(U)$ is isomorphic to $H_C(D^k)$, where D denotes the open unit disc in $\mathbf{C}$, which in turn is isomorphic to $\Lambda_1(\sqrt[k]{n})$. (V. S. Mityagin and G. M. Henkin [20].) So, if U is as above, then Proposition 4.2 (or Example 4.3) implies that $FI^1_C(H_C(U))=\emptyset$.

If $U=\mathbf{C}^k$ for some $k\geqq 1$, then we do not know if $FI^1_C(H_C(\mathbf{C}^k))$ is empty or not. (In connection with this, see also a counterexample on factorization by L. Nachbin [22].)

(2) If E is a (DFM)-space (the strong dual of a Fréchet—Montel space), then E satisfies all the assumptions in Theorem 3.1 and E is also a k-space, so $FI_C(\Omega)=W_C(\Omega)$ for any open subset Ω of E.

It is known that any entire function on E factorizes through a normed linear space (as a function of uniform bounded type; J. F. Colombeau and J. Mujica [6]). So, any (DFM)-space E has the same factorization property for entire functions as the space $\Lambda_1(\alpha)$ in Example 4.3 but nevertheless $FI_C(E)=W_C(E)\neq\emptyset$.

The next Proposition gives a sufficient condition on a Fréchet space E in order that $FI_C(E)\neq\emptyset$.

Proposition 4.4. *Let E be a complex Fréchet space for which each closed, C^∞, (0, 1)-form is exact. Then $FI_C(E)\neq\emptyset$ where $F=\{\mathbf{C}\}$.*

Proof. The proof uses an idea of R. Meise and D. Vogt [19, Lemma 3.1].

Let $(z_n)_{n\in\mathbf{N}}$ be given by $z_n=(n, y_n)\in\mathbf{C}\times F$, where F is a closed subspace of E of codimension 1. Let $(P_n)_{n\in\mathbf{N}}$ be a sequence of continuous complex-valued polynomials on E,

$$P_n(z) = c_n^0+\sum_{i=1}^{N(n)} c_n^i((x, y)-(n, y_n))$$

where $z=(x, y)\in\mathbf{C}\times F$ and c_n^i is homogeneous of degree i, $0\leqq i\leqq N(n)$. As in [19] we now choose $\psi\in C^\infty(\mathbf{R})$ with $\operatorname{supp}\psi\subset\left[-\frac{1}{4}, \frac{1}{4}\right]$ and ψ identically equal to 1 on the interval $\left[-\frac{1}{8}, \frac{1}{8}\right]$. Define ψ_n: $\mathbf{R}\to\mathbf{R}$ and φ_n: $\mathbf{C}\to\mathbf{R}$ by $\psi_n(\xi)=\psi(\xi-n)$ and $\varphi_n(x)=\psi_n(\operatorname{Re} x)$, respectively ($n\in\mathbf{N}$). Next we define g: $(\mathbf{C}\setminus\mathbf{N})\times F\to\mathbf{C}$ by $g(x, y)=\sum_{n\in\mathbf{N}}\varphi_n(x)P_n(z)(x-n)^{-1}$. Then $g\in C^\infty((\mathbf{C}\setminus\mathbf{N})\times F)$ and $\bar\partial g(x, y)\cdot(w_1, w_2)=\sum_{n\in\mathbf{N}}P_n(z)(x-n)^{-1}\psi_n'(\operatorname{Re} x)\bar w_1$, $(w_1, w_2)\in(\mathbf{C}\setminus\mathbf{N})\times F$. The choice of ψ_n shows that $\bar\partial g$ can be extended to a closed $C^\infty(0, 1)$-form ω on E, by setting $\omega(n, y)=0$ for all $n\in\mathbf{N}$, $y\in F$. By our assumption on E, there exists a C^∞ function u on E such that $\bar\partial u=\omega$.

Now we choose a function $v\in H_C(\mathbf{C})$ such that $(v\sim(x-n))(N(n)+1, n)$ for all $n\in\mathbf{N}$. Define f: $E\to\mathbf{C}$ by

$$f(x, y) = \begin{cases} (g(x, y)-u(x, y))v(x), & x \neq n, \\ P_n(n, y), & x = n. \end{cases}$$

It follows that $f\in H_C(E)$ and the k-th derivative $\hat d^k f(z_n)=\hat d^k P_n(z_n)$, $1\leqq k\leqq N(n)$, for all $n\in\mathbf{N}$. Thus $(f\sim P_n)(N(n), z_n)$ for all $n\in\mathbf{N}$ and we get that $(z_n)_{n\in\mathbf{N}}\in FI_C(E)$.

Next we give an example of a space E such that $W_C(E)\neq W_H(E)\neq W_S(E)$ and $FI_C(E)\neq FI_H(E)\neq FI_S(E)$.

Example 4.5. Let $E=l^p(\mathbf{N})$, $1<p<+\infty$, (or any infinite-dimensional, reflexive Banach space) and F = the class of complex Fréchet spaces. Denote by E_σ the space E with its weak topology. By Mackey's theorem E_σ has a countable, fundamental system of absolutely convex bounded sets, namely $(nB)_{n\in\mathbf{N}}$ where B is the unit ball in E. Thus E_σ satisfies the assumptions in Theorem 3.1, so we get

$$W_C(\Omega) \subset FI_H(\Omega) = W_H(\Omega) \subset FI_S(\Omega) = W_S(\Omega)$$

for any open subset Ω of E_σ.

Now, B is compact in E_σ so each $f\in H_H(E_\sigma)$ is bounded on B. Furthermore, $H_C(E)=H_S(E)=H_S(E_\sigma)$. Since the unit ball is not a bounding set for $H_C(E)$ it therefore follows that $W_S(E_\sigma)\neq W_H(E_\sigma)$.

In order to see that $W_C(E_\sigma)\neq W_H(E_\sigma)$ we choose $P_n\in P_C(E)$ such that $P_n(z)=(\varphi_n(z))^n/(n!)^{1/q}$, where $(\varphi_n)_{n\in\mathbf{N}}$ are the unit vectors in $l^q(\mathbf{N})=l^p(\mathbf{N})'$, and define $f(z)=\sum_{n\in\mathbf{N}}P_n(z)$. The series defining f then converges uniformly on compact subsets of E_σ, so $f\in H_H(E_\sigma)$. The sequence $(z_n)_{n\in\mathbf{N}}$ defined by $z_n=n^{1/q}e_n$, where $\varphi_k(e_n)=\delta_k^n$, $k, n\in\mathbf{N}$, is an element of $W_H(E_\sigma)$ $(\lim_{n\to+\infty}|P_n(z_n)|=+\infty)$ but since each $h\in H_C(E_\sigma)$ factorizes through a finite-dimensional quotient space of E_σ, it is easy to check that there is no $h\in H_C(E_\sigma)$ such that $\lim_{n\to+\infty}|h(z_n)|=+\infty$ (in fact, each $h\in H_C(E_\sigma)$ is bounded on some infinite subsequence of $(z_n)_{n\in\mathbf{N}}$). Hence, $W_C(E_\sigma)\neq W_H(E_\sigma)$. The fact that each $h\in H_C(E_\sigma)$ factorizes through a finite-dimensional space also implies that $FI_C(E_\sigma)=\emptyset$.

Summing up we have the following inclusions and equalities:

$$\emptyset = FI_C(E_\sigma) \subsetneqq W_C(E_\sigma) \subsetneqq W_H(E_\sigma) = FI_H(E_\sigma) \subsetneqq W_S(E_\sigma) = FI_S(E_\sigma).$$

Next we give an example of a space satisfying the assumptions in Theorem 3.1 and in Lemma 3.3, but E is not a (DF)- or (DFC)-space.

Example 4.6. Let E be a complex, reflexive, infinite dimensional Banach space. Denote by $s(E'_\beta)$ the nuclear bornology on the strong dual E'_β of E, i.e. a subset of $s(E'_\beta)$ is bounded if and only if it is contained in the convex, balanced hull of a sequence $(z_n)_{n\in\mathbf{N}}$ in E'_β such that $\lim_{n\to+\infty}n^kz_n=0$ for each $k\in\mathbf{N}$. Now equip E with the topology obtained by taking polars of bounded subsets in $s(E'_\beta)$. We denote E with this topology by $(E, s(E, E'_\beta))$. It follows by polarity from [4, Lemma 1, p. 97] that for each sequence $(V_n)_{n\in\mathbf{N}}$ of 0-neighbourhoods in $(E, s(E, E'_\beta))$ there exists a sequence $(\lambda_n)_{n\in\mathbf{N}}$ of positive real numbers such that $\bigcap_{n\in\mathbf{N}}\lambda_nV_n$ is again a 0-neighbourhood. Furthermore, since E is reflexive and

$$\sigma(E, E') \subsetneqq s(E, E'_\beta) \subsetneqq \beta(E, E'),$$

it follows from Mackey's theorem that $(E, s(E, E'_\beta))$ has a countable fundamental system of bounded subsets. In fact, $(E, s(E, E'_\beta))$ is hemicompact. (The fact that $\sigma(E, E')\neq s(E, E'_\beta)$ follows from [14, Prop. 2, p. 218].)

In order to prove that $(E, s(E, E'_\beta))$ is not a (DFC)-space, we assume the contrary and try to obtain a contradiction. So, assume that $(E, s(E, E'_\beta))=(F', \tau_0)$ for some Fréchet space F. Then the bornological dual $(E, s(E, E'_\beta))^\times$ of $(E, s(E, E'_\beta))$ is equal to $(F', \tau_0)^\times$ which in turn equals the space F with its compact bornology (H. Hogbé-Nlend [13]). But $(E, s(E, E'_\beta))^\times=$the space E'_β with its nuclear bornology. Hence, the nuclear and compact bornologies on E'_β coincide. But this is only possible if E has finite dimension (H. Hogbé-Nlend and V. B. Moscatelli [14, Cor. 2, p. 161]). So, we have a contradiction and conclude that $(E, s(E, E'_\beta))$ is not a (DFC)-space.

It remains to be shown that $(E, s(E, E'_\beta))$ is not a (DF)-space. Choose $z_n^k=2^{-n/k}e_n$, where $(e_n)_{n\in\mathbf{N}}$ are the unit vectors in $E'=l^q(\mathbf{N})$. Then (z_n^k) is a rapidly decreasing sequence for each fixed $k\in\mathbf{N}$, i.e. $\lim_{n\to+\infty}n^mz_n^k=0$ for each $m\in\mathbf{N}$. Suppose now that

$(E, s(E, E'_\beta))$ is a (DF)-space in order to get a contradiction. Since $\bar{\Gamma}(\{z_n^k\}_{n\in\mathbf{N}})$ is an equicontinuous subset of $l^q(\mathbf{N})$ for each $k\in\mathbf{N}$ ($\bar{\Gamma}$ = the closed, convex, balanced hull) this implies that

$$B = \bar{\Gamma}\big(\bigcup_{k\in\mathbf{N}} \bar{\Gamma}(\{z_n^k\}_{n\in\mathbf{N}})\big)$$

is compact in $l^q(\mathbf{N})$, since it is bounded. But B = the unit ball in $l^1(\mathbf{N})$ and thus we get a contradiction. Hence, $(E, s(E, E'_\beta))$ is not a (DF)-space.

Remark. Example 4.6 shows that for some spaces E, other than (DF)- and (DFC)-spaces, it is not necessary to apply Lemma 3.2 in the proof of Theorem 3.1, i.e. the conditions in Lemma 3.3 are already fulfilled.

In view of M. Valdivia's result [24, Thm. 10] quoted in the introduction, the following example has a certain interest. (We wish to thank V. B. Moscatelli for pointing out this example.)

Example 4.7. There are (DF)-spaces without continuous norms.

An example of such a space is obtained by taking the strong dual of a Fréchet space constructed by I. Amemiya [1]. (This example can also be found in G. Köthe [16, p. 408].)

Let R be the set of all strictly increasing sequences $\eta=(\eta_n)_{n\in\mathbf{N}}$, $\eta_n>0$. Define on R the linear space E of complex-valued functions f such that

$$p_n(f) = \left(\sum_{\eta\in R} |f(\eta)|^2\eta_n\right)^{1/2} < +\infty, \quad n\in\mathbf{N},$$

and equip E with the topology given by the norms $(p_n)_{n\in\mathbf{N}}$. Then E becomes a Fréchet space. Let $B\subset E$ be a bounded subset, i.e. $\sup_{f\in B} p_n(f)=M_n<+\infty$, $n\in\mathbf{N}$. Choose $\eta=(\eta_n)_{n\in\mathbf{N}}$ such that $\lim_{n\to+\infty} M_n^2/\eta_n=0$. Since $|f(\eta)|^2\eta_n\leqq M_n^2$, $n\in\mathbf{N}$, we get that $f(\eta)=0$ for each $f\in B$. This implies that the polar of B, an open subset of the strong dual E'_β of E, contains a straight line. Since B was arbitrarily chosen, this proves that E'_β lacks continuous norms.

5. An application

In this section we apply Theorem 3.1 and prove that a holomorphically convex, open subset of a (DFC)-space is a domain of existence of a meromorphic function.

In M. Valdivia [24, Cor. 1.8] it is proved that a holomorphically convex, open subset Ω of a (DFC)-space is a domain of existence of a holomorphic function. (See also J. Mujica [21].) The proof of this consists in constructing a function $f\in H_C(\Omega)$ such that, for each open, connected subset U of E with $U\cap\Omega\neq\emptyset$ and $U\not\subset\Omega$, f is unbounded on each component of $U\cap\Omega$. (Note that the existence of such a function f does not apriori imply that Ω is a domain of existence of a meromorphic function.)

Before stating the theorem we need some definitions.

Definition 5.1. *Let E be a complex, locally convex space and Ω an open subset of E. A meromorphic function on Ω is a collection $(V_i, g_i, h_i)_{i\in I}$ where $(V_i)_{i\in I}$ is an open covering of Ω, $g_i, h_i \in H_C(V_i)$ for each $i\in I$, g_i is not identically equal to 0 on any component of V_i and $g_i h_j = g_j h_i$ on $V_i \cap V_j$. We denote the class of meromorphic functions on Ω by $M(\Omega)$.*

It follows from Definition 5.1 that each $(V_i, g_i, h_i)_{i\in I}\in M(\Omega)$ is represented by $f=h_i/g_i$ on V_i, $i\in I$, so we write $f=(V_i, g_i, h_i)_{i\in I}$.

Definition 5.2. *Let E be a complex, locally convex space and Ω an open subset of E. We say that Ω is a domain of existence of a meromorphic function if there exists $f\in M(\Omega)$ such that for any open subsets U_1 and U_2 of E, with $U_2\neq\emptyset$, $U_1\not\subset\Omega$ and $U_2\subset U_1\cap\Omega$, there are no meromorphic functions $f_1\in M(U_1)$ and $f_2\in M(U_2)$ such that f and f_1 are both equal to f_2 on U_2.*

Theorem 5.3. *Let E be a complex (DFC)-space and Ω a holomorphically convex, open subset of E. Then Ω is a domain of existence of a meromorphic function.*

Proof. In M. Valdivia [24, Thm. 8] a double sequence $(z_{nm})_{n,m\in\mathbf{N}}$ is constructed satisfying the following conditions:

(1) $(z_{nm})_{n,m\in\mathbf{N}}$ is a sequence of distinct elements of $\chi(\Omega)$ without accumulation points in $\chi(\Omega)$, where χ is the canonical projection from E onto a quotient space of E and $\chi(\Omega)$ is a hemicompact $k_{\mathbf{R}}$-space, i.e. $\chi(\Omega)$ has a countable, fundam ntal system of compact subsets and a function $f\colon \chi(\Omega)\to\mathbf{R}$ is continuous whenever its restriction to each compact subset of $\chi(\Omega)$ is continuous;

(2) $\lim_{n\to+\infty} z_{nm}=z_m$ for each $m\in\mathbf{N}$, where $(z_m)_{m\in\mathbf{N}}$ is a dense subset of $\partial\chi(\Omega)$;

(3) if W is a connected component of $U\cap\Omega$, where U is open and connected in E, $U\cap\Omega\neq\emptyset$ and $U\not\subset\Omega$, then there exists $x_0\in\partial W\cap\partial\Omega$, $m\geqq 0$ and a sequence (x_{nm}) in W such that $\lim_{n\to+\infty} x_{nm}=x_0$ and $\big(\chi(x_{nm})\big)_{n\in\mathbf{N}}$ is a subsequence of $(z_{nm})_{n\in\mathbf{N}}$;

(4) there exists $G\in H_C\big(\chi(\Omega)\big)$ such that $G(z_{nm})=n+m$ for all $n, m\in\mathbf{N}$.

Applying Theorem 3.1 with $L_k=K_k\cap\{z\in\chi(\Omega)\colon |G(z)|\leqq k\}$, $k\in\mathbf{N}$, where $(K_k)_{k\in\mathbf{N}}$ is a fundamental system of compact subsets of $\chi(\Omega)$, we obtain a function $\hat{f}\in H_C\big(\chi(\Omega)\big)$ such that $\hat{f}$ is not identically equal to 0 on any component $\chi(\Omega)$ and $(\hat{f}\sim 0)(n+m, z_{nm})$ for all $n, m\in\mathbf{N}$. (Note that $\hat{f}$ is continuous since $\chi(\Omega)$ is a $k_{\mathbf{R}}$-space.)

Now we define $f\in H_C(\Omega)$ by $f=\hat{f}\circ\chi$. We claim that Ω is the domain of meromorphy for f. We assume that this is false, in order to obtain a contradiction. I.e. we assume that there exists open subsets U_1 and U_2 of E and two meromorphic functions $f_1\in M(U_1)$ and $f_2\in M(U_2)$ such that $U_2\neq\emptyset$, $U_1\not\subset\Omega$, $U_2\subset U_1\cap\Omega$ and f and f_1 are both equal to f_2 on U_2.

We make some observations in order to shorten the notation. First we observe that by choosing U_1 smaller if necessary, we can obtain that $U_1 \not\subset \Omega$, $U_1 \cap \Omega \neq \emptyset$ and there exists $g, h \in H_C(U_1)$ such that $gf_1 = h$ on U_1, where g is not identically equal to 0 on any component of U_1. Secondly, if W is any component of $U_1 \cap \Omega$ and if $U_2 \cap W \neq \emptyset$, then f_2 has a unique meromorphic extension to W, so we can assume that $W \subset U_2$. (Note that f_2 is in fact holomorphic on U_2 since $f \in H_C(\Omega)$).

It is easy to see that these assumptions on U_2 and U_1 do not affect the generality.

Let W be a component of $U_1 \cap \Omega$ contained in U_2 and choose $x_0 \in \partial W \cap \partial \Omega$ and a sequence $(x_{nm})_{n \in \mathbf{N}}$ in W such that $\lim_{n \to +\infty} x_{nm} = x_0$ and $(\chi(x_{nm}))_{n \in \mathbf{N}}$ is a subsequence of $(z_{nm})_{n \in \mathbf{N}}$ for some $m \in \mathbf{N}$. Since $gf_1 = h$ on U_1 and $f = f_2 = f_1$ on W, we get that $gf = h$ on W. From this and the fact that $(f \sim 0)(n+m, x_{nm})$ for all $n \in \mathbf{N}$, it follows that $(h \sim 0)(n+m, x_{nm})$ for all $n \in \mathbf{N}$. Assuming for a moment that $\lim_{n \to +\infty} \hat{d}^k h(x_{nm}) = \hat{d}^k h(x_0)$ for all $k \in \mathbf{N}$, we get that h has a zero of infinite order at x_0 and from the uniqueness of analytic continuation it then follows that h is identically equal to 0 on W. But neither f nor g is identically equal to 0 on W and thus we get a contradiction. So, it remains to prove e.g. that $\hat{d}^k h \colon U_1 \to L^k(E) =$ the space of symmetric k-linear, continuous forms on E^k, with the topology of uniform convergence on compact subsets of E^k, is continuous. We prove this when $k=1$. (The case $k=0$ is obviously true and the case $k>1$ is treated in analogy with $k=1$.)

Let K be a compact subset of U_1. Since h is continuous there exists a neighbourhood V of K in U_1 such that $\sup_{x \in V} |h(x)| < +\infty$. The fact that K is compact implies that we can choose an absolutely convex 0-neighbourhood W in E such that $K + W \subset V$. Let now B be an absolutely convex, bounded subset of E and choose $\varrho > 0$ such that $\varrho B \subset W$. Using Cauchy-inequalities we get

$$\sup_{z \in B, x \in K} |d\hat{h}(x)(z)| \leqq \sup_{z \in K + \varrho B} |h(z)|/\varrho \leqq \sup_{z \in K+W} |h(z)|_\varrho \leqq \sup_{z \in V} |h(z)|_\varrho < +\infty.$$

This ends the proof of Theorem 5.3.

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