On the growth of subharmonic functions along paths

John Lewis*, John Rossi† and Allen Weitsman*

1. Introduction

Let $\mathbf{C}$ be the complex plane. Then by a path γ tending to ∞, we shall always mean a continuous mapping of $0 \leqq t < 1$ into $\mathbf{C}$ with $\lim_{t \to 1} |\gamma(t)| = +\infty$. If u is subharmonic in $\mathbf{C}$, put $M(r) = M(r, u) = \max_{|z|=r} u(z)$, $0 < r < \infty$. In [7] Huber proved the following theorem:

Theorem A. *Let u be subharmonic in $\mathbf{C}$ and suppose that* $\lim_{r \to \infty} \dfrac{M(r)}{\log r} = +\infty$. *Given $\lambda > 0$ there exists a path, $\Gamma(\lambda)$, tending to ∞ with*

$$\int_{\Gamma(\lambda)} e^{-\lambda u} |dz| < +\infty.$$

In Theorem A, $|dz|$ denotes arc length. Also in [10] Talpur proved

Theorem B. *Let u be subharmonic in $\mathbf{C}$ with* $\lim_{r \to \infty} \dfrac{M(r)}{\log r} = +\infty$. *Then there exists a path Γ tending to ∞ with*

$$\frac{u(z)}{\log |z|} \to \infty \quad as \quad z \to \infty \quad on \quad \Gamma.$$

In this paper, we obtain the following generalization of Theorems A and B, which in fact solves a problem raised by Hayman in [5, p. 12].

Theorem 1. *Let u be subharmonic in $\mathbf{C}$ and suppose that* $\lim_{r \to \infty} \dfrac{M(r)}{\log r} = +\infty$.

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Then there exists a path Γ tending to ∞ with

$$\int_\Gamma e^{-\lambda u}|dz| < +\infty \quad \textit{for each} \quad \lambda > 0, \tag{1.1}$$

$$\frac{u(z)}{\log|z|} \to +\infty \quad \textit{as} \quad z\to\infty \quad \textit{on} \quad \Gamma. \tag{1.2}$$

The important feature of Theorem 1 is that Γ is independent of λ. In the special case $u=\log|f|$ where f is an entire function of finite order, Theorem 1 was proved by Zhang [12]. Our main tool is a version of Hall's lemma which may be stated

Lemma A. *Let w, $0\leqq w\leqq 1$, be subharmonic in $\Delta=\{z\colon |z|<1\}$ with $w(0)=1-\delta$. Then*

$$m(F) \geqq 2\pi - c\delta, \tag{1.3}$$

where

$$F = \{\theta\in[0, 2\pi]\colon w(re^{i\theta}) > 0 \quad \textit{for} \quad 0\leqq r<1\},$$

and m denotes one dimensional Lebesgue measure.

In Lemma A, as in the sequel, c denotes a positive absolute constant, not necessarily the same at each occurrence. Lemma A differs somewhat from Hall's lemma as it is stated for example in [3, ch. 12]. Related versions appear elsewhere (cf. [8, p. 13], [4, p. 193]). We shall sketch a proof in Section 5.

Let $d(G, H)$ denote the distance between the sets G and H. Let $L(\gamma)$ denote the length of a curve γ. Choose δ_0, $0<\delta_0<1$, so that $m(F)\geqq\pi$ in (1.3) when $0<\delta\leqq\delta_0$. Using Lemma A and conformal mapping we obtain in Section 2,

Lemma 1. *Let v, $0\leqq v\leqq 1$ be subharmonic in a simply connected domain D. If $a\in D$ and $v(a)\geqq 1-\delta_0$, then there is a path γ from a to b in ∂D (boundary of D) with $\gamma-\{b\}\subseteqq D$, $v>0$ on $\gamma-\{b\}$, and*

$$L(\gamma) \leqq cd(a, \partial D). \tag{1.4}$$

In Section 3 we apply Lemma 1 inductively in certain components of $\{z\colon u(z)<c\}$ to obtain Theorem 1. In Section 4 we indicate that our method also yields a slightly different form of a theorem due to Davis and Lewis [2]:

Theorem C. *Let D be a simply connected domain with $0\in D$. If u, $0\leqq u\leqq 1$ is subharmonic in D and $u(0)=\varepsilon>0$, then there is a path γ from 0 to a point b on ∂D with $\gamma-\{b\}\subseteqq D$, $u>0$ on $\gamma-\{b\}$, and*

$$L(\gamma) \leqq c\varepsilon^{-c} d(0, \partial D). \tag{1.5}$$

Also in Section 4 we make a remark concerning the smallest exponent for which (1.5) is valid.

Finally, we note that our method avoids all problems arising from the fact that u in Theorem 1 may not be continuous (see [10], [11]), since we work only in components of $\{z\colon u(z)<c\}$, which are open and simply connected.

2. Proof of Lemma 1

Let v, D be as in Lemma 1. We assume, as we may, that $a=0\in D$, since otherwise we change coordinate systems. Let f be the Riemann mapping function from Δ to D with $f(0)=0$, $f'(0)>0$. We put $w=v\circ f$ and apply Lemma A to get a corresponding set F with $m(F)\geqq\pi$. We claim for some $\theta_0\in F$ that

$$\int_0^1 |f'(re^{i\theta_0})|\,dr \leqq cd(0,\partial D). \tag{2.1}$$

Once this claim is proved we can take $\gamma=f(re^{i\theta_0})$, $0\leqq r<1$, to get Lemma 1. To prove (2.1) we first show

$$\iint_\Delta |f'(z)|\,|f(z)|^{-1}\,dx\,dy \leqq c. \tag{2.2}$$

(2.2) is well known (see [9, Thm. 5.2]). For completeness we give the proof of (2.2). Let $g=\dfrac{f}{f'(0)}$. Then, $\dfrac{g'}{g}=\dfrac{f'}{f}$, so it suffices to prove (2.2) for g. We shall need the basic estimate ([9, Thm. 1.6]),

$$|z|(1+|z|)^{-2} \leqq |g(z)| \leqq |z|(1-|z|)^{-2}, \quad z\in\Delta. \tag{2.3}$$

Now

$$\iint_\Delta |g'(z)|\,|g(z)|^{-1}\,dx\,dy = \iint_{\{|g|<1\}} (\ )\,dx\,dy+\iint_{\{|g|\geqq 1\}} (\ )\,dx\,dy = I_1+I_2.$$

From Schwarz's inequality and (2.3),

$$I_1 \leqq \Big(\iint_{\{|g|<1\}} |g'(z)|^2|g(z)|^{-1}\,dx\,dy\Big)^{1/2}\Big(\iint_{\{|g|<1\}} |g(z)|^{-1}\,dx\,dy\Big)^{1/2}$$
$$\leqq \Big(\iint_{|\zeta|\leqq 1} |\zeta|^{-1}\,d\xi\,d\eta\Big)^{1/2}\Big(\iint_\Delta (1+|z|)^2|z|^{-1}\,dx\,dy\Big)^{1/2} \leqq c.$$

Similarly,

$$I_2 \leqq \Big(\iint_{\{|g|>1\}} |g'(z)|^2|g(z)|^{-(9/4)}\,dx\,dy\Big)^{1/2}\Big(\iint_{\{|g|>1\}} |g(z)|^{1/4}\,dx\,dy\Big)^{1/2} \leqq c.$$

Thus (2.2) is valid. Next put

$$h(e^{i\theta}) = \sup_{0<r<1}\,(r^{-1}|f(re^{i\theta})|), \quad 0\leqq\theta\leqq 2\pi.$$

We shall show that

$$(2.4)\qquad m\{\theta\in[0,2\pi]:\ h(e^{i\theta})\geqq Kf'(0)\}\leqq cK^{-1/4},\quad 0<K<+\infty.$$

To do this we use (see [9, Thm. 5.1]),

$$(2.5)\qquad \sup_{0<r<1}\left(r^{-1/4}\int_0^{2\pi}|f(re^{i\theta})|^{1/4}\,d\theta\right)\leqq cf'(0)^{1/4}.$$

Hence, $\frac{f(z)}{z}$ is in the Hardy space, $H^{1/4}$, and so by a theorem of Hardy—Littlewood [3, Thm. 1.9] its radial maximal function, h, is integrable to the $\frac{1}{4}$ power, and satisfies the same type of inequality as (2.5). This inequality and the usual weak type estimate imply (2.4).

From (2.2) and (2.4) we see there exists $\theta_0\in F$ with $h(e^{i\theta_0})\leqq cf'(0)$, and

$$\int_0^1|f'(re^{i\theta_0})|\,|f(re^{i\theta_0})|^{-1}r\,dr\leqq c.$$

We conclude that

$$\int_0^1|f'(re^{i\theta_0})|\,dr\leqq h(e^{i\theta_0})\int_0^1|f'(re^{i\theta_0})|\,|f(re^{i\theta_0})|^{-1}r\,dr\leqq cf'(0).$$

Since (see [9, Cor. 1.4])

$$\tfrac{1}{4}f'(0)\leqq d(0,\partial D)\leqq f'(0),$$

the proof of (2.1) and Lemma 1 is complete.

3. Proof of Theorem 1

Let u be as in Theorem 1 and choose $a_1\in\mathbf{C}$ with $u(a_1)>0$. Let δ_1 be such that $(1-\delta_1)^2=1-\delta_0$, and let D_1 be the component of $\{z:\ u(z)<(1-\delta_1)^{-1}u(a_1)\}$ containing a_1. Since u is upper semicontinuous, D_1 is open, and it follows from the maximum principle for subharmonic functions that D_1 is simply connected. Put

$$v(z)=u(a_1)^{-1}\max\left[(u-\delta_1u(a_1))(1-\delta_1),\,0\right](z),\quad z\in D_1.$$

Note that

$$v(a_1)=(1-\delta_1)^2=(1-\delta_0),$$

and

$$v\leqq[(1-\delta_1)^{-1}-\delta_1](1-\delta_1)=1-\delta_1+\delta_1^2<1.$$

So by Lemma 1, there is a curve γ_1 joining a_1 to a point a_2 on ∂D_1 with $\gamma_1-\{a_2\}\subseteqq D_1$, $v>0$ on $\gamma_1-\{a_2\}$, and

$$L(\gamma_1)\leqq cd(a_1,\partial D_1).$$

Since $v>0$ on $\gamma_1-\{a_2\}$ we see that

$$u \geqq \delta_1 u(a_1) \quad \text{on} \quad \gamma_1-\{a_2\}.$$

From the upper semicontinuity of u observe that

$$u(a_2) \geqq (1-\delta_1)^{-1}u(a_1).$$

We continue by induction. Suppose that $\gamma_1, \ldots, \gamma_n$ $(n\geqq 1)$ have been constructed with endpoints $a_1, a_2, \ldots, a_n, a_{n+1}$, respectively, where γ_k joins a_k to a_{k+1} in D_k, the component of $\{z\colon u(z)<(1-\delta_1)^{-1}u(a_k)\}$ containing a_1, with $a_{k+1}\in\partial D_k$. Also suppose that

$$L(\gamma_k) \leqq cd(a_k, \partial D_k), \quad 1 \leqq k \leqq n, \tag{3.1}$$

$$u \geqq \delta_1 u(a_k) \quad \text{on} \quad \gamma_k-\{a_{k+1}\}, \quad 1 \leqq k \leqq n, \tag{3.2}$$

$$u(a_{k+1}) \geqq (1-\delta_1)^{-1}u(a_k), \quad 1 \leqq k \leqq n. \tag{3.3}$$

Note from (3.3) that $D_i \subseteqq D_j$, when $i\leqq j$. We then let D_{n+1} be the component containing a_1 of $\{z\colon u(z)<(1-\delta_1)^{-1}u(a_{n+1})\}$. As previously, we see that D_{n+1} is open and simply connected. Also by the induction hypothesis, $a_{n+1}\in D_{n+1}$. Put

$$v(z) = u(a_{n+1})^{-1}\max\left[(1-\delta_1)\big(u-\delta_1 u(a_{n+1})\big), 0\right](z), \quad z\in D_{n+1}.$$

Then, $v(a_{n+1})=1-\delta_0$ and $v\leqq 1$. Applying Lemma 1 we get a curve γ_{n+1} joining a_{n+1} to a point $a_{n+2}\in\partial D_{n+1}$ with properties (3.1)—(3.2) for $k=n+1$. (3.3) is also true for $k=n+1$, since u is upper semicontinuous. We conclude by induction that (3.1)—(3.3) is valid for all positive integers.

Put $\Gamma=\bigcup_{k=1}^{\infty}\gamma_k$. At this point we indicate the significance of (3.1)—(3.3). From (3.3) and iteration we find that

$$u(a_{k+1}) \geqq (1-\delta_1)^{-k}u(a_1), \quad k = 0, 1, 2, \ldots. \tag{3.4}$$

From (3.2) it follows that

$$u \geqq \delta_1(1-\delta_1)^{(1-k)}u(a_1) \quad \text{on} \quad \gamma_k, \quad k = 1, 2, \ldots. \tag{3.5}$$

Thus u is large on γ_k when k is large. Moreover, (3.1) implies γ_k is not "too long" as the next argument shows.

To prove (1.1), given a positive integer n, let $m=m(n)$ be the least positive integer such that

$$u(z) > n\log|z|+M(1),$$

at some point $z\in D_m-\Delta$. Then if $k\geqq m$ there exists a sequence $(z_r)_1^{\infty}$ in D_k with $\lim_{r\to\infty} z_r=\zeta$ in ∂D_k, and

$$n\log|z_r| < u(z_r)-M(1) \leqq (1-\delta_1)^{-1}u(a_k)-M(1),$$

by the Phragmén—Lindelöf Maximum Principle, and the fact that $D_m \subseteqq D_k$ for $k \geqq m$. Thus for $k \geqq m$,

$$(3.6) \qquad d(a_1, \partial D_k) \leqq |\zeta| + |a_1| \leqq \exp\{n^{-1}[(1-\delta_1)^{-1}u(a_k) - M(1)]\} + |a_1|.$$

To estimate $L(\gamma_k)$ we use (3.6), (3.1), and an iterative procedure. Then for $k \geqq m$,

$$(3.7) \qquad L(\gamma_k) \leqq c d(a_k, \partial D_k) \leqq c[|a_k - a_1| + d(a_1, \partial D_k)]$$

$$\leqq c\left[\sum_{i=1}^{k-1} L(\gamma_i) + d(a_1, \partial D_k)\right] = c\left[L(\gamma_{k-1}) + \sum_{i=1}^{k-2} L(\gamma_i) + d(a_1, \partial D_k)\right]$$

$$\leqq c\left[c\left\{\sum_{i=1}^{k-2} L(\gamma_i) + d(a_1, \partial D_{k-1})\right\} + \sum_{i=1}^{k-2} L(\gamma_i) + d(a_1, \partial D_k)\right]$$

$$\leqq c(1+c)\left[\sum_{i=1}^{k-2} L(\gamma_i) + d(a_1, \partial D_k)\right] \leqq \ldots \leqq c(1+c)^{k-1} d(a_1, \partial D_k)$$

$$\leqq (1+c)^k B \exp[n^{-1}(1-\delta_1)^{-1}u(a_k)],$$

where

$$B = |a_1| + \exp[-n^{-1}M(1)].$$

Given $\lambda > 0$ we choose $n = n(\lambda)$ so large that

$$n^{-1}(1-\delta_1)^{-1} \leqq \frac{\delta_1 \lambda}{2}.$$

Then from (3.7), (3.2), and (3.4) we have for $k \geqq m(n)$,

$$\int_{\gamma_k} e^{-\lambda u}|dz| \leqq B \exp[k \log(1+c) + n^{-1}(1-\delta_1)^{-1}u(a_k) - \delta_1 \lambda u(a_k)]$$

$$\leqq B \exp[k \log(1+c) - \tfrac{1}{2}\delta_1 \lambda (1-\delta_1)^{(1-k)} u(a_1)].$$

Summing this inequality, we get (1.1).

To prove (1.2) we use (3.2) and (3.7). Then for $k > m(n)$ and $z \in \gamma_k$,

$$|z| \leqq |a_1| + \sum_{i=1}^{k} L(\gamma_i) \leqq k(1+c)^k A \exp[n^{-1}(1-\delta_1)^{-1}u(a_k)]$$

$$\leqq k(1+c)^k A \exp[n^{-1}(1-\delta_1)^{-1}\delta_1^{-1}u(z)],$$

where

$$A = B + |a_1| + \sum_{i=1}^{m(n)} L(\gamma_i).$$

Taking logarithms and using (3.5) we conclude that for $z \in \Gamma$,

$$\liminf_{z \to \infty} \left(\frac{u(z)}{\log |z|}\right) \geqq \delta_1(1-\delta_1)n.$$

Since n is arbitrary, (1.2) is true.

4. Proof of Theorem C

Let u, D be as in Theorem C and let D_1 be the component of

$$D \cap \{z\colon\ u(z) < (1-\delta_0)^{-1} u(0)\}$$

containing 0. As in the proof of Theorem 1, we note that D_1 is open and simply connected. Put

$$v(z) = u(0)^{-1}(1-\delta_0)u(z),\quad z \in D_1.$$

Then v satisfies the conditions of Lemma 1, so there is a path γ_1 joining $z=0$ to a point a_1 in ∂D_1 with $\gamma_1 - \{a_1\} \subseteqq D_1$, $v>0$ on $\gamma_1 - \{a_1\}$, and

$$L(\gamma_1) \leqq c d(0, \partial D_1).$$

If $a_1 \in \partial D$, we quit. Otherwise, we let D_2 be the component of

$$\{z\colon\ u(z) < (1-\delta_0)^{-1} u(a_1)\}$$

containing 0 and continue as in the proof of Theorem 1. After at most k times, where k is the least positive integer such that $\varepsilon(1-\delta_0)^{-k} \geqq 1$, that is,

$$k-1 \leqq \log(\varepsilon)/[\log(1-\delta_0)],$$

we obtain a path $\gamma = \bigcup_{i=1}^{k} \gamma_i$, joining 0 to a point on ∂D with $u>0$ on γ. The length estimate in Lemma 1 implies, as in the proof of (3.7), that γ has length at most

$$k(1+c)^k d(0, \partial D) \leqq c\varepsilon^{-c} d(0, \partial D).$$

This concludes the proof of Theorem C.

We remark that for $D=\Delta$ and subharmonic functions of the form

$$u = \max[\log|f|, 0],\quad f \text{ analytic in } \Delta,\quad |f| \leqq e,$$

it is permissible to take $c=3$ for the exponent in Theorem C. Indeed by a generalization of a theorem of Garnett due to Dahlberg [1], there exists ψ infinitely differentiable in Δ with

$$\iint_\Delta (|\psi_z| + |\psi_{\bar z}|)\, dx\, dy \leqq c\varepsilon^{-2},$$

and $|f-\psi| \leqq \dfrac{\varepsilon}{16}$. From the coarea theorem it follows that

$$\int_0^\infty L(\{|\psi| = t\})\, dt \leqq c \iint_\Delta (|\psi_z| + |\psi_{\bar z}|)\, dx\, dy \leqq c\varepsilon^{-2},$$

so for some ε', $\dfrac{\varepsilon}{4} < \varepsilon' < \dfrac{\varepsilon}{2}$,

$$L(\{|\psi| = 1+\varepsilon'\}) \leqq c\varepsilon^{-3}.$$

Next note that

$$0\in\left\{|f|>1+\frac{9}{16}\varepsilon\right\}\subseteq\{|\psi|>1+\varepsilon'\}\subseteq\left\{|f|>1+\frac{3}{16}\varepsilon\right\},$$

since $|f-\psi|\leqq\dfrac{\varepsilon}{16}$ and $|f(0)|=e^{\varepsilon}>1+\varepsilon$. Hence the closure of the component of $\{|\psi|>1+\varepsilon'\}$ containing zero, contains points in $\partial\Delta$, and the part of its boundary in Δ has length at most $c\varepsilon^{-3}$. Using these facts it is easy to deduce the existence of γ in Theorem C with

$$L(\gamma)\leqq c\varepsilon^{-3}.$$

5. Proof of Lemma A

If $H\subseteq\Delta$, let H^* be the projection (from $z=0$) of $H-\{0\}$ onto $\{z\colon |z|=1\}$. Let w be as in Lemma A and put $\psi=1-w$,

$$\Omega=\{z\colon \psi(z)>\tfrac{1}{2}\}.$$

We note that Ω is open since ψ is lower semicontinuous. Then for the proof of Lemma A it clearly suffices to show

$$m(\Omega^*)\leqq c\psi(0)=c\delta. \tag{5.1}$$

We first prove (5.1) when Ω is replaced by Ω_1, where

$$\Omega_1=\Omega\cap\{z\colon \tfrac{1}{2}<|z|<1\}.$$

We note that (5.1) for Ω_1 can be derived from Hall's lemma using conformal mapping as in [3, ch. 12]. We prefer, however, to use Hall's technique and argue directly. To do this choose a finite collection, $\sigma_1,\dots,\sigma_n$, of open circular arcs (about $z=0$) whose closures are contained in Ω_1, with $\sigma_i^*\cap\sigma_j^*=\emptyset$, $i\neq j$, and

$$\textstyle\sum_{i=1}^n m(\sigma_i^*)\geqq\frac{1}{2}m(\Omega_1^*).$$

This choice is possible since any compact set contained in Ω_1^* can be covered by the projections of a finite number of circular arcs in Ω_1.

Let

$$g(z,\zeta)=\log\left|\frac{1-\bar\zeta z}{z-\zeta}\right|,\quad z,\zeta\in\Delta,$$

be Green's function for Δ with pole at $\zeta\in\Delta$. We shall need the estimates:

$$g(z,\zeta)\leqq c\frac{(1-|z|)(1-|\zeta|)}{|z-\zeta|^2},\quad z,\zeta\in\Delta, \tag{5.2}$$

$$g(z,\zeta)\leqq c\log\frac{(1-|z|)}{|z-\zeta|},\quad 0<|z-\zeta|\leqq\frac{1}{2}(1-|z|). \tag{5.3}$$

Let

$$dv(\zeta) = |d\zeta|(1-|\zeta|)^{-1}, \quad \zeta \in \bigcup_{i=1}^{n} \sigma_i,$$

and $dv(\zeta)=0$, otherwise. Put

$$\varphi(z) = \int_{\Delta} g(z, \zeta)\, dv(\zeta), \quad z \in \Delta.$$

We claim that

$$\varphi \leqq c. \tag{5.4}$$

To prove (5.4), let

$$I_n = \int_{2^{(n-1)}(1-|z|) \leqq |z-\zeta| \leqq 2^n(1-|z|)} g(z, \zeta)\, dv(\zeta),$$

for $n=0, \pm 1, \ldots$. Then from (5.2) it follows easily for $n \geqq 0$ that $I_n \leqq c2^{-n}$, while for $n \leqq -1$, it follows from (5.3) that $I_n \leqq c|n|2^n$. Summing these inequalities we get (5.4).

Next observe from (5.2) that $\lim_{|z| \to 1} \varphi(z)=0$. From this observation, (5.4), the fact that $\psi > \frac{1}{2}$ on the closure of $\bigcup_{i=1}^n \sigma_i$, and the minimum principle for superharmonic functions we deduce $\varphi \leqq c\psi$. Thus,

$$m(\Omega_1^*) \leqq 2m\left(\bigcup_{i=1}^{n} \sigma_i^*\right) \leqq c\int_{\Delta} \log(|\zeta|^{-1})\, dv(\zeta) = c\varphi(0) \leqq c\psi(0).$$

This proves (5.1) for Ω_1.

Finally we show that (5.1) holds for $\Omega_2 = \Omega - \Omega_1$. To do this we use the Riesz representation formula for positive superharmonic functions ([6, Thm. 6.18]) to write $\psi = h+p$, where $h \geqq 0$, is the greatest harmonic minorant of ψ in Δ, and p is a Green's potential. From Harnack's inequality

$$h(z) \leqq ch(0) \leqq c\psi(0), \quad |z| \leqq \tfrac{1}{2}. \tag{5.5}$$

Also, if μ is the positive Borel measure associated with p, then from (5.2), (5.3), we deduce for $|z| \leqq \frac{1}{2}$,

$$\begin{aligned} p(z) &= \int_{|\zeta| \leqq \frac{3}{4}} g(z, \zeta)\, d\mu(\zeta) + \int_{\frac{3}{4} < |\zeta| < 1} g(z, \zeta)\, d\mu(\zeta) \\ &\leqq c\int_{|\zeta| < \frac{3}{4}} \log \frac{4}{|z-\zeta|}\, d\mu(\zeta) + c\int_{|\zeta| < 1} (1-|\zeta|)\, d\mu(\zeta) \\ &= cq(z) + c\int_{|\zeta| < 1} (1-|\zeta|)\, d\mu(\zeta) \leqq cq(z) + c\psi(0). \end{aligned} \tag{5.6}$$

Put

$$\bar{q}(e^{i\theta}) = \sup_{0 \leqq r \leqq \frac{1}{2}} q(re^{i\theta}), \quad 0 \leqq \theta \leqq 2\pi.$$

We shall show that

$$\int_0^{2\pi} \bar{q}(e^{i\theta})\, d\theta \leqq cq(0) \leqq c\psi(0). \tag{5.7}$$

To prove (5.7) we write for $0<r\leqq\frac{1}{2}$,

$$q(re^{i\theta}) = \int_{|\zeta|\leqq\frac{r}{2}} \dots + \int_{\frac{3}{2}r\leqq|\zeta|} \dots + \int_{\frac{r}{2}<|\zeta|<\frac{3}{2}r} \dots .$$

The first two integrals are easily estimated above by $cq(0)$. To estimate the third integral, put $\zeta=\varrho e^{i\varphi}$, and

$$J(e^{i\theta}) = \int_{|\zeta|<\frac{3}{4}} \log[4\,|e^{i\theta}-e^{i\varphi}|^{-1}]\, d\mu(\varrho e^{i\varphi}).$$

Then,

$$\int_{\frac{r}{2}<|\zeta|<\frac{3}{2}r} \log(4\,|re^{i\theta}-\varrho e^{i\varphi}|^{-1})\, d\mu(\varrho e^{i\varphi})$$

$$\leqq c\int_{\frac{r}{2}<|\zeta|<\frac{3}{2}r} \log(4\,|e^{i\theta}-e^{i\varphi}|^{-1})\, d\mu(\varrho e^{i\varphi}) + c\left(\log\frac{1}{r}\right)\mu\left(\left\{z\colon |z|<\frac{3}{2}r\right\}\right)$$

$$\leqq cJ(e^{i\theta})+cq(0).$$

Since the right-hand side of this inequality is independent of r it follows that

$$\bar{q}(e^{i\theta}) \leqq cq(0)+cJ(e^{i\theta}).$$

Integrating this inequality with respect to θ from 0 to 2π, and interchanging the order of integration we get (5.7). From (5.7) and the usual weak type estimates, it follows that

$$m\{\theta\in[0, 2\pi]\colon \bar{q}(e^{i\theta}) \geqq K\} \leqq K^{-1}c\psi(0), \quad 0<K<\infty.$$

Thus from (5.5) and (5.6),

$$m(\Omega_2^*) \leqq m\{\theta\in[0, 2\pi]\colon c\bar{q}(e^{i\theta})+c\psi(0) \geqq \tfrac{1}{2}\} \leqq c\psi(0).$$

We conclude first that (5.1) holds for Ω_2 and then from our earlier work that (5.1) holds for Ω. The proof of Lemma A is now complete.

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John Lewis
Department of Mathematics
University of Kentucky
Lexington, KY 40 506
USA

John Rossi
Department of Mathematics
Virginia Polytechnic Inst.
Blacksburg, VA 24 061
USA

Allen Weitsman
Department of Mathematics
Purdue University
Lafayette, IN 47 907
USA