On determinacy notions for the two dimensional moment problem

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One of the most important notions in the theory of the classical moment problem is that of determinacy. In the multidimensional case this concept is very far from being understood. In addition to the "usual" determinacy, Fuglede [5] proposed two other notions which he called strong determinacy and ultradeterminacy. The aim of the present paper is to contribute to a better understanding of these concepts. We restrict ourselves to the two-dimensional case. In Section 1 we discuss three examples. Among others they answer a question of Fuglede ([5], p. 62) by showing that determinacy does not imply strong determinacy and strong determinacy does not imply ultradeterminacy. In Section 2 we introduce another determinacy concept called strict determinacy and we give sufficient conditions for this.

First we explain some terminology and some notation, cf. [5] for more details. Let $d\in\mathbf{N}$. A **positive semi-definite** d**-sequence** is a real d-sequence $\alpha=(\alpha_k)_{k\in\mathbf{N}_0^d}$ such that $\sum_{r,s=1}^{m}\alpha_{k_r+k_s}\gamma_r\overline{\gamma_s}\geqq 0$ for arbitrary $k_1,\dots,k_m\in\mathbf{N}_0^d$, $m\in\mathbf{N}$ and $\gamma_1,\dots,\gamma_m\in\mathbf{C}$. Let $\mathscr{H}_\alpha$ denote the canonical Hilbert space associated with a positive semi-definite d-sequence α, see e.g. [5], Section 4. Scalar product and norm of $\mathscr{H}_\alpha$ are denoted by $\langle .,.\rangle_\alpha$ and $\|.\|_\alpha$, respectively. For notational simplicity we consider polynomials of the polynomial algebra $\mathbf{C}(x_1,\dots,x_d)$ directly as elements of $\mathscr{H}_\alpha$. (Strictly speaking, we have to take the corresponding equivalence classes.) The multiplication operator by a polynomial p on $\mathbf{C}(x_1,\dots,x_d)$ is denoted by M_p.

Let $M(\mathbf{R}^d)$ be the set of all positive Borel measures μ on $\mathbf{R}^d$ which have finite moments of all order, i.e. $\int|x^k|\,d\mu(x)<\infty$ for all $k\in\mathbf{N}_0^d$. Here we set $x^k:=x_1^{k_1}\dots x_d^{k_d}$ for $x=(x_1,\dots,x_d)$ and $k=(k_1,\dots,k_d)\in\mathbf{N}_0^d$, where $x_j^0:=1$. A real d-sequence $\alpha=(\alpha_k)_{k\in\mathbf{N}_0^d}$ is called a **moment** d**-sequence** if there exists a measure $\mu\in M(\mathbf{R}^d)$ which has the moments α_k, i.e. $\alpha_k=\int x^k\,d\mu$ for $k\in\mathbf{N}_0^d$. In this case we say that μ is a representing measure for α. For $\mu_1,\mu_2\in M(\mathbf{R}^d)$, we write $\mu_1\sim\mu_2$ if μ_1 and μ_2 have the same moments, i.e. $\int x^k\,d\mu_1=\int x^k\,d\mu_2$ for all $k\in\mathbf{N}_0^d$. For a measure μ on $\mathbf{R}^d$ and $j\in\{1,\dots,d\}$, the projection measure μ_{x_j} is defined as the image of μ under the mapping $(x_1,\dots,x_d)\to x_j$.

Suppose α is a moment d-sequence and μ is a representing measure for α. We say that α or μ is **determinate** if μ is the only representing measure for α. Further, α or μ is said to be **strongly determinate** if each multiplication operator M_{x_j}, $j=1, \ldots, d$, on $\mathbf{C}(x_1, \ldots, x_d)$ is essentially self-adjoint in the Hilbert space $\mathscr{H}_\alpha$ ([5], p. 56). The d-sequence α or the measure μ will be called **ultradeterminate** if there are self-adjoint extensions S_j of M_{x_j}, $j=1, \ldots, d$, on the Hilbert space $\mathscr{H}_\alpha$ such that $\mathbf{C}(x_1, \ldots, x_d)$ is dense in the intersection of the domains of S_j with respect to the norm $\|\cdot\|_\alpha + \|S_1 \cdot\|_\alpha + \ldots + \|S_d \cdot\|_\alpha$ ([5], p. 57). We refer to Fuglede's paper [5] for a thorough discussion of these concepts. Let us note that ultradeterminacy implies strong determinacy and strong determinacy implies determinacy.

1. Three Examples

The following simple fact is often used in the sequel.

Lemma 0. *Let $\mu \in M(\mathbf{R}^d)$ and let q be a polynomial such that $q(x) \neq 0$ for $x \in \operatorname{supp} \mu$. Then $\mathbf{C}(x_1, \ldots, x_d)$ is dense in $L^2(|q|^2\mu)$ if and only if $q(x)\mathbf{C}(x_1, \ldots, x_d)$ is dense in $L^2(\mu)$.*

Proof. The map $f \to qf$ is an isometry of $L^2(|q|^2\mu)$ onto $L^2(\mu)$ which maps $\mathbf{C}(x_1, \ldots, x_d)$ onto $q(x)\mathbf{C}(x_1, \ldots, x_d)$. ∎

Example 1. A determinate measure which is not strongly determinate

We take an indeterminate N-extremal measure $\nu_0 \in M(\mathbf{R})$. (Such measures exist, cf. [5], p. 59.) Set $\nu := (1+x^2)^{-1}\nu_0$. Then $\nu \in M(\mathbf{R})$ is determinate, hence the multiplication operator M_x on $\mathbf{C}(x)$ in $L^2(\nu)$ is essentially self-adjoint. Since ν_0 is indeterminate, $\mathbf{C}(x)$ is not dense in $L^2((1+x^2)\nu_0) = L^2((1+x^2)^2\nu)$. By Lemma 0, $(x^2+i)\mathbf{C}(x)$ is not dense in $L^2(\nu)$, so M_x^2 is not essentially self-adjoint. Let $\mu := \pi(\nu)$ be the image of ν under the mapping $\pi(x) = (x, x^2)$, $x \in \mathbf{R}$, of $\mathbf{R}$ into $\mathbf{R}^2$. Obviously, $\mu \in M(\mathbf{R}^2)$.

The multiplication operators M_x and M_y on $\mathbf{C}(x, y)$ in $L^2(\mu)$ are unitarily equivalent to M_x and M_x^2, respectively, on $\mathbf{C}(x)$ in $L^2(\nu)$. Therefore, M_y on $\mathbf{C}(x, y)$ is not essentially self-adjoint, so that μ is not strongly determinate.

We show that μ is determinate. For this let $\tilde{\mu} \in M(\mathbf{R}^2)$ such that $\tilde{\mu} \sim \mu$. By construction, μ is supported on the graph of the parabola $y = x^2$. Since $\int (y-x^2)^2 d\tilde{\mu} = \int (y-x^2)^2 d\mu = 0$, $\operatorname{supp} \tilde{\mu}$ is also contained in the graph of this parabola. It follows that $\tilde{\mu} = \pi(\tilde{\nu})$, where $\tilde{\nu}$ is the projection measure $\tilde{\mu}_x$. Clearly, $\tilde{\nu} \sim \nu$. Since ν is determinate, this yields $\tilde{\nu} = \nu$ and hence $\tilde{\mu} = \mu$, i.e. μ is determinate.

Example 2. A strongly determinate measure which is not ultradeterminate.

For $j=1,2$, let ν_j be a measure from $M(\mathbf{R})$ such that:
(i) M_x on $\mathbf{C}(x)$ in $L^2(\nu_j)$ is essentially self-adjoint.
(ii) M_x^2 is not essentially self-adjoint.
(iii) $\nu_j([-\delta, \delta])=0$ for some $\delta>0$.
(We refer to the first paragraph of Example 1 for the existence of such measures.)

Let μ_j, $j=1,2$, be the images of ν_j under the mappings $\varrho_1(x)=(x,0)$, $\varrho_2(x)=(0,x)$, $x\in\mathbf{R}$. Clearly, $\mu:=\mu_1+\mu_2$ is in $M(\mathbf{R}^2)$. For simplicity, we shall not distinguish between μ_1, μ_2 and its restriction to the x- resp. y-axis and we identify $L^2(\mu)$ and $L^2(\nu_1)\oplus L^2(\nu_2)$.

Statement 1. *The multiplication operators M_x and M_y on $\mathbf{C}(x,y)$ in $L^2(\mu)$ are both essentially self-adjoint, so μ is strongly determinate.*

Proof. We carry out the proof for M_x. It suffices to show that $(x+i)\mathbf{C}(x,y)$ is dense in $L^2(\mu)$. For this let $\varphi\in L^2(\mu)$ and let $\varepsilon>0$. By assumption (i), there is a $p\in\mathbf{C}(x)$ such that $\|\varphi(x,0)-(x+i)p(x)\|^2_{L^2(\nu_1)}<\varepsilon$. Similarly, by (i) and (iii), $y\mathbf{C}(y)$ is dense in $L^2(\nu_2)$, so there exists a $q\in\mathbf{C}(y)$ such that $\|\varphi(0,y)-ip(0)-iyq(y)\|^2_{L^2(\nu_2)}<\varepsilon$. Then we have $\big\|\varphi(x,y)-(x+i)(p(x)+yq(y))\big\|^2_{L^2(\mu)}<2\varepsilon$. ∎

Statement 2. *The polynomial $x+i$ is not in the closure of $(x+i)x\mathbf{C}(x)$ in $L^2(\nu_j)$, $j=1,2$.*

Proof. Fix $j\in\{1,2\}$ and let $\|\cdot\|$ denote the norm of $L^2(\nu_j)$. Assume the contrary. Let $p\in\mathbf{C}(x)$ and let $\varepsilon>0$. We write $p(x)=a+xq(x)$ with $a\in\mathbf{C}$ and $q\in\mathbf{C}(x)$. By our assumption $x+i$ is in the closure of $(x+i)x\mathbf{C}(x)$, so we can choose $r\in\mathbf{C}(x)$ such that $\|(x+i)a-(x+i)xr(x)\|<\varepsilon$. Then $\big\|(x+i)p(x)-(x+i)x(q(x)+r(x))\big\|<\varepsilon$. This shows that the closure of $(x+i)x\mathbf{C}(x)$ in $L^2(\nu_j)$ contains $(x+i)\mathbf{C}(x)$. But $(x+i)\mathbf{C}(x)$ is dense in $L^2(\nu_j)$ by (i), hence $(x+i)x\mathbf{C}(x)$ is also dense in $L^2(\nu_j)$. Thus $\mathbf{C}(x)$ is dense in $L^2((x^4+x^2)\nu_j)$ by Lemma 0 and hence in $L^2((x^4+1)\nu_j)$ by (iii). Again by Lemma 0, $(x^2+i)\mathbf{C}(x)$ is dense in $L^2(\nu_j)$. Therefore, M_x^2 is essentially self-adjoint which contradicts (ii). ∎

Statement 3. *The multiplication operator M_{x+y} is not essentially self-adjoint on $\mathbf{C}(x,y)$ in $L^2(\mu)$ and μ is not ultradeterminate.*

Proof. Assume to the contrary that M_{x+y} is essentially self-adjoint. Then $(x+y+i)\mathbf{C}(x,y)$ is dense in $L^2(\mu)$. By Statement 2, there is a number $\gamma>0$ such that

$$(1) \qquad \|(x+i)-(x+i)xp(x)\|_{L^2(\nu_j)} \geqq \gamma \quad \text{for} \quad j=1,2 \quad \text{and} \quad p\in\mathbf{C}(x).$$

Set $\varphi(x,y)=x+i$ if $y=0$ and $\varphi(x,y)=0$ otherwise. Take a number ε with $\gamma>2\varepsilon>0$.

We choose $p\in\mathbf{C}(x, y)$ such that

$$(2)\qquad \|\varphi-(x+y+i)p(x, y)\|^2_{L^2(\mu)} =$$

$$\|(x+i)-(x+i)p(x, 0)\|^2_{L^2(\nu_1)}+\|(y+i)p(0, y)\|^2_{L^2(\nu_2)} < \varepsilon^2.$$

Write $p(x, 0)=a+xq(x)$ and $p(0, y)=a+yr(y)$. From (1) and from the first summand in (2) we obtain that $\gamma|1-a|<\varepsilon$, so $|a|>\frac{1}{2}$, since $2\varepsilon<\gamma$. The second summand in (2) gives $\|(y+i)(a+yr(y))\|_{L^2(\nu_2)}<\varepsilon$. Hence

$$\|(y+i)+(y+i)yr(y)a^{-1}\|_{L^2(\nu_2)} < \varepsilon|a|^{-1} < 2\varepsilon < \gamma$$

which contradicts (1) in case $j=2$. This proves that M_{x+y} is not essentially self-adjoint.

Since $(x+y+i)\mathbf{C}(x, y)$ is not dense in $L^2(\mu)$ as just shown, $\mathbf{C}(x, y)$ is not dense in $L^2((1+(x+y)^2)\mu)$ by Lemma 0 and hence not in $L^2((1+x^2+y^2)\mu)$. Therefore, μ is not ultradeterminate ([5], p. 58). ∎

Example 3. A strongly determinate measure for which the projection measures are indeterminate.

Let $\nu_1=\sum_{n=0}^{\infty}a_n\delta_{x_n}$ and $\nu_2=\sum_{n=0}^{\infty}b_n\delta_{y_n}$ be two indeterminate N-extremal measures from $M(\mathbf{R})$ such that $\nu_1(\{0\})\neq 0$ and $\nu_2(\{0\})\neq 0$. Without loss of generality we assume that $x_0=y_0=0$ and $x_n\neq 0$, $y_n\neq 0$ for all $n\geqq 1$. Set $\mu_1=\sum_{n=1}^{\infty}a_n\delta_{(x_n, 0)}$ and $\mu_2=\sum_{n=1}^{\infty}b_n\delta_{(0, y_n)}$ and $\mu=\mu_1+\mu_2$. Obviously, $\mu\in M(\mathbf{R}^2)$. Then the measures μ_1 and μ_2 are both determinate ([1], Theorem 3.4 or [2], Theorem 7). Therefore, it follows in a similar way as in Example 2 (see the proof of Statement 1 therein) that the multiplication operators M_x and M_y on $\mathbf{C}(x, y)$ in $L^2(\mu)$ are essentially self-adjoint, hence μ is strongly determinate. The projection measures of μ are

$$\mu_x = \sum_{n=1}^{\infty}a_n\delta_{x_n}+\Big(\sum_{n=1}^{\infty}b_n\Big)\delta_0,\quad \mu_y = \sum_{n=1}^{\infty}b_n\delta_{y_n}+\Big(\sum_{n=1}^{\infty}a_n\Big)\delta_0.$$

We have $\nu_1\leqq\lambda\mu_x$ and $\nu_2\leqq\lambda\mu_y$ with some positive constant λ. Therefore, μ_x and μ_y are both indeterminate, since ν_1 and ν_2 are indeterminate.

2. Some General Results

Proposition 1*. *Let $\alpha=(\alpha_{nm})$ be a positive semi-definite 2-sequence. Suppose that $\mathbf{C}(x)$ is dense in $L^4(\mu_1)$, where μ_1 is a measure of $M(\mathbf{R})$ with moments $\alpha_{n,0}$, $n\in\mathbf{N}_0$. Then α is a moment 2-sequence.*

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* According to Fuglede ([5], p. 62), this is an unpublished result of J. P. R. Christensen.

Proof. Let $m_0 \in \mathbf{N}_0$. The crucial step of the proof is to show that the moment sequence $(\alpha_{n,2m_0}+\alpha_{n,0})_{n\in\mathbf{N}_0}$ is determinate. Let $\nu_{m_0}\in M(\mathbf{R})$ be a representing measure for this sequence. Fix $z\in\mathbf{C}\setminus\mathbf{R}$ and set $\mu:=|x+z|^{-4}\mu_1$. (We avoid the dependence of μ on z in the notation.) For $k=0, 1$ and $q\in\mathbf{C}(x)$, we have

$$(3) \qquad \left|\int (x+z)^{2k} q(x)d\nu_{m_0}\right| = |\langle (x+z)^{2k} q(x), y^{2m_0}+1\rangle_\alpha|$$

$$= |\langle q(x), (x+\bar{z})^{2k}(y^{2m_0}+1)\rangle_\alpha| \leqq \gamma\|q(x)\|_\alpha = \gamma\|q(x)\|_{L^2(\mu_1)} = \gamma\|(x+z)^2 q(x)\|_{L^2(\mu)},$$

where γ is a positive constant. From (3) applied with $k=0$ and from the Hölder inequality we obtain

$$(4) \qquad \left|\int q(x)\,d\nu_{m_0}\right| \leqq \gamma\|q(x)\|_{L^2(\mu_1)} \leqq \tilde{\gamma}\|(x+z)^{-2}q(x)\|_{L^4(\mu_1)}$$

for $q\in\mathbf{C}(x)$, where $\tilde{\gamma}:=\gamma\|(x+z)^2\|_{L^4(\mu_1)}$.

Now take an arbitrary $p\in\mathbf{C}(x)$. Since $\mathbf{C}(x)$ is dense in $L^4(\mu_1)$, there is a sequence (q_n) of polynomials such that $q_n(x)\to(x+z)^{-2}p(x)$ in $L^4(\mu_1)$. Setting $q=(x+z)^2q_n-p$ in (4), we conclude that $\int (x+z)^2 q_n(x)d\nu_{m_0}\to\int p(x)d\nu_{m_0}$. Since $\|(x+z)^2q_n-p\|_{L^2(\mu)}=\|q_n-(x+z)^{-2}p\|_{L^2(\mu_1)}$, we have $(x+z)^2q_n(x)\to p(x)$ in $L^2(\mu)$. Therefore, it follows from (3) applied with $k=1$ that $\left|\int p(x)d\nu_{m_0}\right|\leqq\gamma\|p(x)\|_{L^2(\mu)}$. Hence there exists a function $\psi\in L^2(\mu)$ such that

$$(5) \qquad \int p(x)\,d\nu_{m_0} = \int p(x)\psi(x)\,d\mu, \quad p\in\mathbf{C}(x).$$

Using once more that $\mathbf{C}(x)$ is dense in $L^4(\mu_1)$, there is a sequence (p_n) from $\mathbf{C}(x)$ such that $p_n(x)\to(x+z)^{-1}$ in $L^4(\mu_1)$. From (5) and from the Hölder inequality we obtain

$$\int |(x+z)p_n(x)-1|^2 d\nu_{m_0} = \int |(x+z)p_n(x)-1|^2\psi(x)\,d\mu$$

$$= \int |p_n(x)-(x+z)^{-1}|^2|x+z|^{-2}\psi(x)\,d\mu_1$$

$$\leqq \|p_n(x)-(x+z)^{-1}\|^2_{L^4(\mu_1)}\cdot\|\psi(x)\|_{L^2(\mu)} \to 0,$$

i.e 1 is in the closure of $(x+z)\mathbf{C}(x)$ in $L^2(\nu_{m_0})$. Since $\mathbf{C}(x)=\mathbf{C}\cdot 1+(x+z)\mathbf{C}(x)$, $\mathbf{C}(x)$ is contained in the closure of $(x+z)\mathbf{C}(x)$. If $\mathcal{K}_{m_0}$ denotes the closure of $\mathbf{C}(x)$ in $L^2(\nu_{m_0})$, $(x+z)\mathbf{C}(x)$ is dense in $\mathcal{K}_{m_0}$ for $z\in\mathbf{C}\setminus\mathbf{R}$ and hence M_x is essentially self-adjoint on $\mathbf{C}(x)$ in the Hilbert space $\mathcal{K}_{m_0}$. Thus $(\alpha_{n,2m_0}+\alpha_{n,0})$ is determinate (see e.g. [5], Theorem 7 applied with $n=1$) for each $m_0\in\mathbf{N}_0$. By Lemma 2 below, $(\alpha_{n,2m_0}+\alpha_{n,2(m_0+1)})$ is determinate for every $m_0\in\mathbf{N}_0$. Therefore, by a result of Eskin ([4], Theorem 2), α is a moment 2-sequence. ∎

Lemma 2. *Let $\alpha=(\alpha_{nm})$ be a positive semi-definite 2-sequence and let $m_0\in\mathbf{N}_0$. If the sequence $(\alpha_{n,2(m_0+1)}+\alpha_{n,0})_{n\in\mathbf{N}_0}$ is determinate, then $(\alpha_{n,2m_0}+\alpha_{n,2(m_0+1)})_{n\in\mathbf{N}_0}$ is also determinate.*

Proof. Let ν_{m_0+1} and σ_{m_0} be measures of $M(\mathbf{R})$ with moments $\alpha_{n,2(m_0+1)}+\alpha_{n,0}$ and $\alpha_{n,2m_0}+\alpha_{n,2(m_0+1)}$, respectively. For $p\in\mathbf{C}(x)$, we have

$$(6)\quad \|p(x)\|_{L^2(\sigma_{m_0})}=\|y^{m_0}(y+i)p(x)\|_\alpha\leqq 2\|(y^{m_0+1}+i)p(x)\|_\alpha=2\|p(x)\|_{L^2(\nu_{m_0+1})}.$$

The inequality in (6) can be seen as follows: We extend the multiplication operator M_y on $\mathbf{C}(x,y)$ to a self-adjoint operator in a possibly larger Hilbert space. Since $|y^{m_0}(y+i)|\leqq 2|y^{m_0+1}+i|$ for $y\in\mathbf{R}$, the inequality follows at once from the spectral theorem.

Let $z\in\mathbf{C}\setminus\mathbf{R}$. Since ν_{m_0+1} is determinate by assumption, the function 1 is in the closure of $(x+z)\mathbf{C}(x)$ in $L^2(\nu_{m_0+1})$. From (6) we conclude that 1 is also in the closure of $(x+z)\mathbf{C}(x)$ in $L^2(\sigma_{m_0})$. Arguing in the same way as at the end of the proof of Proposition 1, it follows that σ_{m_0} is determinate. ∎

Remark. Replacing ν_{m_0} by σ_{m_0} in the proof of Proposition 1, the same reasoning shows directly that σ_{m_0} is determinate without using Lemma 2. But the determinacy of ν_{m_0} and Lemma 2 are needed for the proof of Theorem 4 below.

We shall say that a positive semi-definite 2-sequence α is **strictly determinate** if $(x+w)(y+z)\mathbf{C}(x,y)$ is dense in $\mathcal{H}_\alpha$ for $w=\pm i$ and $z=\pm i$. If this is true, it follows easily that the multiplication operators M_x and M_y on $\mathbf{C}(x,y)$ in $\mathcal{H}_\alpha$ are essentially self-adjoint and that the resolvents of their closures commute. Therefore, each strictly determinate positive semi-definite 2-sequence is a strongly determinate moment 2-sequence.

Lemma 3. *A moment 2-sequence α is strictly determinate if and only if $\mathbf{C}(x,y)$ is dense in $L^2((1+x^2)(1+y^2)\mu)$ for some representing measure $\mu\in M(\mathbf{R}^2)$ for α.*

The proof of this lemma is straightforward. We omit the details.

From Lemma 3 we see that if α is strictly determinate, $\mathbf{C}(x,y)$ is dense in $L^2((1+x^2+y^2)\mu)$ and hence α is ultradeterminate ([5]). That the ultradeterminacy does not imply the strict determinacy can be seen as follows. Take an indeterminate N-extremal measure $\nu\in M(\mathbf{R})$ and let μ be the image of $(1+x^2)^{-1}\nu$ under the mapping $\pi(x)=(x,x)$ of $\mathbf{R}$ into $\mathbf{R}^2$. Then it is easy to check that $\mathbf{C}(x,y)$ is dense in $L^2((1+x^2+y^2)\mu)$ and that $\mathbf{C}(x,y)$ is not dense in $L^2((1+x^2)(1+y^2)\mu)$. Therefore, the moment 2-sequence associated with μ is ultradeterminate ([5]), but not strictly determinate by Lemma 3.

Our main result in this section is the following

Theorem 4. *Suppose that $\alpha=(\alpha_{nm})$ is a positive semi-definite 2-sequence such that the sequence (α_{0m}) is determinate. Suppose that one of the following two assumptions is satisfied:*

(i) *$\mathbf{C}(x)$ is dense in $L^4(\mu_1)$, where $\mu_1\in M(\mathbf{R})$ has the moments α_{n0}, $n\in\mathbf{N}_0$.*

(ii) *For every $m_0\in\mathbf{N}_0$, the moment sequence $(\alpha_{n,2m_0}+\alpha_{n,0})_{n\in\mathbf{N}_0}$ is determinate.*

Then α is a strictly determinate moment 2-sequence.

Proof. By the proof of Proposition 1 and by Lemma 2 and Eskin's theorem ([4]), (i) implies (ii) and (ii) implies that α is a moment 2-sequence. Thus it suffices to show that the moment 2-sequence α is strictly determinate provided that (ii) is fulfilled. We assume condition (ii).

Let μ be a representing measure for α. Take functions $f, g \in C_0(\mathbf{R})$. Let $w, z \in \{-i, i\}$ and let $\varepsilon > 0$. The projection measure μ_y is a representing measure for the sequence $(\alpha_{0,m})$. Since the latter is determinate by assumption, there exists a polynomial $q \in \mathbf{C}(y)$ such that $\delta \|(y+z)(g(y)-q(y))\|_{L^2(\mu_y)} < \varepsilon$, where

$$\delta := \sup \{|(x+w)f(x)|;\ x \in \mathbf{R}\}.$$

There are numbers $\gamma > 0$ and $m_0 \in \mathbf{N}_0$ (of course, depending on q and so on ε) such that $|(y+z)q(y)| \leqq \gamma |y^{m_0}+i|$, $y \in \mathbf{R}$. Put $\nu := ((y^{2m_0}+1)\mu)_x$. Clearly, $\nu \in M(\mathbf{R})$ and ν has the moments $\alpha_{n,2m_0} + \alpha_{n,0}$, $n \in \mathbf{N}_0$. By (ii), ν is determinate. Hence there is a $p \in \mathbf{C}(x)$ such that $\gamma \|(x+w)(f(x)-p(x))\|_{L^2(\nu)} < \varepsilon$. Then we have

$$\begin{aligned}
&\|f(x)g(y)-p(x)q(y)\|_{L^2((1+x^2)(1+y^2)\mu)} \\
&= \|(x+w)(y+z)(f(x)g(y)-p(x)q(y))\|_{L^2(\mu)} \\
&\leqq \|(x+w)(f(x)-p(x))(y+z)q(y)\|_{L^2(\mu)} + \|(x+w)f(x)(y+z)(g(y)-q(y))\|_{L^2(\mu)} \\
&\leqq \gamma \|(x+w)(f(x)-p(x))(y^{m_0}+i)\|_{L^2(\mu)} + \delta \|(y+z)(g(y)-q(y))\|_{L^2(\mu)} \\
&= \gamma \|(x+w)(f(x)-p(x))\|_{L^2(\nu)} + \delta \|(y+z)(g(y)-q(y))\|_{L^2(\mu_y)} < 2\varepsilon.
\end{aligned}$$

By Lemma 3 this proves that α is strictly determinate. ∎

Proposition 5. *Suppose $\alpha = (\alpha_{nm})$ is a moment 2-sequence with representing measure $\mu \in M(\mathbf{R}^2)$. Suppose that $(x+i)\mathbf{C}(x)$ is dense in $L^{2+\delta}(\mu_x)$ for some $\delta > 0$ and $(y+i)\mathbf{C}(y)$ is dense in $L^2(\mu_y)$ (or equivalently, $(\alpha_{0,m})$ is determinate). Then α is strictly determinate.*

Proof. Take $k \in \mathbf{N}$ such that $k \geqq \frac{\gamma}{2} := 1 + 2\delta^{-1}$ and keep the notation of the preceding proof. Hölder's inequality yields

$$\|(x+w)(f(x)-p(x))(y^{m_0}+i)\|_{L^2(\mu)} \leqq \left(\int |y^{m_0}+i|^{2k}\, d\mu\right)^{\gamma^{-1}} \|(x+w)(f(x)-p(x))\|_{L^{2+\delta}(\mu_x)}.$$

By the first assumption $p \in \mathbf{C}(x)$ can be chosen such that the right-hand side of this inequality is smaller than ε. Using this and arguing in a similar way as in the proof of Theorem 4, the assertion follows. ∎

3. Some Open Problems

a) Suppose $\alpha=(\alpha_{n,m})$ is a positive semi-definite 2-sequence. A fundamental problem is to find (sufficient) conditions for α to be a moment 2-sequence. Results in this direction were obtained in [3], [4], [6] and in Proposition 1 above. But there is still a strong gap between known affirmative results and counter-examples. For instance, I do not know whether the determinacy of $(\alpha_{n,0})$ (even together with the determinacy of $(\alpha_{0,m})$) implies that α is a moment 2-sequence.

b) Suppose $\alpha=(\alpha_{n,m})$ is a moment 2-sequence such that $(\alpha_{n,0})$ and $(\alpha_{0,m})$ are both determinate. Then α is determinate ([7]). Is α also strongly determinate?

c) Suppose α is a determinate moment 2-sequence with representing measure μ. What does this imply for α or μ? The only **necessary** condition for determinacy in the multidimensional case I know is the denseness of polynomials in $L^1(\mu)$. It is not known whether the polynomials are dense in $L^2(\mu)$. I believe that the latter is not true in general*.

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* In the meantime C. Berg and M. Thill have shown that there exists a determinate measure for which the polynomials are not dense in L^2, cf. their forthcoming paper "Rotation invariant measures".