Ark. Mat., **38** (2000), 381–397

Lazzeri's Jacobian of oriented compact Riemannian manifolds

Elena Rubei

Abstract. The subject of this paper is a Jacobian, introduced by F. Lazzeri (unpublished), associated with every compact oriented Riemannian manifold whose dimension is twice an odd number. We study the Torelli and Schottky problem for Lazzeri's Jacobian of flat tori and we compare Lazzeri's Jacobian of Kähler manifolds with other Jacobians.

1. Introduction

The subject of this paper is a Jacobian, introduced by F. Lazzeri (unpublished), associated with every compact oriented Riemannian manifold of dimension twice an odd number.

In the literature there are already "analogous" Jacobians associated with compact Kähler manifolds: Weil's and Griffiths' Jacobians, introduced in the 50's and the 60's, respectively (see [W] and [G], and also [C], [GH], [Wel], [Gre] and [L1]).

Definition 1.1. Let (M,g) be a compact Kähler manifold of complex dimension m. Let Ω be the (1,1)-form associated with g. Let $p\in\mathbf{N}$ with $p\leq m-1$.

Suppose Ω is rational, then the p^{th} *Weil's Jacobian* is the following abelian variety: the torus

$$H^{2p+1}(M,\mathbf{R})/(H^{2p+1}(M,\mathbf{Z})/\text{torsion})$$

with the complex structure given by $-C$ and the polarization whose real part is $\mathcal{R}(\alpha,\beta)=\int_M \alpha\wedge *\beta$, where Weil's operator $C\colon H^q(M,\mathbf{C})\to H^q(M,\mathbf{C})$ is the linear operator that is multiplication by i^{a-b} on $H^{a,b}$ (it takes $H^q(M,\mathbf{R})$ into $H^q(M,\mathbf{R})$ and, if q is odd, then $C^2=-1$).

The p^{th} *Griffiths' Jacobian* is the complex torus

$$H^{2p+1}(M,\mathbf{C})/(F^{p+1}(M)+H^{2p+1}(M,\mathbf{Z}))=\overline{F^{p+1}(M)}/\pi_{\overline{F^{p+1}},F^{p+1}}(H^{2p+1}(M,\mathbf{Z}))$$

with the complex structure given by i on $\overline{F^{p+1}(M)}$ (we use the notation $F_q^l(M):=\bigoplus_{a+b=q,\ a\geq l} H^{a,b}(M,\mathbf{C})$ and omit the subscript q when no confusion can arise).

More recently Lazzeri introduced the following Jacobian.

Definition 1.2. If (M,g) is a compact oriented Riemannian manifold of dimension $2m=2(2k+1)$, then *Lazzeri's Jacobian* of (M,g) is the following principally polarized abelian variety: the torus

$$H^m(M,\mathbf{R})/(H^m(M,\mathbf{Z})/\text{torsion})$$

with the complex structure given by the operator $*$ and the polarization whose imaginary part is $\mathcal{I}(\alpha,\beta)=-\int_M \alpha\wedge\beta$, $\alpha,\beta\in H^m(M,\mathbf{Z})$.

(The operator $*: H^m(M,\mathbf{R})\to H^m(M,\mathbf{R})$, defined through the isomorphism of $H^m(M,\mathbf{R})$ with the space of harmonic m-forms, has square -1, so it induces a complex structure on $H^m(M,\mathbf{R})$ and one can easily see that our complex torus is really a principally polarized abelian variety since, on $H^m(M,\mathbf{R})$, the form $\int_M \cdot\wedge*\cdot$ is positive definite, $\int_M \cdot\wedge\cdot=\int_M *\cdot\wedge*\cdot$ and $\int_M \cdot\wedge\cdot$ is principal.)

To compare Weil's and Griffiths' Jacobians with Lazzeri's Jacobian we need some notation.

Notation and recalls 1.3. (See [C], [W], [GH], [Wel], [Gre] and [L1].) Let (M,g) be a compact Kähler manifold of complex dimension m. Let Ω be the $(1,1)$-form associated with g.

For all $q\in\mathbf{N}$ odd, we define

$$J_q(M):=\bigoplus_{\substack{a+b=q\\ a-b\equiv 1\ (\mathrm{mod}\ 4)}} H^{a,b}(M,\mathbf{C}),$$

the i-eigenspace of C.

Let L be the operator on the set of forms on M defined by $L\eta=\Omega\wedge\eta$ and Λ the adjoint operator of L. A form η is said to be *primitive* if $\Lambda\eta=0$. The operators L and Λ induce operators, again called L and Λ, on $H^q(M,\mathbf{C})$, and $\alpha\in H^q(M,\mathbf{C})$ is said to be *primitive* if $\Lambda\alpha=0$. Every form ω of degree q can be written uniquely in the form $\omega=\sum_{r\geq(q-m)^+} L^r\omega_r$, where ω_r is a primitive form of degree $q-2r$ and $x^+:=\max\{0,x\}$. Analogously $\alpha\in H^q(M,\mathbf{C})$ can be written uniquely in the form $\alpha=\sum_{r\geq(q-m)^+} L^r\alpha_r$, with α_r primitive in $H^{q-2r}(M,\mathbf{C})$. Define

$$\begin{aligned}K&=K(M,g)\\ &=\left\{\alpha\in H^m(M,\mathbf{C})\ \middle|\ \alpha=\sum_{\substack{r\geq 0\\ r\equiv -m(m+1)/2\ (\mathrm{mod}\ 2)}} L^r\alpha_r,\ \alpha_r\in H^{m-2r}(M,\mathbf{C})\ \text{primitive}\right\},\end{aligned}$$

$$K' = K'(M,g)$$
$$= \left\{ \alpha \in H^m(M,\mathbf{C}) \;\middle|\; \alpha = \sum_{\substack{r \geq 0 \\ r \equiv 1-m(m+1)/2 \,(\mathrm{mod}\, 2)}} L^r \alpha_r,\ \alpha_r \in H^{m-2r}(M,\mathbf{C}) \text{ primitive} \right\}.$$

Let (M,g) be a compact Kähler manifold of dimension $m=2k+1$. Consider the decomposition $H^m(M,\mathbf{C}) = K(M,g) \oplus K'(M,g)$. In Section 3 we see that $*=C$ on K and $*=-C$ on K'. Observe that $H^m(M,\mathbf{R}) = (H^m(M,\mathbf{R}) \cap K) \oplus (H^m(M,\mathbf{R}) \cap K')$. Then the k^{th} Griffiths', the k^{th} Weil's and Lazzeri's Jacobians of (M,g) are the same real torus, $H^m(M,\mathbf{R})/(H^m(M,\mathbf{Z})/\text{torsion})$, with different complex structures:

$-C$ for the k^{th} Weil's Jacobian;

$C|_{H^m(M,\mathbf{R}) \cap K} \oplus -C|_{H^m(M,\mathbf{R}) \cap K'}$ for Lazzeri's Jacobian;

$C|_{\{\eta+\bar{\eta} \mid \eta \in J_m(M) \cap \overline{F^{k+1}(M)}\}} \oplus -C|_{\{\eta+\bar{\eta} \mid \eta \in J_m(M) \cap F^{k+1}(M)\}}$ for the k^{th} Griffiths' Jacobian.

(From this we see that the "change" of the complex structure depends on the complex structure of M for Griffiths' Jacobian and Weil's Jacobian, and on the class of Ω for Lazzeri's Jacobian and Weil's Jacobian, and that, for Kähler manifolds, Lazzeri's Jacobian depends only on the complex structure of M and on Ω).

Lazzeri's Jacobian and the k^{th} Weil's Jacobian are principally polarized abelian varieties(1) (also if Ω is not rational) and the real part of the polarization is the same, $\int_M \cdot \wedge * \cdot$.

If $m=1$, i.e. M is a compact Riemannian surface with a hermitian metric, we have that Lazzeri's Jacobian, Weil's Jacobian and Griffiths' Jacobian (with the polarization given by $-\int_M \cdot \wedge \cdot$ on $H^m(M,\mathbf{Z})$) are the same (in fact, if $m=1$, i.e. $k=0$, we have $K=\{0\}$ and $J_1 \cap \overline{F^1} = \{0\}$) and they are isomorphic to the usual Jacobian.

The advantage of Lazzeri's Jacobian with respect to Weil's and Griffiths' ones is that it is definable not only for compact Kähler manifolds, but for any compact oriented Riemannian manifold whose dimension is twice an odd number. Unfortunately Lazzeri's Jacobian of Kähler manifolds, as Weil's Jacobian, does not vary holomorphically (see Section 3), while Griffiths' one does.

(1) Observe that the polarization of Lazzeri's Jacobian is principal if and only if the polarization of Weil's Jacobian is principal, in fact: let $\mathcal{I}_W$ and $\mathcal{I}_L$ be the imaginary parts of the polarizations of Weil's Jacobian and Lazzeri's Jacobian, respectively, and let $\mathcal{B}$ be a basis of $H^m(M,\mathbf{Z})$. Consider a basis $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ such that $\mathcal{A}_1$ is a basis of $H^m(M,\mathbf{R}) \cap K$ and $\mathcal{A}_2$ is a basis of $H^m(M,\mathbf{R}) \cap K'$ and such that the determinant of the matrix expressing $\mathcal{B}$ as a function of $\mathcal{A}$ is 1. We have $\det_{\mathcal{B}} \mathcal{I}_W = \det_{\mathcal{A}} \mathcal{I}_W = \pm \det_{\mathcal{A}} \mathcal{I}_L = \pm \det_{\mathcal{B}} \mathcal{I}_L$. Thus $|\det_{\mathcal{B}} \mathcal{I}_W| = 1 \Leftrightarrow |\det_{\mathcal{B}} \mathcal{I}_L| = 1$.

The outline of the paper is the following. In Section 2 we study the Torelli and Schottky problems for Lazzeri's Jacobian of flat tori. Section 3 deals with Lazzeri's Jacobian for Kähler manifolds, and in Section 4 we examine Lazzeri's Jacobian of a bundle.

Notation 1.4. Let

$$\mathcal{A}_h := \{\text{principally polarized abelian varieties of dimension } h\}/\text{isomorphisms},$$

let $\mathcal{H}_h$ be the h-Siegel upper half space and $p_h\colon \mathcal{H}_h \to A_h$ be the projection.

Let $k \in \mathbf{N}$ and $m=2k+1$. Let $\mathcal{R} \subset \{(M,g) \mid M$ is an oriented compact C^∞ manifold of dimension $2m$ and g is a Riemannian metric on $M\}$.

Let $(M_1,g_1),(M_2,g_2) \in \mathcal{R}$. We say that $(M_1,g_1) \sim (M_2,g_2)$ if and only if there exists an orientation preserving diffeomorphism $f\colon M_1 \to M_2$ and a C^∞ map $t\colon M_1 \to \mathbf{R}^+$ such that $(M_1, tg_1) \in \mathcal{R}$ and $f^* g_2 = tg_1$. Let

$$T = T_{\mathcal{R}}\colon \mathcal{R}/\sim \longrightarrow \mathcal{A}_{(1/2)b_m(M)}$$

be the map sending the class of (M,g) into Lazzeri's Jacobian of (M,g).

Now fix M and a symplectic basis $\mathcal{S}$ of $H^m(M,\mathbf{Z})/\text{torsion}$ with respect to $-\int_M \cdot\wedge\cdot$ and let $\mathcal{R} \subset \{g \mid g$ is metric on $M\}$. Then $T_{\mathcal{R}}$ can be lifted to a map

$$\widehat{T} = \widehat{T}_{\mathcal{R},\mathcal{S}}\colon \mathcal{R}/\text{conformal equivalence} \longrightarrow \mathcal{H}_{(1/2)b_m(M)}$$

($g_1, g_2 \in \mathcal{R}$ are conformally equivalent if and only if there exists a C^∞ map $t\colon M \to \mathbf{R}^+$ such that $g_1 = tg_2$).

First (in Section 2) we study the case of flat tori.

Definition 1.5. Let $\sim$ be the equivalence defined in Notation 1.4. Define

$$\mathcal{F}_n = \{(\mathbf{R}^n/\Lambda, g) \mid \Lambda \text{ is a lattice and } g \text{ is a flat metric on } \mathbf{R}^n/\Lambda\}/\sim.$$

Observe that here the equivalence $\sim$ becomes $(\mathbf{R}^n/\Lambda, g) \sim (\mathbf{R}^n/\Lambda', g')$ if and only if there exists an orientation preserving map $f\colon \mathbf{R}^n/\Lambda \to \mathbf{R}^n/\Lambda'$ induced by a linear map $\mathbf{R}^n \to \mathbf{R}^n$ such that $f^*g' = cg$ for some $c \in \mathbf{R}^+$, where the orientation is the standard one for $\mathbf{R}^n$ (in fact, if $(\mathbf{R}^n/\Lambda, g)$ and $(\mathbf{R}^n/\Lambda', g')$ are equivalent for $\sim$ through a map φ, then φ is given by an affine map $\mathbf{R}^n \to \mathbf{R}^n$).

To study Lazzeri's Jacobian of flat tori, we first observe that

$$\mathcal{F}_n = \{(\mathbf{R}^n/\mathbf{Z}^n, g) \mid g \text{ is a flat metric on } \mathbf{R}^n/\mathbf{Z}^n\}/\sim.$$

Then we fix a symplectic basis of $H^m(\mathbf{R}^n/\mathbf{Z}^n, \mathbf{Z})$ with respect to $-\int_{\mathbf{R}^n/\mathbf{Z}^n} \cdot\wedge\cdot$ and study the map

$$\widehat{T}: \{(\mathbf{R}^n/\mathbf{Z}^n, g) \mid g \text{ is a flat metric on } \mathbf{R}^n/\mathbf{Z}^n\}/\text{conformal equivalence} \longrightarrow \mathcal{H}_N,$$

where $n=2m=2(2k+1)$ and $N:=\frac{1}{2}\dim \bigwedge^m \mathbf{R}^n$. Using the study of $\widehat{T}$, we study the quotient map

$$T: \mathcal{F}_n = \{(\mathbf{R}^n/\mathbf{Z}^n, g) \mid g \text{ is a flat metric on } \mathbf{R}^n/\mathbf{Z}^n\}/\sim \longrightarrow \mathcal{A}_N.$$

We prove the following theorem.

Theorem A. (i) *Chosen any symplectic basis of $H^m(\mathbf{R}^n/\mathbf{Z}^n, \mathbf{Z})$ with respect to $-\int_{\mathbf{R}^n/\mathbf{Z}^n} \cdot\wedge\cdot$, the map*

$$\widehat{T}: \{g \mid g \textit{ is a flat metric on } \mathbf{R}^n/\mathbf{Z}^n\}/\textit{conformal equivalence} \to \mathcal{H}_N$$

is injective.

(ii) *Lazzeri's Jacobian of a generic flat oriented torus has no nontrivial automorphisms as a principally polarized abelian variety.*

(iii) *The map $T: \mathcal{F}_n \to \mathcal{A}_N$ is generically locally injective.*

Then we study the image of $\widehat{T}$ (and thus the image of T, since obviously $\operatorname{Im} T = p_N(\operatorname{Im} \widehat{T})$), see Theorem B.

In Section 3 we consider Lazzeri's Jacobian of Kähler manifolds. We saw that it depends only on the complex structure and on the class Ω of the $(1,1)$-form associated with the metric. In Proposition C we fix Ω and we study when the map associating Lazzeri's Jacobian with the complex structure is holomorphic and we observe another local "Torelli theorem" for Lazzeri's Jacobian, which holds in particular for another class of Ricci-flat metrics (Kähler–Einstein ones on complex manifolds with trivial canonical bundle). One could conjecture that $\widehat{T}_{\mathcal{R}}$ is locally injective if $\mathcal{R}$ is a set of Ricci-flat metrics on a manifold M.

Finally in Section 4 given a bundle $F \to M$, we study when there is a holomorphic map from Lazzeri's Jacobian of M into Lazzeri's Jacobian of F and we study when it is injective (Proposition D).

We think that one can easily find open problems about Lazzeri's Jacobian, for instance to study Prym–Tyurin varieties for it, the relationship with the theory of degeneration of abelian varieties and to continue with the study of Schottky and Torelli type problems.

2. The case of tori with flat metrics

2.a. Some lemmas of linear algebra

In order to study Lazzeri's Jacobian of flat tori, it is useful to state some lemmas from linear algebra.

Notation 2.1. Let $m \in \mathbf{N}$, V be an $\mathbf{R}$-vector space of dimension $n=2m$ and $\{v_j\}_{j=1,\dots,n}$ be a basis of V. Let $\mathcal{R} := \{(i_1,\dots,i_m) \in \mathbf{N}^m \mid 1 \le i_1 < \dots < i_m \le n\}$. We call the basis $\{v_I = v_{i_1} \wedge \dots \wedge v_{i_m}\}_{I \in \mathcal{R}}$ of $\bigwedge^m V$ *lexicographically ordered* if we order it by ordering the multiindices of $\mathcal{R}$ by the lexicographic order.

Let $\mathcal{I} := \{I = (1, i_2, \dots, i_m) \in \mathbf{N}^m \mid 1 < i_2 < \dots < i_m \le n\}$. If $I \in \mathcal{I}$, we choose one of the multiindices J such that we obtain (I, J) from $(1, \dots, n)$ with an even number of transpositions and we call it $\tilde{I}$. Let $\mathcal{E} = \{\tilde{I} \mid I \in \mathcal{I}\}$. Now take the multiindices in $\mathcal{I}$ in the lexicographic order and call them I_1, $I_2, \dots$. *The symmetric lexicographic order* is the order I_1, $I_2, \dots$, $\tilde{I}_1$, $\tilde{I}_2, \dots$ of the multiindices of $\mathcal{I} \cup \mathcal{E}$. We call the basis $\{v_I\}_{I \in \mathcal{I} \cup \mathcal{E}}$ of $\bigwedge^m V$ *symmetrically lexicographically ordered* if we order it by ordering the multiindices of $\mathcal{I} \cup \mathcal{E}$ by the symmetric lexicographic order.

Lemma 2.2. *Let $m \in \mathbf{N}$ and $n = 2m$. Consider the set of positive scalar products on $\mathbf{R}^n$ up to conformal equivalence. The map, defined on this set, sending the class of a positive definite scalar product into its operator $*$ on $\bigwedge^m (\mathbf{R}^n)^\vee$, is injective.*

Proof. Given a scalar product on $\mathbf{R}^n$ and $v, w \in \mathbf{R}^n$, we have that $v \perp w$ if and only if there exists an m-subspace S of $\mathbf{R}^n$ such that $S \perp w$ and $S \ni v$, and this is equivalent to the existence of $\alpha \in \bigwedge^{m-1} \mathbf{R}^n$, α simple (i.e. of the kind $v_1 \wedge \dots \wedge v_{m-1}$) such that $w \wedge *(\alpha \wedge v) = 0$ and $\alpha \wedge v \neq 0$. Thus $*$ determines the conformal structure. $\square$

Lemma 2.3. *Let Ω be a real upper triangular $n \times n$ matrix, where $n = 2m$. Using the lexicographically ordered set $\mathcal{R}$ as the set of multiindices, the matrix $\bigwedge^m \Omega$ is upper triangular. Moreover, if $\bigwedge^m \Omega = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$, using the symmetrically lexicographically ordered multiindices $\mathcal{I} \cup \mathcal{E}$, then $B = 0$, A is upper triangular and D lower triangular.*

We leave the proof to the reader.

Lemma 2.4. *Let Ω be a real upper triangular $n \times n$ matrix, where $n = 2m$, with $\det \Omega = 1$. Let $\bigwedge^m \Omega = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$, using the multiindices $\mathcal{I} \cup \mathcal{E}$ ordered by first taking the multiindices in $\mathcal{I}$ in any order K_1, $K_2, \dots$, and then taking $\widetilde{K}_1$, $\widetilde{K}_2, \dots$ (e.g. the symmetric lexicographic order). Then we have*

(a) $A^t D = I$;

(b) *for* $I \in \mathcal{I}$ *and* $J \in \mathcal{E}$,

$$(A^{-1}C)_{IJ} = \begin{cases} 0, & \text{if } \tilde{I} \text{ and } J \text{ have more than one index in common,} \\ \pm\dfrac{\Omega_{1,l}}{\Omega_{1,1}}, & \text{if } \tilde{I} \text{ and } J \text{ have only the index } l \text{ in common.} \end{cases}$$

Proof. If P and Q are two multiindices, then $\Omega_{P,Q}$ will denote the determinant of the minor (P,Q) of Ω. We will denote the j^{th} column of Ω by v_j and $v_j - \Omega_{1,j}e_1$ by $\bar{v}_j$.

(a) Let $I=(i_1,\dots,i_m)\in\mathcal{I}$ and $J=(j_1,\dots,j_m)\in\mathcal{E}$. We have

$$\begin{aligned}
(A^tD)_{I,J}e_1\wedge\ldots\wedge e_n &= \sum_{S\in\mathcal{I}} \Omega_{S,I}\Omega_{\tilde{S},J} e_1\wedge\ldots\wedge e_n \\
&= \Big(\sum_{S\in\mathcal{I}} \Omega_{S,I} e_{s_1}\wedge\ldots\wedge e_{s_m}\Big)\wedge\Big(\sum_{T\in\mathcal{E}} \Omega_{T,J} e_{t_1}\wedge\ldots\wedge e_{t_m}\Big) \\
&= \Big(\sum_{S\in\mathcal{I}\cup\mathcal{E}} \Omega_{S,I} e_{s_1}\wedge\ldots\wedge e_{s_m}\Big)\wedge\Big(\sum_{T\in\mathcal{I}\cup\mathcal{E}} \Omega_{T,J} e_{t_1}\wedge\ldots\wedge e_{t_m}\Big) \\
&= v_{i_1}\wedge\ldots\wedge v_{i_m}\wedge v_{j_1}\wedge\ldots\wedge v_{j_m} \\
&= \begin{cases} 0, & \text{if } J\neq\tilde{I}, \\ e_1\wedge\ldots\wedge e_n, & \text{if } J=\tilde{I}, \end{cases}
\end{aligned}$$

where $S=(s_1,\dots,s_m)$, $T=(t_1,\dots,t_m)$, and the third equality holds because $\Omega_{S,I}=0$ for $S\in\mathcal{E}$ and $I\in\mathcal{I}$ and $e_S\wedge e_T=0$ if $S,T\in\mathcal{I}$ or $S,T\in\mathcal{E}$.

(b) Let $\tilde{I}=T=(t_1,\dots,t_m)\in\mathcal{E}$ and $J=(j_1,\dots,j_m)\in\mathcal{E}$, and observe that they necessarily have some index in common. Let $l=t_r=j_s$. We have, using (a),

$$\begin{aligned}
(A^{-1}C)_{TJ}e_1\wedge\ldots\wedge e_n &= (D^tC)_{TJ}e_1\wedge\ldots\wedge e_n \\
&= \Big(\sum_{K\in\mathcal{I}} \Omega_{\tilde{K},T}\Omega_{K,J}\Big) e_1\wedge\ldots\wedge e_n \\
&= \bar{v}_{t_1}\wedge\ldots\wedge\bar{v}_{t_m}\wedge(v_{j_1}\wedge\ldots\wedge v_{j_m} - \bar{v}_{j_1}\wedge\ldots\wedge\bar{v}_{j_m}) \\
&= \bar{v}_{t_1}\wedge\ldots\wedge\bar{v}_{t_m}\wedge v_{j_1}\wedge\ldots\wedge v_{j_m} \\
&= \bar{v}_{t_1}\wedge\ldots\wedge\bar{v}_{t_m}\wedge v_{j_1}\wedge\ldots\wedge v_{j_{s-1}}\wedge\Omega_{1,l}e_1\wedge v_{j_{s+1}}\wedge\ldots\wedge v_{j_m} \\
&= \bar{v}_{t_1}\wedge\ldots\wedge\bar{v}_{t_m}\wedge\bar{v}_{j_1}\wedge\ldots\wedge\bar{v}_{j_{s-1}}\wedge\Omega_{1,l}e_1\wedge\bar{v}_{j_{s+1}}\wedge\ldots\wedge\bar{v}_{j_m} \\
&= \begin{cases} 0, & \text{if } T \text{ and } J \text{ have more than one index in common,} \\ \varepsilon\dfrac{\Omega_{1,l}}{\Omega_{1,1}} e_1\wedge\ldots\wedge e_n, & \text{if } T \text{ and } J \text{ have only the index } l \text{ in common,} \end{cases}
\end{aligned}$$

where ε is the sign of the permutation taking $(t_1,\dots,t_m,j_1,\dots,j_{s-1},1,j_{s+1},\dots,j_m)$ into $(1,\dots,n)$. $\square$

2.b. Proofs of Theorems A and B

Notation 2.5. Let $P_n := \{A \in M(n\times n, \mathbf{R}) \mid A = A^t,\ A>0 \text{ and } \det A = 1\}$ and let $\mathcal{T}_n := \{T \in M(n\times n, \mathbf{R}) \mid T \text{ lower triangular, } T_{j,j}>0 \text{ for all } j \text{ and } \det T = 1\}$.

Remark 2.6. (i) The set

$$\{(\mathbf{R}^n/\mathbf{Z}^n, g) \mid g \text{ is a flat metric on } \mathbf{R}^n/\mathbf{Z}^n\}/\text{conformal equivalence}$$

is in bijection with P_n, and thus with $\mathcal{T}_n$.

(ii) We have that

$$\mathcal{F}_n = \{(\mathbf{R}^n/\mathbf{Z}^n, g) \mid g \text{ is a flat metric on } \mathbf{R}^n/\mathbf{Z}^n\}/\sim = P_n/\operatorname{SL}(n, \mathbf{Z})$$

(where $A \in \operatorname{SL}(n,\mathbf{Z})$ acts on P_n by $P \mapsto A^t P A$, $P \in P_n$) and we endow this set with the quotient topology induced by the set in (i).

Notation 2.7. Let $n := 2m = 2(2k+1)$ and $N := \frac{1}{2}\binom{n}{m}$.

We want to study the map

$$\widehat{T}\colon \{(\mathbf{R}^n/\mathbf{Z}^n, g) \mid g \text{ flat metric on } \mathbf{R}^n/\mathbf{Z}^n\}/\text{conformal equivalence} = P_n = \mathcal{T}_n \longrightarrow \mathcal{H}_N$$

(after fixing a symplectic basis of $H^m(\mathbf{R}^n/\mathbf{Z}^n, \mathbf{Z})$ with respect to $\int_{\mathbf{R}^n/\mathbf{Z}^n} \cdot\wedge\cdot$) and

$$T\colon \mathcal{F}_n = \{(\mathbf{R}^n/\mathbf{Z}^n, g) \mid g \text{ is a flat metric on } \mathbf{R}^n/\mathbf{Z}^n\}/\sim = P_n/\operatorname{SL}(n,\mathbf{Z}) \longrightarrow \mathcal{A}_N.$$

We recall that $H^q(\mathbf{R}^n/\mathbf{Z}^n, \mathbf{R})$ is in bijection with the set of harmonic q-forms, which are the translation invariant q-forms, and thus $H^q(\mathbf{R}^n/\mathbf{Z}^n, \mathbf{R}) = \bigwedge^q (\mathbf{R}^n)^\vee$ and $H^q(\mathbf{R}^n/\mathbf{Z}^n, \mathbf{Z}) = \bigwedge^q(\mathbf{Z}^n)^\vee$ (see [LB] or [M]).

In the next proposition, after choosing a symplectic basis for $H^m(\mathbf{R}^n/\mathbf{Z}^n, \mathbf{Z})$ with respect to $-\int_{\mathbf{R}^n/\mathbf{Z}^n} \cdot\wedge\cdot$, we write explicitly the image of $\widehat{T}$ of the torus $\mathbf{R}^n/\mathbf{Z}^n$ with the flat metric given by the matrix L^tL as a function of L.

Proposition 2.8. *Let $L \in GL(n, \mathbf{R})$.*

Let

$$\bigwedge^{2k+1} (L^{-1})^t = \begin{pmatrix} A(L) & C(L) \\ B(L) & D(L) \end{pmatrix},$$

using the symmetrically lexicographically ordered set $\mathcal{I} \cup \mathcal{E}$ as set of multiindices.

Set

$$\begin{pmatrix} X(L) \\ Y(L) \end{pmatrix} = \begin{pmatrix} A(L) & -B(L) \\ B(L) & A(L) \end{pmatrix}^{-1} \begin{pmatrix} C(L) \\ D(L) \end{pmatrix}$$

and define $Z(L)=X(L)+iY(L)\in M(N\times N,\mathbf{C})$.

Let $\det L>0$. *Let* $\{dx_j\}_{j=1,\dots,n}$ *be the standard basis of* $(\mathbf{R}^n)^\vee$. *Consider the symmetrically lexicographically ordered basis* $\{dx_I=dx_{i_1}\wedge\ldots\wedge dx_{i_{2k+1}}\}_{I\in\mathcal{I}\cup\mathcal{E}}$ *as symplectic basis for* $H^m(\mathbf{R}^n/\mathbf{Z}^n,\mathbf{Z})$ *with respect to* $-\int_{\mathbf{R}^n/\mathbf{Z}^n}\cdot\wedge\cdot$.

Then $\widehat{T}(\mathbf{R}^n/\mathbf{Z}^n,L^tL)=Z(L)$ *(the orientation of* $\mathbf{R}^n/\mathbf{Z}^n$ *is given by the standard one of* $\mathbf{R}^n$*).*

Proof. Let b_i be the columns of L^{-1}, they are an orthonormal basis of $\mathbf{R}^n$ for the metric L^tL. The matrix representing the symmetrically lexicographically ordered basis $\{dx_I\}_{I\in\mathcal{I}\cup\mathcal{E}}$ as a function of the symmetrically lexicographically ordered basis $\{b_I^\vee\}_{I\in\mathcal{I}\cup\mathcal{E}}$ of $H^m(\mathbf{R}^n/\mathbf{Z}^n,\mathbf{R})$ is $\bigwedge^{2k+1}(L^{-1})^t=\left(\begin{smallmatrix}A&C\\B&D\end{smallmatrix}\right)$. Thus the matrix expressing $\{dx_I,*dx_I\}_{I\in\mathcal{I}}$ ($\mathcal{I}$ lexicographically ordered) as a function of the symmetrically lexicographically ordered basis $\{b_I^\vee\}_{I\in\mathcal{I}\cup\mathcal{E}}$ is $\left(\begin{smallmatrix}A&-B\\B&A\end{smallmatrix}\right)$ and the one expressing $\{dx_I\}_{I\in\mathcal{E}}$ as a function of the symmetrically lexicographically ordered basis $\{b_I^\vee\}_{I\in\mathcal{I}\cup\mathcal{E}}$ is $\left(\begin{smallmatrix}C\\D\end{smallmatrix}\right)$. Thus the matrix $\left(\begin{smallmatrix}A&-B\\B&A\end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix}C\\D\end{smallmatrix}\right)$ expresses $\{dx_I\}_{I\in\mathcal{E}}$ as a function of $\{dx_I,*dx_I\}_{I\in\mathcal{I}}$. Thus we have proved that the period matrix of Lazzeri's Jacobian of $(\mathbf{R}^n/\mathbf{Z}^n,L^tL)$, taking the symmetrically lexicographically ordered basis $\{dx_I\}_{I\in\mathcal{I}\cup\mathcal{E}}$ as symplectic basis, is $Z(L)$. □

Proof of Theorem A. We use Proposition 2.8 and its notation.

(i) Since harmonic forms on tori are the translation invariant forms, by Lemma 2.2, we conclude (i).

(ii) Let $S:=\mathrm{Sp}_{2N}(\mathbf{Z})$.(2) If $F_\sigma:=\{x\in\mathcal{H}_N\,|\,\sigma(x)=x\}$ for $\sigma\in S^*:=S\setminus\{\mathrm{Id}\}$, we have to prove that $\mathcal{T}_n\setminus Z^{-1}(\bigcup_{\sigma\in S^*}F_\sigma)$ is an open dense subset of $\mathcal{T}_n$ (Z defined in Proposition 2.8). The openness follows from the fact that S acts properly and discontinuously on $\mathcal{H}_N$ (see [LB], p. 218). To prove the density, it is sufficient (by Baire's theorem) to prove that for all $\sigma\in S^*$, the set $\mathcal{T}_n\setminus Z^{-1}(F_\sigma)$ is an open dense subset of $\mathcal{T}_n$. Since $Z^{-1}(F_\sigma)$ is defined by polynomial equations, we have only to prove that $\mathcal{T}_n\setminus Z^{-1}(F_\sigma)\neq\emptyset$, i.e. that $Z(\mathcal{T}_n)\subset F_\sigma$ implies $\sigma=\mathrm{Id}$.

Let $\sigma\in S$ be the map $Z\mapsto(MZ+N)(PZ+Q)^{-1}$ and suppose that $Z(\mathcal{T}_n)\subset F_\sigma$. Obviously, $Z\in F_\sigma$ if and only if $ZPZ+ZQ=MZ+N$.

Let $L\in\mathcal{T}_n$. For every $c\in\mathbf{R}^+$ let $\tilde{L}(c)$ be the matrix obtained from L by multiplying $L_{1,1}$ by c^{-4k-1} and the other entries of L by c. We have $Z(\tilde{L}(c))=c^{4k+2}Z(L)$, by the definition of Z. Since $Z(\tilde{L}(c))\in F_\sigma$ for all $c\in\mathbf{R}^+$, we have $c^{8k+4}Z(L)PZ(L)+c^{4k+2}(Z(L)Q-MZ(L))-N=0$ for all $c\in\mathbf{R}^+$ and $L\in\mathcal{T}_n$. Thus

(2) We define $\mathrm{Sp}_{2N}(\mathbf{Z}):=\left\{\left(\begin{smallmatrix}\alpha&\beta\\\gamma&\delta\end{smallmatrix}\right)\in M(2N\times 2N,\mathbf{Z})\,|\,\left(\begin{smallmatrix}\alpha&\beta\\\gamma&\delta\end{smallmatrix}\right)^t\left(\begin{smallmatrix}0&-I\\I&0\end{smallmatrix}\right)\left(\begin{smallmatrix}\alpha&\beta\\\gamma&\delta\end{smallmatrix}\right)=\left(\begin{smallmatrix}0&-I\\I&0\end{smallmatrix}\right)\right\}$, which acts on $\mathcal{H}_N$ by $\left(\begin{smallmatrix}\alpha&\beta\\\gamma&\delta\end{smallmatrix}\right)Z=(\delta Z+\gamma)(\beta Z+\alpha)^{-1}$ (see [LB], Chapter 8).

(1) $N=0$;

(2) $Z(L)PZ(L)=0$ for all $L\in\mathcal{T}_n$;

(3) $Z(L)Q-MZ(L)=0$ for all $L\in\mathcal{T}_n$.

Hence, $Q=(M^t)^{-1}$ by (1) and the fact that $\sigma\in S$, and $P=0$ and $M=Q$ by (2) and (3), taking $L=I$. Thus M is orthogonal. From (3) we have $Y(L)M=MY(L)$ and this implies that M is diagonal (take L such that $Y(L)$ is diagonal with the diagonal elements different from each other). Being M orthogonal and diagonal, M must be diagonal with only ±1 on the diagonal. Still from $Y(L)=MY(L)M^{-1}$, taking L such that $Y(L)$ is not diagonal, we obtain $M=I$.

(iii) This follows right away from (i) and (ii), since S acts properly and discontinously on $\mathcal{H}_N$. $\square$

Remark 2.9. The map $T:\mathcal{F}_n\to\mathcal{A}_N$ is not injective.

In fact, considering a diagonal matrix $F\in\mathcal{T}_n$ and its inverse matrix F^{-1}, we have $C(F)=B(F)=C(F^{-1})=B(F^{-1})=0$ and $A(F)=A(F^{-1})^{-1}$, $D(F)=D(F^{-1})^{-1}$ (using the notation of Proposition 2.8). Thus $Y(F^{-1})=Y(F)^{-1}$ and $X(F^{-1})=X(F)=0$. Then $Z(F)=-Z(F^{-1})^{-1}$. Hence the Z's differ by an element of $\mathrm{Sp}_{2N}(\mathbf{Z})$. Thus Lazzeri's Jacobians of $(\mathbf{R}^n/\mathbf{Z}^n, F^2)$ and of $(\mathbf{R}^n/\mathbf{Z}^n, (F^{-1})^2)$ are isomorphic principally polarized abelian varieties.

But we can choose F in such a way that there does not exist $A\in\mathrm{SL}(n,\mathbf{Z})$ such that $A^tF^2A=(F^{-1})^2$. In fact $A^tF^2A=(F^{-1})^2$ is equivalent to FAF being orthogonal and, e.g., if we take as F the diagonal matrix whose diagonal is $\left(1,\dots,1,\frac{1}{2},\frac{1}{3},6\right)$, we have that there does not exist $A\in\mathrm{SL}(n,\mathbf{Z})$ such that FAF is orthogonal, and thus there does not exist $A\in\mathrm{SL}(n,\mathbf{Z})$ such that $A^tF^2A=(F^{-1})^2$. Hence F^2 and $(F^{-1})^2$ $(\in P_n)$ do not represent the same element in $\mathcal{F}_n$ (see Remark 2.6).

Now we will choose a particular symplectic basis and use Proposition 2.8 and Lemmas 2.3 and 2.4 to study the image of $\widehat{T}$ (and then the image of T, since obviously it is equal to $p_N(\mathrm{Im}\,\widehat{T})$).

Theorem B. *Let $\{dx_1,\dots,dx_n\}$ be the standard basis of $(\mathbf{R}^n)^\vee$. Take the symmetrically lexicographically ordered basis $\{dx_I=dx_{i_1}\wedge\ldots\wedge dx_{i_m}\}_{I\in\mathcal{I}\cup\mathcal{E}}$ as a symplectic basis of $(H^m(\mathbf{R}^n/\mathbf{Z}^n,\mathbf{Z}),-\int_{\mathbf{R}^n/\mathbf{Z}^n}\cdot\wedge\cdot)$ and index the entries of the matrices in $\mathcal{H}_N$ by the lexicographically ordered set $\mathcal{I}$.*

Then $\mathrm{Im}\,\widehat{T}=\{X+iY\in\mathcal{H}_N\,|\,(1)$ *and* (2) *hold*$\}$, *where*

(1) $Y=EE^t$, *where* $E_{I,J}=\det(S)_{I,J}$, $I,J\in\mathcal{I}$, *for some upper triangular* $n\times n$ *matrix* S *with determinant* 1 *and* $S_{j,j}\geq0$ *for* $j=1,\dots,n$;

(2) $X_{IJ}=0$, *if* I *and* J *have more than one index in common* $(I,J\in\mathcal{I})$.

Proof. Let $L\in\mathcal{T}_n$. Using Lemmas 2.3 and 2.4 and the notation of Proposition 2.8, we have $\bigwedge^{2k+1}(L^{-1})^t=\begin{pmatrix} A(L) & C(L)\\ 0 & (A(L)^t)^{-1}\end{pmatrix}$. Hence, $X(L)=A(L)^{-1}C(L)$

and Proposition 2.8 and Lemma 2.4 imply the claim for X. Moreover, $Y(L)=A^{-1}(L)(A^{-1}(L))^t$, where $A(L)_{I,J}=\det((L^{-1})^t)_{I,J}$. So $Y(L)=A^{-1}(L)(A^{-1}(L))^t=A(L^{-1})(A(L^{-1}))^t$, where $A(L^{-1})_{I,J}=\det(L^t)_{I,J}$. Taking $E=A(L^{-1})$ and $S=L^t$ we finish the proof by again using Proposition 2.8. □

3. The case of Kähler manifolds

3.a. A lemma and Abel's map

First we prove the lemma we used in the introduction to compare Lazzeri's Jacobian of compact Kähler manifolds with Weil's and Griffiths' Jacobians.

Notation 3.1. Here the operator $*$, defined as usual on real forms, is extended to complex forms by **C**-linearity, as in [C] and [W].

Let M be a complex manifold. Weil's operator C is the linear operator defined on the forms of bidegree (a,b) by $C\eta=i^{a-b}\eta$ (observe that C takes real forms to real forms). If M is of kählerian type, then C induces on $H^q(M,\mathbf{C})$ the operator we already called C in the introduction.

Lemma 3.2. *Let (M,g) be a hermitian manifold of complex dimension m. Let η be an m-form on M. If we write $\eta=\sum_{r\geq 0}L^r\eta_r$ with η_r being primitive $(m-2r)$-forms, we have that*

$$*\eta=\sum_{r\geq 0}(-1)^{(m^2+m)/2+r}L^rC\eta_r,\qquad C\eta=\sum_{r\geq 0}L^rC\eta_r.$$

Then we can decompose the space of m-forms into the two parts

$$\left\{\eta=\sum_{\substack{r\geq 0\\ r\text{ even}}}L^r\eta_r\;\middle|\;\eta_r\text{ primitive }(m-2r)\text{-forms}\right\}$$

and

$$\left\{\eta=\sum_{\substack{r\geq 0\\ r\text{ odd}}}L^r\eta_r\;\middle|\;\eta_r\text{ primitive }(m-2r)\text{-forms}\right\}.$$

On the first part $*=(-1)^{(m^2+m)/2}C$ and on the second part $*=(-1)^{(m^2+m)/2+1}C$.

Proof. In [C], p. 26, or [Wel] it is proved that if ω is a primitive p-form on M and $r\leq m-p$, where $m=\dim_{\mathbf{C}}M$, then

$$*L^r\omega=\frac{(-1)^{p(p+1)/2}r!}{(m-p-r)!}L^{m-p-r}C\omega.$$

Applying it, with $p=m-2r$, to each term $L^r\eta_r$ of the sum $\eta=\sum_{r\geq 0} L^r\eta_r$, we obtain

$$*\eta=*\sum_{r\geq 0} L^r\eta_r=\sum_{r\geq 0} *L^r\eta_r=\sum_{r\geq 0}(-1)^{(m^2+m)/2+r}L^rC\eta_r.$$

Since $CL=LC$, we have $C\eta=\sum_{r\geq 0} CL^r\eta_r=\sum_{r\geq 0} L^rC\eta_r$. □

Corollary 3.3. *Let (M,g) be a compact Kähler manifold of complex dimension m. Consider the operators $*$ and C on $H^m(M,\mathbf{C})$. We have that $*=C$ on K and $*=-C$ on K' (see Notation 1.3).*

Let (M,g) be a compact Kähler manifold of complex dimension $m=2k+1$.

Notation 3.4. If $H^q(M,\mathbf{C})=V\oplus W$, $\pi_{V,W}$ will denote the projection onto V.

Observe that $J_m=(J_m\cap K)\oplus(J_m\cap K')$. Analogously as in the introduction, we have that the k^{th} Weil's Jacobian, the k^{th} Griffiths' Jacobian and Lazzeri's Jacobian are the real torus $J_m(M)/\pi(H^m(M,\mathbf{Z}))$, where π is $\pi_{J_m(M),\overline{J_m(M)}}$, with different complex structures. We describe the situation in Table 1.

Table 1.

Jacobian	$H^m(M,\mathbf{R})/(H^m(M,\mathbf{Z})/\text{torsion})$	$J_m(M)/\pi(H^m(M,\mathbf{Z}))$
k^{th} Weil's	$-C$	$-i$
k^{th} Griffiths'	$C\vert_{\{\eta+\bar\eta\vert\eta\in J_m(M)\cap\overline{F^{k+1}(M)}\}} \oplus -C\vert_{\{\eta+\bar\eta\vert\eta\in J_m(M)\cap F^{k+1}(M)\}}$	$i\vert_{J_m(M)\cap\overline{F^{k+1}(M)}} \oplus -i\vert_{J_m(M)\cap F^{k+1}(M)}$
Lazzeri's	$C\vert_{H^m(M,\mathbf{R})\cap K}\oplus -C\vert_{H^m(M,\mathbf{R})\cap K'}$	$i\vert_{J_m(M)\cap K}\oplus -i\vert_{J_m(M)\cap K'}$

Notation 3.5. Set $J'_m(M):=(K(M,g)\cap J_m(M))\oplus\overline{K'(M,g)\cap J_m(M)}$.

Remark 3.6. (i) We have $T(M,g)=J'_m(M)/\pi_{J'_m(M),\overline{J'_m(M)}}(H^m(M,\mathbf{Z}))$ with the complex structure given by i and the imaginary part of the polarization $(\alpha,\beta)=-\int_M(\alpha+\bar\alpha)\wedge(\beta+\bar\beta)$.

(ii) Seeing $T(M,g)$ as $J'_m(M)/\pi(H^m(M,\mathbf{Z}))$, we can define an Abel–Jacobi map μ for k-cycles also for Lazzeri's Jacobian. Let Z_0 be a k-cycle in M and set $B(Z_0):=\{Z$ k-cycle homologous to $Z_0\}$. Let $\{\psi_1,\dots,\psi_l\}$ be a basis of $J'_m(M)$. If $Z\in B(Z_0)$ and C is a $(2k+1)$-chain with $\partial C=Z-Z_0$, let $\mu(Z):=\left(\int_C\psi_1,\dots,\int_C\psi_l\right)\in T(M)$.

One can easily see that the definition is good (in an analogous way as in [L2], p. 131) and, by the same calculation as in [G], p. 826, that μ is not holomorphic in general.

3.b. Some remarks on holomorphic variation and injectivity of Lazzeri's period map

We saw that Lazzeri's Jacobian of a compact Kähler manifold (M,g) of odd dimension depends only on the complex structure of M and on the class Ω of the $(1,1)$-form associated with g. In this subsection we will fix Ω and we will study when the map $\widetilde{T}$, associating Lazzeri's Jacobian with the complex structure, is holomorphic and we will see that its differential is injective if $K_M=\mathcal{O}$.

Let (M,g) be a compact Kähler manifold of dimension m and let Ω be the class of the $(1,1)$-form associated with g. Consider a smooth deformation of the complex structure $\mathcal{M}\to\Delta$, $\Delta\ni 0$ a polycylinder.

Notation 3.7. We let M_t be the fibre over t and let ϕ be a C^∞ trivialization $\mathcal{M}\to M\times\Delta$ (possibly restricting Δ), ϕ induces diffeomorphisms $\phi_t\colon M_t\to M$. Let $\varrho\colon T_0(\Delta)\to H^1(\Theta)$ be the Kodaira–Spencer map, where $T_0(\Delta)$ is the holomorphic tangent space to Δ at 0 and $\Theta=\theta(T^{10}(M))$.

We recall the definition of Griffiths' and Weil's period maps.

Griffiths' period map $\mathcal{G}_q^p\colon\Delta\to G_{\mathbf{C}}(f_q^p,H^q(M,\mathbf{C}))$ (possibly restricting Δ) is the map

$$t\longmapsto\mathcal{G}_q^p(t)=(\phi_t^{-1})^*F_q^p(M_t),$$

where $f_q^p:=\dim F_q^p(M)$ for $p,q\in\mathbf{N}$.

If $q>0$ is an odd integer, Weil's period map $\mathcal{W}_q\colon\Delta\to G_{\mathbf{C}}\big(\tfrac{1}{2}b_q,H^q(M,\mathbf{C})\big)$ (possibly restricting Δ) is the map

$$t\longmapsto\mathcal{W}_q(t)=(\phi_t^{-1})^*J_q(M_t).$$

By Griffiths' calculation [G], p. 812, (see also [Gre], p. 33), if $\phi(t)$ is a smoothly varying harmonic $(q-r,r)$-form on M_t, $\phi=\phi(0)$ and $\cdot$ is the contraction, we have

$$\frac{\partial\phi(0)}{\partial t}=\varrho\left(\frac{\partial}{\partial t}\right)\cdot\phi,\ \text{which is of type } (q-r-1,r+1),\tag{1}$$

$$\frac{\partial\phi(0)}{\partial\bar t}=\overline{\varrho\left(\frac{\partial}{\partial t}\right)}\cdot\phi,\ \text{which is of type } (q-r+1,r-1).\tag{2}$$

Thus, while $\mathcal{G}_q^p$ is holomorphic and $\operatorname{Im}d\mathcal{G}_q^p(0)$ is in $\operatorname{Hom}(H^{p,q-p},H^{p-1,q-p+1})$, the map $\mathcal{W}_q$ is not holomorphic. More precisely, if $\phi\in F_q^p(M)$, then

$$\frac{\partial\mathcal{G}_q^p(0)}{\partial t}(\phi)=\pi_{\overline{F_q^{q-p+1}(M)},F_q^p(M)}\left(\varrho\left(\frac{\partial}{\partial t}\right)\cdot\phi\right)=\varrho\left(\frac{\partial}{\partial t}\right)\cdot\phi,$$

$$\frac{\partial\mathcal{G}_q^p(0)}{\partial\bar t}(\phi)=\pi_{\overline{F_q^{q-p+1}(M)},F_q^p(M)}\left(\overline{\varrho\left(\frac{\partial}{\partial t}\right)}\cdot\phi\right)=0.$$

While, if q is odd and $\phi \in J_q(M)$, then

$$\frac{\partial \mathcal{W}_q(0)}{\partial t}(\phi) = \pi_{\overline{J_q(M)}, J_q(M)}\left(\varrho\left(\frac{\partial}{\partial t}\right)\cdot\phi\right) = \varrho\left(\frac{\partial}{\partial t}\right)\cdot\phi,$$
$$\frac{\partial \mathcal{W}_q(0)}{\partial \bar{t}}(\phi) = \pi_{\overline{J_q(M)}, J_q(M)}\left(\overline{\varrho\left(\frac{\partial}{\partial t}\right)}\cdot\phi\right) = \overline{\varrho\left(\frac{\partial}{\partial t}\right)}\cdot\phi.$$

Finally we recall that in [G], p. 844, Griffiths proved that if $K_M = \mathcal{O}$, then the maps $H^1(\Theta) \to \mathrm{Hom}(H^{m,0}, H^{m-1,1})$ and $\overline{H^1(\Theta)} \to \mathrm{Hom}(H^{0,m}, H^{1,m-1})$ given by the contraction are injective, then, if also ϱ is injective, $d\mathcal{G}_m^m(0)$ is injective.

Now let m be odd.

Definition 3.8. Let $\Delta' := \{t \in \Delta \mid$ there exists a Kähler metric on M_t whose $(1,1)$-form is of class $\phi_t^*(\Omega)\}$.

Fix a symplectic basis $\mathcal{S}$ of $H^m(M, \mathbf{Z})$ with respect to $-\int_M \cdot\wedge\cdot$ and, for all $t \in \Delta'$, a Kähler metric g_t on M_t whose $(1,1)$-form is of class $\phi_t^*(\Omega)$. Define

$$\widetilde{T}\colon \Delta' \longrightarrow \mathcal{H}_{(1/2)b_m(M)}$$

as the map sending t to the matrix in $\mathcal{H}_{(1/2)b_m(M)}$ representing $T(M_t, g_t)$ with the symplectic basis $\phi_t^*\mathcal{S}$.

Observe that the map $\widetilde{T}$ (which is the composition of the map $t \mapsto (\phi_t^{-1})^* g_t$ with $\widehat{T}_{\mathcal{S},\{(\phi_t^{-1})^* g_t\}}$) does not depend on the choice of g_t, but only on $\mathcal{S}$ and Ω. It describes how Lazzeri's Jacobian depends on the complex structure (after fixing Ω).

To study when $\widetilde{T}$ is holomorphic and injective, set $\mathcal{L}(t) := (\phi_t^{-1})^* J'_m(M_t) \in G_{\mathbf{C}}\left(\frac{1}{2}b_m, H^m(M, \mathbf{C})\right)$ for $t \in \Delta'$ (Lazzeri's period map).

Let $\phi \in J'_m(M)$. By (1) and (2) we have

$$\frac{\partial \mathcal{L}(0)}{\partial t}(\phi) = \pi_{\overline{\mathcal{L}(0)}, \mathcal{L}(0)}\left(\varrho\left(\frac{\partial}{\partial t}\right)\cdot\phi\right) = \varrho\left(\frac{\partial}{\partial t}\right)\cdot\phi,$$
$$\frac{\partial \mathcal{L}(0)}{\partial \bar{t}}(\phi) = \pi_{\overline{\mathcal{L}(0)}, \mathcal{L}(0)}\left(\overline{\varrho\left(\frac{\partial}{\partial t}\right)}\cdot\phi\right) = \overline{\varrho\left(\frac{\partial}{\partial t}\right)}\cdot\phi.$$

See $T(M_t, g_t)$ as $\mathcal{L}(t)/\pi_{\mathcal{L}(t), \overline{\mathcal{L}(t)}(H^m(M,\mathbf{Z}))}$ (see Remark 3.6). Chosen a basis $\{\omega_j(t)\}$ of $\mathcal{L}(t)$, let $\begin{pmatrix} E(t) \\ F(t) \end{pmatrix}$ be the matrix expressing $\omega_j(t)$ as a function of $\mathcal{S}$. Then $\widetilde{T}(t) = -\overline{E(t)F(t)^{-1}}$. Thus $\widetilde{T}$ is holomorphic if and only if $\bar{\mathcal{L}}$ is holomorphic and $\widetilde{T}$ is injective if and only if $\bar{\mathcal{L}}$ is injective. By the remark just before Definition 3.8, if ϱ is injective, the map $d\mathcal{W}(0)$ is injective and then $\mathcal{W}$ is locally injective. Hence also $\bar{\mathcal{L}}$ is locally injective, in fact $(\phi_t^{-1})^*(K(M_t, g_t)) = K(M, g)$ and $(\phi_t^{-1})^*(K'(M_t, g_t)) = K'(M, g)$, thus $\mathcal{W}(t) = (\mathcal{L}(t) \cap K(M, g)) \oplus \overline{\mathcal{L}(t) \cap K'(M, g)}$, and injectivity of $\mathcal{W}$ implies injectivity of $\mathcal{L}$ and $\bar{\mathcal{L}}$. Thus we have the following result.

Proposition C. *Let (M,g) be a compact Kähler manifold of odd complex dimension m and Ω be the class of the form associated with g. Fix a symplectic basis S of $H^m(M,\mathbf{Z})$ and consider a smooth deformation of the complex structure $\mathcal{M}\to\Delta$, for which we use Notation* 3.7 *and* 3.8. *Suppose that Δ' is a neighbourhood of* 0. *Then*

(a) *$\widetilde{T}:\Delta'\to\mathcal{H}$ is holomorphic at* 0 *if and only if $\varrho(T_0(\Delta))\cdot(K(M,g)\cap J_m(M)\oplus \overline{K'(M,g)\cap J_m(M)})=0$, where $\cdot$ is the contraction;*

(b) *if $K_M=\mathcal{O}$ and ϱ is injective then $\widetilde{T}$ is locally injective at* 0.

The example of Mattuck in [G], p. 588, which shows that Weil's Jacobian does not always vary holomorphically, shows also that Lazzeri's Jacobian does not always vary holomorphically (i.e. that the condition in (a) of Proposition C is not always satisfied).

Remark 3.9. If $K_M=\mathcal{O}$ we can take as g_t the Kähler–Einstein metric whose $(1,1)$-form is of class $\phi_t^*(\Omega)$ (it exists uniquely by Calabi–Yau's theorem, see [SP]).

4. Lazzeri's Jacobian of a bundle

Definition 4.1. An element λ of a lattice Λ is *primitive* if there does not exist a $\lambda'\in\Lambda$, $\lambda'\neq\pm\lambda$, such that $\lambda\in\mathbf{Z}\lambda'$.

Proposition D. *Let (M,g_M) and (N,g_N) be Riemannian compact oriented manifolds of dimension $2(2k+1)$ and $2(2s)$, respectively. Let $p:F\to M$ be a bundle with fibre N and structure group $G\subset\mathrm{Diff}(N)$ and suppose that g_N is G-invariant. Consider the metric on F induced by g_M and g_N. Suppose there exists $\lambda\in H^{2s}(N,\mathbf{Z})/\mathrm{torsion}$ such that $\lambda\neq0$, $*_N\lambda=\lambda$ and λ is G-invariant.*

We can define a holomorphic map $e_\lambda:T(M)\to T(F)$ (we omit the metrics) such that

(1) *if θ_F and θ_M are the polarizations of $T(F)$ and $T(M)$, respectively, then $e_\lambda^*\theta_F=(\int_N\lambda\wedge*\lambda)\theta_M$;*

(2) *e_λ is injective if one of the following conditions holds:*

(a) *$\int_N\lambda\wedge*\lambda=1$;*

(b) *$F=M\times N$ and λ is primitive;*

(c) *$H^*(N,\mathbf{Z})$ is free and G-invariant.*

Proof. Let us define $e_\lambda:T(M)\to T(F)$. If $F=M\times N$ we can define e_λ simply as the map induced by the map $H^{2k+1}(M,\mathbf{R})\to H^{2k+1+2s}(M\times N,\mathbf{R})$ defined by $\eta\mapsto\eta\wedge\lambda$. More generally, define $\Lambda\in H^{2s}(F,\mathbf{Z})$ in the following way: if U is a trivializing open subset of M, let $\Lambda|_{p^{-1}(U)}:=\pi^*\lambda$, where π is the composition of a C^∞ trivialization $p^{-1}(U)\to U\times N$ with the projection $U\times N\to N$. Define

$E_\lambda: H^{2k+1}(M,\mathbf{R}) \to H^{2k+1+2s}(F,\mathbf{R})$ by $E_\lambda(\eta) = p^*\eta \wedge \Lambda$. The map E_λ induces a map $e_\lambda: T(M) \to T(F)$.

The fact that $*_N\lambda = \lambda$ implies at once that e_λ is holomorphic.

Let us prove that $e_\lambda^*\theta_F = \left(\int_N \lambda\wedge *\lambda\right)\theta_M$. Let $\{U_\alpha\}_\alpha$ be a trivializing covering of M and let ψ_α be a partition of unity for this covering. Let ϕ_α be the partition of unity for the covering $\{p^{-1}(U_\alpha)\}$ of F, defined by $\phi_\alpha(y) = \psi_\alpha(p(y))$. Let $\omega_1, \omega_2 \in H^{2k+1}(M,\mathbf{R})$. We have

$$\begin{aligned}
\int_F E_\lambda(\omega_1)\wedge E_\lambda(\omega_2) &= \sum_\alpha \int_{p^{-1}(U_\alpha)} \phi_\alpha E_\lambda(\omega_1)\wedge E_\lambda(\omega_2) \\
&= \sum_\alpha \int_{U_\alpha \times N} \phi_\alpha \omega_1 \wedge\lambda\wedge\omega_2\wedge\lambda \\
&= \sum_\alpha \int_{U_\alpha \times N} \phi_\alpha \omega_1 \wedge\omega_2\wedge\lambda\wedge *\lambda \\
&= \sum_\alpha \int_{U_\alpha} \psi_\alpha \omega_1 \wedge\omega_2 \int_N \lambda\wedge *\lambda \\
&= \left(\int_N \lambda\wedge *\lambda\right) \int_M \omega_1\wedge\omega_2.
\end{aligned}$$

If (a) holds then the map e_λ is injective since it is a homomorphism of principally polarized abelian varieties and preserves the polarization. It is easy to verify that (b) implies that e_λ is injective. Finally, if (c) holds, then e_λ is injective by the theorem of Leray–Hirsch (see [S], p. 258). □

Consider Kähler manifolds. A completely analogous statement holds for the k^{th} Weil's Jacobian of M and $(k+s)^{\text{th}}$ Weil's Jacobian of F with the condition $*_N\lambda = \lambda$ replaced by $C_N\lambda = \lambda$. As to Griffiths' Jacobians, if we replace the condition $*_N\lambda = \lambda$ with "λ is of type (s,s)" in the hypotheses, we have that there exists a holomorphic map from the k^{th} Griffiths' Jacobian of M into the $(k+s)^{\text{th}}$ Griffiths' Jacobian of F and it is injective if (b) or (c) hold.

Acknowledgements. I warmly thank F. Catanese for suggesting me to study Lazzeri's Jacobian and for many helpful suggestions. I thank also M. Manetti for some corrections and the referee and the editor for some suggestions to improve the paper.

References

[C] CHERN, S. S., *Complex Manifold*, Instituto de Fisica e Matematica Universidade de Recife, 1959.

[Gre] GREEN, M., Infinitesimal methods in Hodge theory, in *Algebraic Cycles and Hodge Theory* (Albano, A. and Bardelli, E., eds.), Lecture Notes in Math. **1594**, pp. 1–92, Spriger-Verlag, Berlin–Heidelberg, 1994.

[G] GRIFFITHS, P., Periods of integrals on algebraic manifolds, I; II, *Amer. J. Math.* **90** (1968), 568–626; 805–865.

[GH] GRIFFITHS, P. and HARRIS, J., *Principles of Algebraic Geometry*, Wiley, New York, 1978.

[LB] LANGE, H. and BIRKENHAKE, C., *Complex Abelian Varieties*, Springer-Verlag, Berlin, 1992.

[L1] LIEBERMAN, D. I., Higher Picard varieties, *Amer. J. Math.* **90** (1968), 1165–1199.

[L2] LIEBERMAN, D. I., Intermediate Jacobians, in *Algebraic Geometry, Oslo, 1970* (Oort, F., ed.), pp. 125–139, Wolters-Nordhoff, Groningen, 1972.

[M] MUMFORD, D., *Abelian Varieties*, Tata Institute of Fundamental Research, Bombay; Oxford Univ. Press, London, 1974.

[SP] SÉMINAIRE PALAISEAU, *Première classe de Chern et courbure de Ricci: preuve de la conjecture de Calabi,* Astérisque **58**, Soc. Math. France, Paris, 1978.

[S] SPANIER, E., *Algebraic Topology*, McGraw-Hill, New York–Toronto–London, 1966.

[W] WEIL, A., *Introduction à l'étude des variétés kählériennes*, Hermann, Paris, 1958.

[Wel] WELLS R., *Differential Analysis on Complex Manifolds*, Springer-Verlag, Berlin–New York, 1980.

Received January 7, 1998
in revised form August 15, 1999

Elena Rubei
Dipartimento di Matematica
Università di Firenze
Viale Morgagni 67/A
IT-50134 Firenze
Italy
email: rubei@math.unifi.it