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Application of the fundamental principle to complex Cauchy problem

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Abstract. In this paper we give an explicit formula for the solution of the non-homogeneous complex Cauchy problem with Cauchy data given on a bounded smooth strictly convex domain in a non-characteristic hyperplane. These formulas are obtained using the explicit version of the fundamental principle given in terms of residue currents; moreover, we characterize the domain of definition of the solution and we generalize these techniques to the non-homogeneous Goursat problem.

1. Introduction

Throughout this paper, we consider the following problem called the complex Cauchy problem. Let $n\geq 2$ and $\Omega\subset\mathbf{C}^n$ be a bounded smooth strictly convex domain. Let $g\in\mathcal{O}(\bar\Omega)$ be a holomorphic function in a neighbourhood of $\bar\Omega$. Let $P\in\mathbf{C}[z_1,\dots,z_n]$ be a polynomial of degree m of the form $P=\sum_{|\alpha|\leq m}a_\alpha z^\alpha$ with the usual notation. We want to find a unique $f\in\mathcal{O}(\Omega)$ such that

$$\begin{cases} P\left(\dfrac{\partial}{\partial z}\right)f=g & \text{in } \Omega,\\ f(0,z')=h_0(z'), & \\ \vdots & \\ \dfrac{\partial^{m-1}f}{\partial z_1^{m-1}}(0,z')=h_{m-1}(z') & \end{cases}$$

for all $z'\in\omega=\Omega\cap\{z\in\mathbf{C}^n:z_1=0\}$.

The first restriction we will make is assuming that the complex hyperplane $\{z\in\mathbf{C}^n:z_1=0\}$ is non-characteristic for the operator $P(\partial/\partial z)$, i.e. $P_m(1,0,\dots,0)\neq 0$ with $P_m=\sum_{|\alpha|=m}a_\alpha z^\alpha$. In this case, by Cauchy–Kowalewsky's theorem, the complex Cauchy problem has a unique solution f.

If we consider the real Cauchy problem (i.e. the same problem where $\Omega\subset\mathbf{R}^n$, $g\in C^\infty(\bar\Omega)$ and where $z\in\mathbf{C}^n$ and $\partial/\partial z$ are replaced by $t\in\mathbf{R}^n$ and $i\partial/\partial t$), John [14] has proved that, in order to get the same conclusion, we have to assume moreover that the polynomial P is hyperbolic, that is, if $P(t_1,t')=0$ with $t'\in\mathbf{R}^{n-1}$, then $t_1\in\mathbf{R}$. In this case, Herglotz, Petrowsky and Gårding have given an explicit integral expression of the solution f. See [11] and [14].

For the complex Cauchy problem, the first explicit solution was given by Fantappiè [10]. As a result of the development of symbolic calculus in Italy during the 40s, Fantappiè obtained some formal formulas, but there was no study of the domain of definition of the solution f. For example, the formula obtained by Fantappiè for P homogeneous and $n=3$ to the complex Cauchy problem is

$$f(z)=\sum_{k=0}^{m}\frac{1}{(2\pi i)^2}\int_{C_2}\int_{C_3}\int_0^1\frac{\bar p_k(\zeta_2,\zeta_3,z)}{P(1,-\zeta_1/\zeta_2,(\zeta_1-1)/\zeta_3)}\,\frac{d\zeta_1}{\zeta_2\zeta_3}\,d\zeta_3\,d\zeta_2,$$

where $\bar p_k(\zeta_2,\zeta_3,z)=\mathcal{J}^k\bar\varphi_k$ with $\mathcal{J}$ being the operator of integration along the real line with direction coefficients (ζ_2,ζ_3) passing through z between $z_1=0$ and z, and $\bar\varphi_m=g$ and

$$\bar\varphi_k=h_k+\sum_{\alpha\in\mathbf{N}^3}a_\alpha\frac{\partial^{\alpha_2+\alpha_3}h_{m+k-\alpha_2-\alpha_3}}{\partial z_2^{\alpha_2}\partial z_3^{\alpha_3}},\quad k=0,\dots,m-1,$$

and where C_2 and C_3 are curves around the origin such that $\bar p_k$ is regular inside these, and $1/P(1,-\zeta_1/\zeta_2,(\zeta_1-1)/\zeta_3)$ is regular outside. These formulas have been generalized by Leray in [18] and by Henkin in [12].

Other formulas are obtained in Sternin–Shatalov's book [27] in a more general case (since the Cauchy data (h_j) are not necessarily given on a piece of hyperplane, but can be given on a hypersurface X, possibly with some characteristic points, that is some $x\in X$ where $P_m(\partial h/\partial x)=0$ and where h is a holomorphic function in a neighbourhood U of x such that $X\cap U=U\cap h^{-1}(0)$, with $\partial h/\partial x\neq 0$). For example, the Cauchy problem

$$\begin{cases} P\left(-\dfrac{\partial}{\partial x}\right)f=g & \text{on } \Omega,\\ f\equiv 0 \pmod m & \text{on } X\cap\Omega,\end{cases}$$

has a solution if Ω is (P,X)-convex (see [27], p. 371, Definition 5.5 and p. 379, Theorem 5.13) and the solution f is given by

$$f(x)=\left(\frac{i}{2\pi}\right)^{n-1}\int_{h_1(x)}\frac{\partial^{n-1}\bar G}{\partial p_0^{n-1}}(p(y-x),p)f(y)\,dy\wedge\omega(p),$$

where ω is the Leray form $\omega(p)=\sum_{j=1}^{n}(-1)^{j-1}p_j\,dp_1\wedge\ldots\wedge\widehat{dp_j}\wedge\ldots\wedge dp_n$, G is the function $G(p_0,p)=(2\pi i)^{-2}\int_{C(p)} e^{\lambda p_0}\,d\lambda/H(\lambda p)$ and where $C(p)$ is a contour which surrounds all singularities of the integrand and where the class $h_1(x)$ is defined in [27], pp. 372–378.

In this paper, we use the explicit formulation of the fundamental principle given in [7] and [25] to obtain the solution of the holomorphic Cauchy problem. The first approach is, like in [25], to give an explicit representation formula of the solutions f of the equation $P(\partial/\partial z)f=g$ in Ω. This will be done in Theorem 3.4. Setting $\varphi(z)=\sup_{\zeta\in\Omega}\mathrm{Re}\langle\zeta,z\rangle$, we regularize φ in ψ such that $2\partial\psi/\partial\zeta$ maps $\mathbf{C}^n$ into $\bar{\Omega}$. Denoting by $[1/P]$ the principal value current, we obtain the representation formula

$$f(z)=\left[\frac{1}{P(\zeta)}\right]\cdot\left(g\left(2\frac{\partial\psi}{\partial\zeta}\right)e^{\langle\zeta,z-2\partial\psi(\zeta)/\partial\zeta\rangle}\frac{(2\bar\partial\partial\psi)^n}{(2\pi i)^n n!}\right)$$
$$+\bar\partial\left[\frac{1}{P(\zeta)}\right]\cdot\left(p\left(\zeta,\frac{\partial}{\partial\zeta}\right)f\left(2\frac{\partial\psi}{\partial\zeta}\right)e^{\langle\zeta,z-2\partial\psi(\zeta)/\partial\zeta\rangle}\frac{(2\bar\partial\partial\psi)^{n-1}}{(2\pi i)^n(n-1)!}\right),$$

where $p(\zeta,z)$ is the Oka–Hefer differential form defined in Section 2. If we now deform the function ψ so that $2\partial\psi/\partial\zeta$ maps $P^{-1}(0)$ into $\bar\omega$, one can hope to obtain the explicit formula of the Cauchy problem.

The second approach consists in reducing the first problem (using Duhamel's principle and the principle of superposition) to the problem

$$\begin{cases} P\left(\dfrac{\partial}{\partial z}\right)f=0 & \text{in } \Omega,\\ f(0,z')=0, & z'\in\omega,\\ \vdots & \\ \dfrac{\partial^{m-2}f}{\partial z_1^{m-2}}(0,z')=0, & z'\in\omega,\\ \dfrac{\partial^{m-1}f}{\partial z_1^{m-1}}(0,z')=h(z'), & z'\in\omega. \end{cases}$$

By the fundamental principle given in [7], one can write f in the form

$$f(z)=\bar\partial\left[\frac{1}{P(\zeta)}\right]\cdot(e^{\langle\zeta,z\rangle}\,d\zeta_1\wedge\Phi(\zeta')),$$

where $\zeta=(\zeta_1,\zeta')$ and Φ is an $(n-1,n-1)$-form in ζ' and so

$$\frac{\partial^l f}{\partial z_1^l}(0,z')=\bar\partial\left[\frac{1}{P(\zeta)}\right]\cdot(e^{\langle\zeta',z'\rangle}\zeta_1^l\,d\zeta_1\wedge\Phi(\zeta'))$$
$$=\begin{cases} 0, & \text{if } l<m-1,\\ \dfrac{2\pi i}{P_m(1,0,\ldots,0)}\displaystyle\int_{\mathbf{C}^{n-1}} e^{\langle\zeta',z'\rangle}\Phi(\zeta'), & \text{if } l=m-1, \end{cases}$$

where $P_m(1,0,\dots,0)\neq 0$; and so, the choice

$$\Phi(\zeta')=P_m(1,0,\dots,0)h\left(2\frac{\partial\psi}{\partial\zeta'}\right)e^{-\langle\zeta',2\partial\psi/\partial\zeta'\rangle}\frac{(2\bar\partial\partial\psi(\zeta'))^{n-1}}{(2\pi i)^n(n-1)!}$$

gives the solution.

These two approaches combined together give us the solution of the complex Cauchy problem. Moreover, the solution we obtain is defined in the domain of convergence of the integrals we obtain, and this domain coincides with the domains given by Kiselman and Schiltz in [15] and [26]. We prove, in fact, that f is defined in the domain $\widetilde{\Omega}=\Omega\cap\widetilde{\omega}$, where $\widetilde{\omega}$ is the set of z in $\mathbf{C}^n$ such that every complex characteristic hyperplane passing through z meets ω.

The main interesting fact in this approach is that an immediate generalization gives us, in a totally similar way, the solution of the holomorphic Goursat problem with data on a complex vector subspace of bigger complex codimension satisfying a condition of non-characteristicity analogous to those introduced in the case of the complex Cauchy problem, and the results obtained in this case are new. In particular, if $P:\mathbf{C}^n\to\mathbf{C}^N$ is a complete intersection polynomial mapping, we say that a complex subspace H of codimension N is non-characteristic for the differential operator $P(\partial/\partial z)$ if and only if we can write $H=H_1\cap\dots\cap H_N$, where each H_j is a complex hyperplane non-characteristic for the differential operator $P_j(\partial/\partial z)$. In particular, we will see that the Goursat problem

$$\begin{cases} P_1\left(\dfrac{\partial}{\partial z}\right)f(z)=g_1(z),\\ \vdots\\ P_N\left(\dfrac{\partial}{\partial z}\right)f(z)=g_N(z),\end{cases}$$

where $g_1,\dots,g_N$ are holomorphic functions in a neighbourhood of $\overline{\Omega}$, and where f is given on $\omega=\Omega\cap H$, where H is a non-characteristic subspace of codimension N, has got a solution f. Moreover, we prove that this solution is defined on $\Omega\cap\widetilde{\omega}$, where $\widetilde{\omega}$ is the set of z in $\mathbf{C}^n$ such that every complex characteristic subspace of codimension N passing through z meets ω.

Applying these results to the case of the global complex Goursat problem with right-hand sides of exponential type, we obtain, in a different way, the results obtained by Ebenfelt and Shapiro in [8] with simplified representation formula.

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2. Preliminaries

Let $n\in\mathbf{N}^*$. If $z\in\mathbf{C}^n$, we set $z=(z_1,\dots,z_n)$. If $n\geq 2$, we set $z=(z_1,z')$, where $z'=(z_2,\dots,z_n)\in\mathbf{C}^{n-1}$. If $(z,\zeta)\in\mathbf{C}^n\times\mathbf{C}^n$, we let

$$\langle z,\zeta\rangle := \sum_{j=1}^{n} z_j\zeta_j = \langle z_1,\zeta_1\rangle + \langle z',\zeta'\rangle \quad \text{and} \quad |z|^2 := \langle z,\bar{z}\rangle = |z_1|^2+|z'|^2.$$

We set also, for $z\in\mathbf{C}^n$ and a multi-index $l\in\mathbf{N}^n$, $z^l=z_1^{l_1}\dots z_n^{l_n}$ and $|l|=l_1+\dots+l_n$. If $n\geq 2$, we set $l=(l_1,l')$, where $l'=(l_2,\dots,l_n)\in\mathbf{N}^{n-1}$, and thus $z^l=z_1^{l_1}(z')^{l'}$ and $|l|=l_0+|l'|$.

Let P be a polynomial over $\mathbf{C}^n$ of degree m and of the form

$$P(z) = \sum_{\substack{l\in\mathbf{N}^n\\ |l|\leq m}} a_l z^l = \sum_{k=0}^{m} z_1^k b_k(z'). \tag{2.1}$$

We denote by $P(D)$, where $D=\partial/\partial z=(\partial/\partial z_1,\dots,\partial/\partial z_n)$, the holomorphic differential operator obtained from P by replacing all the terms of the form z^l by the operators $\partial^l=\partial^{|l|}/\partial z^l=\partial^{|l|}/\partial z_1^{l_1}\dots\partial z_n^{l_n}$. We denote by P_m the principal part of P, that is $P_m(z)=\sum_{|l|=m} a_l z^l$.

A complex (real) hyperplane H of $\mathbf{C}^n$ is *characteristic for the operator* $P(D)$ if and only if we can write $H=\{z\in\mathbf{C}^n : \langle\zeta,z\rangle=p\}$ $(H=\{z\in\mathbf{C}^n : \operatorname{Re}\langle\zeta,z\rangle=p\})$, where $P_m(\zeta)=0$ (cf. [26]).

We have the following lemma (cf. [9]).

Lemma 2.1. *Let $n\in\mathbf{N}$, $n\geq 2$. If the complex hyperplane $\{z\in\mathbf{C}^n : z_1=0\}$ is non-characteristic for the operator $P(D)$, then there exist positive constants A, B such that*

$$|P(\zeta)|\leq 1 \implies |\zeta_1|\leq A+B|\zeta'|, \quad \zeta\in\mathbf{C}^n.$$

Proof. If the complex hyperplane $\{z\in\mathbf{C}^n : z_1=0\}$ is non-characteristic for the differential operator $P(D)$, then $a_{(m,0,\dots,0)}=P_m(1,0,\dots,0)\neq 0$ and we have

$$P(z)=P_m(1,0,\dots,0)z_1^m+\sum_{l_1=0}^{m-1}\sum_{|l'|\leq m-l_1} a_l z_1^{l_1}(z')^{l'}.$$

Lemma 2.1 is now a consequence of the following classical lemma applied to the polynomial $Q(z_1)=P(z_1,z')-\alpha$, where $|\alpha|\leq 1$. $\square$

Lemma 2.2. *If* $Q(z_1)=z_1^m+\sum_{k=1}^m a_k z_1^{m-k}$, *then* $Q(z_1)=0$ *implies*

$$|z_1|\le 2\max_{k=1,\dots,m}|a_k|^{1/k}.$$

Proof. If $Q(z_1)=0$ for a $z_1\neq 0$ then $-1=\sum_{k=1}^m a_k/z_1^k$, and so

$$|z_1|>2\max_{k=1,\dots,m}|a_k|^{1/k}$$

implies $1\le\sum_{k=1}^m |a_k|/|z_1|^k<\sum_{k=1}^m 1/2^k<1$, which is a contradiction. □

To the polynomial P, one can associate a family of Hefer–Oka–Weil polynomials satisfying the identities

$$P(z)-P(\zeta)=\sum_{k=1}^n p_k(\zeta,z)(z_k-\zeta_k),\quad \zeta,z\in\mathbf{C}^n.$$

One can take, for example,

$$p_1(\zeta,z)=\frac{P(\zeta)-P(z_1,\zeta')}{\zeta_1-z_1},\quad \dots,\quad p_n(\zeta,z)=\frac{P(z_1,\dots,z_{n-1},\zeta_n)-P(z)}{\zeta_n-z_n}.$$

We introduce the polynomial differential form $p(\zeta,z):=\sum_{k=1}^n p_k(\zeta,z)\,d\zeta_k$. In particular, $p(\zeta,D)f(w)$ denotes the form obtained from $p(\zeta,z)$ by considering its coefficients as polynomials in z and replacing z^l by $\partial^{|l|}f(w)/\partial z^l$ for all $w\in\mathbf{C}^n$.

We define the residue currents as in [7] and [24]. Let χ be a smooth function such that $\chi(x)=0$, if $x\le c_1$, and $\chi(x)=1$, if $x\ge c_2$, where $0<c_1<c_2<1$. The currents $R=[1/P]$ and $\bar\partial R=\bar\partial[1/P]$ are the weak limits

$$\lim_{\varepsilon\to 0}\frac{\chi_\varepsilon(\zeta)}{P(\zeta)}\quad\text{resp.}\quad\lim_{\varepsilon\to 0}\frac{\bar\partial\chi_\varepsilon(\zeta)}{P(\zeta)},$$

where $\chi_\varepsilon(\zeta)=\chi(|P(\zeta)|/\varepsilon)$. By [24], we know that these residue currents may act on differential forms defined on the projective space $\mathbf{P}^n$. We use the following proposition which is a direct consequence of Theorem 1 in [1].

Proposition 2.3. *Let f be a holomorphic function in $\mathbf{C}^n$. Suppose that there exist smooth functions $Q^k\colon\mathbf{C}^n\times\mathbf{C}^n\to\mathbf{C}^n$, $k=1,\dots,p$, and a function G on $\mathbf{C}^p$ which is holomorphic in a neighbourhood of the image of $\mathbf{C}^n\times\mathbf{C}^n$ by the mapping $\langle z-\zeta,Q\rangle$ defined by*

$$(\zeta,z)\longmapsto(\langle z-\zeta,Q^1(\zeta,z)\rangle,\dots,\langle z-\zeta,Q^p(\zeta,z)\rangle)$$

and satisfying $G(0)=1$.

We set, for $\alpha\in\mathbf{N}^p$ *and* z *fixed in* $\mathbf{C}^n$, $D^\alpha G$ *to be the functions obtained by composing* $\langle z-\zeta, Q\rangle$ *with derivatives of* G. *We also let* $(\bar\partial q)^\alpha := (\bar\partial q^1)^{\alpha_1}\wedge\ldots\wedge(\bar\partial q^p)^{\alpha_p}$, *where* $q^k(\zeta,z):=\sum_j Q_j^k(\zeta,z)\,d\zeta_j$.

Suppose, moreover, that the form

$$f(\zeta)\sum_{|\alpha|=n}\frac{D^\alpha G}{\alpha!}(\bar\partial q)^\alpha$$

is integrable over $\mathbf{C}^n$. *Then the following representation formula holds,*

$$f(z)=\frac{1}{(2\pi i)^n}\int_{\mathbf{C}^n} f(\zeta)\sum_{|\alpha|=n}\frac{D^\alpha G}{\alpha!}(\bar\partial q)^\alpha.$$

3. The fundamental principle

Let Ω be a strictly smooth convex domain in $\mathbf{C}^n$ defined by

$$\Omega := \{z\in\mathbf{C}^n : \varrho(z)<1\},$$

where ϱ is a convex function, smooth except possibly at the origin and such that, for every $z\in\mathbf{C}^n$ and every $\lambda\geq 0$, we have $\varrho(\lambda z)=\lambda\varrho(z)$. In particular, $0\in\Omega$. We define the *support function* φ of Ω by

$$\varphi(\zeta) := \sup_{z\in\Omega}\operatorname{Re}\langle z,\zeta\rangle.$$

As proved by Berndtsson in [4], we have the following proposition.

Proposition 3.1. *The open domain* $\Omega^*=\{\zeta\in\mathbf{C}^n:\varphi(\zeta)<1\}$ *is a smooth strictly convex domain in* $\mathbf{C}^n$, φ *is smooth on* $\mathbf{C}^n\setminus\{0\}$, *and* $2\partial\varphi/\partial\zeta:\partial\Omega^*\to\partial\Omega$ *is a diffeomorphism with inverse* $2\partial\varrho/\partial z:\partial\Omega\to\partial\Omega^*$.

In addition, we have the following proposition (cf. [25], Lemma 1).

Proposition 3.2. *There exists a smooth convex function* ψ *on* $\mathbf{C}^n$ *such that* $\psi\equiv\varphi$ *on* $\mathbf{C}^n\setminus\Omega^*$, *and such that* $2\partial\psi/\partial\zeta$ *is a diffeomorphism on* $\Omega^*\setminus\{0\}$ *into* $\Omega\setminus\{0\}$. *Moreover,* $2\partial\psi(0)/\partial\zeta=0$, $2\partial\psi(\mathbf{C}^n\setminus\Omega^*)/\partial\zeta=\partial\Omega$ *and* $2\partial\psi(\mathbf{C}^n)/\partial\zeta=\bar\Omega$.

We simply write $2\partial\psi/\partial\zeta$ for $2\partial\psi(\zeta)/\partial\zeta$ and $\omega_k(\zeta)$ for $(2\pi i)^{-n}(2\bar\partial\partial\psi(\zeta))^k/k!$. We have the following proposition.

Proposition 3.3. *If Ω is a domain in $\mathbf{C}^n$ satisfying the above conditions, and $f\in\mathcal{O}(\bar{\Omega})$ then we have the representation formula*

$$f(z)=\int_{\mathbf{C}^n} f\left(2\frac{\partial\psi}{\partial\zeta}\right)e^{\langle\zeta,z-2\partial\psi/\partial\zeta\rangle}\omega_n(\zeta),\quad z\in\Omega. \tag{3.1}$$

Proof. For this, we do like in [4], [5] and [7]. We first show that

$$\int_{\mathbf{C}^n} f\left(2\frac{\partial\psi}{\partial\zeta}\right)e^{\langle\zeta,z-2\partial\psi/\partial\zeta\rangle}\omega_n(\zeta) = \frac{1}{(2\pi i)^n}\int_{\mathbf{C}^n\setminus\{0\}} f\left(2\frac{\partial\varphi}{\partial\zeta}\right)e^{\langle\zeta,z-2\partial\varphi/\partial\zeta\rangle}\frac{(2\bar{\partial}\partial\varphi(\zeta))^n}{n!}.$$

For this, we remark that

$$\int_{\mathbf{C}^n} f\left(2\frac{\partial\psi}{\partial\zeta}\right)e^{\langle\zeta,z-2\partial\psi/\partial\zeta\rangle}\omega_n(\zeta)=\frac{1}{(-2\pi i)^n}\int_{\Lambda_\psi} f(\bar{w})e^{\langle\zeta,z-\bar{w}\rangle}\frac{(d\zeta\wedge d\bar{w})^n}{n!},$$

where $\Lambda_\psi=\{(\zeta,w)\in(\mathbf{C}^n\setminus\{0\})\times\mathbf{C}^n:\bar{w}=2\partial\psi/\partial\zeta\}$ is a subvariety of $\mathbf{C}^{2n}$ and the form $d\zeta\wedge d\bar{w}=\sum_{j=1}^n d\zeta_j\wedge d\bar{w}_j$; and so, one has just to verify that

$$\int_{\Lambda_\psi} f(\bar{w})e^{\langle\zeta,z-\bar{w}\rangle}\frac{(d\zeta\wedge d\bar{w})^n}{n!}=\int_{\Lambda_\varphi} f(\bar{w})e^{\langle\zeta,z-\bar{w}\rangle}\frac{(d\zeta\wedge d\bar{w})^n}{n!}.$$

Since the form $f(\bar{w})e^{\langle\zeta,z-\bar{w}\rangle}(d\zeta\wedge d\bar{w})^n/n!$ is closed on $U\times\mathbf{C}^n$, where U is an open domain on which the function $w\mapsto f(\bar{w})$ is defined, one has just to verify that the form satisfies some integrability conditions. This is just a direct consequence of the identity (cf. [25])

$$|e^{\langle\zeta,z-2\partial\psi/\partial\zeta\rangle}|=e^{\operatorname{Re}\langle\zeta,z-2\partial\psi/\partial\zeta\rangle}$$

and

$$\begin{aligned}\operatorname{Re}\left\langle\zeta,z-2\frac{\partial\psi}{\partial\zeta}\right\rangle &\le \operatorname{Re}\langle\zeta,z\rangle-\psi(\zeta)+\psi(0)\le\varrho(z)\frac{\operatorname{Re}\langle\zeta,z\rangle}{\varrho(z)}-\varphi(\zeta)+C\\ &\le(\varrho(z)-1)\varphi(\zeta)+C\le\varepsilon(\varrho(z)-1)|\zeta|+C,\end{aligned}$$

where ε is such that the ball of center 0 and radius ε is included in Ω. We then show that

$$\frac{1}{(2\pi i)^n}\int_{\mathbf{C}^n\setminus\{0\}} f\left(2\frac{\partial\varphi}{\partial\zeta}\right)e^{\langle\zeta,z-2\partial\varphi/\partial\zeta\rangle}\frac{(2\bar{\partial}\partial\varphi(\zeta))^n}{n!}=f(z).$$

For this, we do as in [7]: we write $\mathbf{C}^n\setminus\{0\}=\partial\Omega^*\times(0,+\infty)$ and put $\zeta=2s\partial\varrho(w)/\partial w$, so $\varphi(\zeta)=s$ and $w=2\partial\varphi(\zeta)/\partial\zeta$. We have $(2\bar\partial\partial\varphi)^n=(-1)^n n s^{n-1}\,ds\wedge 2\partial\varrho\wedge(2\bar\partial\partial\varrho)^{n-1}$ and so

$$\begin{aligned}\frac{1}{(2\pi i)^n}&\int_{\mathbf{C}^n\setminus\{0\}} f\left(2\frac{\partial\varphi}{\partial\zeta}\right)e^{\langle\zeta,z-2\partial\varphi/\partial\zeta\rangle}\frac{(2\bar\partial\partial\varphi(\zeta))^n}{n!}\\&=\frac{1}{(-2\pi i)^n}\int_{(0,+\infty)\times\partial\Omega} f(w)e^{s\langle 2\partial\varrho/\partial w,z-w\rangle}\frac{s^{n-1}ds}{(n-1)!}\wedge 2\partial\varrho(w)\wedge(2\bar\partial\partial\varrho)^{n-1}\\&=\frac{1}{(2\pi i)^n}\int_{\partial\Omega}\frac{f(w)\partial\varrho(w)\wedge(\bar\partial\partial\varrho)^{n-1}}{\langle\partial\varrho/\partial w,w-z\rangle^n}=f(z),\end{aligned}$$

by the Cauchy–Fantappiè–Leray formula. □

Using this proposition, we obtain the following theorem.

Theorem 3.4. *If Ω is a domain in $\mathbf{C}^n$ satisfying the above conditions, then for every $g\in\mathcal{O}(\bar\Omega)$, there exists $f\in\mathcal{O}(\bar\Omega)$ such that*

$$P(D)f=g\quad\text{on }\Omega \tag{3.2}$$

and the following representation formula holds for every $z\in\Omega$:

$$\begin{aligned}f(z)=R\cdot&\left(g\left(2\frac{\partial\psi}{\partial\zeta}\right)e^{\langle\zeta,z-2\partial\psi/\partial\zeta\rangle}\omega_n(\zeta)\right)\\&+\bar\partial R\cdot\left(p(\zeta,D)f\left(2\frac{\partial\psi}{\partial\zeta}\right)e^{\langle\zeta,z-2\partial\psi/\partial\zeta\rangle}\omega_{n-1}(\zeta)\right).\end{aligned}\tag{3.3}$$

In fact, the first term is a solution of (3.2), and the operator $T\colon f\in\mathcal{O}(\bar\Omega)\mapsto T(f)$ defined by the second term is a projection operator onto the space of solutions of the homogeneous equation

$$P(D)f=0\quad\text{on }\Omega.$$

Proof. Let us first remark that (3.2) has a solution $f\in\mathcal{O}(\bar\Omega)$, by Martineau's theorem (cf. [13]), and this solution satisfies (3.1). Let $z\in\Omega$ and let $\alpha>0$ be such that $z\in(1-\alpha)\Omega$. Applying Proposition 2.3 to the holomorphic function $\zeta\mapsto e^{\langle\zeta,z\rangle}$, with $p=2$, $G(\lambda_1,\lambda_2)=e^{\lambda_1}(1+\lambda_2)$, $Q^1(\sigma,\zeta)=2(1-\alpha)\partial\psi/\partial\sigma$ and $Q^2(\sigma,\zeta)=\chi_\varepsilon(\sigma)(p_1(\sigma,\zeta),\dots,p_n(\sigma,\zeta))/P(\sigma)$, and letting $\varepsilon\to 0$ yields

$$\begin{aligned}e^{\langle\zeta,z\rangle}&=P(\zeta)R\cdot(e^{\langle\sigma,z\rangle+\langle 2(1-\alpha)\partial\psi/\partial\sigma,\zeta-\sigma\rangle}(1-\alpha)^n\omega_n(\sigma))\\&\quad+\bar\partial R\cdot(p(\sigma,\zeta)\wedge e^{\langle\sigma,z\rangle+\langle 2(1-\alpha)\partial\psi/\partial\sigma,\zeta-\sigma\rangle}(1-\alpha)^{n-1}\omega_{n-1}(\sigma)).\end{aligned}\tag{3.4}$$

Note that this is possible because of the convergence factor $e^{\langle \sigma, z-2(1-\alpha)\partial\psi/\partial\sigma\rangle}$. By substituting (3.4) into (3.1) and reversing the order of integration and current action, we get

$$f(z)=R(\sigma)\cdot\left(g\left(2(1-\alpha)\frac{\partial\psi}{\partial\sigma}\right)e^{\langle\sigma,z-2(1-\alpha)\partial\psi/\partial\sigma\rangle}(1-\alpha)^n\omega_n(\sigma)\right)$$
$$+\bar{\partial}R(\sigma)\cdot\left(p(\sigma,D)f\left(2(1-\alpha)\frac{\partial\psi}{\partial\sigma}\right)e^{\langle\sigma,z-2(1-\alpha)\partial\psi/\partial\sigma\rangle}(1-\alpha)^{n-1}\omega_{n-1}(\sigma)\right)$$

which follows from the identity

$$Q(D)f(z)=\frac{1}{(2\pi i)^n}\int_{\mathbf{C}^n}Q(\zeta)f\left(2\frac{\partial\psi}{\partial\zeta}\right)e^{\langle\zeta,z-2\partial\psi/\partial\zeta\rangle}\omega_n(\zeta),\quad z\in\Omega.$$

If we let $\alpha\to 0$, we obtain the theorem. □

Corollary 3.5. *If $g\in\mathrm{Exp}(\mathbf{C}^n)$ is a holomorphic function on $\mathbf{C}^n$ of exponential type (i.e. there exists some constants $A,B>0$ such that, for all $z\in\mathbf{C}^n$, we have $|g(z)|\le Ae^{B|z|}$) then there exists $f\in\mathrm{Exp}(\mathbf{C}^n)$ such that*

$$P(D)f=g \quad \textit{on } \mathbf{C}^n$$

and the following representation formula holds for every $z\in\mathbf{C}^n$:

$$f(z)=\frac{1}{(2\pi i)^n}R\cdot\left(g(\bar{\zeta})e^{\langle\zeta,z-\bar{\zeta}\rangle}\frac{(\bar{\partial}\partial|\zeta|^2)^n}{n!}\right)$$
$$+\frac{1}{(2\pi i)^n}\bar{\partial}R\cdot\left(p(\zeta,D)f(\bar{\zeta})e^{\langle\zeta,z-\bar{\zeta}\rangle}\frac{(\bar{\partial}\partial|\zeta|^2)^{n-1}}{(n-1)!}\right).$$

Proof. There exists a solution $f\in\mathrm{Exp}(\mathbf{C}^n)$ of $P(D)f=g$, see [21]. We follow the preceding proof line by line, but we replace $\psi(\zeta)$ by $\frac{1}{2}|\zeta|^2$ and $(1-\alpha)\psi(\sigma)$ by $\frac{1}{2}|\sigma|^2$. □

4. The homogeneous Cauchy problem

Let now $n\in\mathbf{N}$, $n\ge 2$, and let ω be a domain in $\mathbf{C}^{n-1}$ satisfying the above conditions. We write as before

$$2\frac{\partial\psi}{\partial\zeta'}=2\left(\frac{\partial\psi(\zeta')}{\partial\zeta_2},\dots,\frac{\partial\psi(\zeta')}{\partial\zeta_n}\right)\quad\text{and}\quad\omega_k'(\zeta')=\frac{(2\bar{\partial}_{\zeta'}\partial_{\zeta'}\psi(\zeta'))^k}{(2\pi i)^n k!},$$

where ψ is the function associated with ω, i.e. ψ is the regularization of $\varphi(\zeta')=\sup_{z'\in\omega}\operatorname{Re}\langle\zeta',z'\rangle$ defined in Proposition 3.2.

Let P be a polynomial of the form (2.1), and let $h_0,\dots,h_{m-1}\in\mathcal{O}(\bar\omega)$; this means that $h_0,\dots,h_{m-1}$ are holomorphic functions defined in an open neighbourhood of $\bar\omega$ in $\mathbf{C}^{n-1}$. We want to find a holomorphic function f defined on an open domain $\widetilde\omega\subset\mathbf{C}^n$ containing ω and such that

$$(4.1)\qquad \begin{cases} P\left(\dfrac{\partial}{\partial z}\right)f(z)=0 & \text{on } \widetilde\omega,\\ f(0,z')=h_0(z'), & z'\in\omega,\\ \vdots & \\ \dfrac{\partial^{m-1}f}{\partial z_1^{m-1}}(0,z')=h_{m-1}(z'), & z'\in\omega. \end{cases}$$

Definition 4.1. Let ω be a bounded smooth strictly convex domain in $\mathbf{C}^{n-1}$. We define $\widetilde\omega$ to be the set of $z\in\mathbf{C}^n$ such that every real characteristic hyperplane passing through z meets ω.

We now obtain the main theorem of the paper.

Theorem 4.2. *Let n and ω satisfy the above conditions. Let $P:\mathbf{C}^n\to\mathbf{C}$ be a polynomial of degree m such that the complex hyperplane $\{z\in\mathbf{C}^n:z_1=0\}$ is non-characteristic for $P(D)$. Let $h_0,h_1,\dots,h_{m-1}\in\mathcal{O}(\bar\omega)$. Then there exists $f\in\mathcal{O}(\widetilde\omega)$ satisfying the holomorphic Cauchy problem* (4.1). *Moreover, for every $z\in\widetilde\Omega$, we have the representation formula*

$$(4.2)\qquad f(z)=\bar\partial R\cdot\left(\sum_{k=0}^{m-1}p_{1,k}(\zeta)h_k\left(2\frac{\partial\psi}{\partial\zeta'}\right)e^{\langle\zeta_1,z_1\rangle+\langle\zeta',z'-2\partial\psi/\partial\zeta'\rangle}\omega'_{n-1}(\zeta')\wedge d\zeta_1\right),$$

where $p_1(\zeta,z_1)=\sum_{k=0}^{m-1}p_{1,k}(\zeta)z_1^k$.

Proof. Let us consider the function f defined by (4.2). We will show first that, for z_1 close to 0, this formula makes sense. To do this, it is sufficient to prove that the form defined in the brackets makes sense as a form defined on $\mathbf{P}^n$ for z_1 close to 0. Let us fix z' in ω. The support of the residue current $\bar\partial R$ is included in $P^{-1}(0)$. So, by Lemma 2.1, we see that (4.2) holds if we replace the form in the brackets by the same form multiplied by the smooth function $\theta(\zeta)=\chi(|\zeta_1|^2-A-B|\zeta'|^2)$, where $\chi(x)=1$, if $x\le-1$, and $\chi(x)=0$, if $x\ge0$. An elementary estimate as above gives us

$$\operatorname{Re}\left\langle\zeta',z'-2\frac{\partial\psi}{\partial\zeta'}\right\rangle\le\varepsilon(\varrho(z')-1)|\zeta'|+C,$$

and so

$$|\theta(\zeta)e^{\langle\zeta_1,z_1\rangle+\langle\zeta',z'-2\partial\psi/\partial\zeta'\rangle}|\le\theta(\zeta)e^{A'|\zeta'||z_1|+(\varrho(z')-1)\varepsilon|\zeta'|+C}.$$

For $|z_1|$ sufficiently small, we obtain, with $\alpha,\beta>0$

$$|\theta(\zeta)e^{\langle\zeta_1,z_1\rangle+\langle\zeta',z'-2\partial\psi/\partial\zeta'\rangle}|\le\theta(\zeta)e^{-\alpha|\zeta'|+C}\le e^{-\beta|\zeta|+C'}$$

and since the different canonical coordinate systems on $\mathbf{P}^n$ are connected by rational transformations, the formula (4.2) makes sense for $z'\in\omega$ and z_1 sufficiently small.

Consider the function

$$f_k(z)=\bar{\partial}R\cdot\left(p_{1,k}(\zeta)h_k\left(2\frac{\partial\psi}{\partial\zeta'}\right)e^{\langle\zeta_1,z_1\rangle+\langle\zeta',z'-2\partial\psi/\partial\zeta'\rangle}\omega'_{n-1}(\zeta')\wedge d\zeta_1\right)$$

which is, of course, an evident solution of $P(\partial/\partial z)f=0$ on $\widetilde{\omega}$. We shall prove that

$$\frac{\partial f_k}{\partial z_1^l}(0,z')=\begin{cases}0, & \text{if } l\neq k,\\ h_k(z'), & \text{if } l=k.\end{cases}$$

Without loss of generality, we may assume that we have $b_m(z')=1$ and $\deg b_j(z')\le m-j$, so that we have $p_{1,k}(\zeta)=\sum_{j=k+1}^{m}\zeta_1^{j-1-k}b_j(\zeta')$. We now use the following lemma.

Lemma 4.3. *Under the above conditions, if $\Phi(\zeta')$ is an $(n-1,n-1)$-form depending on ζ' and rapidly decreasing on $\mathbf{C}^{n-1}$, then*

$$\bar{\partial}\left[\frac{1}{P(\zeta)}\right]\cdot(\zeta_1^l\,d\zeta_1\wedge\Phi(\zeta'))=\begin{cases}0, & \text{if } l<m-1,\\ 2\pi i\displaystyle\int_{\mathbf{C}^{n-1}}\Phi(\zeta'), & \text{if } l=m-1.\end{cases}$$

Proof. We have

$$\begin{aligned}\bar{\partial}\left[\frac{1}{P(\zeta)}\right]\cdot(\zeta_1^l\,d\zeta_1\wedge\Phi(\zeta'))&=\lim_{\varepsilon\to0}\int_{\mathbf{C}^n}\frac{\zeta_1^l\bar{\partial}\chi_\varepsilon(\zeta)\wedge d\zeta_1\wedge\Phi(\zeta')}{P(\zeta)}\\&=\lim_{\varepsilon\to0}\int_{\mathbf{C}^{n-1}}\int_{\mathbf{C}}\frac{\zeta_1^l\bar{\partial}_{\zeta_1}\chi_\varepsilon(\zeta_1,\zeta')\wedge d\zeta_1}{P(\zeta)}\Phi(\zeta').\end{aligned}$$

By Stokes' theorem, we have

$$\int_{\mathbf{C}}\frac{\zeta_1^l\bar{\partial}_{\zeta_1}\chi_\varepsilon(\zeta_1,\zeta')\wedge d\zeta_1}{P(\zeta)}=\lim_{R\to\infty}\int_{|\zeta_1|=R}\frac{\zeta_1^l\,d\zeta_1}{P(\zeta_1,\zeta')}=\begin{cases}0, & \text{if } l<m-1,\\ 2\pi i, & \text{if } l=m-1.\end{cases}\quad\square$$

And so, by this lemma, if $l<k$, we obtain

$$\frac{\partial f_k}{\partial z_1^l}(0,z')=\bar{\partial}R\cdot\left(p_{1,k}(\zeta)\,\zeta_1^l\,h_k\left(2\frac{\partial\psi}{\partial\zeta'}\right)e^{\langle\zeta',z'-2\partial\psi/\partial\zeta'\rangle}\omega'_{n-1}(\zeta')\wedge d\zeta_1\right)=0.$$

If $l=k$, we obtain

$$\begin{aligned}\frac{\partial f_k}{\partial z_1^l}(0,z')&=\bar{\partial}R\cdot\left(p_{1,k}(\zeta)\zeta_1^k h_k\left(2\frac{\partial\psi}{\partial\zeta'}\right)e^{\langle\zeta',z'-2\partial\psi/\partial\zeta'\rangle}\omega'_{n-1}(\zeta')\wedge d\zeta_1\right)\\&=2\pi i\int_{\mathbf{C}^{n-1}}h_k\left(2\frac{\partial\psi}{\partial\zeta'}\right)e^{\langle\zeta',z'-2\partial\psi/\partial\zeta'\rangle}\omega'_{n-1}(\zeta')=h_k(z').\end{aligned}$$

If $m>l>k$ then

$$\zeta_1^l p_1(k,\zeta)=\sum_{j=k+1}^{m}\zeta_1^{j-1-k+l}b_j(\zeta')=\zeta_1^{l-k-1}P(\zeta)-\sum_{j=0}^{k}\zeta_1^{l-k-1+j}b_j(\zeta').$$

The degree of the polynomial $\sum_{j=0}^k\zeta_1^{l-k-1+j}b_j(\zeta')$ in ζ_1 is less than $l-1\le m-2$ and since the current $\bar{\partial}R\times P(\zeta)$ is the current 0, we obtain

$$\frac{\partial f_k}{\partial z_1^l}(0,z')=0.$$

We show now that (4.2) makes sense if $z\in\widetilde{\omega}$. Observe first that, if a real hyperplane H intersects the domain ω, then every real hyperplane H' sufficiently close to H meets the domain ω. We fix z in $\widetilde{\omega}$; then there exists $\alpha\in(0,1)$ such that $z/(1-\alpha)\in\widetilde{\omega}$. If a real hyperplane $\{\sigma:\operatorname{Re}\langle\zeta,\sigma-z\rangle=0\}$ intersects the domain ω in a point $\sigma(\zeta)$, then the real hyperplane $\{\sigma:\operatorname{Re}\langle\zeta,\sigma-z/(1-\alpha)\rangle=0\}$ meets the domain ω in $\sigma(\zeta)/(1-\alpha)$, and so the real hyperplane $\{\sigma:\operatorname{Re}\langle\zeta,\sigma-z\rangle=0\}$ intersects the domain $(1-\alpha)\omega$. In particular, for every $\zeta\in\mathbf{C}^n$, $P_m(\zeta)=0$ implies that $\{\sigma:\operatorname{Re}\langle\zeta,\sigma-z\rangle=0\}$ meets $(1-\alpha)\omega$. We remark now that there exists a positive constant A such that

$$P(\zeta)=0\quad\Longrightarrow\quad\left|P_m\left(\frac{\zeta}{|\zeta|}\right)\right|\le\frac{A}{|\zeta|}$$

and by the property

$$|P(z)|\le1\quad\Longrightarrow\quad d(z,P_m^{-1}(0))\text{ is bounded}$$

(cf. [2], [15], [19]), there exists a positive constant B such that

$$P(\zeta)=0\quad\Longrightarrow\quad d\left(\frac{\zeta}{|\zeta|},P_m^{-1}(0)\right)\le\frac{B}{|\zeta|^{1/m}}.$$

This shows that there exists a constant C such that for every $\zeta\in P^{-1}(0)$, $|\zeta|\geq C$ implies that the real hyperplane $\{\sigma:\operatorname{Re}\langle\zeta,\sigma-z\rangle=0\}$ meets the domain $\left(1-\frac{1}{2}\alpha\right)\omega$. In particular, if $\zeta\in P^{-1}(0)$ with $|\zeta|\geq C$, and if we take a point

$$\sigma'(\zeta)\in\{\sigma:\operatorname{Re}\langle\zeta,\sigma-z\rangle=0\}\cap\left(1-\tfrac{1}{2}\alpha\right)\omega,$$

then we obtain

$$\begin{aligned}|\theta(\zeta)e^{\langle\zeta_1,z_1\rangle+\langle\zeta',z'-2\partial\psi/\partial\zeta'\rangle}| &= |\theta(\zeta)e^{\langle\zeta',\sigma'(\zeta)-2\partial\psi/\partial\zeta'\rangle}|\\ &\leq\theta(\zeta)e^{-\alpha\varphi(\zeta')/2+D}\leq e^{-\beta|\zeta|+E},\end{aligned}$$

with $\beta>0$, and thus the expression (4.2) makes sense for $z\in\widetilde{\omega}$, which proves the theorem. □

Remark 4.4. Kiselman has proved in [15] that, if f is a solution of $P(\partial/\partial z)f=0$ in an open set $U\subset\mathbf{C}^n$, then f has a holomorphic extension in $\Gamma(U)$, which is the set of $z\in\mathbf{C}^n$ for which every complex characteristic hyperplane passing through z meets U. In particular, due to the Cauchy–Kowalewsky theorem, there exists a small neighbourhood U of ω on which the solution f of the homogeneous Cauchy problem (4.1) is defined. Hence, if $\Gamma(\omega)$ is the set of $z\in\mathbf{C}^n$ for which every characteristic complex hyperplane passing through z meets ω, then clearly $\Gamma(\omega)\subset\widetilde{\omega}$.

We will prove that, in fact, the preceding inclusion is an equality. For this, suppose the contrary: there exists thus a $z\in\widetilde{\omega}$ and a complex characteristic hyperplane $H_{\mathbf{C}}^{z,\zeta}=\{\sigma:\langle\zeta,z-\sigma\rangle=0\}$ which does not intersect ω with $P_m(\zeta)=0$. Due to the homogeneity of P_m and the fact that $z\in\widetilde{\omega}$, we know that, for every $\alpha\in\mathbf{C}$, the real hyperplane $H_{\mathbf{R}}^{z,\alpha\zeta}=\{\sigma:\operatorname{Re}\langle\alpha\zeta,z-\sigma\rangle=0\}$ intersects ω. But $H_{\mathbf{C}}^{z,\zeta}=H_{\mathbf{R}}^{z,\zeta}\cap H_{\mathbf{R}}^{z,i\zeta}$. Moreover, $(H'_{\mathbf{R}})^{z,\zeta}=H_{\mathbf{R}}^{z,\zeta}\cap\mathbf{C}^{n-1}$ and $(H'_{\mathbf{R}})^{z,i\zeta}=H_{\mathbf{R}}^{z,i\zeta}\cap\mathbf{C}^{n-1}$ are real orthogonal hyperplanes in $\mathbf{C}^{n-1}$, so $(H'_{\mathbf{R}})^{z,\zeta}\cap(H'_{\mathbf{R}})^{z,i\zeta}=(H'_{\mathbf{C}})^{z,\zeta}=H_{\mathbf{C}}^{z,\zeta}\cap\mathbf{C}^{n-1}$. If this complex hyperplane does not intersect ω in $\mathbf{C}^{n-1}$, then by the convexity of ω and the Hahn–Banach theorem, there exists a real hyperplane H'_0 in $\mathbf{C}^{n-1}$ which contains $(H'_{\mathbf{C}})^{z,\zeta}$ and which does not intersect ω in $\mathbf{C}^{n-1}$, and moreover, there exists $\alpha\in\mathbf{C}$ such that $H'_0=(H'_{\mathbf{R}})^{z,\alpha\zeta}$, which is a contradiction.

Remark 4.5. By a different argument, Schiltz has proved in [26] that, if a real characteristic hyperplane H is tangent to $\partial\omega$ in $\mathbf{C}^{n-1}$, then it separates $\mathbf{C}^n$ into two half-spaces, of which one, H_+, contains ω and that $\widetilde{\omega}$ is the intersection of all these half-spaces. One can verify that this definition coincides with our definition: for this, if $\widetilde{\omega}^s$ is the set described by Schiltz, and if $z\notin\widetilde{\omega}^s$, then there exists a real characteristic hyperplane H in $\mathbf{C}^n$, tangent to $\partial\Omega$ such that $z\in\mathbf{C}^n\setminus H_+$. In particular, the real characteristic hyperplane passing through z and parallel to H does not intersect Ω, which implies that $z\notin\widetilde{\omega}$. In an analogous way, if $z\notin\widetilde{\omega}$, then there exists a real characteristic hyperplane containing z which does not intersect ω. We know (cf. [26]) that there exists two real characteristic hyperplanes parallel to H and tangent to $\partial\omega$ in $\mathbf{C}^{n-1}$; one of these hyperplanes, H', does separate ω and $H\cap\mathbf{C}^{n-1}$ in $\mathbf{C}^{n-1}$ so $H\subset\mathbf{C}^n\setminus H'_+$, which implies that $z\notin H_+$. And so, Schiltz characterization coincides with our characterization.

Example 4.6. Let us consider the following Cauchy problem. If ω is a smooth strictly convex domain in $\mathbf{C}$ then the solution f of the holomorphic Cauchy problem

$$\begin{cases} \dfrac{\partial f}{\partial z_1}-\dfrac{\partial f}{\partial z_2}=0, \\ f(0,z_2)=h(z_2), \qquad z_2\in\omega, \end{cases}$$

is defined on $\widetilde{\omega}=\{z\in\mathbf{C}^2 : z_1+z_2\in\omega\}$ and is given by $f(z)=h(z_1+z_2)$.

Proof. Of course, we have $P(\zeta)=\zeta_1-\zeta_2$. In addition, the definition of $\widetilde{\omega}$ in this case is immediate and we obtain

$$\begin{aligned} f(z) &= \frac{1}{(2\pi i)^2}\bar{\partial}\left[\frac{1}{\zeta_1-\zeta_2}\right]\cdot\left(h\left(2\frac{\partial\psi}{\partial\zeta_2}(\zeta_2)\right)e^{\langle\zeta_1,z_1\rangle+\langle\zeta_2,z_2-2\partial\psi/\partial\zeta_2\rangle}(2\bar{\partial}\partial\psi(\zeta_2))\wedge d\zeta_1\right) \\ &= \frac{1}{2\pi i}\int_{\mathbf{C}} h\left(2\frac{\partial\psi}{\partial\zeta_2}(\zeta_2)\right)e^{\langle\zeta_2,z_1\rangle+\langle\zeta_2,z_2-2\partial\psi/\partial\zeta_2\rangle}(2\bar{\partial}\partial\psi(\zeta_2)) \\ &= h(z_1+z_2), \end{aligned}$$

since $1/(2\pi i)\bar{\partial}[1/(\zeta_1-\zeta_2)]\wedge(d\zeta_1-d\zeta_2)$ is the current of integration over $\zeta_1=\zeta_2$. □

Example 4.7. Let us consider the following Cauchy problem: If ω is a smooth strictly convex domain in $\mathbf{C}$ then the solution f of the holomorphic Cauchy problem

$$\begin{cases} \dfrac{\partial f}{\partial z_1}-\dfrac{\partial f}{\partial z_2}-f(z)=0, \\ f(0,z_2)=h(z_2), \qquad z_2\in\omega, \end{cases}$$

is defined on $\widetilde{\omega}=\{z\in\mathbf{C}^2 : z_1+z_2\in\omega\}$ and is given by $f(z)=h(z_1+z_2)e^{z_1}$.

Proof. Of course, we have $P(\zeta)=\zeta_1-\zeta_2-1$. In addition, the definition of $\widetilde{\omega}$ in this case is immediate and we obtain

$$\begin{aligned} f(z) &= \frac{1}{(2\pi i)^2}\bar{\partial}\left[\frac{1}{\zeta_1-\zeta_2-1}\right]\cdot\left(h\left(2\frac{\partial\psi}{\partial\zeta_2}(\zeta_2)\right)e^{\langle\zeta_1,z_1\rangle+\langle\zeta_2,z_2-2\partial\psi/\partial\zeta_2\rangle}(2\bar{\partial}\partial\psi(\zeta_2))\wedge d\zeta_1\right) \\ &= \frac{1}{2\pi i}\int_{\mathbf{C}} h\left(2\frac{\partial\psi}{\partial\zeta_2}(\zeta_2)\right)e^{\langle\zeta_2+1,z_1\rangle+\langle\zeta_2,z_2-2\partial\psi/\partial\zeta_2\rangle}(2\bar{\partial}\partial\psi(\zeta_2)) \\ &= e^{z_1}h(z_1+z_2), \end{aligned}$$

since $(1/2\pi i)\bar{\partial}[1/(\zeta_1-\zeta_2-1)]\wedge(d\zeta_1-d\zeta_2)$ is the current of integration over the complex line $\zeta_1=\zeta_2+1$. □

5. The non-homogeneous Cauchy problem

We now consider the following situation. Let Ω be a bounded smooth strictly convex domain in $\mathbf{C}^n$. Let P be a polynomial over $\mathbf{C}^n$ of degree m such that the complex hyperplane $\{z\in\mathbf{C}^n:z_1=0\}$ is non-characteristic for the operator $P(D)$. Let $g\in\mathcal{O}(\bar{\Omega})$ and let $h_0,\dots,h_{m-1}\in\mathcal{O}(\bar{\omega})$ where $\omega=\Omega\cap\{z\in\mathbf{C}^n:z_1=0\}$. We want to solve the holomorphic non-homogeneous Cauchy problem

$$\tag{5.1} \begin{cases} P\left(\dfrac{\partial}{\partial z}\right)f(z)=g(z), & z\in\Omega,\\ f(0,z')=h_0(z'), & z'\in\omega,\\ \vdots & \\ \dfrac{\partial^{m-1}f}{\partial z_1^{m-1}}(0,z')=h_{m-1}(z'), & z'\in\omega. \end{cases}$$

The idea we use for solving this problem is to apply Theorem 4.1 combined with the *Duhamel's principle* (cf. [14]).

Proposition 5.1. *Let f be a solution of the holomorphic Cauchy problem* (5.1) *with $h_0,\dots,h_{m-1}=0$. Then we can write*

$$f(z)=\int_0^{z_1}F(z_0,z_1-z_0,z')\,dz_0,$$

where $F(z_0,z_1,z')$ is, for each z_0, a solution of $P(D)F=0$ with initial data

$$P_m(1,0,\dots,0)\frac{\partial^k F}{\partial z_1^k}(z_0,0,z')=\begin{cases}0, & \text{if } k\le m-2,\\ g(z_0,z'), & \text{if } k=m-1.\end{cases}$$

Proof. Let f be defined as before. If $k\ge 1$ then

$$\frac{\partial^k f}{\partial z_1^k}(z)=\sum_{j=0}^{k-1}\frac{\partial^k F}{\partial z_1^j}(z_1,0,z')+\int_0^{z_1}\frac{\partial^k F}{\partial z_1^k}(z_0,z_1-z_0,z')\,dz_0$$

so that

$$\frac{\partial^k f}{\partial z_1^k}(0,z')=0,\quad k=0,\dots,m-1,$$

and

$$P(D)f(z)=P_m(1,0,\dots,0)\frac{\partial^{m-1}F}{\partial z_1^{m-1}}(z_1,0,z')=g(z). \qquad \square$$

A direct application of this proposition gives us the following theorem.

Theorem 5.2. *Let f be a solution of the holomorphic Cauchy problem* (5.1). *Then f is defined on $\widetilde{\Omega}=\Omega\cap\widetilde{\omega}$, and we have the representation formula, for every $z\in\widetilde{\Omega}$,*

$$f(z)=\bar{\partial}R\cdot\left(\hat{g}\left(z_1,\zeta_1,2\frac{\partial\psi}{\partial\zeta'}\right)e^{\langle\zeta_1,z_1\rangle+\langle\zeta',z'-2\partial\psi/\partial\zeta'\rangle}\omega'_{n-1}(\zeta')\wedge d\zeta_1\right)$$

$$+\bar{\partial}R\cdot\left(\sum_{k=0}^{m-1}p_{1,k}(\zeta)h_k\left(2\frac{\partial\psi}{\partial\zeta'}\right)e^{\langle\zeta_1,z_1\rangle+\langle\zeta',z'-2\partial\psi/\partial\zeta'\rangle}\omega'_{n-1}(\zeta')\wedge d\zeta_1\right)$$

with

$$\hat{g}\left(z_1,\zeta_1,2\frac{\partial\psi}{\partial\zeta'}\right)=\int_0^{z_1}g\left(z_0,2\frac{\partial\psi}{\partial\zeta'}\right)e^{-\langle\zeta_1,z_0\rangle}\,dz_0.$$

Remark 5.3. A simpler approach to this problem is the following. We know that the function f_0 defined by

$$f_0(z)=R\cdot\left(g\left(2\frac{\partial\psi}{\partial\zeta}\right)e^{\langle\zeta,z-2\partial\psi/\partial\zeta\rangle}\omega_n(\zeta)\right)$$

is a particular solution of the equation $P(D)f_0=g$. It remains to write $f=f_0+f'$ with f' being the solution of the Cauchy problem

$$\begin{cases} P\left(\dfrac{\partial}{\partial z}\right)f'=0 & \text{in } \Omega,\\ f'(0,z')=h'_0(z'), & \\ \vdots & \\ \dfrac{\partial^{m-1}f'}{\partial z_1^{m-1}}(0,z')=h'_{m-1}(z'), & \end{cases}$$

with

$$h'_j(z')=h_j(z')-\frac{\partial^j f_0}{\partial z_0^j}(0,z').$$

Remark 5.4. In the case when $\Omega=\mathbf{C}^n$ and the functions g, $h_0,\dots,h_{m-1}$ are holomorphic of exponential type, we can see that the solution f of the holomorphic Cauchy problem (5.1) is defined on $\mathbf{C}^n$ and of exponential type. For this, it is sufficient, by Duhamel's principle, to prove that the solution of a homogeneous Cauchy problem with data of exponential type is also of exponential type. Let us consider $z\in\mathbf{C}^n$; there exists $R>0$ such that, if $\mathbf{B}_{n-1}$ is the unit ball in $\mathbf{C}^{n-1}$, then

$z\in\widetilde{R\mathbf{B}}_{n-1}$. One has to estimate the solution f with the help of formula (4.2); we obtain, by Lemma 2.1,

$$|f(z)|\le\sum_{k=1}^{q}A_k|z|^{\alpha_k}\sup_{|\zeta_1|\le A+B|\zeta'|}|\zeta|^{\beta_k}H_k\left(R\frac{\bar{\zeta}'}{|\zeta'|}\right)|e^{\langle\zeta,z\rangle-R|\zeta'|}|,$$

where A_k, α_k and β_k are positive constants independent of z and where H_k are holomorphic functions of exponential type such that $|H_k(z')|\le B_k e^{C_k|z'|}$ for all $z'\in\mathbf{C}^{n-1}$, where B_k and C_k are positive constants. And so

$$|f(z)|\le\sum_{k=1}^{q}A_k|z|^{\alpha_k}B_k e^{C_kR}\sup_{|\zeta_1|\le A+B|\zeta'|}|\zeta|^{\beta_k}e^{|\zeta||z|-R|\zeta'|}.$$

If $|\zeta_1|\le A+B|\zeta'|$, we obtain the existence of $\gamma>0$ depending only on A and B, such that $\gamma|\zeta|\le|\zeta'|\le|\zeta|$. And so, if we take $R=2|z|/\gamma$, we deduce that (4.2) has a meaning, and

$$\begin{aligned}|f(z)|&\le\sum_{k=1}^{q}A_k|z|^{\alpha_k}B_k e^{2C_k|z|/\gamma}\sup_{x\in\mathbf{R}^+}x^{\beta_k}e^{-|z|x/\gamma}\\&=\sum_{k=1}^{q}A_k|z|^{\alpha_k}B_k e^{2C_k|z|/\gamma}\left(\frac{\beta_k\gamma}{|z|}\right)^{\beta_k}e^{-\beta_k},\end{aligned}$$

which implies that $f\in\operatorname{Exp}(\mathbf{C}^n)$. And so, we can write

$$f(z)=\int_{\mathbf{C}^n}f(w)e^{\langle\bar{w},z-w\rangle}\omega_n(w),$$

where $\omega_n(w)=(2\pi i)^{-n}(\bar{\partial}\partial|w|^2)^n/n!$. By inverting w and $\bar{w}$, we obtain

$$f(z)=\int_{\mathbf{C}^n}f(\bar{w})e^{\langle w,z-\bar{w}\rangle}\omega_n(w);\tag{5.2}$$

and so

(5.3)
$$P\left(\frac{\partial}{\partial z}\right)f(z)=g(z)=\int_{\mathbf{C}^n}P(w)f(\bar{w})e^{\langle w,z-\bar{w}\rangle}\omega_n(w)=\int_{\mathbf{C}^n}g(\bar{w})e^{\langle w,z-\bar{w}\rangle}\omega_n(w).$$

But, we have the following division formula

$$\begin{aligned}e^{\langle w,z\rangle}&=P(w)\left[\frac{1}{P(\zeta)}\right]\cdot(e^{\langle\zeta,z\rangle+\langle\psi^0(\zeta),w-\zeta\rangle}\theta_n(\zeta))\\&\quad+\bar{\partial}\left[\frac{1}{P(\zeta)}\right]\cdot(p(\zeta,w)\wedge e^{\langle\zeta,z\rangle+\langle\psi^0(\zeta),w-\zeta\rangle}\theta_{n-1}(\zeta)),\end{aligned}\tag{5.4}$$

where $\psi^0\colon \mathbf{C}^n \to \mathbf{C}^n$, $\Psi^0(\zeta) := \sum_{k=1}^n \psi_k^0(\zeta)\, d\zeta_k$ and $\theta_k(\zeta) := (2\pi i)^{-n}(\bar\partial \Psi^0(\zeta))^k/k!$, and where ψ^0 is chosen so that (5.4) is well defined and such that the image of $P^{-1}(0)$ by ψ^0 is included in the hyperplane $\{z\in\mathbf{C}^n : z_1=0\}$.

Let χ be as in the preceding section and let $\chi_1(\zeta) := \chi(|P(\zeta)|)$.

By Lemma 2.1, one can take

$$\psi^0(\zeta) := (\chi_1(\zeta)\bar\zeta_1, \bar\zeta_2, \dots, \bar\zeta_n).$$

And so, by inserting (5.4) into (5.2), by exchanging residue currents and integration, and by using (5.3), we have

$$\begin{aligned} f(z) = {} & \left[\frac{1}{P(\zeta)}\right] \cdot (g(\psi^0(\zeta)) e^{\langle \zeta, z-\psi^0(\zeta)\rangle} \theta_n(\zeta)) \\ & + \bar\partial \left[\frac{1}{P(\zeta)}\right] \cdot \left(\sum_{k=0}^{m-1} p_k(\zeta) h_k(\psi^0(\zeta)) e^{\langle \zeta, z-\psi^0(\zeta)\rangle} \theta_{n-1}(\zeta) \right). \end{aligned}$$

In particular, for $g=0$, we obtain an answer to a question of Passare [23], who has proved this formula in the particular case

$$P(z) = (z_1 - L_1(z'))^{m_1} \dots (z_1 - L_k(z'))^{m_k},$$

where each L_j is linear, and who has conjectured that this formula is true for every polynomial containing αz_1^m with $\alpha\in\mathbf{C}^*$ and $m=\deg P$.

6. The Goursat problem

In this section, we generalize the preceding results to the case of systems. Let $P\colon \mathbf{C}^n \to \mathbf{C}^N$ be a complete intersection polynomial mapping (that is $\dim_{\mathbf{C}} P^{-1}(0) = n-N$), hence in particular $N \le n$.

Definition 6.1. We say that a complex plane of complex codimension N is *non-characteristic* for the differential operator $P(D)$ if we cannot write $H = H_1 \cap \dots \cap H_N$, where each H_j is a complex characteristic hyperplane for the differential operator $P_j(D)$. In other terms, we say that a complex plane H of complex codimension N is *characteristic* for the differential operator $P(D)$ if we can write $H = H_1 \cap \dots \cap H_N$, where each H_j is a complex characteristic hyperplane for the differential operator $P_j(D)$. If now H is a real plane of real codimension N, we say that H is *characteristic* if we can write $H = H_1 \cap \dots \cap H_N$, where each H_j is a real characteristic hyperplane for the differential operator $P_j(D)$.

If each P_j is a polynomial on C^n of degree m_j, we first want to solve the homogeneous Goursat problem, that is to find a holomorphic function f such that

$$(6.1) \qquad \begin{cases} P_1\left(\frac{\partial}{\partial z}\right) f(z) = 0, \\ \vdots \\ P_N\left(\frac{\partial}{\partial z}\right) f(z) = 0 \end{cases}$$

and such that

$$(6.1') \qquad \frac{\partial^{j_1+\ldots+j_N} f}{\partial z_1^{j_1} \ldots \partial z_N^{j_N}}(0, \ldots, 0, z^{(N)}) = h_{(j_1,\ldots,j_N)}(z^{(N)}), \quad z^{(N)} \in \omega,$$

where, for every $k \in \{1, \ldots, N\}$, we have $0 \le j_k \le m_k - 1$, and where the $h_{(j_1,\ldots,j_N)}$ are holomorphic functions in a neighbourhood of $\bar{\omega}$ which is a smooth strictly convex domain in $\mathbf{C}^{n-N}$, with the notation $\mathbf{C}^n \ni z = (z_1, \ldots, z_N, z^{(N)})$, where $z^{(N)} \in \mathbf{C}^{n-N}$, and where the complex plane $\{z \in \mathbf{C}^n : z_1 = \ldots = z_N = 0\}$ is non-characteristic for the differential operator; moreover, we will suppose that, for every $k \in \{1, \ldots, N\}$, the complex hyperplane is non-characteristic for the differential operator $P_k(D)$.

We associate with P, as in [7] and [25], a Weil–Oka–Hefer matrix $(p_{(j,k)})$ consisting of polynomials $p_{(j,k)}(\zeta, z)$ which satisfy

$$P_j(z) - P_j(\zeta) = \sum_{k=1}^{n} p_{(j,k)}(\zeta, z)(z_k - \zeta_k), \quad j = 1, \ldots, N.$$

More precisely, we choose them such that we have

$$p_{(1,1)}(\zeta, z) = \frac{P_1(\zeta) - P_1(z_1, \zeta')}{\zeta_1 - z_1},$$

$$\vdots$$

$$p_{(k,k)}(\zeta, z) = \frac{P_k(\zeta) - P_k(\zeta_1, \ldots, \zeta_{k-1}, z_k, \zeta^{(k)})}{\zeta_k - z_k},$$

$$\vdots$$

$$p_{(N,N)}(\zeta, z) = \frac{P_N(\zeta) - P_N(\zeta_1, \ldots, \zeta_{N-1}, z_N)}{\zeta_N - z_N}.$$

We write $p_{(k,k)}(\zeta, z) = \sum_{j=0}^{m_k - 1} p_{(k,k,j)} z_k^j$. We denote by ψ the function associated with ω, depending only on $\zeta^{(N)} \in \mathbf{C}^{n-N}$. We next introduce some more notation.

If $I\subset\mathbf{N}_n$, we denote by $|I|$ the cardinality of I and let $\mu(I)=(-1)^{\max(0,|I|-1)}$. If $I\subset\mathbf{N}_n$ and $J\subset\mathbf{N}_n$, then $P_{I,J}$ will denote the form defined by

$$P_{I,J}(\zeta,z):=\bigwedge_{k=1}^{q} g_{i_k}(\zeta,z)\times\prod_{l=1}^{r}P_{j_l}(z),$$

where $I=\{i_1,\dots,i_q\}$ and $J=\{j_1,\dots,j_r\}$, and $i_1<\dots<i_q$.

Let χ be a smooth function such that $\chi(x)=0$, if $x\le c_1$, and $\chi(x)=1$, if $x\ge c_2$, where $0<c_1<c_2<1$. The $(0,q)$-current

$$R_I=\bar\partial\left[\frac{1}{P_I}\right]\left[\frac{1}{P_J}\right]:=\bar\partial\left[\frac{1}{P_{i_1}}\right]\wedge\dots\wedge\bar\partial\left[\frac{1}{P_{i_q}}\right]\left[\frac{1}{P_{j_1}}\right]\cdots\left[\frac{1}{P_{j_{N-q}}}\right],$$

where

$$I\cup J=\mathbf{N}_N,\quad I\cap J=\emptyset,\quad I:=\{i_1,\dots,i_q\},\quad J:=\{j_1,\dots,j_{N-q}\},\quad i_1<\dots<i_q,$$

is by definition the weak limit

$$\lim_{\varepsilon\to 0}\frac{\bar\partial\chi^{\varepsilon}_{i_1}(\zeta)}{P_{i_1}(\zeta)}\wedge\dots\wedge\frac{\bar\partial\chi^{\varepsilon}_{i_q}(\zeta)}{P_{i_q}(\zeta)}\frac{\chi^{\varepsilon}_{j_1}(\zeta)}{P_{j_1}(\zeta)}\cdots\frac{\chi^{\varepsilon}_{j_{N-q}}(\zeta)}{P_{j_{N-q}}(\zeta)},$$

where $\chi^{\varepsilon}_j(\zeta):=\chi(|P_j(\zeta)|/\varepsilon_j(\varepsilon))$ and where the ε_j are strictly positive functions such that

$$\lim_{\varepsilon\to 0}\varepsilon_j(\varepsilon)=0 \text{ and } \lim_{\varepsilon\to 0}\frac{\varepsilon_j(\varepsilon)}{\varepsilon^q_{j+1}(\varepsilon)}=0,\quad q\in\mathbf{N},$$

(cf. [24]). These limits depend on the choice of χ and ε, except for the current $R_{\mathbf{N}_n}$. If we want to obtain more canonical residue currents, we have to suppose, moreover, that, for every $I\subset\mathbf{N}_N$, the polynomial mapping $P_I=(P_{i_1},\dots,P_{i_q})$ is a complete intersection.

We generalize Theorem 4.1 in such a way.

Theorem 6.2. *Let us consider $\tilde\omega$, which is the set of $z\in\mathbf{C}^n$ such that every subspace of real codimension N, characteristic for the differential operator $P(D)$, passing through z meets the domain ω. Then, under the above conditions, there exists a solution $f\in\mathcal{O}(\tilde\omega)$ of (6.1) and (6.1′). Moreover, we have the following representation formula, for every $z\in\tilde\omega$,*

$$f(z)=\bar\partial\left[\frac{1}{P_1(\zeta)}\right]\wedge\dots\wedge\bar\partial\left[\frac{1}{P_N(\zeta)}\right]\cdot\Bigg(\sum_{k_1=0}^{m_1-1}\cdots\sum_{k_N=0}^{m_N-1}p_{(1,1,k_1)}(\zeta)\cdots p_{(N,N,k_N)}(\zeta)$$
$$\times h_{(k_1,\dots,k_N)}\left(2\frac{\partial\psi}{\partial\zeta^{(N)}}(\zeta^{(N)})\right)e^{\langle\zeta_1,z_1\rangle+\dots+\langle\zeta_N,z_N\rangle+\langle\zeta^{(N)},z^{(N)}-2\partial\psi/\partial\zeta^{(N)}\rangle}\Bigg)$$
$$\wedge\frac{(2\bar\partial_{\zeta^{(N)}}\partial_{\zeta^{(N)}}\psi(\zeta^{(N)}))^{n-N}}{(2\pi i)^n(n-N)!}\wedge d\zeta_N\wedge\dots\wedge d\zeta_1.$$

Proof. The proof is similar to the proof of Theorem 4.1. □

If we now want to solve the non-homogeneous Goursat problem, we need an explicit version of the fundamental principle, which is a direct generalization of Theorem 3.4, analogous to the main theorem of [25]. We have the following theorem.

Theorem 6.3. *Let $P: \mathbf{C}^n \to \mathbf{C}^N$ be a complete intersection polynomial mapping. Let us consider $g_1, \dots, g_N$, holomorphic functions in a neighbourhood of $\bar{\Omega}$ such that*

$$P_j(D) g_k = P_k(D) g_j.$$

Then there exists a holomorphic function f in a neighbourhood of $\bar{\Omega}$ that is a solution of the system of differential equations

$$P_j(D) = g_j, \quad j = 1, \dots, N.$$

Moreover, we can write

$$f(z) = S_1(g_1)(z) + \dots + S_N(g_N)(z) + T(f)(z), \quad z \in \Omega,$$

where

$$S_1(g_1)(z) = \frac{1}{(2\pi i)^n} \sum_{1 \notin I \subset \mathbf{N}_n} \mu(I) R_I \cdot \left(P_{I, \mathbf{N}_n \setminus I \setminus \{1\}}(\zeta, D) g_1 \left(2 \frac{\partial \psi}{\partial \zeta} \right) e^{\langle \zeta, z - 2\partial\psi/\partial\zeta \rangle} \frac{(2\bar{\partial}\partial\psi)^{n-|I|}}{(n-|I|)!} \right),$$

$$S_2(g_2)(z) = \frac{1}{(2\pi i)^n} \sum_{\substack{1 \in I \\ 2 \notin I \subset \mathbf{N}_n}} \mu(I) R_I \cdot \left(P_{I, \mathbf{N}_n \setminus I \setminus \{2\}}(\zeta, D) g_2 \left(2 \frac{\partial \psi}{\partial \zeta} \right) e^{\langle \zeta, z - 2\partial\psi/\partial\zeta \rangle} \frac{(2\bar{\partial}\partial\psi)^{n-|I|}}{(n-|I|)!} \right),$$

$$\vdots$$

$$S_{m-1}(g_{m-1})(z) = \frac{1}{(2\pi i)^n} \sum_{\substack{1, \dots, m-2 \in I \\ m-1 \notin I \subset \mathbf{N}_n}} \mu(I) R_I \cdot \left(P_{I, \mathbf{N}_n \setminus I \setminus \{m-1\}}(\zeta, D) g_{m-1} \left(2 \frac{\partial \psi}{\partial \zeta} \right) e^{\langle \zeta, z - 2\partial\psi/\partial\zeta \rangle} \frac{(2\bar{\partial}\partial\psi)^{n-|I|}}{(n-|I|)!} \right),$$

$$S_m(g_m)(z) = \frac{1}{(2\pi i)^n} \mu(\mathbf{N}_{m-1}) R_{\mathbf{N}_{m-1}} \cdot \left(P_{\mathbf{N}_{m-1}, \emptyset}(\zeta, D) g_m \left(2 \frac{\partial \psi}{\partial \zeta} \right) e^{\langle \zeta, z - 2\partial\psi/\partial\zeta \rangle} \frac{(2\bar{\partial}\partial\psi)^{n-m+1}}{(n-m+1)!} \right),$$

$$T(f)(z)=\frac{1}{(2\pi i)^n}\mu(\mathbf{N}_m)R_{\mathbf{N}_m}$$
$$\cdot\left(P_{\mathbf{N}_m,\emptyset}(\zeta,D)f\left(2\frac{\partial\psi}{\partial\zeta}\right)e^{\langle\zeta,z-2\partial\psi/\partial\zeta\rangle}\frac{(2\bar\partial\partial\psi)^{n-m}}{(n-m)!}\right).$$

Proof. The proof is analogous to the proof of Theorem 3.4. If the solution f exists, then it satisfies (3.1). We also apply Proposition 2.3 to the holomorphic function $\zeta\mapsto e^{\langle\zeta,z\rangle}$ with $p=N+1$ and the weights

$$Q^1(\sigma,\zeta)=2(1-\alpha)\frac{\partial\psi}{\partial\sigma},$$
$$Q^2(\sigma,\zeta)=\frac{\chi_1^\varepsilon(\zeta)(p_{(1,1)}(\sigma,\zeta),\dots,p_{(1,n)}(\sigma,\zeta))}{P_1(\sigma)},$$
$$\vdots$$
$$Q^{N+1}(\sigma,\zeta)=\frac{\chi_N^\varepsilon(\zeta)(p_{(N,1)}(\sigma,\zeta),\dots,p_{(N,n)}(\sigma,\zeta))}{P_N(\sigma)}.$$

After this we follow the proof of Theorem 3.4 line by line.

And so, we have just to prove that the solution f exists. For simplicity, we assume that $N=2$ (the proof is similar for larger N). Let g_1 and g_2 be holomorphic functions in a neighbourhood of $\bar\Omega$, such that $P_1(\partial/\partial z)g_2=P_2(\partial/\partial z)g_1$. We define

$$\begin{aligned}F(z)&=S_1(g_1)(z)+S_2(g_2)(z)\\
&=\left[\frac{1}{P_1(\zeta)}\right]\left[\frac{1}{P_2(\zeta)}\right]\left(P_2\left(\frac{\partial}{\partial z}\right)g_1\left(\frac{2\partial\psi}{\partial\zeta}\right)e^{\langle\zeta,z-2\partial\psi/\partial\zeta\rangle}\frac{(2\bar\partial\partial\psi)^n}{(2\pi i)^n n!}\right)\\
&\quad+\left[\frac{1}{P_1(\zeta)}\right]\bar\partial\left[\frac{1}{P_2(\zeta)}\right]\left(p_2\left(\zeta,\frac{\partial}{\partial z}\right)g_1\left(\frac{2\partial\psi}{\partial\zeta}\right)e^{\langle\zeta,z-2\partial\psi/\partial\zeta\rangle}\frac{(2\bar\partial\partial\psi)^{n-1}}{(2\pi i)^n(n-1)!}\right)\\
&\quad+\left[\frac{1}{P_2(\zeta)}\right]\bar\partial\left[\frac{1}{P_1(\zeta)}\right]\left(p_1\left(\zeta,\frac{\partial}{\partial z}\right)g_2\left(\frac{2\partial\psi}{\partial\zeta}\right)e^{\langle\zeta,z-2\partial\psi/\partial\zeta\rangle}\frac{(2\bar\partial\partial\psi)^{n-1}}{(2\pi i)^n(n-1)!}\right),\end{aligned}\tag{6.2}$$

F is then holomorphic in Ω and

$$\begin{aligned}P_1\left(\frac{\partial}{\partial z}\right)F(z)&=\left[\frac{1}{P_1(\zeta)}\right]\left[\frac{1}{P_2(\zeta)}\right]\left(P_1(\zeta)P_2\left(\frac{\partial}{\partial z}\right)g_1\left(\frac{2\partial\psi}{\partial\zeta}\right)e^{\langle\zeta,z-2\partial\psi/\partial\zeta\rangle}\frac{(2\bar\partial\partial\psi)^n}{(2\pi i)^n n!}\right)\\
&\quad+\left[\frac{1}{P_1(\zeta)}\right]\bar\partial\left[\frac{1}{P_2(\zeta)}\right]\left(P_1(\zeta)p_2\left(\zeta,\frac{\partial}{\partial z}\right)g_1\left(\frac{2\partial\psi}{\partial\zeta}\right)e^{\langle\zeta,z-2\partial\psi/\partial\zeta\rangle}\frac{(2\bar\partial\partial\psi)^{n-1}}{(2\pi i)^n(n-1)!}\right)\\
&\quad+\left[\frac{1}{P_2(\zeta)}\right]\bar\partial\left[\frac{1}{P_1(\zeta)}\right]\left(P_1(\zeta)p_1\left(\zeta,\frac{\partial}{\partial z}\right)g_2\left(\frac{2\partial\psi}{\partial\zeta}\right)e^{\langle\zeta,z-2\partial\psi/\partial\zeta\rangle}\frac{(2\bar\partial\partial\psi)^{n-1}}{(2\pi i)^n(n-1)!}\right).\end{aligned}$$

We use the identity $\bar{\partial}[1/P_1(\zeta)]P_1(\zeta)=0$ to obtain

$$\begin{aligned}P_1\Big(\frac{\partial}{\partial z}\Big)F(z)&=\Big[\frac{1}{P_2(\zeta)}\Big]\bigg(P_2\Big(\frac{\partial}{\partial z}\Big)g_1\Big(\frac{2\partial\psi}{\partial\zeta}\Big)e^{\langle\zeta,z-2\partial\psi/\partial\zeta\rangle}\frac{(2\bar{\partial}\partial\psi)^n}{(2\pi i)^n n!}\bigg)\\&\quad+\bar{\partial}\Big[\frac{1}{P_2(\zeta)}\Big]\bigg(p_2\Big(\zeta,\frac{\partial}{\partial z}\Big)g_1\Big(\frac{2\partial\psi}{\partial\zeta}\Big)e^{\langle\zeta,z-2\partial\psi/\partial\zeta\rangle}\frac{(2\bar{\partial}\partial\psi)^{n-1}}{(2\pi i)^n(n-1)!}\bigg)\\&=g_1(z).\end{aligned}$$

In a similar way, $P_2(\partial/\partial z)F(z)=g_2(z)$, using the identities $P_2(\partial/\partial z)g_1=P_1(\partial/\partial z)g_2$ in (6.2). □

Now, to solve the non-homogeneous Goursat problem, one can do as in Remark 5.3.

Remark 6.4. In fact, the Goursat problem has a unique solution f by Cauchy–Kowalewsky–Lednev's theorem (cf. [16], [17]). Using this result, if the functions g_j and h_J are in $\mathrm{Exp}(\mathbf{C}^n)$ and $\mathrm{Exp}(\mathbf{C}^{n-N})$, respectively, then, in $R\mathbf{B}_n\cap\widetilde{R\mathbf{B}}_{n-N}$, we obtain the solution f of the holomorphic non-homogeneous Goursat problem. Estimations like those in Remark 5.4 give us that $f\in\mathrm{Exp}(\mathbf{C}^n)$. We can also derive an analogous formula for f, by taking

$$\psi^0(\zeta)=(\chi_1(\zeta)\bar{\zeta}_1,\dots,\chi_N(\zeta)\bar{\zeta}_N,\bar{\zeta}^{(N)}),$$

so that ψ^0 maps $P^{-1}(0)$ into $\mathbf{C}^{n-N}$.

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