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To what extent does the dual Banach space E' determine the polynomials over E?

Silvia Lassalle and Ignacio Zalduendo

Abstract. We show that under conditions of regularity, if E' is isomorphic to F', then the spaces of homogeneous polynomials over E and F are isomorphic. Some subspaces of polynomials more closely related to the structure of dual spaces (weakly continuous, integral, extendible) are shown to be isomorphic in full generality.

1. Introduction

In a recent paper [DD] Díaz and Dineen show that if E' is isomorphic to F', and E' has the Schur property and the approximation property, then for any n the spaces of n-homogeneous polynomials over E and F are isomorphic. Thus, in a sense, the dual spaces determine the polynomials over the spaces. The purpose of this note is to investigate further conditions assuring the existence of isomorphisms between spaces of polynomials. We also look into the preservation or non-preservation of some classes of polynomials by these isomorphisms, since it has seemed to us that the mere fact that two spaces of polynomials are isomorphic does not do justice to the rich structure of such spaces (note for example that $\mathcal{P}({}^k\mathbf{R}^n)$ is isomorphic to $\mathcal{P}({}^{n-1}\mathbf{R}^{k+1})$).

In Section 2 we show that we can assure the existence of an isomorphism under regularity conditions on the spaces. Recall that a Banach space E is called Arens-regular if all linear operators $E\to E'$ are weakly compact, and symmetrically Arens-regular if this is so for all symmetric linear operators (see [AGGM] and [GI]). In Section 3 we investigate the preservation of some subspaces of polynomials under the maps defined in Section 2. We find—with no condition on E or F—that if E' is isomorphic to F', then the spaces of n-homogeneous integral polynomials over E and F are isomorphic; and the same holds true for the spaces of weakly continuous polynomials. Finally, in Section 4 we give some examples.

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Added in proof. We have learnt that F. Cabello Sánchez, J. M. F. Castillo and R. García have independently obtained Theorem 4, see Theorem 1 in [CCG].

2. $\bar{s}$ and the Aron–Berner extension

Before going into the problem, we discuss some preliminaries and fix notation. Throughout, E and F will be real or complex Banach spaces. By J_F we will denote the canonical inclusion of F in its bidual F''. By $\mathcal{P}(^nE)$ we denote the space of all n-homogeneous continuous scalar-valued polynomials over E. This is easily seen to be isomorphic to the space of all continuous symmetric n-linear forms, which we denote by $\mathcal{L}_s^n(E)$. Thus each polynomial P has an associated symmetric multilinear form A such that $P(x)=A(x,\dots,x)$. There are many interesting subclasses of polynomials. Among them: $\mathcal{P}_w(^nE)$, the class of those polynomials which are weakly continuous on bounded sets; $\mathcal{P}_I(^nE)$, the class of integral polynomials; and $\mathcal{P}_{\mathrm{wsc}}(^nE)$, the class of weakly sequentially continuous polynomials. For more on polynomials over Banach spaces, see [D], [M] and [GJL].

Our approach to the problem will be via the following construction. Any continuous linear morphism $s\colon E'\to F'$ induces a continuous linear map

$$\bar{s}\colon \mathcal{P}(^nE)\longrightarrow \mathcal{P}(^nF)$$

in the following way. If y is an element of F, define the linear morphism

$$\tilde{y}\colon \mathcal{L}_s^k(E)\longrightarrow \mathcal{L}_s^{k-1}(E)$$

by $\tilde{y}(B)(x_1,\dots,x_{k-1})=s(B_{x_1,\dots,x_{k-1}})(y)$, where $B_{x_1,\dots,x_{k-1}}$ is the element of E' obtained by fixing the $k-1$ variables $x_1,\dots,x_{k-1}$. Now if P is an n-homogeneous polynomial over E, and A is its associated symmetric n-linear form, A can be assigned an n-linear form $\tilde{s}(A)$ over F by setting

$$\tilde{s}(A)(y_1,\dots,y_n)=(\tilde{y}_1\circ\dots\circ\tilde{y}_n)(A).$$

Note that $\tilde{s}(A)$ need not be symmetric. We can, however, define an n-homogeneous polynomial over F by putting $\bar{s}(P)(y)=\tilde{s}(A)(y,\dots,y)$.

In the case $s=J_{E'}\colon E'\to E'''$ (the natural inclusion), the morphism $\bar{s}$ obtained is the well-known Aron–Berner extension of polynomials from a Banach space E to its bidual ([AB], [Z2]). In this particular case we will use the notation $\overline{P}$ and $\bar{A}$

for $\bar{s}(P)$ and $\bar{s}(A)$ respectively. The lack of symmetry of $\bar{A}$ is at the heart of the matter that concerns us here, so we take a moment to refer to some properties of $\bar{A}$. First, although $\bar{A}$ need not be symmetric, the elements of E and those of E'' can always be permuted in the variables of $\bar{A}$. Also, $\bar{A}$ is always weak* continuous in its first variable. It can be seen ([Ü], [ACG]) that symmetry of $\bar{A}$ is equivalent to its weak* continuity in all variables, and also to the weak compactness of the operator $E \to \mathcal{P}(^{n-1}E)$ associated to P. Symmetry of $\bar{A}$ is always obtained if the space E has the property of symmetric regularity mentioned above.

Since the Aron–Berner extension is a well-studied object, we will find it very convenient to write $\bar{s}$ and $\tilde{s}$ in terms of the Aron–Berner extension. This is what we do in the following lemma.

Lemma 1. *For any $y_1, \dots, y_n$ in F, and all symmetric n-linear forms A over E, $\bar{s}(A)(y_1, \dots, y_n) = \bar{A}(s'(J_F(y_1)), \dots, s'(J_F(y_n)))$. In particular, $\bar{s}(P) = \overline{P} \circ s' \circ J_F$.*

Proof. We proceed by induction on n. If $A \in E'$, we have $\bar{s}(A)(y) = s(A)(y) = J_F(y)(s(A)) = s'(J_F(y))(A) = \bar{A}(s'(J_F(y)))$. Now suppose that the result is true for $(n-1)$-linear forms. We first show that the $(n-1)$-linear form over E'' obtained from $\bar{A}$ by fixing $s'(J_F(y_n))$ in the last variable coincides with $\overline{\tilde{y}_n(A)}$:

Let $z_1, \dots, z_{n-1} \in E''$. We have

$$\bar{A}(z_1, \dots, z_{n-1}, (s'(J_F(y_n))) = (\tilde{z}_1 \circ \dots \circ \tilde{z}_{n-1})((s'(J_F(y_n))(A))^{\sim})$$

and

$$\overline{\tilde{y}_n(A)}(z_1, \dots, z_{n-1}) = (\tilde{z}_1 \circ \dots \circ \tilde{z}_{n-1})(\tilde{y}_n(A))$$

(where the tildes over elements of E'' refer to $J_{E'}$). Thus it will be enough to check that the $(n-1)$-linear forms over E, $(s'(J_F(y_n))(A))^{\sim}$ and $\tilde{y}_n(A)$, coincide. Let $x_1, \dots, x_{n-1} \in E$. Then

$$\begin{aligned}(s'(J_F(y_n))(A))^{\sim}(x_1, \dots, x_{n-1}) &= s'(J_F(y_n))(A_{x_1,\dots,x_{n-1}}) = J_F(y_n)(s(A_{x_1,\dots,x_{n-1}})) \\ &= s(A_{x_1,\dots,x_{n-1}})(y_n) = \tilde{y}_n(A)(x_1, \dots, x_{n-1}).\end{aligned}$$

Now, using our inductive hypothesis

$$\begin{aligned}\tilde{s}(A)(y_1, \dots, y_n) &= (\tilde{y}_1 \circ \dots \circ \tilde{y}_n)(A) = (\tilde{y}_1 \circ \dots \circ \tilde{y}_{n-1})(\tilde{y}_n(A)) \\ &= \tilde{s}(\tilde{y}_n(A))(y_1, \dots, y_{n-1}) = \overline{\tilde{y}_n(A)}(s'(J_F(y_1)), \dots, s'(J_F(y_{n-1}))) \\ &= \bar{A}(s'(J_F(y_1)), \dots, s'(J_F(y_n))). \quad \square\end{aligned}$$

In what follows we will usually write y instead of $J_F(y)$, for elements $y \in F$, even when we consider the elements of F'' via the natural inclusion.

Corollary 2. *If $\bar{A}$ is symmetric, then $\overline{\tilde{s}(A)}=\bar{A}\circ(s'\times\ldots\times s')$. Thus $\overline{\tilde{s}(A)}$ is also symmetric, and if P is the homogeneous polynomial associated to A, then $\overline{\bar{s}(P)}=\bar{P}\circ s'$.*

Proof. Clearly $\tilde{s}(A)$ is symmetric, for

$$\tilde{s}(A)(y_1,\ldots,y_n)=\bar{A}(s'(y_1),\ldots,s'(y_n)).$$

Let $y_1,\ldots,y_n$ denote elements of F, and $w_1,\ldots,w_n$ elements of F''. We will show, by induction on k, that

$$\overline{\tilde{s}(A)}(w_1,\ldots,w_k,y_{k+1},\ldots,y_n)=\bar{A}(s'(w_1),\ldots,s'(w_k),s'(y_{k+1}),\ldots,s'(y_n)).$$

Recall that the Aron–Berner extension of any symmetric n-linear form is weak* continuous in its first variable, and furthermore elements of F and F'' always commute. Also, s' is weak*-weak* continuous. Consider a net (y_α) of elements of F, weak* converging to w_1. Then, for $k=1$,

$$\begin{aligned}\overline{\tilde{s}(A)}(w_1,y_2,\ldots,y_n)&=\lim_\alpha\overline{\tilde{s}(A)}(y_\alpha,y_2,\ldots,y_n)=\lim_\alpha\tilde{s}(A)(y_\alpha,y_2,\ldots,y_n)\\&=\lim_\alpha\bar{A}(s'(y_\alpha),s'(y_2),\ldots,s'(y_n))=\bar{A}(s'(w_1),s'(y_2),\ldots,s'(y_n)).\end{aligned}$$

Now suppose the equality holds for $k-1$. We have

$$\begin{aligned}\overline{\tilde{s}(A)}(w_1,\ldots,w_k,y_{k+1},\ldots,y_n)&=\lim_\alpha\overline{\tilde{s}(A)}(y_\alpha,w_2,\ldots,w_k,y_{k+1},\ldots,y_n)\\&=\lim_\alpha\overline{\tilde{s}(A)}(w_2,\ldots,w_k,y_\alpha,y_{k+1},\ldots,y_n)\\&=\lim_\alpha\bar{A}(s'(w_2),\ldots,s'(w_k),s'(y_\alpha),s'(y_{k+1}),\ldots,s'(y_n))\\&=\lim_\alpha\bar{A}(s'(y_\alpha),s'(w_2),\ldots,s'(w_k),s'(y_{k+1}),\ldots,s'(y_n))\\&=\bar{A}(s'(w_1),s'(w_2),\ldots,s'(w_k),s'(y_{k+1}),\ldots,s'(y_n)).\end{aligned}$$

Therefore $\overline{\tilde{s}(A)}(w_1,\ldots,w_n)=\bar{A}(s'(w_1),\ldots,s'(w_n))$, and $\overline{\bar{s}(P)}(w)=\bar{P}(s'(w))$. $\square$

Corollary 3. *Let $P\in\mathcal{P}(^nE)$, and let A be its associated symmetric n-linear form, and suppose that $s\colon E'\to F'$ is an isomorphism. Then if $\bar{A}$ is symmetric, $(\overline{s^{-1}}\circ\bar{s})(P)=P$.*

Proof. Note that $\overline{\tilde{s}(A)}$ is symmetric. Thus for $x_1,\ldots,x_n\in E$, we have

$$\begin{aligned}\widetilde{s^{-1}}(\tilde{s}(A))(x_1,\ldots,x_n)&=\overline{\tilde{s}(A)}((s^{-1})'(x_1),\ldots,(s^{-1})'(x_n))\\&=\bar{A}(s'((s^{-1})'(x_1)),\ldots,s'((s^{-1})'(x_n)))\\&=\bar{A}(x_1,\ldots,x_n)=A(x_1,\ldots,x_n).\end{aligned}$$

Since $\tilde{s}(A)$ is the symmetric n-linear form associated to $\bar{s}(P)$, for any $x\in E$ we have $\overline{s^{-1}}(\bar{s}(P))(x)=P(x)$. $\square$

Theorem 4. *If E and F are symmetrically Arens-regular, and E' and F' are isomorphic (resp. isometric), then for any n, $\mathcal{P}(^nE)$ and $\mathcal{P}(^nF)$ are isomorphic (resp. isometric).*

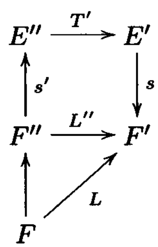
Proof. Let $s: E' \to F'$ be the isomorphism. Since E is symmetrically Arens-regular, for any symmetric n-linear form A over E we have that $\bar{A}$ is symmetric. Thus for any n-homogeneous polynomial P over E, $(\overline{s^{-1}} \circ \bar{s})(P) = P$. Analogously, for any n-homogeneous polynomial Q over F, $(\bar{s} \circ \overline{s^{-1}})(Q) = Q$. Note also that

$$\|\bar{s}(P)\| = \|\overline{P} \circ s' \circ J_F\| \le \|P\| \, \|s\|^n,$$

and the same for s^{-1} and Q. □

Remark 1. If one has morphisms $s: E' \to F'$ and $t: F' \to G'$, and A is the symmetric form associated to a homogeneous polynomial P, then symmetry of $\bar{A}$ implies $(\bar{t} \circ \bar{s})(P) = \overline{t \circ s}(P)$.

Remark 2. It is easy to see that if E' and F' are isomorphic, and one is Arens-regular, then so is the other. Indeed, say $s: E' \to F'$ is an isomorphism, and F is Arens-regular. If $T: E \to E'$ is a linear map, consider the diagram



where $L = s \circ T' \circ s' \circ J_F$. Since L is weakly compact, its bitranspose L'', has range in F' by Gantmacher's theorem. Thus $T' = s^{-1} \circ L'' \circ (s')^{-1}$ is weakly compact, since L'' is. Then so is T. Thus symmetric regularity of both spaces in the theorem can be replaced by regularity of one of them.

Remark 3. Note that for $s = J_{E'}$, Corollary 2 recovers the following result of [AGGM]: If E is symmetrically regular, then $\overline{\overline{P}} = \overline{P} \circ \varrho$, where $\varrho: E^{iv} \to E''$ is the restriction map; thus there are no new 'evaluations' beyond E''.

3. $\bar{s}$ and some subspaces of polynomials

Since $s: E' \to F'$ is a morphism between dual Banach spaces, it is natural to expect that those types of polynomials over E which are more closely related to E'

will be preserved by the morphism $\bar{s}: \mathcal{P}(^nE) \to \mathcal{P}(^nF)$. The formula $\bar{s}(P) = \overline{P} \circ s' \circ J_F$ proved in the previous section shows immediately that finite type, nuclear, and approximable polynomials are all preserved by $\bar{s}$. In this section we show that the same is true for weakly continuous and integral polynomials, and that if E' and F' are isomorphic, then $\mathcal{P}_w(^nE)$ is isomorphic to $\mathcal{P}_w(^nF)$, and $\mathcal{P}_I(^nE)$ is isomorphic to $\mathcal{P}_I(^nF)$, with no further assumptions on E or F. Perhaps surprisingly, the class of weakly sequentially continuous polynomials is not, in general, preserved by $\bar{s}$. We provide examples of this situation in the last section.

Recall that weakly continuous polynomials over E can be characterized ([AG], [T]) as those for which there exists a compact set K contained in E' for which

$$|P(x)| \leq c \sup_{\gamma \in K} |\gamma(x)|^n$$

holds for all $x \in E$. We will denote the seminorm on the right-hand side of the inequality by $\|x\|_K$, and say that P is K-bounded. The smallest possible c is called the K-norm of P and denoted $\|P\|_K$.

Lemma 5. *If P is K-bounded, then $\bar{s}(P)$ is $s(K)$-bounded and*

$$\|\bar{s}(P)\|_{s(K)} \leq \|P\|_K.$$

Proof. It was proved in [AG], [CDDL] that $\overline{P}$ is K-bounded if P is and that $\|\overline{P}\|_K = \|P\|_K$. Thus for any $y \in F$,

$$\begin{aligned} |\bar{s}(P)(y)| &= |(\overline{P} \circ s' \circ J_F)(y)| \leq \|\overline{P}\|_K \sup_{\gamma \in K} |\gamma((s' \circ J_F)(y))|^n \\ &= \|P\|_K \sup_{\gamma \in K} |s(\gamma)(y)|^n = \|P\|_K \|y\|^n_{s(K)}. \quad \square \end{aligned}$$

Thus $\bar{s}$ preserves weakly continuous polynomials.

Proposition 6. *If $s: E' \to F'$ is an isomorphism (resp. isometry), then*

$$\bar{s}: \mathcal{P}_w(^nE) \longrightarrow \mathcal{P}_w(^nF)$$

is an isomorphism (resp. isometry).

Proof. Given $P\in\mathcal{P}_w({}^nE)$, its associated linear operator $E\to\mathcal{P}({}^{n-1}E)$ is compact ([AHV]). Thus the Aron–Berner extension of its associated n-linear form is symmetric and we have, by the results of the previous section, $(\overline{s^{-1}}\circ\bar{s})(P)=P$. Analogously, for $Q\in\mathcal{P}_w({}^nF)$, one has $(\bar{s}\circ\overline{s^{-1}})(Q)=Q$. Note also that

$$\|\bar{s}(P)\|=\|\overline{P}\circ s'\circ J_F\|\leq\|P\|\,\|s\|^n,$$

and the same for s^{-1} and Q. □

Recall that a polynomial $P\in\mathcal{P}({}^nE)$ is called integral if there is a regular Borel measure μ on $B_{E'}$ (the unit ball of E' in its weak* topology) such that for all $x\in E$,

$$P(x)=\int_{B_{E'}}\gamma(x)^n\,d\mu(\gamma).$$

The measure μ is said to represent P, and the infimum of the total variations of the measures representing P is the integral norm of P, denoted $\|P\|_I$. It was proved in [CZ] that the Aron–Berner extension of an integral polynomial is integral. Indeed, if μ is a measure representing P, and one defines

$$U\colon L^1(\mu)\longrightarrow E'$$

by

$$U(f)(x)=\int_{B_{E'}}f(\gamma)\gamma(x)\,d\mu(\gamma),$$

then U is a norm-one map and the Aron–Berner extension of P may be written

$$\overline{P}(z)=\int_{B_{E'}}U'(z)^n\,d\mu.$$

With this notation, we prove the following lemma.

Lemma 7. *If P is integral, then $\bar{s}(P)$ is integral, and $\|\bar{s}(P)\|_I\leq\|s\|^n\|P\|_I$.*

Proof. For $y\in F$,

$$\bar{s}(P)(y)=\overline{P}(s'(J_F(y)))=\int_{B_{E'}}U'(s'(J_F(y)))^n\,d\mu=\int_{B_{E'}}[(s\circ U)'\circ J_F]^n(y)\,d\mu.$$

Thus, $\bar{s}(P)$ is integral, and

$$\|\bar{s}(P)\|_I\leq\|(s\circ U)'\circ J_F\|^n|\mu|\leq\|s\|^n|\mu|.$$

Since this holds for any measure representing P, $\|\bar{s}(P)\|_I\leq\|s\|^n\|P\|_I$. □

Proposition 8. *If $s: E' \to F'$ is an isomorphism (resp. isometry), then*

$$\bar{s}: \mathcal{P}_I({}^nE) \longrightarrow \mathcal{P}_I({}^nF)$$

is an isomorphism (resp. isometry).

Proof. By Lemma 7, $\bar{s}$ and $\overline{s^{-1}}$ are both morphisms between the spaces of integral polynomials. We show now that $\overline{\bar{s}(P)} = \bar{P} \circ s'$. Clearly $\bar{P} \circ s'$ coincides with $\bar{s}(P)$ when restricted to F. Thus it will be enough ([Z1]) to see that the first-order differentials of $\bar{P} \circ s'$ have the properties

(a) for all $y \in F$, $D(\bar{P} \circ s')(y)$ is weak* continuous;

(b) for all $w \in F''$ and $(y_\alpha) \subset F$, weak* converging to w,

$$D(\bar{P} \circ s')(w)(y_\alpha) \to D(\bar{P} \circ s')(w)(w).$$

But since $(\bar{P} \circ s')(w) = \int_{B_{E'}} U'(s'(w))^n \, d\mu$, upon differentiating we have

$$D(\bar{P} \circ s')(w)(v) = n \int_{B_{E'}} U'(s'(w))^{n-1} U'(s'(v)) \, d\mu,$$

which is weak* continuous of the variable v.

Now

$$\overline{s^{-1}}(\bar{s}(P)) = \overline{\bar{s}(P)} \circ (s^{-1})' \circ J_E = \bar{P} \circ s' \circ (s^{-1})' \circ J_E = \bar{P} \circ J_E = \bar{P}|_E = P.$$

Analogously, for $Q \in \mathcal{P}_I({}^nF)$, one has $(\bar{s} \circ \overline{s^{-1}})(Q) = Q$. The norms of $\bar{s}$ and $\overline{s^{-1}}$ are controlled by the inequality in Lemma 7. □

Remark 4. Note that the same type of result as in the propositions can be obtained for any subspace of polynomials as long as their associated linear operators $E \to \mathcal{P}({}^{n-1}E)$ are weakly compact, and the Aron–Berner extension preserves the class.

In particular, we obtain the following results for extendible polynomials (see [KR], [C] and [Z2] for the pertinent definitions).

Lemma 9. *If P is extendible, then $\bar{s}(P)$ is extendible, and* $\|\bar{s}(P)\|_e \leq \|s\|^n \|P\|_e$. □

Proposition 10. *If $s: E' \to F'$ is an isomorphism (resp. isometry), then*

$$\bar{s}: \mathcal{P}_e({}^nE) \longrightarrow \mathcal{P}_e({}^nF)$$

is an isomorphism (resp. isometry). □

4. Examples

In this section we give three examples. The first shows that $\overline{\bar{s}(P)}$ may differ from $\bar{P}\circ s'$. The second that $\overline{t\circ s}$ may differ from $\bar{t}\circ\bar{s}$, and the third that even when $s\colon X'\to Y'$ is an isomorphism, $\bar{s}$ may not preserve the class of weakly sequentially continuous polynomials.

We mention a few facts and fix some notation before going into the examples. If X is a Banach space, consider the 2-homogeneous polynomial P defined over $X\times X'$ by $P(x,x')=x'(x)$. It is easily seen that P is weakly sequentially continuous if and only if X has the Dunford–Pettis property. Also, one may check that the Aron–Berner extension of P to $X''\times X'''$ is

$$\bar{P}(x'',x''')=\tfrac{1}{2}[x'''(x'')+x''(\varrho(x'''))],$$

where $\varrho\colon X'''\to X'$ is the restriction (i.e., the transpose of the natural inclusion $J_X\colon X\to X''$). In the first two examples we will use the notation

$$E=X\times X',\quad F=X''\times X',\quad G=X''\times X'''$$

and over these spaces we consider the polynomials

$$P(x,x')=x'(x),\quad Q(x'',x')=x''(x')$$

and morphisms

$$s\colon E'\longrightarrow F',\quad t\colon F'\longrightarrow G'$$

given by $s=J_{X'}\oplus \mathrm{id}_{X''}$, $t=\mathrm{id}_{X'''}\oplus J_{X''}$. Also, we let $r=J'_{X'}$.

Example 1. $\overline{\bar{s}(P)}\neq\bar{P}\circ s'$.

We first calculate $\bar{s}(P)$. For $(x'',x')\in F$, we have

$$\begin{aligned}\bar{s}(P)(x'',x')&=(\bar{P}\circ s')(J_{X''}(x''),J_{X'}(x'))=\bar{P}(rJ_{X''}(x''),J_{X'}(x'))\\&=\bar{P}(x'',J_{X'}(x'))=\tfrac{1}{2}[J_{X'}(x')(x'')+x''(\varrho(J_{X'}(x')))]\\&=\tfrac{1}{2}[x''(x')+x''(x')]=Q(x'',x').\end{aligned}$$

Thus $\bar{s}(P)=Q$. Now calculate $\overline{\bar{s}(P)}$ and $\bar{P}\circ s'$,

$$\begin{aligned}\overline{\bar{s}(P)}(x^{iv},x''')&=\bar{Q}(x^{iv},x''')=\tfrac{1}{2}[x^{iv}(x''')+x'''(r(x^{iv}))],\\(\bar{P}\circ s')(x^{iv},x''')&=\bar{P}(r(x^{iv}),x''')=\tfrac{1}{2}[x'''(r(x^{iv}))+r(x^{iv})(\varrho(x'''))].\end{aligned}$$

Thus $\overline{\bar{s}(P)}=\bar{P}\circ s'$ if and only if $x^{iv}(x''')=r(x^{iv})(\varrho(x'''))$, but this only holds for reflexive X.

Note that we have seen that $\bar{s}(P)=Q$. If X has the Dunford–Pettis property but X' does not, then P is weakly sequentially continuous, but Q is not. We also have the following result.

Corollary 11. *If X is an infinite-dimensional Banach space with the Dunford–Pettis property, then $X\times X'$ contains l^1.*

Proof. If $E=X\times X'$ did not contain l^1 then every weakly sequentially continuous polynomial would be weakly continuous [FGL]. Thus the operator $E\to E'$ associated to the polynomial P above would be compact, its corresponding $\bar{A}$ would be symmetric, and $\overline{s(P)}=\bar{P}\circ s'$ forcing X to be reflexive, and therefore finite-dimensional. □

Example 2. $\overline{t\circ s}\neq\bar{t}\circ\bar{s}$.

First, calculate $t'\circ J_G$,

$$(t'\circ J_G)(x'',x''')=t'(J_{X''}(x''),J_{X'''}(x'''))=(J_{X''}(x''),J'_{X''}(J_{X'''}(x''')))$$
$$=(J_{X''}(x''),x''').$$

We have already calculated $\overline{s(P)}$, so using this we have

$$(\bar{t}\circ\bar{s})(P)(x'',x''')=(\overline{s(P)}\circ t'\circ J_G)(x'',x''')=\overline{s(P)}(J_{X''}(x''),x''')$$
$$=\tfrac{1}{2}[J_{X''}(x'')(x''')+x'''(r(J_{X''}(x'')))]=x'''(x'').$$

Then

$$(\overline{t\circ s})(P)(x'',x''')=(\bar{P}\circ(t\circ s)'\circ J_G)(x'',x''')=(\bar{P}\circ s')((t'\circ J_G)(x'',x'''))$$
$$=(\bar{P}\circ s')(J_{X''}(x''),x''')=\bar{P}(r(J_{X''}(x'')),x''')$$
$$=\bar{P}(x'',x''')=\tfrac{1}{2}[x'''(x'')+x''(\varrho(x'''))],$$

thus $\overline{t\circ s}=\bar{t}\circ\bar{s}$ if and only if $x''(\varrho(x'''))=x'''(x'')$, but again, this only happens for reflexive X.

Example 3. An isomorphism $s\colon X'\to Y'$ such that $\bar{s}$ does not preserve the class of weakly sequentially continuous polynomials.

We begin with the well-known example of [S] of Banach spaces X and Y with isomorphic duals, such that X has the Dunford–Pettis property and Y does not:

$$X=\Big(\sum_{n\geq 1} l_2^n\Big)_1 \quad\text{and}\quad Y=X\oplus l_2.$$

Call the isomorphism $s\colon X'\to Y'$, and consider the 2-homogeneous polynomial Q over Y defined by $Q(x,a)=\sum_{n\geq 1}a_n^2$. The operator $Y\to Y'$ associated to Q sends (x,a) to $(0,a)$ and is therefore weakly compact. Then $(\bar{s}\circ\overline{s^{-1}})(Q)=\overline{s\circ s^{-1}}(Q)=Q$. Since X has the Dunford–Pettis property, all polynomials over X are weakly sequentially continuous [R], in particular $(\overline{s^{-1}})(Q)$ is. Thus $\bar{s}$ sends this weakly sequentially continuous polynomial onto Q, which is not weakly sequentially continuous ($Q(0,e_n)=1$ for all n).

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Silvia Lassalle
Departamento de Matemática (FCEN)
Universidad de Buenos Aires
Pab. I - Ciudad Universitaria
(1428) Buenos Aires
Argentina

Ignacio Zalduendo
Departamento de Economía y Matemática
Universidad de San Andrés
CC 35 (1644) Victoria
Buenos Aires
Argentina