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Removability theorems for Sobolev functions and quasiconformal maps

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Abstract. We establish several conditions, sufficient for a set to be (quasi)conformally removable, a property important in holomorphic dynamics. This is accomplished by proving removability theorems for Sobolev spaces in $\mathbf{R}^n$. The resulting conditions are close to optimal.

1. Introduction

The object of this paper is to provide a few conditions, sufficient for a set to be (quasi)conformally and Sobolev removable. Such results are useful in dynamics, since they provide tools for establishing conformal conjugacy between two topologically conjugate holomorphic dynamical systems. Particularly, our Theorem 1 (see its dynamical reformulation in Section 4) is used in [GS2] to establish conformal removability of a large class of Julia sets. This problem was also studied earlier in [Jo] to provide tools for applications in dynamics.

We are mostly interested in the planar case, but all our theorems work in $\mathbf{R}^n$, where the notion of quasiconformal removability makes sense.

Definition 1. We say that a compact set $K \subset U$ is *(quasi)conformally removable inside a domain* U, if any homeomorphism of U, which is (quasi)conformal on $U \setminus K$, is (quasi)conformal on U.

In the dynamics literature such sets are often called "holomorphically removable". We prefer to use the term "conformally removable", because it can be essential that one considers maps and not just holomorphic functions. See [AB], [Be] and [C] for some of the related problems concerned with functions, rather than maps. An easy application of the measurable Riemann mapping theorem (see, e.g., [A])

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shows that for planar sets properties of conformal and quasiconformal removability are equivalent. It is not difficult to see that the property of quasiconformal removability is quasiconformally invariant. Another easy argument shows that Sobolev $W^{1,2}$-removability ($W^{1,n}$ in $\mathbf{R}^n$) is a stronger property.

Definition 2. We say that a compact set $K \subset U$ is *$W^{1,p}$-removable inside a domain U*, if any function, continuous in U and belonging to $W^{1,p}(U\setminus K)$, belongs to $W^{1,p}(U)$.

Note that our definition of Sobolev removability assumes that functions under consideration are continuous, so perhaps it is more appropriate to call such sets *$W^{1,p}$-removable for continuous functions.* It is not known, whether or not Sobolev $W^{1,2}$ and conformal removability for planar sets are equivalent or not. It seems that all known methods of proving (quasi)conformal removability apply to Sobolev removability, and no full geometric characterization of removable sets is known in either case.

It is not difficult to show that any set of σ-finite length is conformally removable, whereas any set of positive area is not. Namely, compacts of σ-finite length are removable for continuous analytic functions by [Be] and hence conformally removable. Also by [U], a compact set has zero area if and only if it is removable for Lipschitz functions, analytic off it, and from an exceptional Lipschitz function f analytic off the set one easily constructs an exceptional homeomorphism $g(z):=z+\varepsilon f(z)$, conformal off the set. Those conditions turn out to be the best possible (sufficient and necessary correspondingly), expressible in purely metric terms. In fact, it is not difficult to see (by constructing an exceptional homeomorphism, quasiconformal off the set) that a Cartesian product of an uncountable set with an interval is not conformally removable. A much stronger statement is proved in [Ka], namely that such a set contains a non-removable graph, see also [G] and [C]. On the other hand, the Cartesian product of two sets of zero length is conformally removable (ACL arguments as below easily apply to such a set since almost every line parallel to the coordinate axes does not touch it, see also Theorem 10 in [AB]). See [Bi] for further discussion of related problems.

Standard extension theorems show that (quasi)conformal removability of a compact K inside U is equivalent to its removability inside V (provided, of course, that K is contained in both domains), and that the union of two disjoint compacta is (quasi)conformally removable if and only if both of them are. However, the latter does not seem to be known if their intersection is non-empty. These statements are much easier to check for Sobolev removability, and one does not need to assume that the compacta are disjoint.

Since removability properties do not depend on the reference domain U, below by removability we will mean removability inside $\mathbf{C}$ or $\mathbf{R}^n$. We will work with a compact set K which is the boundary of some connected domain Ω (the latter has nothing to do with the domain U). Our conditions can be translated to the case when K is the boundary of a union of finitely many domains as well. To simplify the proofs, Ω will be assumed to be bounded. In the case of unbounded domains one has to restrict summation and integration in the proofs below to some bounded part of Ω containing $K=\partial\Omega$, or consider the intersection of Ω with some big ball as a new domain. We consider the Whitney decomposition $\mathcal{W}=\{Q\}$ of Ω. For an integrable function ϕ we denote by $\phi(Q)$ its mean value on the cube Q, by $|Q|$ the volume and by $l(Q)$ the side length of the cube Q.

Definition 3. Fix a family Γ of curves starting at a fixed point $z_0\in\Omega$ (or just a fixed distance ε away from $\partial\Omega$) and accumulating to $\partial\Omega$ such that their accumulation sets cover $\partial\Omega$. We consider the "shadow" cast by a cube Q if a light source is placed at z_0: namely the *shadow* $\mathcal{SH}(Q)$ is the closure in $\partial\Omega$ of the union of all curves $\gamma\in\Gamma$ starting at z_0 and passing through Q.

Denote by $s(Q)$ the diameter of the shadow $\mathcal{SH}(Q)$, and define a non-negative function ϱ on Ω by setting $\varrho|_Q:=s(Q)/l(Q)$ for $Q\in\mathcal{W}$. The function ϱ is well-defined on interiors of Whitney cubes and hence almost everywhere in Ω.

Remark 1. One can think of the curves from Γ as of quasihyperbolic geodesics starting at z_0. Condition (1) below involves shadows and hence depends on the family Γ, and it seems that in most of its applications it is optimal to use quasihyperbolic (or hyperbolic for planar domains) geodesics as curves in Γ: in the situations under consideration they satisfy the requirements placed on Γ and for such a family condition (1) is easier to check.

Since the quasihyperbolic metric $\operatorname{dist}_{\mathrm{qh}}(\cdot,\cdot)$ also appears in a few conditions below, we recall that it is the metric on Ω with the element $|dz|/\operatorname{dist}(z,\partial\Omega)$. It behaves much like the hyperbolic metric in the planar domains, e.g. it is a geodesic metric—see the expository paper of P. Koskela [Ko] for this and other properties.

Note that if we take a sufficiently small size Δ of the Whitney cubes, then any curve from Γ passes through at least one Whitney cube of that size. Hence $\partial\Omega$ is completely covered by the shadows of those cubes, whose number is finite.

We will be interested in domains $\Omega\subset\mathbf{R}^n$ satisfying (for some family Γ) the geometric condition

$$\sum_{Q\in\mathcal{W}} s(Q)^n<\infty, \tag{1}$$

or equivalently $\varrho\in L^n(\Omega,m)$, where m denotes n-dimensional Lebesgue measure. In Section 3 it will be shown that (1) follows from other conditions. Also note, that it is sufficient to include in the sum above only cubes Q contained in some neighborhood of K, or, equivalently, ϱ is integrated over the intersection of Ω with some neighborhood of K (because of that there is no loss of generality in the assumption that Ω is bounded). Covering $\partial\Omega$ by the shadows of small Whitney cubes implies that boundaries of such domains have zero Lebesgue volume (area in the planar case). It is also easy to see that for domains satisfying (1) every curve from Γ, starting at z_0 and approaching $\partial\Omega$ has exactly one landing point, that every $z\in\partial\Omega$ is a landing point of such a curve, and if $z\in\mathcal{SH}(Q)$ there is such a curve passing through Q.

Remark 2. For a simply connected planar domain Ω, whose Riemann uniformization map is $\phi:\mathbf{D}\to\Omega$, and a family Γ consisting of the images of the radii (same as hyperbolic geodesics) our condition corresponds to

$$\sum_I \operatorname{diam}^2(\phi(I))<\infty,$$

where the sum is taken over all dyadic arcs of the unit circle $\partial\mathbf{D}$.

Our main theorem shows that the geometric condition (1) is sufficient for quasiconformal removability.

Theorem 1. *If Ω satisfies condition* (1) *then $K=\partial\Omega$ is quasiconformally removable.*

We will use this theorem to deduce other conditions, sufficient for removability. Particularly, take Γ to be the family of quasihyperbolic geodesics. It is almost immediate that boundaries of John and Hölder domains are removable.

Corollary 1. *Boundaries of John domains are quasiconformally removable.*

Proof. For John domains one has $s(Q)<Cl(Q)$, with C depending on the John constant. Hence

$$\sum_{Q\in\mathcal{W}} s(Q)^n<C\sum_{Q\in\mathcal{W}}|Q|\le C\operatorname{Volume}(\Omega)<\infty,$$

and the desired condition (1) is satisfied. $\square$

See [Jo] for an earlier proof of the corollary above and the definition of John domains. A simply connected domain in the complex plane is called Hölder if the Riemann uniformization map is Hölder-continuous in the closed unit disc. This property is weaker than being a John domain. In the multiply connected case and in $\mathbf{R}^n$ the latter definition can be substituted by a proper quasihyperbolic boundary condition, as e.g. in [GS1], [Ko].

Corollary 2. *Boundaries of Hölder domains are quasiconformally removable.*

Proof. Since we will be proving the stronger Theorem 3 and Corollary 4 later on, here we just give a brief sketch of the proof in the planar case modulo ideas from [JM].

Assume that the domain Ω is bounded. Let $\{Q_j^k\}_j$ denote the collection of all Whitney cubes whose hyperbolic distance to z_0 is comparable to k. By the Hölder property one has $s(Q_j^k) \le Ce^{-ck}$, and by arguments of [JM], independently of k,

$$\sum_j s(Q_j^k)^{2-\varepsilon} < M < \infty.$$

Combining these two estimates we obtain (1):

$$\sum_{Q\in\mathcal{W}} s(Q)^2 < \sum_k \sum_j s(Q_j^k)^{\varepsilon} s(Q_j^k)^{2-\varepsilon} < \sum_k C^{\varepsilon} e^{-\varepsilon ck} M < \infty. \quad \square$$

Corollary 2 has an immediate application in dynamics: by [GS1] Julia sets of Collet–Eckmann polynomials bound Hölder domains, and we arrive at the following conclusion.

Corollary 3. *Julia sets of Collet–Eckmann polynomials are (quasi)conformally removable.*

In [PR] the rigidity of such Julia sets was shown, which means that they are removable for conjugations, arising from dynamics. An improvement of this corollary (using Theorem 5) appears in [GS2].

Even much weaker (than being Hölder) conditions on the regularity of the domain Ω appear to be sufficient for removability, consult [Ko] for other similar conditions and their implications.

Theorem 2. *If for some fixed point $z_0 \in \Omega$ a domain $\Omega \subset \mathbf{R}^n$ satisfies*

$$\operatorname{dist}_{\mathrm{qh}}(\cdot, z_0) \in L^n(\Omega_K, m), \tag{2}$$

then $K = \partial\Omega$ is quasiconformally and $W^{1,n}$-removable.

By Ω_K above we mean some neighborhood of K inside Ω, since only integrability near K is needed. To prove Theorem 2 we will show that (2) implies (1), and even more:

$$\operatorname{dist}_{\mathrm{qh}}(\cdot, z_0) \in L^n(\Omega_K, m) \quad \Longrightarrow \quad (1) \quad \Longrightarrow \quad \operatorname{dist}_{\mathrm{qh}}(\cdot, z_0) \in L^1(\Omega_K, m).$$

It is interesting to note that the statement above is sharp in the following sense: one cannot replace L^n by $L^{n-\varepsilon}$ or L^1 by $L^{1+\varepsilon}$. Such integrability conditions already appeared in the paper [Je] of D. Jerison about domains admitting Poincaré-type inequalities.

Theorem 3. *If a domain Ω satisfies the quasihyperbolic boundary condition*

$$(3)\qquad \operatorname{dist}(x,\partial\Omega) < \exp(-(\operatorname{dist}_{\mathrm{qh}}(x,z_0)^{n-1}\log\operatorname{dist}_{\mathrm{qh}}(x,z_0))^{1/n}/o(1)),$$

as $x\in\Omega$ tends to $\partial\Omega$ for some fixed $z_0\in\Omega$, then $K=\partial\Omega$ is quasiconformally and $W^{1,n}$-removable.

We will reduce Theorem 3 to Theorem 2 by showing that (3) implies (2). In fact, our proof shows that (3) implies an even stronger property: $\operatorname{dist}_{\mathrm{qh}}(\cdot,z_0)\in L^p(\Omega_K,m)$ for any $p<\infty$.

Corollary 4. *If a planar domain Ω is simply connected and the modulus of continuity of the Riemann uniformization map $\phi\colon\mathbf{D}\to\Omega$ satisfies*

$$(4)\qquad \omega_\phi(t) < \exp\left(-\sqrt{\log\frac{1}{t}\log\log\frac{1}{t}}\Big/o(1)\right),$$

as $t\to 0$, then $K=\partial\Omega$ is conformally and $W^{1,2}$-removable.

Proof. For a point $x=\phi(\zeta)$ close to the boundary, and a fixed reference point z_0, one clearly has $\operatorname{dist}_{\mathrm{qh}}(x,z_0)\asymp\log 1/(1-|\zeta|)$ and

$$\operatorname{dist}(x,\partial\Omega)\le\omega_\phi(\operatorname{dist}(\zeta,\partial\mathbf{D}))=\omega_\phi(1-|\zeta|).$$

The corollary readily follows. □

Remark 3. By [JM] we know a sharp condition on ω_ϕ, sufficient for the conclusion that the boundary of a planar domain Ω has zero area:

$$\int_0\left|\frac{\log\omega_\phi(t)}{\log t}\right|^2\frac{dt}{t}=\infty.$$

Considering conditions that stop at the $\log\log$ term, we conclude, that

$$(5)\qquad \omega_\phi(t) < \exp\left(-\sqrt{\log\frac{1}{t}\Big/\log\log\frac{1}{t}}\right),$$

is sufficient for $\partial\Omega$ to have zero area, whereas for any $\varepsilon>0$ there exist domains satisfying

$$\omega_\phi(t) < \exp\left(-\sqrt{\log\frac{1}{t}\Big/\left(\log\log\frac{1}{t}\right)^{1+\varepsilon}}\right),$$

with $\partial\Omega$ having positive area, and hence non-removable. This shows that our Theorem 3 is very close to being best possible. Unfortunately, there is a small gap between conditions (4) and (5), and we do not know whether it is possible to close it.

Our proof of Theorem 1 uses the fact that quasiconformal maps belong to the Sobolev space, and easily translates to show the following result.

Theorem 4. *If for some $p \geq 1$ a domain $\Omega \subset \mathbf{R}^n$ satisfies the condition*

$$\sum_{Q \in \mathcal{W}} (s(Q)/l(Q))^{p'(n-1)} |Q| < \infty, \tag{6}$$

or equivalently $\varrho \in L^{p'(n-1)}(\Omega, m)$, with $1/p + 1/p' = 1$, then $K = \partial\Omega$ is $W^{1,p}$-removable.

Repeating the proofs of Theorems 2 and 3, one can also deduce, from the theorem above, quasihyperbolic boundary conditions sufficient for Sobolev removability, though they look artificially complicated.

Throughout the paper we will denote by *const* various positive finite constants (which depend on the equations they appear in). The inequality $A \lesssim B$ will stand for $A \leq \text{const}\, B$, while $A \asymp B$ will mean $\text{const}^{-1}\, B \leq A \leq \text{const}\, B$.

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2. Absolute continuity on lines

Recall, that a function f is called *absolutely continuous on lines* (ACL for short), if for almost every line l, parallel to the coordinate axes, the restriction $f|_l$ is absolutely continuous. It is a well-known fact that to check quasiconformality of a homeomorphism f, it is sufficient to check that it is ACL and quasiconformal almost everywhere. By the latter one means that for almost all $x \in \mathbf{R}^n$ the homeomorphism f is differentiable and satisfies

$$\max_{\alpha} \partial_\alpha f(x) \leq C \min_{\alpha} \partial_\alpha f(x) \tag{7}$$

for some constant $C>0$. Here we take min and max of directional derivatives $\partial_\alpha f$ for all directions α. See Section 34 of [V] or Section II.B of [A] for precise conditions ensuring K-quasiconformality with specific K.

Hence the following proposition implies Theorems 1 and 4.

Proposition 1. *If Ω satisfies condition* (6), *then any continuous f, which belongs to $W^{1,p}$ for bounded subsets of K^c, is* ACL.

In fact, to deduce Theorem 1, take a domain Ω, satisfying condition (1). The coordinate functions of any homeomorphism f quasiconformal in K^c belong to $W^{1,n}$ for bounded subsets of K^c, and by the proposition above (with $p=n$) to ACL.

Recalling that K has zero area by (1), we obtain that f is ACL and quasiconformal outside K, hence is differentiable and satisfies (7) in K^c, i.e. almost everywhere in $\mathbf{R}^n$. By Section II.B "The analytic definition" from [A] or Theorem 34.6 from [V] we deduce that f is quasiconformal in $\mathbf{R}^n$, which proves Theorem 1. To deduce Theorem 4 one uses Theorem 2.1.4 from [Z]. Now we can turn to the proof of Proposition 1.

Proof. We will fix a bounded domain U, containing K, and prove that for any direction λ

$$\iint_U |\partial_\lambda f| = \iint_{U\setminus K} |\partial_\lambda f|, \tag{8}$$

where $\partial_\lambda f$ denotes the directional derivative of f in the sense of distributions. Here and below we will use $\iint_U$ to denote a double integral, where one first integrates along a line, parallel to λ, and then over all such lines. Hence the identity (8) means that on almost every line l parallel to λ the total variation of f is equal to $\int_{l\cap U\setminus K} |\partial_\lambda f|$. Since $f\in W^{1,p}(U\setminus K)\subset W^{1,1}(U\setminus K)$, this implies (by Fubini) that $\partial_\lambda f$ restricted to almost every line l parallel to λ is in fact an integrable function. By taking all possible directions λ and domains U we derive that $f\in \mathrm{ACL}(\mathbf{R}^n)$, thus proving the proposition.

We fix a direction λ, and some line l, parallel to it (with an intent to integrate over all such lines and apply Fubini's theorem at the end).

We denote the total variation of f on $l\cap U$ by $\int_{l\cap U} |\partial_\lambda f|$, and note that it can be arbitrarily closely approximated by expressions of the form

$$\sum_j |f(x_j)-f(y_j)| + \int_{l\cap U\setminus \bigcup_j [x_j,y_j]} |\partial_\lambda f|, \tag{9}$$

where the pairwise disjoint intervals $[x_j, y_j]$ cover $l\cap K$ with $x_j, y_j \in l\cap K$.

By condition (1), as the Whitney cubes get smaller the diameters of their shadows tend to zero. Hence we can choose such a small size Δ that no shadow of a Whitney cube of this or smaller size intersects more than one interval $[x_j, y_j]$, all such cubes are contained in U, and the shadows of the cubes of this size cover K.

Fix one interval $[x_j, y_j]$. Since there are only finitely many cubes of the size Δ, and the set $[x_j, y_j]\cap K$ is covered by their shadows (which are compact sets), one can cut $[x_j, y_j]$ into finitely many intervals $[u_i, u_{i+1}]$ so that $u_0 = x_j$ and $u_n = y_j$. By an easy compactness argument this can be done in such a way, that for every i either $(u_i, u_{i+1})\subset K^c$ or u_i and u_{i+1} belong to the same shadow $\mathcal{SH}(Q_i)$, and there are curves from Γ, joining u_i and u_{i+1} to Q_i, that do not intersect cubes of larger or equal size.

In the first case we just write

$$|f(u_i)-f(u_{i+1})| \le \int_{[u_i,u_{i+1}]} |\partial_\lambda f|.$$

In the latter case we can join u_i and u_{i+1} by a curve γ_i, which follows one "Γ-curve" from u_i to the cube Q_i and then switches to another "Γ-curve" from Q_i to u_{i+1}.

Recall that for an integrable function ϕ we denote by $\phi(Q)$ its mean value on the cube Q. For any two adjacent Whitney cubes Q and Q' (i.e. such that they have the same side length and share a face, or one of them has twice the side length of the other and they share a face of the smaller one) one can easily show that

$$|f(Q)-f(Q')| \le 2^{n-1}(|\partial f|(Q)l(Q)+|\partial f|(Q')l(Q')).$$

The exact value of the constant above is not important, we use 2^{n-1}, but any finite positive number would be sufficient for our reasoning below. Taking the Whitney cubes intersecting the curve γ_i and excluding some of them one can choose a biinfinite sequence of cubes, such that its tails converge to u_i and u_{i+1} correspondingly, and any two consecutive cubes are adjacent. Applying the inequality above to this sequence, one obtains

$$|f(u_i)-f(u_{i+1})| \le 2^n \sum_{Q\cap\gamma_i\neq\emptyset} |\partial f|(Q)l(Q),$$

where the sum is taken over all Whitney cubes intersecting γ_i (even at a single point). All the cubes in the estimate above have size at most Δ, and by the choice of Δ they belong to U.

Now, adding up the estimates for all i, we obtain

$$\begin{aligned} |f(x_j)-f(y_j)| &\le \sum_i |f(u_i)-f(u_{i+1})| \\ &\le \sum_{[u_i,u_{i+1}]\subset K^c} \int_{[u_i,u_{i+1}]} |\partial_\lambda f| + 2^n \sum_{[u_i,u_{i+1}]\not\subset K^c} \sum_{Q\cap\gamma_i\neq\emptyset} |\partial f|(Q)l(Q). \end{aligned}$$

The first term can be simply estimated by $\int_{[x_j,y_j]\setminus K} |\partial_\lambda f|$. Note that all Whitney cubes we come up with in the second term have one of the points u_i in their shadow and are of the size at most Δ. As the following reasoning shows, for the purpose of estimating $|f(x_j)-f(y_j)|$, we can assume that no cube appears twice in the sums. In fact, if there is a Whitney cube Q, entered by two curves γ_k and γ_l, $k<l$, then

we can make a new curve out of them, connecting u_k directly to u_{l+1}, and thus improving the estimate above, writing

$$|f(x_j)-f(y_j)|\leq\sum_{i<k}|f(u_i)-f(u_{i+1})|+|f(u_k)-f(u_{l+1})|+\sum_{i>l}|f(u_i)-f(u_{i+1})|$$

and obtaining fewer cubes in the resulting estimate. If necessary, we can repeat this procedure a few times, and thus assume that no Whitney cube is entered by two different curves.

Therefore, we can rewrite our estimate as

$$|f(x_j)-f(y_j)|\leq\int_{[x_j,y_j]\setminus K}|\partial_\lambda f|+2^n\sum_{\mathcal{SH}(Q)\cap[x_j,y_j]\neq\emptyset}|\partial f|(Q)l(Q). \tag{10}$$

Recalling that by the choice of Δ no shadow of a cube of that or smaller size intersects more than one interval $[x_j,y_j]$, we conclude that every cube Q in the estimates (10) appears for at most one j and has shadow intersecting l. Now, summing (10) over all j, we obtain the following estimate of the expression (9):

$$\begin{aligned}\sum_j|f(x_j)-f(y_j)|&+\int_{l\cap U\setminus\bigcup_j[x_j,y_j]}|\partial_\lambda f|\\ &\leq\sum_j\Bigl(\int_{[x_j,y_j]\setminus K}|\partial_\lambda f|+2^n\sum_{\mathcal{SH}(Q)\bigcap[x_j,y_j]\neq\emptyset}|\partial f|(Q)l(Q)\Bigr)+\int_{l\cap U\setminus\bigcup_j[x_j,y_j]}|\partial_\lambda f|\\ &\leq 2^n\sum_{\mathcal{SH}(Q)\cap l\neq\emptyset}|\partial f|(Q)l(Q)+\int_{l\cap U\setminus K}|\partial_\lambda f|.\end{aligned}$$

Since $\int_l|\partial_\lambda f|$ can be approximated by expressions of the form (9), we arrive at the estimate

$$\int_{l\cap U}|\partial_\lambda f|\leq 2^n\sum_{\mathcal{SH}(Q)\cap l\neq\emptyset}|\partial f|(Q)l(Q)+\int_{l\cap U\setminus K}|\partial_\lambda f|. \tag{11}$$

Moreover, only Whitney cubes of size at most Δ (which we can choose to be arbitrarily small) are included in the latter estimate.

Notice that a Whitney cube Q participates in the estimate only if the line l intersects its shadow, and the measure of the set of such lines is at most $s(Q)^{n-1}$. Integrating (11) over all lines l, parallel to the direction λ, (here μ denotes the transversal measure on those lines), and applying Fubini's theorem we obtain

$$\begin{aligned}\iint_U |\partial_\lambda f| &\stackrel{\text{def}}{=} \int \left(\int_{l\cap U} |\partial_\lambda f| \right) d\mu(l) \\ &\le \int \left(2^n \sum_{\mathcal{SH}(Q)\cap l\neq\emptyset} |\partial f|(Q)|Q| + \int_{l\cap U\setminus K} |\partial_\lambda f| \right) d\mu(l) \\ &\le 2^n \sum_{\mathcal{SH}(Q)\cap l\neq\emptyset} |\partial f|(Q) l(Q) s(Q)^{n-1} + \iint_{U\setminus K} |\partial_\lambda f|.\end{aligned}$$

The first series in the resulting estimate is convergent: in fact, by Hölder's inequality,

$$\begin{aligned}\sum |\partial f|(Q) l(Q) s(Q)^{n-1} &\le \left(\sum |\partial f|(Q)^p |Q|\right)^{1/p} \left(\sum (s(Q)/l(Q))^{p'(n-1)}|Q|\right)^{1/p'} \\ &< \infty,\end{aligned} \tag{12}$$

where the first sum on the right is (by Hölder again) at most $\sum |\partial f|^p(Q)|Q|$, and hence bounded by the Sobolev norm of f in $W^{1,p}(U\setminus K)$. The second is finite by condition (6). As before, we can assume that only Whitney cubes shadowed by cubes of a small size Δ (which we are free to choose) participate in this series, and thus (with $\Delta\to 0$) its sum can be taken to be arbitrarily small, and we can just drop it from the estimate. Therefore we arrive at

$$\iint_U |\partial_\lambda f| \le \iint_{U\setminus K} |\partial_\lambda f|,$$

and clearly those quantities are equal, thus proving the desired equality (8) and hence the proposition.

Note the similarity between estimate (12) and the proof in [KW]. The latter can be thought of as an application of the extreme case of the Hölder inequality with $p=1$ and $p'=\infty$ (i.e. $\sum |\partial f| l(Q) s(Q)^{n-1} \le \sum |\partial f| l(Q)^n \sup(s(Q)/l(Q))^{n-1}$ in our notation). $\square$

3. Quasihyperbolic boundary conditions

Proof of Theorem 2. For a simply connected planar domain we can take Γ to be all hyperbolic geodesics starting at some fixed point z_0 and accumulating at $\partial\Omega$. In the general case take z_0 to be the center of some Whitney cube $Q(z_0)$, and set $q(Q(z_0)):=0$. For any two adjacent (i.e. sharing at least a part of a face) Whitney cubes, join their centers by an interval, and let $q(Q)$ be the number of intervals in the shortest chain joining the centers of $Q(z_0)$ and Q. Clearly we can remove redundant intervals so that $q(Q)$ is preserved for all $Q\in\mathcal{W}$ and the resulting collection of intervals is a tree. Note that $q(Q)\asymp\mathrm{dist}_{\mathrm{qh}}(Q,z_0)$ for any $Q\in\mathcal{W}$, and hence by the assumption of Theorem 2

$$\sum_{Q\in\mathcal{W}} q(Q)^n l(Q)^n < \infty,$$

(if Ω is unbounded we include only cubes close to K in the sum above).

Define Γ' to contain all chains of intervals starting from z_0 (if Ω is unbounded one should take only the chains not escaping to infinity). Then all curves in Γ' have uniformly bounded finite length, and even better: the lengths of their "tails" tend uniformly to zero. Indeed, if some curve $\gamma\in\Gamma'$ corresponds to the chain $\{Q_j\}$ of cubes with $Q_0=Q(z_0)$ and $j=q(Q_j)$, introduce the "tail" by $\gamma^k:=\gamma\setminus\bigcup_{j<k}Q_j$. Then by Hölder's inequality one has

$$\mathrm{length}(\gamma^k)\asymp\sum_{j\geq k} l(Q_j)\leq\Bigl(\sum_{j\geq k} j^n l(Q_j)^n\Bigr)^{1/n}\Bigl(\sum_{j\geq k}\frac{1}{j^{n/(n-1)}}^{(n-1)/n}\Bigr)\lesssim\frac{1}{\sqrt[n]{k}}<\infty.$$

Take Γ to be those curves in Γ' which contain infinitely many intervals, or equivalently accumulate to K (and hence land at some point in K, since the length is bounded).

To show that the family Γ satisfies the requirements it is sufficient to show that any point in K is a landing point of at least one curve from Γ. Take any $z\in K$. There are Whitney cubes arbitrarily close to z, and their centers are joined to z_0 by some curves from Γ', so there are curves from Γ' which terminate arbitrarily close to z. Since any Whitney cube intersects only finitely many other Whitney cubes, we can apply Cantor's diagonal argument to find a sequence of curves $\gamma_j\in\Gamma'$ terminating at points z_j and a curve $\gamma\in\Gamma$ such that $\lim_{j\to\infty}|z-z_j|=0$ and for any j the first j cubes in γ_j and γ coincide. Because of the latter, the tails γ_j^j and γ^j start at the same point, and hence their union joins z_j to the landing point $y\in K$ of γ. Therefore

$$|y-z|=\lim_{j\to\infty}|y-z_j|\leq\limsup_{j\to\infty}\mathrm{length}(\gamma^j\cup\gamma_j^j)\lesssim\limsup_{j\to\infty}\frac{2}{\sqrt[n]{j}}=0,$$

so $y=z$ and the curve $\gamma\in\Gamma$ lands at z. Hence every point in K is a landing point of some curve from Γ, and this family satisfies the requirements.

Now for every cube $Q\in\mathcal{W}$ find a curve from Γ going through it, such that its length is comparable to $s(Q)$. Taking all the Whitney cubes it passes through, we obtain a sequence of cubes $Q_j=Q_j(Q)$, such that $Q_0=Q$, $j+q(Q)=q(Q_j)$, the cube Q_j is shadowed by Q, and $s(Q)\lesssim\sum_{j=1}^{\infty} l(Q_j)$. Note also that by the construction of Γ a given cube $\widetilde{Q}$ is shadowed by exactly $q(\widetilde{Q})$ cubes $\{Q^i\}_{i=0}^{k-1}$ with $q(Q^i)=i$.

Now applying Hölder's inequality, we obtain

$$\begin{aligned}
\sum_{Q\in\mathcal{W}} s(Q)^n &\lesssim \sum_{Q\in\mathcal{W}}\Bigl(\sum_{j=1}^{\infty} l(Q_j(Q))\Bigr)^n \\
&\le \sum_{Q\in\mathcal{W}}\sum_{j=1}^{\infty} l(Q_j)^n q(Q_j)^{n-1/n}\Bigl(\sum_{j=1}^{\infty} q(Q_j)^{-(n+1)/n}\Bigr)^{n-1} \\
&= \sum_{Q\in\mathcal{W}}\sum_{j=1}^{\infty} l(Q_j)^n q(Q_j)^{n-1/n}\Bigl(\sum_{i=q(Q)+1}^{\infty} i^{-(n+1)/n}\Bigr)^{n-1} \\
&\asymp \sum_{Q\in\mathcal{W}}\sum_{j=0}^{\infty} l(Q_j)^n q(Q_j)^{n-1/n}(q(Q)+1)^{-(n-1)/n} \\
&= \sum_{\widetilde{Q}\in\mathcal{W}} l(\widetilde{Q})^n q(\widetilde{Q})^{n-1/n}\sum_{Q:\widetilde{Q}=Q_j(Q)} (q(Q)+1)^{-(n-1)/n} \\
&= \sum_{\widetilde{Q}\in\mathcal{W}} l(\widetilde{Q})^n q(\widetilde{Q})^{n-1/n}\sum_{i=1}^{q(\widetilde{Q})} i^{-(n-1)/n} \\
&\asymp \sum_{\widetilde{Q}\in\mathcal{W}} l(\widetilde{Q})^n q(\widetilde{Q})^{n-1/n} q(\widetilde{Q})^{1/n} \\
&= \sum_{\widetilde{Q}\in\mathcal{W}} l(\widetilde{Q})^n q(\widetilde{Q})^n \\
&\asymp \int_\Omega \operatorname{dist}_{\mathrm{qh}}(z,z_0)^n\,dm(z),
\end{aligned}$$

thus proving the theorem.

We also promised to show that (1) $\Rightarrow$ $\operatorname{dist}_{\mathrm{qh}}(\cdot\,,z_0)\in L^1(\Omega_K,m)$. Indeed, since the total volume of the cubes shadowed by a given cube Q is $\lesssim s(Q)^n$, one can write

$$\begin{aligned}
\int_\Omega \operatorname{dist}_{\mathrm{qh}}(z,z_0)\,dm(z) &\asymp \sum_{Q\in\mathcal{W}} |Q|\operatorname{dist}_{\mathrm{qh}}(Q,z_0)\asymp \sum_{Q\in\mathcal{W}} |Q|\#\{Q':Q\prec Q'\} \\
&= \sum_{Q'\in\mathcal{W}}\sum_{Q\prec Q'} |Q| \lesssim \sum_{Q'\in\mathcal{W}} s(Q')^n.
\end{aligned}$$

Above the notation $Q\prec Q'$ means that the cube Q' shadows the cube Q. $\square$

Proof of Theorem 3. We will follow the ideology of [JM], preserving the notation.

Recall the definitions

$$\begin{aligned} d(x,r) &:= \max\{\delta(z) : z\in\Omega,\ |x-z|=r\}, \\ \delta(x) &:= \operatorname{dist}(x,\partial\Omega). \end{aligned}$$

We fix some point $z_0\in\Omega$ and write $q(x)$ for $\operatorname{dist}_{\mathrm{qh}}(x,z_0)$.

Denote by $\mathcal{D}$ the collection of all sets of large logarithmic density:

$$\mathcal{D} := \left\{\Delta\subset[0,1] : \text{ for all } r<r_0 \text{ one has } \int_{\Delta\cap[r,1]} \frac{dt}{t} \geq \frac{1}{2}\log\frac{1}{r}\right\},$$

where r_0 is some fixed small number, and define the Marcinkiewicz integral $\tilde{I}_0$ as

$$\tilde{I}_0(x) := \inf\left\{\int_\Delta \frac{d(x,t)^{n-1}}{t^n}\,dt : \Delta\in\mathcal{D}\right\} \quad \text{for } x\notin\Omega.$$

Then Theorem 2.5 in [JM] states that

$$\operatorname{Volume}(\{z\in\Omega^c : \tilde{I}_0(z)>\lambda\}) \leq C\operatorname{Volume}(\Omega)e^{-c\lambda},$$

with absolute constants (it is proven there for planar domains, but the general case is similar). Thus there is a constant $\sigma>0$ such that

$$\int_{\Omega^c} \exp(\sigma\tilde{I}_0(x))\,dm(x) < \infty \tag{13}$$

for any domain Ω.

It is immediate from the definition of the quasihyperbolic metric that for a point x away from z_0 one has $q(x)\gtrsim\int_{[0,1]} dt/d(x,t)$. Applying Hölder's inequality, we obtain for any point $x\in\mathbf{R}^n$ away from z_0, any positive number $r<1$, and any set $\Delta\in\mathcal{D}$ the estimate

$$\begin{aligned} \left(\frac{1}{2}\log\frac{1}{r}\right)^n &\leq \left(\int_{\Delta\cap[r,1]} \frac{dt}{t}\right)^n \leq \left(\int_{\Delta\cap[r,1]} \frac{dt}{d(x,t)}\right)^{n-1} \int_{\Delta\cap[r,1]} \frac{d(x,t)^{n-1}}{t^n}\,dt \\ &\lesssim q(x)^{n-1} \int_{\Delta\cap[r,1]} \frac{d(x,t)^{n-1}}{t^n}\,dt. \end{aligned} \tag{14}$$

Make a new domain Ω' by cutting out of Ω for every Whitney cube Q a cube $\frac{1}{3}Q$ with the same center and side length $\frac{1}{3}l(Q)$. It is easy to see that for $x\in\frac{1}{3}Q$ and

$r > nl(Q)$ one has $d(x,r) = d_\Omega(x,r) \asymp d_{\Omega'}(x,r)$, and hence the Marcinkiewicz integral I_0' for the new domain satisfies

$$I_0'(x) \asymp \inf\left\{ \int_{\Delta\cap[r,1]} \frac{d(x,t)^{n-1}}{t^n}\, dt : \Delta \in \mathcal{D} \right\},$$

if x belongs to a small cube Q with $l(Q) < r_0$. Thus taking infimum in (14) over $\Delta \in \mathcal{D}$ leads to

$$\left(\log \frac{1}{r}\right)^n \le \operatorname{const} q(x)^{n-1} I_0'(x) \tag{15}$$

for $x \in \frac{1}{3}Q$ and $r = nl(Q)$. By the assumption (3),

$$r \asymp \delta(x) \le \exp\left(\frac{-(q(x)^{n-1}\log q(x))^{1/n}}{o(1)} \right),$$

as $r \to 0$, which can be rewritten as

$$\left(\log \frac{1}{r}\right)^n \ge \frac{q(x)^{n-1}\log q(x)}{o(1)}.$$

Combining the latter estimate with (15) we infer that $I_0'(x) \ge \log q(x)/o(1)$. Thus for x close to the boundary of Ω (i.e. for small r)

$$q(x)^n \le \exp(\sigma I_0'(x)),$$

and we deduce (using (13) for the domain Ω') that

$$\begin{aligned} \int_\Omega q(x)^n\, dm(x) = \sum_{Q\in\mathcal{W}} \int_Q q(x)^n\, dm(x) &\lesssim \operatorname{const} + \sum_{\text{small } Q\in\mathcal{W}} \int_{\frac{1}{3}Q} \exp(\sigma I_0'(x))\, dm(x) \\ &\le \operatorname{const} + \int_{\mathbf{C}\setminus\Omega'} \exp(\sigma I_0'(x))\, dm(x) < \infty, \end{aligned}$$

reducing Theorem 3 to Theorem 2. $\square$

4. Applications in dynamics

We will reformulate condition (1) in the following dynamical setting: suppose that F is a polynomial, Ω is the domain of attraction to ∞, and $J_F = K = \partial\Omega$ is the Julia set of F. Suppose that $\{B_j\}$ is a finite collection of domains whose closure covers J_F, denote by $\{P_i^n\}$ the collection of all components of connectivity of pullbacks $F^{-n}B_j$, and by $N(P_i^n)$ the degree of F^n restricted to P_i^n.

One can write the geometric condition

$$\sum_{i,n} N(P_i^n)\operatorname{diam}(P_i^n)^2 < \infty. \tag{16}$$

Theorem 5. *In the setting above condition* (16) *implies* (1), *and is therefore sufficient for the conformal removability of the Julia set.*

Proof. The idea of the proof is that the hyperbolic metric is almost preserved near the Julia set by the dynamics, so, roughly speaking, Whitney cubes are pulled back to Whitney cubes, and their shadows to shadows.

More rigorously, for every sufficiently small Whitney cube Q take the minimal n such that Q is well inside some P_i^n, which is equivalent to taking the maximal n such that $F^n(Q)$ is well inside some B_j. Then (the required distortion estimates are provided by Lemma 7 in [GS1]) $s(Q) \leq \operatorname{diam}(P_i^n)$, and the number of cubes Q corresponding to a fixed P_i^n is $\lesssim N(P_i^n)$. Our theorem follows. □

References

[A] AHLFORS, L. V., *Lectures on Quasiconformal Mappings*, Van Nostrand Mathematical Studies **10D**, Van Nostrand, Princeton, N. J., 1966.

[AB] AHLFORS, L. and BEURLING, A., Conformal invariants and function-theoretic null-sets, *Acta Math.* **83** (1950), 101–129.

[Be] BESICOVITCH, A. S., On sufficient conditions for a function to be analytic and on behavior of analytic functions in the neighborhood of non-isolated points, *Proc. London Math. Soc.* **32** (1931), 1–9.

[Bi] BISHOP, C. J., Some homeomorphisms of the sphere conformal off a curve, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **19** (1994), 323–338.

[C] CARLESON, L., On null-sets for continuous analytic functions, *Ark. Mat.* **1** (1951), 311–318.

[G] GEHRING, F. W., The definitions and exceptional sets for quasiconformal mappings, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **281** (1960), 1–28.

[GS1] GRACZYK, J. and SMIRNOV, S., Collet, Eckmann and Hölder, *Invent. Math.* **133** (1998), 69–96.

[GS2] GRACZYK, J. and SMIRNOV, S., Non-uniform hyperbolicity in complex dynamics, *Preprint*, 1997–2000.

[Je] JERISON, D., The Poincaré inequality for vector fields satisfying Hörmander's condition, *Duke Math. J.* **53** (1986), 503–523.

[Jo] JONES, P. W., On removable sets for Sobolev spaces in the plane, in *Essays on Fourier Analysis in Honor of Elias M. Stein* (Fefferman, C., Fefferman, R. and Wainger, S., eds.), Princeton Math. Ser. **42**, pp. 250–267, Princeton Univ. Press, Princeton, N. J., 1995.

[JM] JONES, P. W. and MAKAROV, N. G., Density properties of harmonic measure, *Ann. of Math.* **142** (1995), 427–455.

[Ka] KAUFMAN, R., Fourier–Stieltjes coefficients and continuation of functions, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **9** (1984), 27–31.

[KW] KAUFMAN, R. and WU, J.-M., On removable sets for quasiconformal mappings, *Ark. Mat.* **34** (1996), 141–158.

[Ko] Koskela, P., Old and new on the quasihyperbolic metric, in *Quasiconformal Mappings and Analysis* (Duren, P., Heinonen, J., Osgood, B. and Palka, B., eds.), pp. 205–219, Springer-Verlag, New York, 1998.

[PR] Przytycki, F. and Rohde, S., Rigidity of holomorphic Collet–Eckmann repellers, *Ark. Mat.* **37** (1999), 357–371.

[U] Uy, N. X., Removable sets of analytic functions satisfying a Lipschitz condition, *Ark. Mat.* **17** (1979), 19–27.

[V] Väisälä, J., *Lectures on n-dimensional Quasiconformal Mappings*, Lecture Notes in Math. **229**, Springer-Verlag, Berlin–Heidelberg, 1971.

[Z] Ziemer, W. P., *Weakly Differentiable Functions*, Grad. Texts in Math. **120**, Springer-Verlag, New York, 1989.

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