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On the Cauchy problem for finitely degenerate hyperbolic equations of second order

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Abstract. This paper is devoted to the study of the Cauchy problem in C^∞ and in the Gevrey classes for some second order degenerate hyperbolic equations with time dependent coefficients and lower order terms satisfying a suitable Levi condition.

1. Introduction

In this paper we shall consider the Cauchy problem

$$\begin{cases} L(t,\partial_t,\partial_x)u(t,x)=0, \\ u(0,x)=u_0(x), \\ \partial_t u(0,x)=u_1(x), \end{cases} \tag{1}$$

on $[0,T]\times\mathbf{R}^n_x$, where

$$L(t,\partial_t,\partial_x)=\partial_t^2-L_2(t,\partial_x)-L_1(t,\partial_x),$$
$$L_2(t,\partial_x)=\sum_{i,j=1}^n a_{ij}(t)\partial^2_{x_i x_j},$$
$$L_1(t,\partial_x)=\sum_{j=1}^n b_j(t)\partial_{x_j},$$

under the weak hyperbolicity condition

$$\sum_{i,j=1}^n a_{ij}(t)\xi_i\xi_j\geq 0 \quad \text{for all } (t,\xi)\in\mathbf{R}\times S^{n-1}. \tag{2}$$

Let us define

$$a(t,\xi)=\sum_{i,j=1}^{n} a_{ij}(t)\frac{\xi_i\xi_j}{|\xi|^2}, \tag{3}$$

$$b(t,\xi)=\sum_{j=1}^{n} b_j(t)\frac{\xi_j}{|\xi|}. \tag{4}$$

We shall assume from now on that $a_{ij}\in C^\infty(\mathbf{R})$ and $b_j\in C^0(\mathbf{R})$. It is well known that the Cauchy problem (1) can fail to be C^∞-well posed, even if $b_j\equiv 0$, due to too fast oscillating coefficients (see [CS]); or, on the other hand, when the Levi condition is not satisfied by L_1, even if the coefficients a_{ij} are constants (see, e.g., [M]).

On the contrary, if L_2 is effectively hyperbolic, then (1) is C^∞-well posed for any choice of L_1 (see [N2] and its bibliography). We observe that in this simple case the effective hyperbolicity of L_2 means that if for some $(\bar t,\bar\xi)\in[0,T]\times S^{n-1}$ we have $a(\bar t,\bar\xi)=0$, then

$$\partial_t^2 a(\bar t,\bar\xi)>0. \tag{5}$$

The aim of this paper is to study the Cauchy problem (1) when the condition (5) is weakened to an assumption of finite degeneracy, and under a very precise *Levi condition* on the lower order term L_1. More precisely, we shall prove the following theorem.

Theorem 1. *Assume that*

$$\sum_{j=0}^{\infty}|\partial_t^j a(t,\xi)|\neq 0 \quad \textit{for all } (t,\xi)\in[0,T]\times S^{n-1}. \tag{6}$$

Let k be the minimal integer satisfying

$$\sum_{j=0}^{k}|\partial_t^j a(t,\xi)|\neq 0 \quad \textit{for all } (t,\xi)\in[0,T]\times S^{n-1}. \tag{7}$$

Suppose that there exist $C>0$ and $\gamma\in\left[0,\frac12\right]$ such that

$$|b(t,\xi)|\le Ca(t,\xi)^\gamma \quad \textit{for all } (t,\xi)\in[0,T]\times S^{n-1}. \tag{8}$$

Then, if

$$\gamma+1/k<\tfrac12, \tag{9}$$

the Cauchy problem (1) *is well posed in $\gamma^{(s)}$ for*

$$s\le\frac{1-\gamma}{\frac12-(\gamma+1/k)}. \tag{10}$$

On the contrary, if

$$\gamma + 1/k \geq \tfrac{1}{2}, \tag{11}$$

the Cauchy problem (1) *is C^∞-well posed.*

We can easily show that under the assumption (6), some k exists for which (7) is satisfied, thanks to the regularity of $a(t,\xi)$ and the compactness of $[0,T]\times S^{n-1}$. We denote by $\gamma^{(s)}$ the (projective) Gevrey class with exponent s (≥ 1), that is, the set of all functions $f \in C^\infty(\mathbf{R}^n)$ such that for any $r>0$ there is a constant $C_r>0$ fulfilling

$$\sup_{x\in\mathbf{R}^n} |\partial_x^\alpha f(x)| \leq C_r r^{|\alpha|}(\alpha!)^s$$

for every multi-index $\alpha \in \mathbf{Z}_+^n$.

We remark that some hyperbolic second order equations finitely degenerating at a point are studied in [K] and in [IO], who consider the coefficients depending also on x, but under more restricted conditions. More precisely, in [IO] it is proved that if

$$a(t,\xi) \geq \delta t^{2l}, \tag{12}$$

$$|b(t,\xi)| \leq C\big(t^\nu + \sqrt{a(t,\xi)}\,\big) \tag{13}$$

for some positive constants δ and C, and for l and ν with $0 \leq \nu < l-1$, then the Cauchy problem (1) is $\gamma^{(s)}$ well posed for $s \leq s_0 = (2l-\nu)/(l-\nu-1)$. It is easy to see that if (12) and (13) are satisfied, then we can apply Theorem 1 with $k=2l$ and $\gamma=\nu/2l$, obtaining the same Gevrey exponent; but, conversely, the assumptions of Theorem 1 are more general. In fact, under the hypothesis (7) an inequality like (12) is not true in general, even for t near 0; moreover the condition (13) is more restricted than (8), as shown by the following examples.

Example 1. Let us consider, in the case $n=2$, the following coefficients:

$$a_{11}(t) = t^6, \quad a_{12}(t) = a_{21}(t) = 0, \quad a_{22}(t) = t^2,$$
$$b_1(t) = t, \qquad b_2(t) = t^{1/3}.$$

Owing to Theorem 1 with $k=6$, $\gamma=\frac{1}{6}$, we know that the Cauchy problem (1) is well posed in $\gamma^{(s)}$ for $s \leq 5$, meanwhile by Theorem 1.2 of [IO] we get $s \leq \frac{17}{5}$.

Example 2. Let us now consider:

$$a_{11}(t) = t^4, \quad a_{12}(t) = a_{21}(t) = 0, \quad a_{22}(t) = t^2,$$
$$b_1(t) = t, \qquad b_2(t) = t^{1/2}.$$

In this case (11) is fulfilled and so (1) is C^∞-well posed.

Finally we remark that in the case of one space variable, if (12) is satisfied, then the Gevrey exponent given by Theorem 1 coincides with the one of [IO], see the example below, studied in [I].

Example 3. For the operator

$$L=\partial_t^2-t^{2l}\partial_x^2+\sqrt{-1}\,t^{\nu}\partial_x$$

it is proved in [I] that the Cauchy problem (1) is well posed in $\gamma^{(s)}$ if and only if $s\leq s_0=(2l-\nu)/(l-\nu-1)$.

The techniques used in the present paper are in part similar to those of [CDS] and [CJS], but we also require the following precise estimates; their proofs are inspired by [N1].

Lemma 1. *Let us consider $a(t,\xi)$ defined by (3), satisfying (7). Then there exist M and ε_0 positive such that for any $\varepsilon\in(0,\varepsilon_0]$ we have*

$$\int_0^T \frac{|\partial_t a(t,\xi)|}{a(t,\xi)+\varepsilon}\,dt\leq M\log\frac{1}{\varepsilon}. \tag{14}$$

Lemma 2. *Let us consider $a(t,\xi)$ defined by (3) and let k be given by (7). Then for any $\eta\geq 0$ there exist M_η and ε_0 positive such that for any $\varepsilon\in(0,\varepsilon_0]$ we have*

$$\int_0^T \frac{1}{(a(t,\xi)+\varepsilon)^{\eta}}\,dt\leq\begin{cases} M_\eta, & \text{if } \eta<1/k,\\ M_\eta\log\dfrac{1}{\varepsilon}, & \text{if } \eta=1/k,\\ M_\eta\varepsilon^{1/k-\eta}, & \text{if } \eta>1/k.\end{cases}$$

2. Proofs of Lemmas 1 and 2

Proof of Lemma 1. Let us fix $(\bar{t},\bar{\xi})\in[0,T]\times S^{n-1}$ and let $\bar{k}\leq k$ be an even integer such that

$$\partial_t^j a(\bar{t},\bar{\xi})=0,\ j=0,\dots,\bar{k}-1,\quad\text{and}\quad \partial_t^{\bar{k}}a(\bar{t},\bar{\xi})\neq 0. \tag{15}$$

Then, by virtue of the Malgrange preparation theorem (see, for instance, [H], Theorem 7.5.5), we can write

$$a(t,\xi)=e(t,\xi)[(t-\bar{t})^{\bar{k}}+b_1(\xi)(t-\bar{t})^{\bar{k}-1}+\dots+b_{\bar{k}}(\xi)]=e(t,\xi)p(t,\xi) \tag{16}$$

for $(t,\xi)\in U=\{(t,\xi)\in\mathbf{R}^{1+n}:|t-\bar{t}|\le\delta,\ |\xi-\bar{\xi}|\le\delta\}$, where $e(t,\xi)$ and $b_j(\xi)$ are C^∞ functions with $e(\bar{t},\bar{\xi})\ne 0$ and $b_j(\bar{\xi})=0$, $j=1,\dots,\bar{k}$, respectively. Further, due to (2), we may suppose that $e(t,\xi)$ is positive in U and so that $p(t,\xi)$ is nonnegative in U. For $(t,\xi)\in U$ we can factorize

$$p(t,\xi)=(t-t_1(\xi))(t-t_2(\xi))\dots(t-t_{\bar{k}}(\xi)). \tag{17}$$

Let C_0, C_1, C_2, C_3 and C_4 be positive constants satisfying

$$C_0\le e(t,\xi)\le C_1,\quad |\partial_t e(t,\xi)|\le C_2,\quad \sum_{j=1}^{\bar{k}}\prod_{i\ne j}|t-t_i(\xi)|\le C_3,\quad \max_{j=1,\dots,\bar{k}}|t_j(\xi)|\le C_4$$

in U. Then we have, for $|\xi-\bar{\xi}|\le\delta$ $(\le\frac{1}{2}T)$

$$\begin{aligned}\int_{\bar{t}-\delta}^{\bar{t}+\delta}\frac{|\partial_t a(t,\xi)|}{a(t,\xi)+\varepsilon}\,dt&\le\int_{\bar{t}-\delta}^{\bar{t}+\delta}\frac{|\partial_t e|p}{ep+\varepsilon}\,dt+\int_{\bar{t}-\delta}^{\bar{t}+\delta}\frac{e|\partial_t p|}{ep+\varepsilon}\,dt\\&\le\frac{C_2}{C_0}T+\frac{C_1}{C_0}\int_{\bar{t}-\delta}^{\bar{t}+\delta}\frac{|\partial_t p|}{p+\varepsilon/C_0}\,dt.\end{aligned}\tag{18}$$

Here, noting that

$$\partial_t p(t,\xi)=\sum_{j=1}^{\bar{k}}\prod_{i\ne j}(t-t_i(\xi))$$

and taking $\delta\le C_4$ and $\varepsilon_0\le C_0C_3(T+2C_4)\le 1/2\varepsilon_0$, we find

$$\begin{aligned}\int_{\bar{t}-\delta}^{\bar{t}+\delta}\frac{|\partial_t p|}{p+\varepsilon/C_0}\,dt&\le\sum_{j=1}^{\bar{k}}\int_{\bar{t}-\delta}^{\bar{t}+\delta}\frac{1}{|t-t_j(\xi)|+\varepsilon/C_0C_3}\,dt\\&\le\sum_{j=1}^{\bar{k}}\int_{-\delta}^{T+\delta}\frac{1}{|t-\operatorname{Re}t_j|+\varepsilon/C_0C_3}\,dt\\&\le\bar{k}\int_{-2C_4}^{T+2C_4}\frac{1}{|t|+\varepsilon/C_0C_3}\,dt\\&\le 2\bar{k}\int_0^{T+2C_4}\frac{1}{t+\varepsilon/C_0C_3}\,dt=2\bar{k}\log\left(1+\frac{C_0C_3(T+2C_4)}{\varepsilon}\right)\\&\le 4\bar{k}\log\frac{1}{\varepsilon}\end{aligned}\tag{19}$$

for $\varepsilon\in(0,\varepsilon_0]$. Therefore, by repeating the calculations in (18) and (19), thanks to the compactness of S^{n-1}, we obtain

$$\int_{\bar{t}-\bar{\delta}}^{\bar{t}+\bar{\delta}}\frac{|\partial_t a(t,\xi)|}{a(t,\xi)+\varepsilon}\,dt\le M\log\frac{1}{\varepsilon}$$

for some $\bar{\delta}>0$, $M>0$, all $\xi\in S^{n-1}$ and $\varepsilon\in(0,\varepsilon_0]$ (ε_0 is retaken small enough, if necessary). Finally we conclude (14) due to the compactness of $[0,T]$. □

Proof of Lemma 2. Let $\eta \geq 0$ be fixed. We also fix $(\bar{t}, \bar{\xi}) \in [0,T] \times S^{n-1}$ and, with the same notation as in the proof of Lemma 1, we can deduce

$$\begin{aligned}\int_{\bar{t}-\delta}^{\bar{t}+\delta} \frac{1}{(a(t,\xi)+\varepsilon)^{\eta}}\,dt &\leq \frac{1}{C_0^{\eta}} \int_{\bar{t}-\delta}^{\bar{t}+\delta} \frac{1}{(p(t,\xi)+\varepsilon/C_0)^{\eta}}\,dt \\ &\leq \frac{1}{C_0^{\eta}} \sum_{j=1}^{\bar{k}} \int_{\bar{t}-\delta}^{\bar{t}+\delta} \frac{1}{(|t-\mathrm{Re}\,t_j|^{\bar{k}}+\varepsilon/C_0)^{\eta}}\,dt \\ &\leq \frac{1}{C_0^{\eta}} \bar{k} \int_{-C_4}^{T+C_4} \frac{1}{(|t|^{\bar{k}}+\varepsilon/C_0)^{\eta}}\,dt \qquad (20) \\ &\leq \frac{2}{C_0^{\eta}} k \left(\int_0^1 \frac{1}{(t^k+\varepsilon/C_0)^{\eta}}\,dt + T + C_4 \right) \\ &\leq \frac{2}{C_0^{\eta}} k \left(\left(\frac{\varepsilon}{C_0}\right)^{1/k-\eta} + \int_{(\varepsilon/C_0)^{1/k}}^{1} \frac{1}{t^{k\eta}}\,dt + T + C_4 \right).\end{aligned}$$

Hence, by using a compactness argument as at the end of the proof of Lemma 1, Lemma 2 immediately follows from (20). □

3. Proof of Theorem 1

First of all, if $k=0$, then L is strictly hyperbolic; moreover obviously k is even and so we may assume that $k \geq 2$. Since the case $s=1$ is well known (see [CDS]), we suppose that $s>1$ and u_0 and u_1 are compactly supported. Then the Cauchy problem (1) has a unique solution $u \in C^2([0,T]; \mathcal{D}^{(s)'})$ for $1<s<2$ (see [CJS]). Here $\mathcal{D}^{(s)'}$ is defined as the dual space of $\mathcal{D}^{(s)}$. Thus we need only check the regularity of the solution with respect to x variables. For this purpose, denoting the partial Fourier transform of u in x by

$$v(t,\xi) = \int_{\mathbf{R}^n} u(t,x) \exp(-\sqrt{-1}\,x \cdot \xi)\,dx,$$

it will be sufficient to estimate the growth order of $v(t,\xi)$ with respect to ξ. The function $v(t,\xi)$ solves the ordinary differential equations in t, depending on the parameter ξ,

$$\partial_t^2 v + a(t,\xi)|\xi|^2 v + \sqrt{-1}\, b(t,\xi)|\xi| v = 0. \qquad (21)$$

With the same method in [CJS], we define

$$a_{\varepsilon}(t,\xi) = a(t,\xi) + \varepsilon$$

and introduce the ε-approximate energy

$$E_\varepsilon(t,\xi)=a_\varepsilon(t,\xi)|\xi|^2|v|^2+|\partial_t v|^2. \tag{22}$$

Differentiating $E_\varepsilon(t,\xi)$ in t and taking (21) into account, we enjoy

$$\frac{d}{dt}E_\varepsilon(t,\xi)\le\left(\frac{|\partial_t a(t,\xi)|}{a(t,\xi)+\varepsilon}+\frac{\varepsilon|\xi|}{\sqrt{a(t,\xi)+\varepsilon}}+\frac{|b(t,\xi)|}{\sqrt{a(t,\xi)+\varepsilon}}\right)E_\varepsilon(t,\xi)$$

and, Gronwall's inequality and (8) yield

$$\begin{aligned} E_\varepsilon(t,\xi)\le E_\varepsilon(0,\xi)\exp\bigg(&\int_0^T\frac{|\partial_t a(t,\xi)|}{a(t,\xi)+\varepsilon}\,dt+\varepsilon|\xi|\int_0^T\frac{1}{\sqrt{a(t,\xi)+\varepsilon}}\,dt\\ &+\int_0^T\frac{C}{(a(t,\xi)+\varepsilon)^{1/2-\gamma}}\,dt\bigg). \end{aligned}\tag{23}$$

Here, putting $|\xi|=\varepsilon^{-\sigma}$, we distinguish two cases.

(i) If $\gamma+1/k\ge\frac{1}{2}$, choosing $\sigma=(k+2)/2k$, we obtain by Lemmas 1 and 2

$$E_\varepsilon(t,\xi)\le E_\varepsilon(0,\xi)\exp(C\log|\xi|) \tag{24}$$

for some $C>0$ and for $|\xi|$ large enough.

(ii) If $\gamma+1/k<\frac{1}{2}$, then we select $\sigma=1-\gamma$ and hence we get by Lemmas 1 and 2

$$E_\varepsilon(t,\xi)\le E_\varepsilon(0,\xi)\exp(C\log|\xi|+C|\xi|^{[1/2-(\gamma+1/k)]/(1-\gamma)}) \tag{25}$$

for some $C>0$ and for $|\xi|$ large enough.

Thus we arrive at the conclusion from (24) and (25) by using arguments similar to the ones in [CDS] and [CJS]. □

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