On the separation properties of the duals of general topological vector spaces

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Consider a non locally convex topological vector space E. Suppose there is a non-zero point x in the subset $E_1 \subset E$ of all points which cannot be separated from the origin by any continuous linear form on E. One might ask whether x can be separated from the origin by a continuous linear form which is defined only on E_1. It will be shown by means of examples that this may be the case. Since E_1 is a linear subspace — namely, the intersection $\bigcap_{f \in E'} f^{-1}(0)$ of all closed hyperplanes through the origin — this fact gives rise to a more general question: Let E_2 be the subspace in E_1 of points which cannot be separated from the origin by any continuous linear form; and then define recursively subspaces $E_3 \supset E_4 \supset \ldots$ In a natural way we thus get a transfinite decreasing »sequence» (indexed by the ordinals) of closed linear subspaces of E. Obviously, this sequence must become stationary at some ordinal $\alpha(E)$. The observation just mentioned shows that this need not happen at once, i.e., we may have $E_1 \neq E_2$, so that $\alpha(E) \geq 2$. Thus we ask: Which ordinal values can be assumed by $\alpha(\cdot)$? Our aim in II below is to answer this question by constructing examples to show that even for locally bounded spaces, $\alpha(\cdot)$ may assume *any* ordinal value.

In I below, we investigate some general properties of the transfinite »sequence» mentioned above.

As a by-product of the constructive methods employed in II, we obtain (Section II.3) a certain isometric imbedding of metric spaces into p-normed spaces, which has universal and functorial properties.

For brevity, we will write tvs for topological vector space(s). Further, E' will denote the (topological) dual of a tvs E, and ω is as usual the least transfinite ordinal. All tvs are supposed to have the same scalar field, which may be the real or the complex number field. A linear subspace of a tvs is always topologized by the subspace topology.

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I. General considerations

1. *Definition.* For a tvs E we write

$$\Lambda E = \bigcap_{f \in E'} f^{-1}(0) .$$

By transfinite induction we define for every ordinal ν a closed linear subspace $\Lambda^\nu E$ so that

$$\Lambda^\nu E = \Lambda\Big(\bigcap_{\nu' < \nu} \Lambda^{\nu'} E\Big) .$$

1.1. PROPOSITION. *Under* $\supset$, *the class* $\{\Lambda^\nu E\}_\nu$ *is well-ordered and has a last element. There is a least ordinal* ν_0 *such that* $\Lambda^{\nu_0} E = \Lambda^{\nu_0 + 1} E$.

1.2. *Definition.* For a tvs E, the ordinal ν_0 of 1.1 is denoted $\alpha(E)$.

1.3. *Remark.* To see that Definitions 1 and 1.2 are worth-while, it is of course essential to show that there are tvs E with $\alpha(E) \geq 2$. As pointed out in the introduction, it will be seen below that $\alpha(\cdot)$ may assume any ordinal value (see II.2.1 and cf. also Section 7 of I).

2. PROPOSITION. *For tvs* E *and* F *and a continuous linear mapping* $f: E \to F$, *we have, for every ordinal* ν, *that* $f(\Lambda^\nu E) \subset \Lambda^\nu F$. *If* f *is open and* $f^{-1}(0) \subset \Lambda^{\nu_0} E$ *for some* ν_0, *we have* $f(\Lambda^\nu E) = \Lambda^\nu F$ *for* $\nu \leq \nu_0$.

Proof. We see that

$$\begin{aligned} f(\Lambda E) &= f\Big(\bigcap_{\varphi \in E'} \varphi^{-1}(0)\Big) \\ &\subset f\Big(\bigcap_{\psi \in F'} f^{-1}\psi^{-1}(0)\Big) = \bigcap_{\psi \in F'} \psi^{-1}(0) = \Lambda F , \end{aligned}$$

which gives the first part in the case $\nu = 1$. If f is open and $f^{-1}(0) \subset \Lambda E$, every element in E' is of the form $\psi \circ f$ with $\psi \in F'$; hence we may replace the inclusion by equality.

For arbitrary ν, the relations are now obtained by straightforward transfinite induction.

2.1. COROLLARY. *Let* E *be a tvs. For every quotient space* E/H, *say, such that* $H \subset \Lambda^{\alpha(E)} E$, *we have* $\alpha(E/H) = \alpha(E)$.

2.2 COROLLARY. *A tvs E has a quotient space E/H with $\alpha(E/H) = \nu$ whenever $\nu \leq \alpha(E)$.*

Proof. Take $H = \Lambda^\nu E$.

3. PROPOSITION. *A tvs E has precisely one subspace $L \subset E$ such that*
(i) $L' = \{0\}$ *and*
(ii) *every subspace* $H \subset E/L$, $H \neq \{0\}$, *has* $H' \neq \{0\}$.

Proof. 1° Uniqueness: Suppose (i) and (ii) to be satisfied for $L = L_1$ as well as for $L = L_0$, say. If $L_1 \not\subset L_0$, the subspace $(L_0 + L_1)/L_0$ of E/L_0 is distinct from $\{0\}$ and has a non-vanishing continuous linear form f on account of (ii) for $L = L_0$. Then f induces a non-vanishing continuous linear form on L_1, and thus raises contradiction against (i) for $L = L_1$. Hence $L_1 \subset L_0$, and similarly $L_0 \subset L_1$.

2° Existence: Let L_0 be the linear hull of all subspaces with dual $\{0\}$. A given continuous linear form on L_0 vanishes on all subspaces with dual $\{0\}$, and thus on each of a class of sets spanning L_0, and so on L_0. Hence $L_0' = \{0\}$. The subspace L_0 is also the largest one with dual $\{0\}$.

Suppose H is any subspace $\neq \{0\}$ of E/L_0. Its canonical inverse image H_1 in E has a non-vanishing continuous linear form f by the preceding paragraph (since $H_1 \setminus L_0 \neq \emptyset$). But $f(L_0) = 0$, so f induces a non-vanishing continuous linear form on $H = H_1/L_0$. Hence (i) and (ii) are satisfied when $L = L_0$.

The proof also gives the next proposition (same notation).

3.1. PROPOSITION. *The subspace L is the largest one which has dual $\{0\}$.*

3.2. THEOREM. *When E and L as in 3, we have $L = \Lambda^{\alpha(E)}E$.*

Proof. Since $\Lambda^{\alpha(E)}E$ clearly has dual $\{0\}$, we get $\Lambda^{\alpha(E)}E \subset L$ from 3.1.

Conversely, we show by transfinite induction that $L \subset \Lambda^\nu E$ for every ν. The assertion is trivially valid for $\nu = 1$; thus assume that $L \subset \Lambda^{\nu'} E$ whenever $\nu' < \nu$. But $L \subset \bigcap_{\nu' < \nu} \Lambda^{\nu'} E = P_\nu$, say, certainly implies $f(L) = 0$ whenever $f \in P_\nu'$. Hence $L \subset \Lambda^\nu E$ by Definition 1.

3.3. PROPOSITION. *When E and L as in 3, we have, for every ν, that $(\Lambda^\nu E)/L = \Lambda^\nu(E/L)$. Further, $\alpha(E) = \alpha(E/L)$.*

Proof. Follows from 3.2, 2, and 2.1.

3.4. *Definition.* A tvs will be called an *A-space* if every subspace distinct from $\{0\}$ has dual distinct from $\{0\}$.

3.5. *Remark.* When E and L as in 3, notice that (i) E has dual $\{0\}$ if and only if $E = L$, and that (ii) E is an A-space if and only if $L = \{0\}$.

Furthermore, L is the smallest subspace for which E/L is an A-space. (Follows from 2.)

We finally state yet another characterization of L.

3.6. PROPOSITION. *When E and L as in 3, every continuous linear mapping $f: E \to F$ into an A-space may be canonically decomposed according to*

$$E \xrightarrow{g} E/L \xrightarrow{h} F ,$$

where g and h are continuous and linear. Moreover, L is the largest subspace for which such a decomposition may be accomplished for all f and F.

Proof. If the image in F of L is distinct from $\{0\}$, it has a non-vanishing continuous linear form φ, say. Then $\varphi \circ f$ does not vanish on L, which is contradictory.

The last statement now follows if we take $F = E/L$ and let f be the canonical mapping.

4. Consider an A-space E. If E is not locally convex, a continuous linear form may of course not always be extended from a subspace to E. However, if E has separating dual — that is, if $\alpha(E) = 1$ — a continuous linear form defined on a *finite-dimensional* subspace may always be extended to E. But if $\alpha(E) \geq 2$, not even that is true; just let the given form be defined and non-vanishing on a one-dimensional subspace of ΛE. The best we can accomplish in such a case is to examine whether the functional may be extended to certain subspaces or not. Thus we shall say that a class $\mathcal{C}$ of subspaces of E is an *extension-class* if the following condition is satisfied:

For every subspace K and non-vanishing continuous linear form f defined on it, there is a space $C \in \mathcal{C}$ such that

(i) $f(K \cap C) \neq \{0\}$ and

(ii) for every finite-dimensional subspace F of $K \cap C$, the restriction of f to F can be extended to C.

Of course, the class of all *one-dimensional* subspaces forms a very trivial example of an extension-class. To obtain something more significant, we restrict our attention to extension classes that are *totally ordered* under inclusion. If $\alpha(E)$ is finite, there is such an extension-class, i.e., the class $\mathcal{L} = \{\Lambda^{\nu}E\}_{\nu} \cup \{E\}$. Namely, if f and K are given, take $C = \Lambda^{n_0}E$, when n_0 is the least integer such that $f(K \cap \Lambda^{n_0}E) \neq \{0\}$.

Before turning to a more general situation, we introduce another new notion. We say that E is an *AP-space* if every 0-neighbourhood contains all but finitely many of the spaces of $\mathcal{L}$. — Notice that E may be an AP-space only if $\alpha(E) \leq \omega$

and $\bigcap_{\nu<\omega} \Lambda^\nu E = \{0\}$; and further, the topology on E may be coarsened to an AP-space topology if and only if the latter conditions are fulfilled. If $\alpha(E)$ is finite, E is trivially an AP-space.

Furthermore, let us by an *A'-space* understand an A-space which has a 0-neighbourhood base of which each element U has the property that $\bigcap_{\varrho>0} \varrho U$ is a linear subspace. This latter condition is not automatically fulfilled in a non locally convex tvs; however, it is, for instance, in the case of a *supremum topology of locally bounded* topologies. (In view of 7.7 or II.2.2, it is clear from the remark of the preceding paragraph that there are, indeed, *AP-spaces* of the last-mentioned kind with $\alpha = \omega$.)

4.1. PROPOSITION. *For an A'-space E, the following statements are equivalent:*
(i) *E is an AP-space,*
(ii) *E has an extension-class that is well-ordered under $\supset$, and*
(iii) *E has an extension-class that is a decreasing sequence under inclusion.*

Proof. (i) *implies* (iii): The argument above for $\alpha(E) < \omega$ is applicable also in the case $\alpha(E) = \omega$. For suppose f and K to be given. Then f is bounded on the intersection of K with some 0-neighbourhood U, say. Since E is an AP-space, U contains $\Lambda^{n_0}E$ for some $n_0 < \omega$. Then $f(K \cap \Lambda^{n_0}E) = \{0\}$, and we can proceed as before. So $\mathscr{L}$ is an extension-class.

(iii) *implies* (ii): Trivial.

(ii) *implies* (i): Let $\mathscr{C}$ be a well-ordered extension-class.

Aiming at a contradiction, we assume E not to be an AP-space. On account of the well-ordering, the class $\mathscr{C}$ has then, for each $n < \omega$, a largest space C_n which does not contain $\Lambda^n E$. Then C_n contains $\Lambda^{n+1}E$; otherwise, the defining conditions for an extension-class would not be fulfilled if we let K be a one-dimensional subspace of $\Lambda^{n+1}E$ not in C_n. For if $C^0 \in \mathscr{C}$, then C^0 has intersection $\{0\}$ with K — and thus we have $f(K \cap C^0) = 0$ for every f — *except* if $C^0 \setminus C_n \neq \emptyset$; and in that case, we have $C^0 \supset \Lambda^n E$, so a non-vanishing f on $K \subset \Lambda^{n+1}E$ can certainly not be extended to C^0.

Since E is an A'-space that is not an AP-space, we can now find a 0-neighbourhood U such that $N = \bigcap_{\varrho>0} \varrho U$ is a linear subspace which does not contain $\Lambda^n E$ for any $n < \omega$. So we can take elements $z_n \in \Lambda^n E \setminus N$, $n < \omega$; then we take for K the linear hull of $N \cup \{z_n | n < \omega\}$. We may assume that $N \supset \bigcap_{n<\omega} C_n$. (Otherwise, replace U by $U + \bigcap_{n<\omega} C_n$.) For each $n < \omega$, we define a continuous linear form g_n on K so that $g_n(z_n) \neq 0$ and so that $g_n(K \cap \Lambda^{n+1}E) = 0$; this is possible, since $K \cap \Lambda^{n+1}E$ has finite codimension in K. Further, by the choice of U and z_n, we see that each of the g_n is bounded on $U \cap K$; so take $\gamma_n > 0$ so that $\gamma_n |g_n| \leq 1/2^n$ on $U \cap K$. By taking $f = \Sigma_n \gamma_n g_n$ on K, we shall have our contradiction against the defining properties of an extension-class. First, f is continuous, since $|f| \leq 1$ on $U \cap K$. Then, on account of $f(\bigcap_{n<\omega} C_n) = 0$ (seen from $f(N) = 0$, which follows from $|f| \leq 1$ on $U \cap K$), it is sufficient to show that f cannot be extended to $C^0 \supset C_n$, say, from the one-dimensional subspace spanned by $z_{n+2} \in C_n \cap K$. This is so, for $C^0 \supset C_n \supset \Lambda^{n+1}E$ and $z_{n+2} \in \Lambda^{n+2}E$.

To illuminate the situation somewhat, we state the following simple proposition.

4.2. PROPOSITION. *For a totally ordered extension-class* $\mathcal{C}$ *of an* A*-space* E, *we have* card $\mathcal{C} \geq$ card min $\{\alpha(E), \omega\}$.

4.3. *Remark.* The notion of AP-space clearly has the following significance. An A-space E is an AP-space if and only if it is the inverse limit of some A-spaces with finite α-values (namely, $E/\Lambda^n E$ where $n < \omega$; cf. 2 above).

5. For a tvs E, it is natural to ask whether the $\Lambda^\nu E$-spaces may be characterized in terms not involving these spaces. A first step in this direction was taken in 3.2 above, where we characterized $\Lambda^{\alpha(E)}E$. Also in 4—4.1 our pursuit was to enlighten this question; namely, the AP-space property, which is defined as a property of the class $\{\Lambda^\nu E\}_\nu$, is there related to the existence of certain extension-classes; and the latter notion is defined otherwise. And notice that the case $\alpha(E) = \omega$ there appears as »critical». — The rest of the question we have to leave as an open problem.

6. This is an auxiliary section.

6.1. *Definition.* Let E be a vector space. In accordance with Landsberg [3], we say that

a) a functional $x \to \|x\|$ is a *p-norm*, where $0 < p \leq 1$, if it satisfies

(i) $\|x + y\| \leq \|x\| + \|y\|$,

(ii) $\|\lambda x\| = |\lambda|^p \|x\|$, and

(iii) $\|x\| = 0$ implies $x = 0$, whenever $x, y \in E$ and λ scalar;

b) if a p-norm $\|\cdot\|$ on E is given and if thereby E is considered as a tvs with 0-neighbourhood base $\{\{x \mid \|x\| < \lambda\} \mid \lambda > 0\}$ (this is certainly a 0-neighbourhood base of a vector space topology; cf. Landsberg [3]), then E is a *p-normed space.*

6.2. *Examples.* We remind of some common examples of p-normed spaces:

a) the spaces l^p of sequences $x = (\xi^1, \xi^2, \ldots)$ such that $\sum_1^\infty |\xi^k|^p < \infty$, with p-norm $\|x\| = \sum_1^\infty |\xi^k|^p$. For $p < 1$, we here have simple examples of non locally convex spaces that are dual-separated (Tychonoff [7]; cf. also Landsberg [3]).

b) the spaces $L^p(0, 1)$ of measurable functions $x(\tau)$ on $(0, 1)$ such that

$\int_0^1 |x(\tau)|^p d\tau < +\infty$ (or, properly, certain equivalence classes of such functions) with p-norm $\|x\| = \int_0^1 |x(\tau)|^p d\tau$. The $L^p(0, 1)$ with $p < 1$ are known not to have any non-vanishing continuous linear forms (Day [1]).

c) the quotient space E/H of a given p-normed space E with p-norm $\|\cdot\|$, say, for a closed subspace H. Then the quotient space topology is given by the quotient space p-norm $\|\hat{x}\|_\sim = \inf\{\|y\| | y \in x + H\}$, if the canonical mapping $E \to E/H$ is written $x \to \hat{x}$.

Also, we introduce:

d) the p-normed direct sum of a class $\{E_\iota\}$ of p-normed spaces with p-norms $\{\|\cdot\|_\iota\}$, say, as a *p-normed space* S as follows. Consider the algebraic direct sum $\oplus_\iota E_\iota$ of the linear spaces E_ι, i.e., the linear space of all formal finite sums $\sum_\iota x_\iota$, such that $x_\iota \in E_\iota$ and all but finitely many of the x_ι are zero. Then the p-normed space S is defined as the linear space $\oplus_\iota E_\iota$ endowed with the p-norm $\|\cdot\|$ given by $\|\sum_\iota x_\iota\| = \sum_\iota \|x_\iota\|_\iota$.

6.3. *Example.* We also recall the space $S(0, 1)$ of measurable functions on $(0, 1)$ with the topology of convergence in measure. This topology is also given by the metric $d(x, y) = \int_0^1 \frac{|x(\tau) - y(\tau)|}{1 + |x(\tau) - y(\tau)|} d\tau$. It is well known that this space has dual $\{0\}$ (cf. [5]).

6.4. We now introduce some notation to be used henceforth. By $\operatorname{co} X$ we denote the convex hull of a set X in a linear space E; and $\operatorname{lin} X$ will be the linear hull. If E is a tvs, $\overline{\operatorname{co}}\, X$ and $\overline{\operatorname{lin}}\, X$ will stand for the closed convex resp. linear hull. Further, we will make extensive use of the following simple consequence of Hahn-Banach's theorem:

A point in a tvs E belongs to the convex hull of every 0-neighbourhood if and only if it is not separated from 0 by E' (LaSalle [4]).

Otherwise stated: Let $\mathscr{B}$ be a 0-neighbourhood base in E. Then

$$\bigcap_{U \in \mathscr{B}} \operatorname{co} U = \Lambda E .$$

In this connection, notice that Definition 1 above may thus be rephrased in terms of convex hulls of 0-neighbourhoods in subspaces:

6.5. PROPOSITION. *Let E be a tvs. For a class $\mathscr{A}$ of subsets of E, write $ic\mathscr{A}$ for the intersection of the convex hulls of the sets of $\mathscr{A}$. Further, let $\mathscr{B}$ be a 0-neighbourhood base of E; and define inductively*

$$icn^1 = ic\mathcal{B}$$
$$icn^\nu = ic\{U \cap icn^{\nu'} | U \in \mathcal{B}\ ,\ \ \nu' < \nu\}$$

for all ordinals ν. *Then*

$$icn^\nu = \Lambda^\nu E$$

for every ordinal ν.

Proof. By transfinite induction.

7. We now turn to the problem of finding tvs E with $\alpha(E) \geq 2$. First we prove a general assertion on sufficient conditions for a given tvs to have a subspace with α-value ≥ 2.

7.1. THEOREM. *Let E be a metrizable tvs and $K \subset E$ a closed subspace which is separable. Suppose there is a decreasing sequence $\{E_i\}_{i=1}^\infty$ of closed subspaces of E such that*

(i) $\bigcap\limits_{i=1}^\infty E_i = K$

(ii) $\Lambda E_i \supset K,\ \ i \geq 1$.

Then E has a closed separable subspace $K_1 \supset K$ such that $\Lambda K_1 = K$.

In particular, if K is dual-separated, we get K_1 with $\alpha(K_1) = 2$; and, more generally, if $\alpha(K)$ is finite, we get $\alpha(K_1) = \alpha(K) + 1$.

Proof. Let $\{V_k\}_{k=1}^\infty$ be a countable 0-neighbourhood base of E and $\{z_k\}_{k=1}^\infty$ a sequence which spans K. By condition (ii) and 6.4 above, we can take $x_{ij} \in E$ for $i \geq 1$ and $1 \leq j \leq k_i$, say, so that

1° $x_{ij} \in E_i \cap V_i$

2° $z_n \in \operatorname{co}\{x_{ij}\}_{j=1}^{k_i},\ \ 1 \leq n \leq i$,

for each $i \geq 1$ and suitable k_i.

Now take $K_1 = \overline{\operatorname{lin}}\,(\{x_{ij}\} \cup K)$. By construction, every point of K is in the convex hull of every 0-neighbourhood (for $\{z_n\}$ spans K and $\{V_n\}$ is a base); so by 6.4, $K \subset \Lambda K_1$. On the other hand, every point x in $K_1 \setminus K$ is outside ΛK_1; for, by (i), the point x is outside E_i for some $i \geq 2$. By construction, $E_i \cap K_1$ has finite codimension in K_1; so the dual of K_1 separates x from 0, as required.

We will make particular use of a special case:

7.2. COROLLARY. *If E is a metrizable tvs and K a closed separable subspace which has separating dual and which is intersection of a decreasing sequence of subspaces $E_1 \supset E_2 \supset \ldots$, each of which has dual $\{0\}$, then there is a closed (separable) subspace K_1 such that $\Lambda K_1 = K$ (and thus $\alpha(K_1) = 2$).*

7.3. Corollary. *Let E be a metrizable tvs such that for every closed subspace K which is an A-space, there is an (isomorphic) imbedding $i: K \to E$ such that E satisfies the conditions of 7.1 with $i(K)$ in the place of K. Then, for any $n < \omega$, the space E has a closed subspace K_n with $\Lambda^n K_n$ one-dimensional (and thus $\alpha(K_n) = n + 1$).*

Proof. Start with a one-dimensional subspace K and find K_1; then take K_1 in the place of K to find K_2; and then go on step by step.

7.4. *Example.* Consider the space $S(0, 1)$ of measurable functions (cf. 6.3 above). We can then apply 7.2 in the case when K is the one-dimensional space of all constant functions. Namely, let E_k be the subspace of all periodic functions with period 2^{-k} $(k \geq 1)$. Clearly, K is the intersection of all the E_k, and each E_k has dual $\{0\}$ (since E_k is essentially $S(0, 2^{-k})$, which space is isomorphic to $S(0, 1)$ and thus has dual $\{0\}$). *Hence $S(0, 1)$ has a closed subspace K_1 with ΛK_1 one-dimensional (and thus $\alpha(K_1) = 2$).*

(See also 7.9 below.)

7.5. *Example.* Similarly, $L^p(0, 1)$ *with* $0 < p < 1$ *has a closed subspace* K_1 *with* ΛK_1 *one-dimensional.*

7.6. *Examples* (continuation). We will now proceed and show that also the conditions of 7.3 are fulfilled for $E = S(0, 1)$ [resp. $L^p(0, 1)$].

To that end, first notice that $(0, 1)$ has the same measure space structure as $I^2 = (0, 1) \times (0, 1)$; so $S(0, 1)$ [resp. $L^p(0, 1)$] may be identified with the corresponding space $S(I^2)$ [resp. $L^p(I^2)$] of measurable functions on I^2. Thus, we can define the required imbeddings $i: K \to E$ of 7.3 as imbeddings $S(0, 1) \supset K \xrightarrow{i} S(I^2)$ [from now on we omit the phrase »resp. (corresponding for) L^p« after statements like this one].

So let K be a closed subspace of $S(0, 1)$. We will find $i: S(0, 1) \to S(I^2)$ and subspaces $E_1 \supset E_2 \supset \ldots$ of $S(I^2)$ such that their intersection equals $i(K)$ and such that each of them has dual $\{0\}$. Consider the elements of $S(0, 1)$ as functions $f(\theta)$ with θ ranging over $(0, 1)$ and the elements of $S(I^2)$ as functions $f(\theta_1, \theta_2)$ with θ_1 and θ_2 ranging over $(0, 1)$. (Thus, we denote the coordinates of I^2 by θ_1 and θ_2 resp.) Whatever K may be, we define the imbedding i by means of

$$S(0, 1) \supset K \ni f(\theta) \xrightarrow{i} f(\theta_1) \in S(I^2);$$

thus, we let i range in the subspace of functions that are constant in the variable θ_2. Clearly i is an isomorphic imbedding. It remains to give the E_k, which is done by

$$E_k = \overline{\operatorname{lin}} \{g(\theta_1) \cdot h(\theta_2) | g \in K; \ h \text{ is periodic with period } 2^{-k}\}.$$

Then E_k has dual $\{0\}$ (cf. 7.4); we must show that $\bigcap_k E_k = i(K)$. First, by definitions, the intersection contains $i(K)$; conversely, an element $f_0(\theta_1, \theta_2)$ of the intersection will be shown to belong to $i(K)$. Since any function of E_k has period 2^{-k} in θ_2 (for almost all θ_1-values), the function $f_0(\theta_1, \theta_2)$ is essentially constant with respect to θ_2; so write $f_0(\theta_1, \theta_2) = g_0(\theta_1)$. Then g_0 — regarded as an element of $S(0, 1)$ — is to be recognized as an element of K. It can be, for if $f(\theta_1, \theta_2)$ belongs to E_1, say, then, for almost every fixed θ_2^0, the function $\theta_1 \to f(\theta_1, \theta_2^0)$ — regarded as an element of $S(0, 1)$ — belongs to K; in particular, $g_0 \in K$.

Thus, 7.3 applies and gives:

For every positive integer n, the space $S(0, 1)$ [resp. $L^p(0, 1)$] has a closed subspace K_n with $\Lambda^n K_n$ one-dimensional (and thus $\alpha(K_n) = n + 1$).

7.7. *Examples* (continuation). We can now easily take one more step: First, notice that the spaces $S_k = S(2^{-k}, 2^{-k+1})$, where $k \geq 1$, may be regarded as subspaces of $S(0, 1)$; further, there are natural continuous and open projections $\pi_k : S(0, 1) \to S_k$ (defined as the restrictions of functions in $S(0, 1)$ to $(2^{-k}, 2^{-k+1})$ [resp. corresponding for L^p]. By 7.6, take for each $k \geq 1$ a space $K_k^\circ \subset S_k$ such that $\Lambda^k K_k^\circ$ is one-dimensional; and let $K^\circ \subset S(0, 1)$ be the closed linear hull of the K_k°. Now, proposition 2 — applied to π_k and to the imbedding $S_k \to S(0, 1)$ — says that $\pi_k(\Lambda^\nu K^\circ) = \Lambda^\nu K_k^\circ$ for all ν and k. This gives on one hand $\Lambda^n K^\circ \neq \{0\}$ for $n < \omega$, and on the other hand $\bigcap_{n<\omega} \Lambda^n K^\circ \subset \bigcap_k \pi_k^{-1}(0) = \{0\}$. Thus we find that $S(0, 1)$ *[resp. $L^p(0, 1)$] has a closed subspace K° with $\alpha(K^\circ) = \omega$.*

7.8. *Remark.* Why are we particularly interested in finding *closed* subspaces in 7.1—7.7? — Because these are complete when we start with a complete space E (such as $S(0, 1)$ or $L^p(0, 1)$); and in general, $\alpha(\cdot)$ *is not invariant under completion.* (See II.2.3 below.)

7.9. *Remark.* Consider the situation in 7.4. It might be interesting to see explicitly what the found subspace K_1 may look like in this particular case. To that end, let us go back and examine the proof of 7.1. What we want is then to give elements x_{ij} which are of the kind mentioned there and which thus, together with K, span K_1.

Noticing that K is generated by the constant function 1 on $(0, 1)$, we thus need functions $x_{ij}(\tau)$ on $(0, 1)$ which satisfy

(1′) $x_{ij} \in E_i$

(1″) $x_{ij} \to 0$ (independently of j) as $i \to \infty$ (convergence in measure)

(2) $1 \in \operatorname{co}\{x_{ij}\}_j$ for every $i \geq 1$,

where the E_i are as defined in 7.4. Denoting by

$$F_N(\tau) = \frac{1}{N+1} \frac{\sin^2 \frac{N+1}{2} \tau}{\sin^2 \frac{\tau}{2}}, \quad N \geq 1,$$

the Fejér kernel on the unit circle, we claim that we may take

$$x_{ij}(\tau) = F_i\left(2\pi\left(2^i\tau + \frac{j}{i}\right)\right), \quad 1 \leq j \leq i, \quad i \geq 1 .$$

Namely, (1′) and (1″) follow from well-known properties of the Fejér kernel; for (2), notice that we have $1 = (1/i)(x_{i1} + \ldots + x_{ii})$ from the well-known relations

$$F_N(\tau) = 1 + \frac{2}{N+1} \sum_1^N (N + 1 - n) \cos n\tau \quad \text{and}$$

$$\sum_{j=1}^{i} \cos 2\pi k\left(\theta + \frac{j}{i}\right) = 0 \quad \text{for integer} \quad k \neq 0 \quad \text{and any} \quad \theta.$$

We have shown that the space K_1 found in 7.4 [resp. 7.5] may be the space

$$\overline{\operatorname{lin}}\left\{1, F_i\left(2\pi\left(2^i\tau + \frac{j}{i}\right)\right)\right\} i \geq 1, \quad 1 \leq j \leq i .$$

II. Constructive methods

In this chapter, we shall first give a general method to construct p-normed spaces with certain properties. Then we will use this to get the announced example which will show that definitions I 1 and I 1.2 are meaningful for all ordinals. The constructive method to be employed will be presented in a somewhat stronger form than required for our chief purpose. For the sake of completeness it may be remarked, that to fulfil this purpose, we could have got away with a simpler but less illuminating device.

1. Suppose there to be given, for a fixed number p $(0 < p \leq 1)$

(i) a set $\mathscr{E}$ of p-normed spaces,

(ii) for each $E \in \mathscr{E}$, an element $e_E \in E$,

(iii) a metric space M (with no linear structure), and

(iv) a function $\Psi : \mathscr{E} \to M \times M$ such that $d(\mathrm{pr}_1\Psi(E),\ \mathrm{pr}_2\Psi(E)) = \|e_E\|$, and such that $\mathrm{Im}(\mathrm{pr}_1\Psi) \cup \mathrm{Im}(\mathrm{pr}_2\Psi) = M$, where $\mathrm{pr}_k : M \times M \to M$ is the projection onto the first resp. second factor and where $d(\cdot\, , \cdot)$ is the distance on M.

We introduce the following notation. For $(m', m'') \in M \times M$, we write $-(m', m'') = (m'', m')$. Further, by a *cycle* we understand a finite sequence of elements in $M \times M$ of the form

$$(m_0, m_1), (m_1, m_2), (m_2, m_3), \ldots, (m_{n-1}, m_n), (m_n, m_0) .$$

We shall now define a space $\Pi(\mathcal{E}, \{e_E\}, M, \Psi)$, which we also denote $\Pi(\mathcal{E})$ or Π when there is no risk for confusion. (Further, if $\mathcal{E}' \subset \mathcal{E}$ and if $M' = \operatorname{Im}(\mathrm{pr}_1 \Psi) \cup \operatorname{Im}(\mathrm{pr}_2 \Psi)$, we denote also $\Pi(\mathcal{E}', \{e_E\}, M', \Psi)$ by $\Pi(\mathcal{E}')$ — in spite of the change from M to M'.)

1.1. PROPOSITION. *Given the objects of* (i) *to* (iv), *there is a unique complete p-normed space $\Pi = \Pi(\mathcal{E}, \{e_E\}, M, \Psi)$ such that*

1° *for each $E \in \mathcal{E}$, there is an isometric imbedding $i_E : E \to \Pi$,*

2° *there is an injection $j : M \to \Pi$ related to the i_E as*

$$i_E(e_E) = j(\mathrm{pr}_2 \Psi(E)) - j(\mathrm{pr}_1 \Psi(E)), \quad \textit{and}$$

3° *if Π^1 is an arbitrary complete p-normed space satisfying* 1° *and* 2° *with i_E^1 and j^1, say, in the places of i_E and j resp., there is a linear mapping $\Phi : \Pi \to \Pi^1$ for which the diagram*

$$\begin{tikzcd} & E \arrow[dl, "i_E"'] \arrow[dr, "i_E^1"] & \\ \Pi \arrow[rr, "\Phi"] & & \Pi^1 \end{tikzcd}$$

is commutative for each $E \in \mathcal{E}$, and which is continuous with operator p-norm not larger than one (i.e., $\sup_{\|x\|=1} \|\Phi(x)\| \le 1$). (Universal property.)

[Concerning 3°, also notice: if $\operatorname{Im} \Psi$ is connected as graph (cf. 1.4), the diagram

$$\begin{tikzcd} & M \arrow[dl, "j"'] \arrow[dr, "j^1"] & \\ \Pi \arrow[rr, "\Phi"] & & \Pi^1 \end{tikzcd}$$

is commutative up to translation (i.e., $j^1 = \Phi \circ j + c$ for some constant element c in Π^1); this follows from the condition that the i_E^1 and j^1 satisfy 2°.]

Before proving 1.1, we state some properties of Π which will be consequences of the construction.

1.2. Proposition. *With notation and assumptions as in 1—1.1 and with* E_1, $E_2 \in \mathscr{E}$, *we have*

$$i(E_1) \cap i(E_2) = \begin{cases} \{0\} & \text{if } \Psi(E_1) \neq \Psi(E_2) \text{ and } \neq -\Psi(E_2) \\ \operatorname{lin}\{e_{E_1}\} & \text{if } \Psi(E_1) = \Psi(E_2) \text{ or } = -\Psi(E_2)\,. \end{cases}$$

1.3. Proposition. *With notation and assumptions as in 1—1.1 (notice especially the last paragraph before 1.1), let* $\mathscr{E}'$ *be a subset of* $\mathscr{E}$ *consisting of all except finitely many of the elements of* $\mathscr{E}$. *Then there is a canonical imbedding* χ *(which is isomorphic but in general not isometric) of* $\Pi(\mathscr{E}')$ *onto a closed subspace of* $\Pi(\mathscr{E})$.

(»Canonical» here means that the diagram

$$\begin{tikzcd} & E \arrow[dl, "i_E"'] \arrow[dr, "i_E"] & \\ \Pi(\mathscr{E}') \arrow[rr, "\chi"'] & & \operatorname{Im}\chi \subset \Pi(\mathscr{E}) \end{tikzcd}$$

is commutative whenever $E \in \mathscr{E}'$.)

1.4. *Remark.* In the sequel, we will not distinguish between E and $i(E)$. Thus, the spaces of $\mathscr{E}$ will be regarded as *subspaces* of Π.

Notice that it is an immediate consequence of the universal property (3° of 1.1) that

$$\overline{\operatorname{lin}}\{i(E)\,|\,E \in \mathscr{E}\} = \Pi\,.$$

Intuitively, we may think of $\operatorname{Im}\Psi$ as the set of edges of a *graph* in M. What we intend to do is to paste the spaces of $\mathscr{E}$ at this graph by identifying the vectors e_E with the (oriented) edges $\Psi(E)$.

Notice that in general the injection $j : M \to \Pi$ is not isometric; however, we know that the distances $d(\mathrm{pr}_1\,\Psi(E), \mathrm{pr}_2\,\Psi(E))$ are preserved, since these equal the p-norm values $\|e_E\|$ (cf. (iv) of 1 and 1° of 1.1).

1.5. *Proof of 1.1—1.3.* To define $\Pi(\mathscr{E}, \{e_E\}, M, \Psi)$, we start with the p-normed direct sum S (cf. I 6.2.d) of the spaces of $\mathscr{E}$. Consider the subspace $H \subset S$ given by

$$H = \operatorname{lin}\{\sum_{k=1}^{n} \varepsilon_k e_{E_k} \,|\, \varepsilon_1 \Psi(E_1), \ldots, \varepsilon_n \Psi(E_n) \text{ is a cycle for some } \varepsilon_k = \pm 1,\ n \geq 2\}\,. \quad \text{(a)}$$

Then we take Π as the completion of the p-normed quotient space S/H.

To give the p-norm $\|\cdot\|_\sim$ on S/H more explicitly, we denote by x the canonical image in S/H of an element $x \in S$. (This tilde notation will not be used except in this proof.) Then $\|\cdot\|_\sim$ is defined by

$$\|\widetilde{\sum x_k}\| = \inf\left(\sum \|y_k\| \mid \widetilde{\sum x_k} = \widetilde{\sum y_k}\right) \tag{b}$$

where the x_k and y_k belong to different spaces of $\mathscr{E}$, and where the infimum is taken over all preimages of $\widetilde{\sum x_k}$ under the canonical mapping $S \to S/H$. (Cf. the definitions in I 6.2.)

From this construction, we see at once how to define the i_E of 1°; for each $E \in \mathscr{E}$, let i_E be the composition of the inclusion map $E \to S$ and the canonical mapping $S \to S/H$. Before the somewhat lengthy verification that these i_E are isometric, we prove the other statements.

First, 1.2 follows from this definition; for, that $\varepsilon_1\Psi(E_1)$, $\varepsilon_2\Psi(E_2)$ is a cycle means precisely that $\varepsilon_1\Psi(E_1) = -\varepsilon_2\Psi(E_2)$. By the definition of H, it means also precisely that $\varepsilon_1\tilde{e}_{E_1} = -\varepsilon_2\tilde{e}_{E_2}$. And (a) further shows that for $x_k \in E_k$, we can have $\tilde{x}_1 = \tilde{x}_2$ *only* if $x_k = \pm \lambda e_{E_k}$ for $k = 1, 2$ and some λ.

Second, for 1.3 we consider the case when $\mathscr{E} \setminus \mathscr{E}'$ is a singleton $\{E_0\}$, say; the general assertion then follows by induction. Let S_0 be the subspace of S which is spanned by the spaces of $\mathscr{E}'$, and let $H_0 \subset S_0$ be the subspace defined by (a) with the spaces $E_1, \dots, E_n$ ranging over $\mathscr{E}'$. Denote by $\varphi : S \to S/H$ the canonical mapping. Since $H_0 = S_0 \cap H$, the identity mapping $S \to S$ induces a bijective linear mapping $\chi_0 : S_0/H_0 \to \varphi(S_0)$, which will be shown to be an isomorphism. If $x \in S_0$ and if $\tilde{\tilde{x}}$ and $\|\cdot\|_{\approx}$ is the canonical image of x in, resp. the quotient p-norm on, S_0/H_0, we have by (b)

$$\|\tilde{x}\|_\sim = \inf\Big(\sum_{x \equiv \Sigma y_k \bmod H} \|y_k\| \mid y_k \text{ in spaces of } \mathscr{E}\Big) \text{ and}$$

$$\|\tilde{\tilde{x}}\|_\approx = \inf\Big(\sum_{x \equiv \Sigma y_k \bmod H_0} \|y_k\| \mid y_k \text{ in spaces of } \mathscr{E}'\Big).$$

Since the first infimum is taken over a larger set, it is smaller, so $\|\tilde{x}\|_\sim \le \|\tilde{\tilde{x}}\|_\approx$. Conversely, let y_k be vectors in spaces of $\mathscr{E}$ so that $x \equiv \sum y_k \bmod H$. Two cases may occur: (1) all y_k are in spaces of $\mathscr{E}'$, and then $x \equiv \sum y_k \bmod H_0$ (since $H_0 = S_0 \cap H$); and (2) $\sum y_k = s_0 + \lambda e_{E_0}$, where $s_0 \in S_0$ and $\lambda \neq 0$ a scalar. In case (2), $\Psi(E_0)$ must be an element of a cycle $\mathscr{L}$ of which all the other elements are in $\pm\Psi(\mathscr{E}')$. (The last statements are consequences of the definition of H.) So

$$s_0 + \lambda e_{E_0} \equiv s_0 + \lambda \sum_{\substack{\varepsilon_k \psi(E_k) \in \mathscr{L} \\ E_k \neq E_0}} \varepsilon_k e_{E_k} \bmod H\,,$$

wherein the second member is an element in S_0 which is congruent modulo H_0 to x (since $H_0 = S_0 \cap H$). Let the sum in the second member have the length $c\|e_{E_0}\|$. Then, for *any* $x \in S_0$, and $\sum y_k \equiv x \bmod H$ the second member has length at most c times the length of the first member. So in case (2) as in case (1), it is seen that for every $x' \equiv x \bmod H$, there is $x'' \in S_0$ with $x'' \equiv x \bmod H_0$ and $\|x''\| \leq c'\|x'\|$ for some constant $c' > 0$ (which is independent of x); and so $\|\cdot\|_{\approx} \leq c'\|\cdot\|_{\sim}$.

Then we extend χ_0 by continuity to an imbedding $\chi : \Pi(\mathscr{E}') \to \Pi(\mathscr{E})$; clearly $\operatorname{Im} \chi$ is closed in $\Pi(\mathscr{E})$, for $\Pi(\mathscr{E}')$ is complete.

Finally, we verify 1°—3° of 1.1.

For 2°: Take $j(m_0) = 0$ for some $m_0 \in M$. To define j for the other points in M, we use the following notation: By a *Ψ-path* from m_1 to m_n we understand a sequence

$$(m_1, m_2), (m_2, m_3), \ldots, (m_{n-2}, m_{n-1}), (m_{n-1}, m_n)$$

where all the (m_k, m_{k+1}) belong to $\operatorname{Im} \Psi \cup (-\operatorname{Im} \Psi)$ and where $n \geq 1$. Let $m \in M$ be any element for which there is a Ψ-path from m_0 to m — or, as we can say, that lies in the same *Ψ-path-component* of M as m_0. Let $\varepsilon_1\Psi(E_1), \ldots, \varepsilon_n\Psi(E_n)$ $(\varepsilon_k = \pm 1, n \geq 1)$ be a Ψ-path from m_0 to m. Then we define $j(m) = \sum_{k=1}^{n} \varepsilon_k e_{E_k}$. This definition is consistent. For, let $\varepsilon_1^1\Psi(E_1^1), \ldots, \varepsilon_{n^1}^1\Psi(E_{n^1}^1)$ be another Ψ-path from m_0 to m; then

$$\varepsilon_1\Psi(E_1), \ldots, \varepsilon_n\Psi(E_n), -\varepsilon_1^1\Psi(E_1^1), \ldots, -\varepsilon_{n^1}^1\Psi(E_{n^1}^1) \tag{c}$$

is a cycle, so by construction (cf. (a) above) we have

$$\sum_1^n \varepsilon_k e_{E_k} - \sum_1^{n^1} \varepsilon_n^1 e_{E^1{}_k} = 0 . \tag{c'}$$

This relation says that the definition of $j(m)$ is independent of the choice of path from m_0 to m. So, if C_0 is the Ψ-path-component of m_0 in M, we have consistently defined $j_{C_0} : C_0 \to \Pi$ — the restriction to C_0 of the required j. And j_{C_0} is injective; for if $j(m') = j(m'')$ and if $\varepsilon_1\Psi(E_1), \ldots, \varepsilon_n\Psi(E_n)$ and $\varepsilon_1^1\Psi(E_1^1), \ldots, \varepsilon_{n^1}^1\Psi(E_{n^1}^1)$ are Ψ-paths from m_0 to m' resp. m'', relation (c′) is satisfied. By construction, (c) must be a cycle, which means that the two paths go from m_0 to the same point, i.e., $m' = m''$.

We have to extend j_{C_0} to all of M. So, for each Ψ-path-component C we define an injection $j_C : C \to \Pi$ in precisely the same manner as before. But the definition of Π (cf. (a)) shows that for a suitable class of subspaces each two of which meet only in $\{0\}$, each space of the class contains precisely one set $\operatorname{Im} j_C$ as a proper subset. So by composing each j_C with a suitable translation, we get injective mappings j_C with disjoint images; these together define an injection j on all of M.

For 3°: The mappings i_E^1 together induce a linear mapping $\overline{i^1} : S \to \Pi^1$ (defined by $\overline{i^1}(\sum x_k) = \sum i_{E_k}^1(x_k)$ for $\sum x_k \in S$, $x_k \in E_k$). First notice that $\overline{i^1}$ is continuous with operator p-norm at most 1, for

$$\|\bar{i}^1(\sum x_k)\| = \|\sum i^1_{E_k}(x_k)\| \leq \sum \|i^1_{E_k}(x_k)\| = \sum \|x_k\| = \|\sum x_k\| ,$$

where $\sum x_k \in S$, with $x_k \in E_k$ and the spaces E_k distinct.

Now, if $\varepsilon_1 \Psi(E_1), \ldots, \varepsilon_n \Psi(E_n)$ $(\varepsilon_k = \pm 1,\ n \geq 1)$ is a cycle, we have for any $m_0 \in M$ occurring in any $\Psi(E_k)$,

$$0 = j^1(m_0) - j^1(m_0) = \sum_{k=1}^{n} \varepsilon_k i^1_{E_k}(e_{E_k}) ,$$

since the i^1_E and j^1 are supposed to satisfy 2°. Thus, any element in the set spanning H and given in the second member of (a) above is mapped to 0 by $\bar{i}^1$; hence $\bar{i}^1(H) = 0$. The mapping $\Phi : \Pi \to \Pi^1$ is then defined as the mapping which is induced by $\bar{i}^1$. From the definition of quotient p-norm we see that Φ has operator p-norm at most 1, for $\bar{i}^1$ has. The commutativity of the diagrams, finally, is an immediate consequence of the definition of Φ.

For 1°, at last, let x be an element in E_0, say, $E_0 \in \mathscr{E}$. By definition, $\|\tilde{x}\|_{\sim} \leq \|x\|$. To show that $\|\tilde{x}\|_{\sim} \geq \|x\|$, on the other hand, let an arbitrary canonical pre-image of $\tilde{x}$ in S be written

$$x' = y + \lambda r ,$$

where $y = x - \lambda e_0$ (notice that $y \in E_0$), λ is a scalar, and $r = \sum_{F \in \mathscr{F}} \gamma_F e_F \equiv e_0 \bmod H$, for a suitable finite subset $\mathscr{F} \subset \mathscr{E} \setminus \{E_0\}$ and suitable numbers γ_k, n. We must then show that $\|x'\| \geq \|x\|$. Notice that, by definition, $\|x'\| = \|y\| + |\lambda|^p \|r\|$. Now, by the definition of H, we know that r is a linear combination of n, say, sums

$$s_k = \sum_{l=1}^{n_k} \varepsilon_{kl} e_{E_{kl}}, \quad 1 \leq k \leq n , \tag{d}$$

such that

$$-\ \Psi(E_0),\ \ \varepsilon_{k1}\Psi(E_{k1}), \ldots, \varepsilon_{kn_k}\Psi(E_{kn_k})$$

is a cycle $\mathscr{L}_k$ for each k. Or, if $\Psi(E_0) = (m_0, m_1)$ say, we may express this in terms of the notation of the proof of 2°: for every k, $\varepsilon_{k1}\Psi(E_{k1}), \ldots, \varepsilon_{kn_k}\Psi(E_{kn_k})$ is a Ψ-path from m_0 to m_1. Also notice that $e_0 \equiv \sum_l \varepsilon_{kl} e_{kl} \bmod H$. To avoid trivial complications, we exclude the case when some E_{kl} equals E_0 (notice that $E_0 \notin \mathscr{F}$).

We first treat the case when there is only one such sum (i.e., $n = 1$) and when no two terms of this sum are multiples of the same vector e_E. In other words, we suppose that $r = s_1$ and that $\Psi(E_{1l'}) \neq \pm \Psi(E_{1l''})$ when $l' \neq l''$. Assumption (iv) and the triangle inequality in M give

$$\begin{aligned} \|x'\| &= \|y\| + |\lambda|^p \|r\| = \|y\| + |\lambda|^p \sum_{l=1}^{n_1} \|e_{E_{1l}}\| \\ &= \|y\| + |\lambda|^p \sum_{l=1}^{n_1} d(\mathrm{pr}_1\ \Psi(E_{1l}),\ \ \mathrm{pr}_2\ \Psi(E_{1l})) \end{aligned}$$

$$\begin{aligned}
&= \|y\| + |\lambda|^p(d(m_0, m_1') + d(m_1', m_2') + \ldots + \\
&\quad + d(m_{n_1-1}', m_1)), \text{ suitable } m_k';\ m_0, m_1 \text{ as above} \\
&\geq \|y\| + |\lambda|^p d(m_0, m_1) = \|y\| + \|\lambda e_0\| \\
&\geq \|y + \lambda e_0\| = \|x\|,
\end{aligned}$$

since, by assumption, $\varepsilon_{11}\Psi(E_{11}), \ldots, \varepsilon_{1n_1}\Psi(E_{1n_1})$ is a Ψ-path from m_0 to m_1; so the assertion follows in this case.

In the general case ($n \geq 1$), we want to reduce the problem to the case $n = 1$. That is, if $x' = y + \lambda r$ is an arbitrary canonical pre-image in S of $\tilde{x}$, we want to find another pre-image $x'' = y + \lambda s'$, where s' is a sum of type (d), such that $\|x''\| \leq \|x'\|$. Since we always have $\|y + \lambda r\| = \|y\| + |\lambda|^p\|r\|$ (by the definition of $\|\cdot\|$ on S), this means that we must find s' with $\|s'\| \leq \|r\|$. This is done in five steps.

First step: We show that if we are given $r \equiv e_0 \bmod H$ and write $r = \sum_{F\in\mathcal{F}} \gamma_F e_F$ as above, then *there is a Ψ-path from m_0 to m_1, consisting of elements* in $\pm\,\Psi(\mathcal{F})$ *only.* This is not quite obvious; though we know — from definition of s_k — that there are such Ψ-paths consisting of elements in $\pm\,\Psi(\{E_{kl}\}_{l=1}^{n_k})$. For, one might ask: can it happen that so many $e_{E_{kl}}$-terms are cancelled in the linear combination of the s_k that a Ψ-path of the required kind is impossible (since we can take $\mathcal{F}$ so small that $\gamma_F \neq 0$ for every F)? We will show that it cannot.

We write $r = \sum_{k=1}^n \theta_k s_k$. Then

$$\sum_{k=1}^{n} \theta_k = 1\,, \tag{e}$$

for $r \equiv s_k \equiv e_0 \bmod H$. To proceed, we need a simple graph-theoretic device. Let Z be a subset of M, and take $\hat{Z} = \{(m', m'') | m' \in Z \text{ and } m'' \in \complement Z\}$. (Thus, $\hat{Z}$ consists of the edges connecting Z and $\complement Z$ in the graph defined by $\operatorname{Im}\Psi$ and mentioned in 1.4 above. Further, the orientation of the edges of $\hat{Z}$ is in the direction out from Z.) For each cycle $\mathcal{L}_k$ (defined in connection to (d)), we introduce a scalar valued function f_k defined on Z. If an element $z \in \hat{Z}$ occurs r times in $\mathcal{L}_k$, and if $-z$ occurs s times, we define

$$f_k(z) = (r - s)\theta_k\,;$$

notice that this situation may be described intuitively as follows. When we go around one round in $\mathcal{L}_k$ (in the natural direction), we pass s times into Z through z, and r times out of Z through z. An elementary consideration shows that

$$\sum_{z\in\hat{Z}} f_k(z) = 0$$

(for, loosely spoken, when we make our round in $\mathcal{L}_k$, we must, totally, pass into Z precisely as many times as we pass out there of). Now define $f(z) = \sum_{k=1}^n f_k(z)$; summing the relations just stated, we get

$$\sum_{z \in \hat{Z}} f(z) = 0 \,. \tag{f}$$

We will now use this result. Assume, for a while, that Ψ is an injection. Then, if $\Psi(E) = \varepsilon z$, write $e_z' = \varepsilon e_E$ (where $E \in \mathcal{E}$, $\varepsilon = \pm 1$, $z \in \hat{Z}$). By definitions, we see that the product of θ_k and the coefficient of e_z' in s_k is precisely $f_k(z)$; and the coefficient of e_z' in r is $f(z)$ (or more strictly, $r = \sum_{\substack{F \in \mathcal{F} \\ e_F \neq \pm e'_z}} \gamma_F e_F + f(z) e_z'$). Also, if $z_0 = (m_0, m_1) \in \hat{Z}$, we get $f(z_0) = \sum_k - \theta_k = -1$, by (e), since $-z_0$ appears exactly once in each $\mathcal{L}_k$, and z_0 in no $\mathcal{L}_k$. Now, let Z be the set of points in M which can be reached from m_0 by a Ψ-path, all of whose elements are in $\pm \Psi(\mathcal{F})$. If the assertion to be shown were not valid, m_1 would be outside Z. Aiming at a contradiction, we assume it to be so. Then $z_0 = (m_0, m_1) \in \hat{Z}$, and $f(z_0) = -1$, as we just remarked. Let $z_1 \in \hat{Z}$ be distinct from z_0; if z_1 is outside $\pm \operatorname{Im} \Psi$, we have defined $f(z_1) = 0$, and if z_1 belongs to $\pm \operatorname{Im} \Psi$, the definition of Z says that e_{z_1}' is certainly outside $\{\pm e_F\}_{F \in \mathcal{F}}$, so the coefficient $f(z_1)$ of e_{z_1}' must vanish. Thus, $f(z) = 0$ whenever $z \neq z_0$, so that

$$\sum_{z \in \hat{Z}} f(z) = f(z_0) = -1 \,,$$

contradicting (f).

Second step: Notice that the assumption that Ψ is injective is not essential. For, define $H_1 \subset S$ as $H_1 = \operatorname{lin} \{e_{E_1} - \varepsilon e_{E_2} | \Psi(E_1) = \varepsilon \Psi(E_2), \ \varepsilon = \pm 1\}$. Then the canonical mapping $S \to S/H_1$ is immediately seen to preserve length of vectors in spaces of $\mathcal{E}$ (regarded as subspaces of S). We now regard S/H as a quotient space of S/H_1 — instead of regarding it as a quotient space of S. And now the canonical images of the e_E in S/H_1 are in biunivocal correspondence with the values $\Psi(E)$, so the argument may be carried out as before.

Third step: We show that *either* r *may be written as just one sum of type* (d), *or the* $\tilde{e}_F$ *with* $F \in \mathcal{F}$ *are linearly dependent.* By what was shown in the first step, there is a path $\varepsilon_1 \Psi(F_1), \ldots, \varepsilon_{n'} \Psi(F_{n'})$, $F_k \in \mathcal{F}$, $\varepsilon_k = \pm 1$, $n' \geq 1$, from m_0 to m_1. Then we can form a sum of type (d), namely $s = \sum_k e_k \varepsilon_{F_k}$, all of whose terms are in $\{\pm e_F\}_{F \in \mathcal{F}}$. Since now $\tilde{e}_0 = \tilde{r} = \tilde{s}$, we get $\tilde{r} - \tilde{s} = 0$, so that either $r = s$ or $\sum_{F \in \mathcal{F}} \beta_F \tilde{e}_F = 0$, for some β_F which do not all vanish.

Fourth step: We show that *if the* $\tilde{e}_F$ *with* $F \in \mathcal{F}$ *are linearly dependent, then there is an* $r' = \sum_{F \in \mathcal{F}} \gamma_F' e_F \equiv e_0 \bmod H$ *such that at least one coefficient* γ_F' *vanishes and such that* $\|r'\| \leq \|r\|$. Thus, more expressively, we can change r in such a way that the number of terms in the e_F-expansion of r is reduced by one unit, and that the length does not increase.

So let $\sum_{F \in \mathcal{F}} \beta_F \tilde{e}_F = 0$ for some β_F which do not all vanish. For any scalar τ, define

$$r(\tau) = r + \tau \sum_{F \in \mathcal{F}} \beta_F e_F = \sum_{F \in \mathcal{F}} (\gamma_F + \tau \beta_F) e_F \,.$$

Notice that $r(0) = r$ and that $r(\tau) \equiv e_0 \bmod H$ for any τ. We will show that $r(\tau)$ assumes minimal length when some coefficient $\gamma_F + \tau\beta_F$ vanishes; this will give the assertion. By definition of the p-norm on S,

$$\|r(\tau)\| = \sum |\gamma_F + \tau\beta_F|^p \|e_F\| .$$

Since the function $\tau \to |\tau|^p$ is concave for $\tau \neq 0$ (since $p \leq 1$), the function $\lambda \to \|r(\lambda)\|$ is concave in the domain where all coefficients are distinct from zero; so the minimum must be assumed in the complement of this domain, as required. (For the real case, cf. fig. 1.).

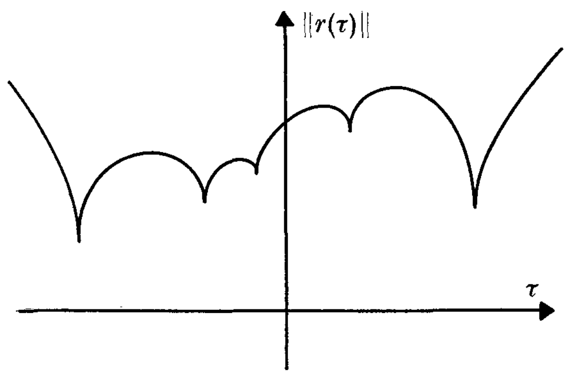


Fig. 1.
($p<1$)

Fifth step: We conclude: Let the canonical pre-image $x' = y + \lambda r$, $r = \sum_{F \in \mathcal{F}} \gamma_F e_F$, of $\tilde{x}$ be given. If the vectors e_F, $F \in \mathcal{F}$, are *linearly dependent*, we can, by what was shown in the fourth step, find another pre-image $x_1 = y + \lambda r_1$, for which $r_1 = \sum_{F \in \mathcal{F}_1} \gamma_F^1 e_F$ with $\mathcal{F}_1$ consisting of the elements in $\mathcal{F}$ except one and such that $\|r_1\| \leq \|r\|$, i.e., $\|x_1\| \leq \|x'\|$. Now we apply the same argument to x_1 as we did to x'; and then we proceed step by step. Thus, we get successively new pre-images of $\tilde{x}$ in such a way that $\mathcal{F}$ is replaced by smaller and smaller subsets; and a new pre-image is never longer than any of the previous ones. Since $\mathcal{F}$ is finite, the procedure must stop some time. So, ultimately, we get $x'' = y + \lambda r''$, where r'' is a linear combination of *linearly independent* vectors e_E, and $\|x''\| \leq \|x'\|$. By what was shown in the third step, r'' can be written as one sum of type (d). Since the terms of this sum are linearly independent, any two of them are certainly not proportional to each other; and we are back to the case first treated. Hence, we know that $\|x''\| \geq \|x\|$, and a fortiori $\|x'\| \geq \|x\|$.

We state explicitly yet another property of Π to be used in the sequel.

1.6. Proposition. *With notations and assumptions as in 1—1.1, let E_1, E_2 be distinct spaces of $\mathcal{E}$ such that the pairs $\Psi(E_1)$ and $\Psi(E_2)$ of elements of M have one element, $m = \mathrm{pr}_2\, \Psi(E_1) = \mathrm{pr}_1\, \Psi(E_2)$, say, in common, such that it belongs* *to no other element pair* $\Psi(E)$. *Let* f_k *be continuous linear forms defined on subspaces* $K_k \subset E_k$ *with* $e_k \in E_k$ *for* $k = 1, 2$. *Further, let* $\mathcal{K}$ *be the set obtained by replacing* E_k *by* K_k *in* $\mathcal{E}$, $k = 1, 2$. *Then, if* $f_1(e_{E_1}) + f_2(e_{E_2}) = 0$, *the forms* f_k *have a common extension to* $\Pi(\mathcal{K})$. (*Notice that by 1.3,* $\Pi(\mathcal{K})$ *may be regarded as a subspace of the tvs* $\Pi(\mathcal{E})$.)

Proof. First extend f_1 and f_2 to forms $\bar{f}_1$ and $\bar{f}_2$ on $K_1 \oplus K_2$ which are 0 on K_2 resp. K_1. By 1.3, $K_3 = \Pi(\mathcal{E} \setminus \{K_1, K_2\})$ can be considered as a closed subspace of $\Pi(\mathcal{K})$. So the form $\bar{h} = \bar{f}_1 + \bar{f}_2$ on $K_1 \oplus K_2 \subset \Pi(\mathcal{K})$ may be extended to the latter space by means of an extension $\bar{\bar{h}}$ which is 0 on K_3. This definition is consistent, since $\bar{h}(e_{E_1} + e_{E_2}) = f_1(e_{E_1}) + f_2(e_{E_2}) = 0$ and $e_{E_1} + e_{E_2}$ spans the space $(K_1 \oplus K_2) \cap K_3$ if this is $\neq \{0\}$. For first, Lemma 1.7 below will show that $\operatorname{lin}\{e_{E_k}\} \supset K_k \cap K_3$ $(k = 1, 2)$. Further, since m is in no other pair $\Psi(E)$ than $\Psi(E_1)$ and $\Psi(E_2)$, the defining relation (a) in the beginning of 1.5 gives $\operatorname{lin}\{e_{E_1}, e_{E_2}\} \cap K_3 \subset \operatorname{lin}\{e_{E_1} + e_{E_2}\}$, whence the assertion.

1.7. LEMMA. *With notation and assumptions as in 1.3, we have for any* $E_0 \in \mathcal{E} \setminus \mathcal{E}'$ *that*

$$\operatorname{Im}(\chi) \cap i_{E_0}(E_0) = \begin{cases} \operatorname{lin}\{i_{E_0}(e_{E_0})\} & \textit{if there is a cycle, one element of which is } \Psi(E_0) \\ & \textit{and the others belong to } \pm \Psi(\mathcal{E}') \\ \{0\} & \textit{otherwise.} \end{cases}$$

Proof. By 1.2 and 1.3, we see that the intersection is contained in $\operatorname{lin}\{i_{E_0}(e_{E_0})\}$. Further, (a) in the beginning of 1.5 shows that $i_{E_0}(e_{E_0})$ is outside $\operatorname{Im}\chi_0$ (notation as in the proof of 1.3 in 1.5) if and only if $\Psi(E_0)$ fails to be in any cycle of which the other elements are in $\pm \Psi(\mathcal{E}')$. But by construction we clearly have $i_{E_0}(e_{E_0})$ at the same distance from $\operatorname{Im}\chi_0$ as from $\operatorname{lin}\{i_E(e_E) | E \in \mathcal{E}'\} (\subset \operatorname{Im}\chi_0)$. But since this space has finite codimension in $\operatorname{lin}\{i_E(e_E) | E \in \mathcal{E}\}$, the vector $i_{E_0}(e_{E_0})$ is seen to be outside thereof — i.e., outside $\operatorname{Im}\chi_0$ — if and only if the mentioned distance is positive, that is, if and only if $i_{E_0}(e_{E_0})$ is outside $\overline{\operatorname{Im}\chi_0} = \operatorname{Im}\chi$. So the assertion follows.

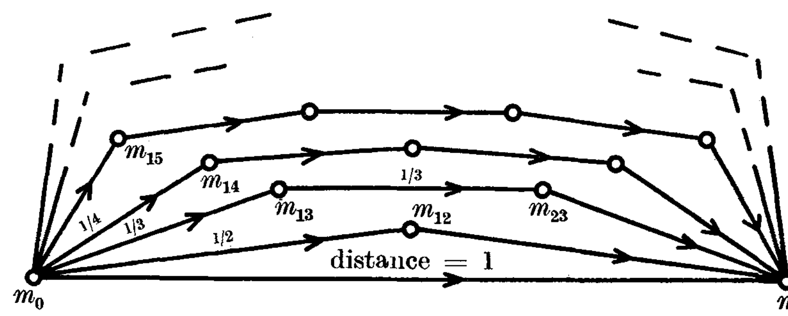
2. *Example.* When given any ordinal ν and any number p, $0 < p < 1$, we are now ready for the actual *construction of a* p*-normed and complete tvs* E *such that* $\Lambda^\nu E$ *is one-dimensional.*

We use transfinite induction.

A. Assume that $\nu = \nu' + 1$ is any ordinal which is of the first kind and ≥ 2, and that we have a space E^0 for which $\Lambda^{\nu'} E^0$ is one-dimensional. We are going to form E by means of the method of 1. Let E_{kl}, $1 \leq k \leq l$, $l \geq 1$, be copies of E^0, and let, for all k, l, the vector e_{kl} be a generator of length $1/l$ of $\Lambda^{\nu'} E_{kl}$. Further, let M be a metric space with points m_0, m_1, and m_{kl} for all k, l with

$1 \leq k < l$. Write $m_0 = m_{0l}$, $m_1 = m_{ll}$, $l \geq 1$; then if $d(.\,,.)$ is the distance, assume that $d(m_{k-1,l}, m_{kl}) = 1/l$ for all k and l. Define $\Psi : \{E_{kl}\}_{kl} \to M \times M$ by $\Psi(E_{kl}) = (m_{k-1,l}, m_{kl})$. We may describe this as follows: M is the set of nodes of the infinite graph in fig. 2, and Im Ψ consists of those pairs which form end-points of some edge.

Fig. 2.



Notice that we have not defined M uniquely. However, this is inessential; cf. 1.4.

We now take $E = \Pi(\{E_{kl}\}, \{e_{kl}\}, M, \Psi)$. We want to show that *$\Lambda^{\nu}E$ is the one-dimensional space which is spanned by* e_{11} (the vector corresponding to the pair (m_0, m_1)). First, $e_{11} = (1/l)\ (le_{1l} + \ldots + le_{ll})$, wherein $\|le_{kl}\| = |l|^p \cdot \|e_{kl}\| = l^{p-1} \to 0$ as $l \to \infty$. But we also know that $e_{kl} \in \Lambda^{\nu'}E$, since $\Lambda^{\nu'}E_{kl} \subset \Lambda^{\nu'}E$ (cf. I.2). Thus $e_{11} \in \mathrm{co}\,(U \cap \Lambda^{\nu'}E)$ for every 0-neighbourhood U; so $e_{11} \in \Lambda\Lambda^{\nu'}E = \Lambda^{\nu}E$.

Conversely, we have to show that any point outside the linear hull of e_{11} is also outside $\Lambda^{\nu}E$. First, we have $E_{kl} \cap \Lambda^{\nu'}E = \Lambda^{\nu'}E_{kl} = \mathrm{lin}\,\{e_{kl}\}$. For the first equality is shown by transfinite induction; suppose that we know that $\bigcap_{\nu'''<\nu''} (E_{kl} \cap \Lambda^{\nu'''}E) = \bigcap_{\nu'''<\nu''} \Lambda^{\nu'''}E_{kl}$ and want to show that $E_{kl} \cap \Lambda^{\nu''}E = \Lambda^{\nu''}E_{kl}$, for some $\nu'' \leq \nu'$. Now, any given point in $\bigcap_{\nu'''<\nu''} \Lambda^{\nu'''}E_{kl} \setminus \Lambda^{\nu''}E_{kl}$ may be separated from $\Lambda^{\nu''}E_{kl}$ by a continuous linear form f on $\bigcap_{\nu'''<\nu''} \Lambda^{\nu'''}E_{kl}$. Since $e_{kl} \in \Lambda^{\nu''}E_{kl}$, we have $f(e_{kl}) = 0$; but E_{kl} has intersection $\mathrm{lin}\,\{e_{kl}\}$ with the closed linear hull E^1 of all the $E_{k'l'}$ with $(k', l') \neq (k, l)$. (This follows from Lemma 1.7.) So we may extend f linearly to $\bar{f}$, say, on $\mathrm{lin}\,(\bigcap_{\nu'''<\nu''} \Lambda^{\nu'''}E_{kl}, E^1) \subset E$, by taking $\bar{f}(E^1) = 0$. The hypothesis now shows that the domain of $\bar{f}$ includes $\bigcap_{\nu'''<\nu''} \Lambda^{\nu'''}E$. Since thus any point in the intersection of the latter space with E_{kl} can be separated from $\Lambda^{\nu''}E_{kl}$, as soon as it is outside this space, by such an $\bar{f}$, it follows that $E_{kl} \cap \Lambda(\bigcap_{\nu'''<\nu''} \Lambda^{\nu'''}E) = E_{kl} \cap \Lambda^{\nu''}E \subset \Lambda^{\nu''}E_{kl}$. The converse inclusion follows, however, from I.2 applied to the inclusion map.

Thus, since now $E_{kl} \cap \Lambda^{\nu'}E = \mathrm{lin}\,\{e_{kl}\}$, it remains to show that any e_{kl} with $(k, l) \neq (1, 1)$ is separated from 0 by a continuous linear form which is defined on some space containing $\Lambda^{\nu'}E$ — i.e., containing all the $e_{k'l'}$ when k' and l' range over all possible values. But notice that the indices $(k, l) \neq (1, 1)$ are those for which the pair $\Psi(E_{kl})$ has at least one point which it has in common with just one other pair. So it follows from 1.6 that any linear form on the one dimensional space spanned by e_{kl} has a continuous extension to a subspace of E containing all the $e_{k'l'}$.

B. If ν is a limit number, we can use a quite analogous argument. We just have to take a larger M; having m_0 and m_1, we then take, for each ordinal $\nu' < \nu$, points $m_{kl}^{\nu'}$ in just the same way as we took m_{kl} before. Further, we take, for each $\nu' < \nu$, spaces $E_{kl}^{\nu'}$ such that $\Lambda^{\nu'} E_{kl}^{\nu'}$ are one-dimensional. The argument in A now carries over verbatim except for some obvious translations — e.g., this time we show that $e_{11} \in \operatorname{co}(U \cap \bigcap_{\nu'<\nu} \Lambda^{\nu'} E)$ (for all 0-neighbourhoods U) rather than $e_{11} \in \operatorname{co}(U \cap \Lambda^{\nu'} E)$.

C. For $\nu = 1$, we can also use the argument in A; but this time we take the E_{kl} one-dimensional. Further, the role of the $\Lambda^{\nu'} E$ and $\Lambda^{\nu'} E_{kl}$ in A is now played by E resp. E_{kl}. (So the assertion of the second paragraph from the end in A now becomes trivial.)

2.1. Proposition. *If* $0 < p < 1$, *then* $\alpha(\cdot)$ *may assume any ordinal value for complete* p*-normed spaces.*

Proof. From the example follows that $\alpha(\cdot)$ may equal any ordinal value of the first kind for such spaces. But if ν is a limit number, consider a complete p-normed space E_0 with $\alpha(E_0) = \nu + 1$. From I.2 it follows that if we take $E = E_0 / \Lambda^{\nu} E$, we get $\alpha(E) = \nu$.

2.2. *Remark.* The space which we get in C of example 2 turns out to be isomorphic to a space given by Peck [6] as an example of a space which fails to have separating dual though it is quotient space of the dual separated space l^p.

2.3. *Example.* Consider the subspace $\operatorname{lin}\{e_{11} + e_{kl} | k \geq 2\}$ of the space E which we get in C of example 2. Since it does not contain e_{11}, it is separated by its dual. On the other hand, e_{11} is in its closure (for $\|e_{kl}\| \to 0$ and so $e_{11} + e_{kl} \to e_{11}$ as $l \to \infty$). Then also all e_{kl} are in the closure, which hence equals E. Thus, we have a simple example of a tvs which is separated by its dual but whose completion is not. Cf. Klee [2] and the question of A. and W. Robertson quoted and answered there. — This shows that it is worthwhile to emphasize completeness in the examples above (cf. I 7.8 and 2.1).

3. Having carried out the program of this paper, we now devote this brief section to another application of the construction of Section 1.

3.1. Proposition. *For any number* p *with* $0 < p \leq 1$, *every metric space* M *(without linear structure) has an isometric imbedding* $j: M \to J(M)$ *into a complete* p*-normed space* $J(M)$ *with the following universal property:*

If $j^1 : M \to E$ is an arbitrary isometric imbedding into a complete p-normed space, then there is a linear mapping $\Phi : J(M) \to E$ for which the diagram

$$\begin{tikzcd} & M \arrow[dl, "j"'] \arrow[dr, "j^1"] & \\ J(M) \arrow[rr, "\Phi"] & & E \end{tikzcd}$$

is commutative up to translation [cf. the remark just after 1.1] and which is continuous with operator p-norm not greater than one.

(Notice that $J(M)$ is unique up to isometric isomorphism.)

Proof. This is a simple special case of 1.1. Namely, let N be a subset of $M \times M$ such that for each element (m', m'') of this product, precisely one of the elements (m', m'') and (m'', m') belongs to N; and then, let Ψ be a bijective mapping from a set $\mathscr{Q}$ of one-dimensional spaces. Let each of these spaces be metrized by means of a p-norm so that condition (iv) is fulfilled for suitable generators $\{e_L\}_{L \in \mathscr{Q}}$ of the spaces of $\mathscr{Q}$. Then $J(M) = \Pi(\mathscr{Q}, \{e_L\}, M, \Psi)$ is the required space; for, to say that j is isometric is the same as to say that each of the spaces of $\mathscr{Q}$ is isometrically imbedded in $J(M)$ (by the metrization of these spaces); since this argument is valid for j^1 as well as for j, we get the universal property of $J(M)$ from 3° of 1.1.

Now, the space Π of 1.1 does, indeed, satisfy a somewhat stronger universal property than the one given in 3° of 1.1; for instance, for the special case considered in the preceding proof, the very same argument as that for 3° in 1.5 gives the following. If $\varkappa : M \to N$ is a contraction (i.e., a mapping that is never length-increasing) into another metric space N, there is a linear mapping $\Phi : J(M) \to J(N)$ for which the diagram

$$\begin{tikzcd} M \arrow[r, "\varkappa"] \arrow[d, "j"] & N \arrow[d, "j"] \\ J(M) \arrow[r, "\Phi"] & J(N) \end{tikzcd}$$

is commutative up to translation and which has operator p-norm at most one. Thus we get:

3.2. PROPOSITION. *For fixed p, the map $M \to J(M)$ is a functorial map from the category of metric spaces and contractions to the category of p-normed spaces and linear mappings with p-norm at most one.*

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