On the uniqueness of summable trigonometric series and integrals

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1. Introduction

The aim of the present work is to give some generalizations of a well-known theorem by Zygmund and Verblunsky, which in one of its original forms can be stated as follows [13, p. 352]:

Let $\sum_{-\infty}^{\infty} a_n e^{inx}$ *be a trigonometric series with complex coefficients and suppose that* $a_n = o(n)$, $n \to \infty$. *If the series is Abel summable to an everywhere finite function* $f \in L^1(\mathbf{T})$ *then the given series is the Fourier series of* f.

This theorem has been generalized in various directions. Verblunsky [13, p. 356] proved it under weaker conditions on the Abel sums of the series and Shapiro [8] has obtained analogous results in higher dimensions under the assumption that $\sum_{R \le |m| \le R+1} |a_m| = o(R)$ as R tends to infinity. Results have also been obtained for trigonometric integrals of one variable by Verblunsky [12] and of several variables by Shapiro [7]. In these theorems summation by the Abel-Poisson or by the Cesàro method is used.

In this paper the theorem will be formulated in a more general form and proved for a class of summation methods which contains the Abel-Poisson method as a special case. Many of the earlier results will appear as corollaries.

The method of proof will consist in replacing summation using a given kernel H by summation using a kernel K, which is a certain linear combination of H and its dilatations. The properties of K will be such that we then can deduce our results directly.

I wish to express my gratitude to professor Yngve Domar, who suggested the problem, for his kind interest and support.

2. Notations and definitions

The points in the n-dimensional Euclidean space $\mathbf{R}^n$ or in the n-dimensional torus $\mathbf{T}^n$ will be denoted by $x = (x_1, x_2, \ldots, x_n)$. We shall write $|x| = r = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}$.

If F is any given function on $\mathbf{R}^n$ and if $R > 0$ then we define the function F_R by the relation $F_R(x) = R^n F(Rx)$ for all $x \in \mathbf{R}^n$.

If a function F has derivatives almost everywhere then we denote those derivatives by $[\partial F/\partial x_1]$ etc. By $\partial F/\partial x_1$ etc. we mean derivatives in distributional sense.

We denote the Laplacian by Δ, i.e.

$$\Delta f = \sum \frac{\partial^2 f}{\partial x_i^2} \text{ and } [\Delta f] = \sum \left[\frac{\partial^2 f}{\partial x_i^2}\right].$$

We write $B(x, \varrho)$ for the open ball with centre x and radius ϱ, $|B(\varrho)|$ will denote its volume.

A real-valued function H on $\mathbf{R}^n$ is said to belong to the class $\mathcal{H}$ if:

(i) H is non-negative and twice continuously differentiable

(ii) H is radial, i.e. there exists a function H_0 on $[0, +\infty)$ such that $H(x) = H_0(|x|)$ for all $x \in \mathbf{R}^n$

(iii) H is integrable and $\int_{\mathbf{R}^n} H\, dx = 1$

(iv) dH_0/dr is non-positive

(v) $d^2H_0/dr^2 = O(r^{-(n+2+\varepsilon)})$ as $r \to \infty$ for some $\varepsilon > 0$.

In the sequel we shall not use the notation H_0, but write abusively $H(x) = H(r)$ and H' for $H_0' = dH_0/dr$.

A locally integrable funcdion F on $\mathbf{R}^n$ is said to belong to the class $\mathcal{M}$ if:

$$\frac{1}{|B(R)|}\int_{B(0,R)} |F| dx = O(1) \quad \text{as} \quad R \to \infty .$$

δ denotes the Dirac measure, i.e. the measure given by the unit mass at the origin.

Throughout G will stand for the function defined by the following relation

$$G(x) = c_n \cdot \frac{1}{|x|^{2-n}} \text{ if } n \geq 3$$
$$= c_2 \cdot \log |x| \text{ if } n = 2$$

where the constants c_n are chosen so that $\Delta G = \delta$.

The Fourier transform of a function F or a measure μ is denoted by $\hat{F}$ and $\hat{\mu}$ respectively, both when the transform is defined in the ordinary sense and in the distributional sense.

3. Some lemmas

Before we can prove our main theorems we need some lemmas. The first one is fundamental for the method of proof in this paper and contains the main new idea.

LEMMA 3.1. *Suppose that H is a given function in the class $\mathcal{H}$. Consider the function K defined by the relation $K(x) = 2\int_1^\infty H_t(x)\,\frac{dt}{t^3}$ for all $x \in \mathbf{R}^n\setminus\{0\}$. Then K satisfies the following conditions:*

(i) *K is non-negative and twice continuously differentiable for $x \neq 0$.*
(ii) *K is radial.*
(iii) *K is integrable and $\int_{\mathbf{R}^n} K\,dx = 1$.*
(iv) *$\Delta K = A - M\delta$, where A is the function defined by $A(x) = -2H'(|x|)/|x|$ for $x \neq 0$ and where $M = \int_{\mathbf{R}^n} A\,dx$.*

Proof. (i)—(iii) are obvious. To prove (iv) we choose an arbitrary function $\varphi \in C_0^\infty(\mathbf{R}^n)$. The result then follows from the following equalities:

$$\langle \Delta K, \varphi\rangle = \langle K, \Delta\varphi\rangle = 2\int_{\mathbf{R}^n} \Delta\varphi(x)dx \int_1^\infty H_t(|x|)\,\frac{dt}{t^3} =$$

$$= 2\lim_{T\to\infty}\int_1^T dt \int_{\mathbf{R}^n} \Delta\varphi(x)H(|x|)\,\frac{dx}{t^3} =$$

$$= 2\lim_{T\to\infty}\int_{\mathbf{R}^n} \varphi\,dx \int_1^T \Delta\,\frac{H_t(|x|)}{t^3}\,dt =$$

$$= 2\lim_{T\to\infty}\int_{\mathbf{R}^n} \varphi\,dx \int_1^T \frac{d}{dt}\left(\frac{t^{n-1}H'(t|x|)}{|x|}\right)dt =$$

$$= \lim_{T\to\infty}\int_{\mathbf{R}^n} (-\varphi A_T + \varphi A)dx = \langle A - M\delta, \varphi\rangle$$

with A and M defined as in (iv) above.

In our next lemma we assume that $n \geq 2$, but the result also holds for $n = 1$ with obvious modifications of the statement and its proof.

LEMMA 3.2. *Let $B \subset \mathbf{R}^n$ be an open ball, let G be the function defined in section 2, let $H \in \mathcal{H}$ and define, as in lemma 3.1, the function K by $K(x) = 2 \int_1^\infty H_t(x)\, dt/t^3$ for all $x \in \mathbf{R}^n \setminus \{0\}$.*

If $F \in \mathcal{M}$ satisfies the following conditions:

(i) *F is upper semi-continuous in B*

(ii) *$\overline{\lim}_{R\to\infty} (F * \Delta K_R)(x) \geq f(x)$ for all $x \in B$, where f is integrable, is finite in the ball B and has compact support,*

then $\Delta(F - f * G) \geq 0$ in B, i.e. $F - f * G$ is an almost subharmonic function in B.

Proof. Let F_0 be defined by $F_0(x) = F(x)$ for $x \in B$ and zero otherwise. A trivial estimate gives

$$[\Delta K](x) = -\frac{2H'(r)}{r} = O(r^{-(n+2+\varepsilon)})$$

as $r \to \infty$ and we hence easily obtain that

$$\lim_{R\to\infty} (\Delta K_R * (F - F_0))(x) = 0 \quad \text{for all} \quad x \in B\,.$$

F_0 therefore also satisfies (i) and (ii) and we may hence without loss of generality assume that F vanishes outside B.

Assume now that (ii) holds with strict inequality for all $x \in B$ and that f is upper semi-continuous. Let $\bar{K}_R$ be defined by $\bar{K}_R(x) = K_R(x)$ for all x in some ball $B(0, \varrho)$ with ϱ greater than two times the radius of B and zero otherwise. Then

$$\overline{\lim}\, F * \Delta\bar{K}_R = \overline{\lim}\, F * \Delta K_R \quad \text{for all} \quad x \in B\,.$$

Since f is upper semi-continuous and finite there exists for each $x_0 \in B$ and each $\varepsilon > 0$ a $\delta > 0$ such that $f(x) \leq f(x_0) + \varepsilon$ whenever $|x - x_0| < \delta$. Hence

$$\Delta\bar{K}_R * (f * G)(x_0) = (\bar{K}_R * f)(x) \leq f(x_0) + 2\varepsilon$$

for all R larger than some R_0 depending upon x_0 and ε.

Using the finiteness of f again we conclude that

$$\overline{\lim_{R\to\infty}}\, ((F - f * G) * \Delta\bar{K}_R)(x) > 0 \quad \text{for all} \quad x \in B\,. \tag{3.1}$$

Furthermore, since f is bounded above, we can easily prove that $-f * G$ and hence also $F - f * G$ is upper semi-continuous. Suppose, still on the assumption that (ii) holds with strict inequality, that $F - f * G$ is not subharmonic in B.

Then we can find a harmonic function u such that $\Phi = F - f * G + u$ has a maximum in some interior point x_0 of B. But then (iv) of lemma 3.1 gives $\overline{\lim}_{R\to\infty} (\Phi * \Delta\bar{K}_R)(x_0) \leq 0$ which contradicts (3.1) since $\overline{\lim}_{R\to\infty} (u * \Delta\bar{K}_R)(x_0) = 0$. The result now follows in the particular case we have considered.

To prove the general case we choose, using the Vitali-Carathéodory theorem [4, p. 75], a non-decreasing sequence $\{f_k\}_{k=1}^{\infty}$ of upper semicontinuous functions satisfying:

(i) $f_k \to f$ a.e. and in L^1-norm as $k \to \infty$

(ii) $f_k < f$ in B.

The first part of this proof tells us that $\Delta(F - f_k * G) \geq 0$ in B. But $f_k * G$ converges to $f * G$ in the distributional sense and thus $\Delta(F - f * G) \geq 0$, which proves the lemma.

The following two lemmas are simple generalizations of lemmas of Shapiro [9, p. 69—70] and our proofs are essentially the same as his.

Lemma 3.3. *Let F be an arbitrary function in $\mathcal{M}$, let F be its Fourier transform and assume $H \in \mathcal{H}$.*

Define the functions $\varkappa$ and U by

$$\varkappa(x) = -\frac{2H'(|x|)}{M|x|} \text{ for } x \in \mathbf{R}^n\setminus\{0\}$$

and

$$U(x) = \begin{cases} C & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

where the constants C and M are chosen such that $\hat{U}(0) = \hat{\varkappa}(0) = 1$.

If $\hat{F}$ is a measure such that $|\hat{F}|(\{x; N \leq |x| < 2N\}) = o(1)$ as $N \to \infty$ then $(F * \varkappa_R - F * U_R)$ tends to zero uniformly in $\mathbf{R}^n$ as $R \to \infty$.

Proof. Put $V = \varkappa - U$. Since $rV \in L^1(\mathbf{R}^n)$ we know that $\hat{V} \in C^1(\mathbf{R}^n)$ and since $\hat{V}(0) = 0$ we immediately see that $\hat{V}(r) = O(r)$ as $r \to 0$. Furthermore, since $H \in \mathcal{H}$, it follows that $d\varkappa/dr = d/dr(-2H'(r)/Mr) \in L^1(\mathbf{R}^n)$. Since dU/dr is a bounded measure we can conclude that $\hat{V}(r) = O(r^{-1})$ as $r \to \infty$.

Put $\mu = \hat{F}$. Then

$$|F * V_R(y)| = \left| \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \hat{V}\left(\frac{x}{R}\right) e^{i(x,y)} d\mu(x) \right| \leq$$

$$\leq \frac{1}{(2\pi)^n} \sum_{N=0}^{\infty} \int_{2^N \leq |x| < 2^{N+1}} \left| \hat{V}\left(\frac{x}{R}\right) \right| d|\mu|(x) + o(1)\,.$$

Using the fact that $\hat{V}(r) = O(r)$ as $r \to 0$ we obtain

$$\sum_{2^N \le R} \int_{2^N \le |x| < 2^{N+1}} \left| \hat{V}\left(\frac{x}{R}\right) \right| d|\mu|(x) = \sum_{2^N \le R} o\left(\frac{2^{N-1}}{R}\right) = o(1) \quad \text{as} \quad R \to \infty .$$

Further because $\hat{V}(r) = O(r^{-1})$ as $r \to \infty$ it follows that

$$\sum_{2^N > R} \int_{2^N \le |x| < 2^{N+1}} \left| \hat{V}\left(\frac{x}{R}\right) \right| d|\mu|(x) = \sum_{2^N > R} o\left(\frac{R}{2^N}\right) = o(1) \quad \text{as} \quad R \to \infty ,$$

and the lemma is proved.

LEMMA 3.4. *With the same assumptions and notations as in lemma 3.3*

$$\lim_{R \to \infty} \sup_{|x-y| \le R^{-1}} |F * U_R(x) - F * U_R(y)|) = 0 .$$

The proof is similar to the preceding one and is therefore omitted.

4. The main theorems

As before let H be an arbitrary function in the class $\mathcal{H}$. Write $f_*(x)$ and $f^*(x)$ for the lower and upper limits respectively of $(F * \Delta H_R)(x)$ as R tends to infinity.

THEOREM 4.1. *Let F be a bounded continuous function on $\mathbf{R}^n$. If*
(i) f_* and f^* are finite for all $x \in \mathbf{R}^n$
(ii) $f_* \ge \chi$, where χ is a locally integrable function,
then $f_* = f^*$ a.e. and $\Delta F = f_*$.

Proof. It is sufficient to prove that $f_* = f^*$ a.e. and $\Delta F = f_*$ in an arbitrary open ball $B \subset \mathbf{R}^n$. We can without loss of generality assume that F vanishes outside B, because if we define a function F_0 by $F_0(x) = F(x)$ for $x \in B$ and $F_0 = 0$ otherwise, then $\lim_{R \to \infty} ((F - F_0) * \Delta H_R)(x) = 0$ for all $x \in B$. Clearly we can also assume that χ has compact support and we define as before the functions K and $\varkappa$ by the relations

$$K(x) = 2 \int_1^\infty H_t(x) \frac{dt}{t^3} , \quad x \in \mathbf{R}^n \setminus \{0\}$$

$$\varkappa(x) = - \frac{2H'(|x|)}{M|x|} , \quad x \in \mathbf{R}^n \setminus \{0\}$$

where we choose M so that $\int_{\mathbf{R}^n} \varkappa \, dx = 1$.

Then the following useful relation holds.

$$2\int_1^T (F * \Delta H_t)\,\frac{dt}{t^3} = M(F * \varkappa - F * \varkappa_T) = F * [\Delta K] - MF * \varkappa_T\,. \tag{4.1}$$

The proof is a direct application of Fubini's theorem:

$$2\int_1^T (F * \Delta H_t)\,\frac{dt}{t^3} = 2\int_1^T \frac{dt}{t^3}\int_{\mathbf{R}^n} F(x-y)\Delta H_t(y)dy =$$

$$= 2\int_{\mathbf{R}^n} F(x-y)dy\int_1^T \Delta H_t(y)\,\frac{dt}{t^3} = M\int_{\mathbf{R}^n} F(x-y)(\varkappa(y)-\varkappa_T(y))dy$$

where the last equality follows from the proof of lemma 3.1.

Relation (4.1) also holds if $F \in \mathcal{M}$, which will be of use later.

If we now in (4.1) let T tend to infinity it follows from the continuity of F in B that the last expression in (4.1) tends to $(F * \Delta K)(x)$ for all $x \in B$. After a change of scale we obtain

$$2R^2\int_R^\infty (F * \Delta H_t)(x)\,\frac{dt}{t^3} = (F * \Delta K_R)(x) \quad \text{for} \quad x \in B\,.$$

From this we conclude that $\chi \leq f_* \leq \varphi_* \leq \varphi^* \leq f^*$ in B, where $\varphi_*(x)$ and $\varphi^*(x)$ are the lower and upper limits of $F * \Delta K_R(x)$ as R tends to infinity.

We can assume that χ is finite for all $x \in B$. Lemma 3.2 then shows that the function $\Phi = F - \chi * G$ satisfies $\Delta\Phi \geq 0$.

Hence $\Delta F = \chi + \mu$ where $\mu = \Delta\Phi$ is a positive measure (see e.g. [5, p. 29]) and hence $F * \Delta H_R = (\mu + \chi) * H_R$.

But $\lim_{R\to 0} \chi * H_R = \chi$ a.e. We also claim that $\lim \mu * H_R$ exists a.e. and equals some locally integrable function g.

To prove this we shall use the wellknown fact [4, p. 149] that

$$\lim_{r\to 0} \frac{1}{|B(r)|}\int_{B(x,r)} d\mu$$

exists for almost all x and equals a locally integrable function, which we denote by g.

By a simple calculation we obtain

$$\mu * H_R(x) = \int_{\mathbf{R}^n} H(y)d\mu(x-y) = \int_0^\infty H_R(r)d\left(\int_{|x-y|\leq r} d\mu(y)\right) =$$

$$= \int_0^\infty H_R'(r)dr\int_{|x-y|\leq r} d\mu(y) = |B(1)|\int_0^\infty R^{n+1}H'(Rr)r^n\left(\frac{1}{B(r)}\int_{|x-y|\leq r} d\mu(y)\right)dr\,.$$

Using the Lebesgue dominated convergence theorem we now easily get that

$$\lim_{R\to\infty} \mu * H_R(x) = g(x) \quad \text{a.e.}$$

From this it follows that $f_* = f^*$ a.e. in B and that f_* and f^* are locally integrable and so lemma 3.2 gives

$$\Delta(F - f_* * G) = 0 \quad \text{in} \quad B$$

and, since B is arbitrary, the theorem follows.

Remark 4.1. The above theorem remains true even if condition (ii) is replaced by (ii′) *$f^* \geq \chi$, where χ is a locally integrable function.*

For a method of proof see [13, p. 358].

In our next theorem we shall replace the condition F continuous by a condition on the Fourier transform of F. This will give us a generalization of the results of Verblunsky [13, p. 352] and Shapiro [8, Theorem 1].

THEOREM 4.2. *Let F belong to the class $\mathcal{M}$ and let $\hat{F}$ denote the Fourier transform of F (in the distributional sense). If*
(i) *$\hat{F}$ is a measure satisfying $|\hat{F}|(\{x; N \leq |x| < 2N\}) = o(1)$ as $N \to \infty$,*
(ii) *f_* and f^* are finite for all $x \in \mathbf{R}^n$,*
(iii) *$f_* \geq \chi$, where χ is a locally integrable function,*
then $f_* = f^*$ a.e. and $\Delta F = f_*$.

Proof. Let $B \subset \mathbf{R}^n$ be an arbitrary ball. It is sufficient to prove that $f_* = f^*$ a.e. and $\Delta F = f^*$ in B.

Let B_0 be a bounded neighbourhood of B. We claim that it suffices to consider an F which vanishes outside B_0. To show this we choose a function $h \in C_0^\infty(\mathbf{R}^n)$ which equals 1 on B and vanishes outside B_0. Define $F_0 = Fh$. We then get $\hat{F}_0 = \hat{F} * \hat{h}$ where $\hat{h} \in C^\infty(\mathbf{R}^n)$ and where

$$\hat{h}(x) = O(|x|^{-p}) \quad \text{as} \quad x \to \infty \quad \text{for all} \quad p\,. \tag{4.3}$$

Let $E_N = \{x \in \mathbf{R}^n; 2^N \leq |x| < 2^{N+1}\}$ and denote the measure $\hat{F}$ by μ. The function $\hat{F}_0$ then satisfies

$$\int_{E_N} |\hat{F}_0| dx = o(1) \quad \text{as} \quad N \to \infty\,, \tag{4.4}$$

since

$$0 \leq \int_{E_N} |\hat{F}_0| dx \leq \int_{\mathbf{R}^n} d|\mu|(y) \int_{E_N} |\hat{h}(x-y)| dx =$$

$$= \int_{|y|<2^{N-1}} d|\mu| \int_{E_N} |\hat{h}(x-y)| dx + \int_{2^{N-1} \leq |y| < 2^{N+2}} d|\mu| \int_{E_N} |\hat{h}(x-y) dx +$$

$$\int_{|y| \geq 2^{N+2}} d|\mu| \int_{E_N} |\hat{h}(x-y)| dx = I_1 + I_2 + I_3 ,$$

where, using (i) and (4.3)

$$I_1 \leq |\mu|(\bigcup_1^{N-2} E_k) \sup_{|x| \geq 2^{N-1}} |\hat{h}(x)| = o(1) \quad \text{as} \quad N \to \infty$$

$$I_2 \leq |\mu|(\bigcup_{N-1}^{N+1} E_k) \sup_{\mathbf{R}^n} |\hat{h}(x)| = o(1) \quad \text{as} \quad N \to \infty$$

$$I_3 \leq \sum_{k=N+2}^{\infty} (\mu(E_k) \sup_{|x| \geq 2^{k-1}} |\hat{h}(x)|) = o(1) \quad \text{as} \quad N \to \infty .$$

We can thus as claimed take F and χ to vanish outside B_0.

As we noted above, relation (4.1) holds also for $F \in \mathcal{M}$. Furthermore, (ii) shows that $2 \int_1^T (F * \Delta H_t)\, dt/t^3$ converges to a finite limit for all $x \in B$ as T tends to infinity. From (4.1) it follows that $\lim_{T\to\infty} (F * \varkappa_T)(x)$ must exist and take finite values for all $x \in B$. But we also know that $(F * \varkappa_T)(x)$ tends to $F(x)$ for almost all x. We can therefore assume that $\lim_{T\to\infty} F * \varkappa_T = F$ for all x and that F is finite everywhere.

If we again set $\varphi^*(x)$ and $\varphi_*(x)$ equal to the upper and lower limits respectively of $(F * \Delta K_R)(x)$ then we get as before $\chi \leq f_* \leq \varphi_* \leq \varphi^* \leq f^*$ in B.

The remaining part of the proof is now an adaptation of the arguments used by Verblunsky and Shapiro.

Choose an everywhere finite upper semi-continuous function $u \leq \chi$.

If $\Phi = F - u * G$ were upper semi-continuous we could as in lemma 3.2 conclude that Φ was subharmonic.

Let therefore D be the set of all points in B where Φ is not upper semi-continuous. We claim that D is the empty set.

Assume that D is non-empty. By the Baire category theorem there exists a ball $B' = B(x_0, 2d)$ such that $x_0 \in D$ and such that $F * \Delta H_R(x)$ is uniformly bounded for $x \in \bar{D} \cap B'$ (where $\bar{D}$ denotes the closure of D).

From this it follows by relation (4.1) that $F * \varkappa_T$ converges uniformly in $\bar{D} \cap B'$ and hence that the restriction $F|_{\bar{D} \cap B'}$ is continuous.

Choose now an arbitrary number $M > (-u * G)(x_0)$.

Since $-u * G$ is upper semi-continuous and since $F|_{\bar{D} \cap B'}$ is continuous we can assume that d has been chosen so that

$$\Phi(x) \leq M + F(x_0) \quad \text{for all} \quad x \in \bar{D} \cap B' . \tag{4.5}$$

Let x be an arbitrary point in $B'' = B(x_0, d)$ not belonging to $\bar{D}$ and let $x' \in \bar{D} \cap B'$ be one of the points minimizing the distance to x from $\bar{D} \cap B'$. Put $R = |x - x'|^{-1}$ and let U be as in lemma 3.3.

Let $\varepsilon > 0$ be given. Since $R \geq d^{-1}$ we can assume that d has been chosen so that

$$\begin{aligned} |F * U_R(x) - F(x')| \leq |F * U_R(x) - F * U_R(x')| + \\ |F * U_R(x') - F * \varkappa_R(x')| + |F * \varkappa_R(x') - F(x')| < \varepsilon . \end{aligned} \tag{4.6}$$

That this assumption is allowed follows from lemmas 3.3 and 3.4 and from the fact that $F * \varkappa_R$ converges uniformly to F on $\bar{D} \cap B'$. We may also assume that

$$|F(x_0) - F(y)| < \varepsilon \quad \text{for all} \quad y \in \bar{D} \cap B' \tag{4.7}$$

and, since $-u * G$ is upper semi-continuous, that

$$(-u * G) * U_R(y) \leq M \quad \text{for all} \quad y \in B' . \tag{4.8}$$

Since Φ is upper semi-continuous in the ball $B(x, |x - x'|)$ we have that Φ is subharmonic there and hence that

$$\Phi(x) \leq \Phi * U_R(x) = F * U_R(x) - (u * G) * U_R(x) . \tag{4.9}$$

From (4.6)—(4.9) it follows that

$$\Phi(x) \leq M + F(x_0) + 2\varepsilon \quad \text{for all} \quad x \in C\bar{D} \cap B$$

and together with (4.5) this gives a contradiction. We conclude that Φ is upper semi-continuous in B.

Thus $\Delta\Phi \geq 0$ in B and using the same argument as in the proof of theorem 4.1 we get that $f_* = f^*$ a.e., that f_* and f^* are locally integrable and that $\Delta(F - f_* * G) = 0$ in B.

This proves the theorem.

5. Application on trigonometric series and integrals

The preceding theorems can be applied to the theory of summation of trigonometric series and integrals. In the particular case when H is the Poisson kernel they will give us some of the classical results by Zygmund, Verblunsky and Shapiro.

We shall denote the points in $\mathbf{Z}^n$ by $m = (m_1, \dots m_n)$ and write

$$|m| = (m_1^2 + m_2^2 + \dots + m_n^2)^{1/2}, \quad (m, x) = \sum_1^n m_i x_i.$$

We start by defining the summation methods in question and by making some remarks.

Definition 5.1. *Let $H \in \mathcal{H}$ and let $\sum_{m \in \mathbf{Z}^n} a_m e^{i(m,x)}$ be a given trigonometric series with complex coefficients.*

The series is said to be summable-(H) at the point $x \in \mathbf{R}^n$ if

$$\lim_{R \to \infty} \sum_{m \in \mathbf{Z}^n} a_m \hat{H}\left(\frac{m}{R}\right) e^{i(m,x)}$$

exists and is finite.

If we have the condition $a_m = O(|m|^k)$ for some k the series defines a periodic distribution on $\mathbf{R}^n$ which we denote by f. We can write

$$\sum a_m \hat{H}\left(\frac{m}{R}\right) e^{i(m,x)} = f * H_R$$

In general this should be interpreted as an equality between two distributions. In all cases we shall consider, however, the series on the left hand side will be absolutely convergent for all $R > 0$ and we simply define $f * H_R$ as its sum.

We shall let $f_*(x)$ and $f^*(x)$ denote the lower and upper limits respectively of $f * H_R(x)$ as R tends to infinity.

Now we can formulate some corollaries to theorems 4.1 and 4.2.

Theorem 5.1 (cf. [14]). *Let $\sum_{m \in \mathbf{Z}^n} a_m e^{i(m,x)}$ be a trigonometric series with complex coefficients, let $H \in \mathcal{H}$ and suppose that $\Sigma\, |a_m \hat{H}(m/R)| < \infty$ for all $R > 0$. If*

(i) $\sum_{m \neq 0} - a_m |m|^{-2} e^{i(m,x)}$ *is the Fourier series of a continuous function* F,

(ii) f_* *and* f^* *are finite everywhere,*

(iii) $f_* \geq \chi$ *for some integrable function* χ

then $f_* = f^*$ a.e. and the given series is the Fourier series of f_*.

Proof. If we suppose that $a_0 = 0$ (as we may, without loss of generality) then the conditions of theorem 4.1 are satisfied and the result follows.

Theorem 5.2 (cf. [13], p. 356]). *Let $\Sigma\, a_m e^{i(m,x)}$ and H be as in theorem 5.1 If (ii) and (iii) of that theorem hold and if in addition*

(i′) $\sum_{N \leq |m| \leq 2N} |a_m| = o(N^2)$ *as* $N \to \infty$

then the conclusion of theorem 5.1 still holds.

Proof. Assume that $a_0 = 0$ and let $F \in L^2(\mathbf{T}^n)$ be the function whose Fourier series is $\sum_{m \neq 0} - |a_m|\, |m|^2 e^{i(m,x)}$. Consider F as a tempered distribution on $\mathbf{R}^n$. Its Fourier transform consists of point masses at the points $\{2\pi m\}$. Condition (i′) implies that this measure satisfies condition (i) of theorem 4.2 from which the theorem follows.

Next we shall prove a result concerning trigonometric integrals.

THEOREM 5.3. *Let $\hat{f}$ be a given function in the Schwartz class $\mathcal{S}'$ of tempered distributions and let $H \in \mathcal{H}$ be such that $fH_R \in L^1(\mathbf{R}^n)$ for all $R > 0$.*

Write $\hat{f} = \hat{f}_0 + \hat{f}_1$, where $\hat{f}_0$ has compact support and where $\hat{f}_1$ vanishes in a neighbourhood of the origin.

Let $f \in \mathcal{S}'$ be the inverse Fourier transform of $\hat{f}$ and let $f_*(x)$ and $f^*(x)$ denote the lower and upper limits of $f * H_R(x)$ as R tends to infinity.

If

(i) $-|y|^{-2}\hat{f}(y)$ *is the Fourier transform of a continuous and bounded function* F_1
(ii) f_* *and* f^* *are finite for all* $x \in \mathbf{R}^n$
(iii) $f_* \geq \chi$, *where* χ *is a locally integrable function,*
then $f_* = f^* = f$ *a.e.*

Proof. Let f_i be the inverse Fourier transform of $\hat{f}_i$, $i = 0, 1$. Then $f_0 \in C^\infty(\mathbf{R}^n)$ and hence $\lim f_0 * H_R = f_0$ for all x. From theorem 4.1 it follows that $\Delta F_1 = f_1$ and the theorem follows.

Remark 5.1 (cf. [7]). Condition (i) is satisfied if for instance

(a) $\hat{f}(y)(1 + |y|^2)^{-1} \in L^1$

or if

(b) $\hat{f}(y)(1 + |y|^2)^{-1} \in L^2$ and $\displaystyle\int_{B(0,R)} \hat{f}(y)(1 + |y|^2)^{-1} e^{i(x,y)}\, dy$

converges pointwise to a continuous function as R tends to infinity.

By the same method, but using theorem 4.2 instead of theorem 4.1, we obtain the following result.

THEOREM 5.4. *Let $\hat{f}$, f, f_*, f^* and H be as in theorem 5.3.*

If (ii) and (iii) of theorem 5.3 hold and if in addition

(i′) $\displaystyle\int_{N \leq |y| \leq 2N} |\hat{f}|\, dy = o(N^2)$ *as* $N \to \infty$

then $f_* = f^* = f$ *a.e.*

6. Exceptional sets

In the preceding sections we have throughout assumed that $f * H_R$ is bounded as a function of R at all points x. This condition can be somewhat weakened and results corresponding to those obtained by Verblunsky [13, p. 356] and Shapiro [8, theorem 2], [7] can be proved.

If $n = 1$ it is for instance sufficient both in theorems 4.1 and 4.2 to assume that f_* and f^* are finite except in a denumerable set E if in addition we know that

$$f * H_R(x) = o(R) \quad \text{as} \quad R \to \infty \quad \text{for all} \quad x \in E\,.$$

If $n \geq 2$ it is sufficient in theorem 4.1 to assume that f_* and f^* are finite except in a set of zero capacity with respect to the kernel $-G$. In theorem 4.2 it is sufficient to assume for $n \geq 2$ that f_* and f^* are finite except in a set without finite cluster points.

The following result seems to be new. For the sake of simplicity we do not formulate it in full generality.

THEOREM 6.1. *Suppose $n \geq 2$. Let $\sum_{m \in \mathbf{Z}^n} a_m e^{i(m,x)}$ be a given trigonometric series with complex coefficients and assume that $H \in \mathcal{H}$ is such that $\Sigma\, |a_m \hat{H}(m/R)| < \infty$ for all $R > 0$. Assume further that E is a bounded closed set of capacity zero with respect to the kernel $-G$.*

If

(i) $\sum_{N \leq |m| \leq 2N} |a_m| = o(N^2)$ *as* $N \to \infty$

(ii) $\lim \sum_{R \to \infty} a_m H(m/R) e^{i(m,x)} = C$ *for* $x \notin E$, *where* C *is a constant,*

(iii) *there exists* $\delta > 0$ *such that* $\Sigma\, a_m H(m/R) e^{i(m,x)} = O(R^{2-\delta})$ *as* $R \to \infty$ *for* $x \in E$,

then $a_m = 0$, $m \neq 0$, *and* $a_0 = C$.

Proof. We may without loss of generality assume that $a_0 = 0$.

Let $F_0 \in L^2(\mathbf{T}^n)$ be the function whose Fourier series is $\Sigma - a_m |m|^{-2} e^{i(m,x)}$. Consider F_0 as a periodic function on $\mathbf{R}^n$ and let $B \subset \mathbf{R}^n$ be an arbitrary open ball containing E. As in the proof of theorem 4.2, we can multiply F_0 by a function $h \in C_0^\infty(\mathbf{R}^n)$ which equals 1 on B and which vanishes outside a neighbourhood B_0 of B. It is sufficient to prove that the function $F = hF_0$ equals C a.e. on B. Put $\chi = Ch$ and let as before $\varkappa$ denote the function $-2H'(r)/Mr$.

From (ii) and (iii) it follows that

$$\int_1^\infty (F * \Delta H_t)(x) dt/t^3$$

converges both when $x \in B \setminus E$ and when $x \in B \setminus CE$ and hence, using (4.1), that $F * \varkappa_T(x)$ tends to a finite limit for all $x \in B$ as T tends to infinity.

We can therefore assume that F has been chosen finite everywhere and such that $\lim_{T\to\infty} F * \varkappa_T = F$ for all $x \in B$.

Let D be the set of points where F is not continuous and suppose that D is non-empty. Using the Baire category theorem we then can prove that there exists a ball $B' = B(x_0, 2d)$ such that its centre x_0 belongs to D and such that $R^{\delta-2}(F * \Delta H_R)(x)$ is uniformly bounded for $x \in \bar{D} \cap B'$ and for all $R \geq 1$. By (4.1) this implies that $F * \varkappa_T$ converges uniformly in $\bar{D} \cap B'$ and hence that $F|_{\bar{D} \cap B'}$ is continuous.

Write $\Phi = F - \chi * G$ and note that $\chi * G$ is continuous. Proceeding as in the proof of theorem 4.2 we get that Φ is continuous in a neighbourhood of x_0 and thus that F is continuous in B.

Using (4.1) once more we see that $\lim (F * \Delta K_R)(x) = C$ for all $x \in B \cap CE$ and we can conclude from lemma 3.2 that the support of $\Delta\Phi$ is contained in E. Since E has capacity zero and since Φ is continuous in B it follows from a classical theorem [3, theorem VII.1] that Φ is harmonic in the whole of B, which gives the theorem.

7. A pointwise saturation theorem

In this section we shall consider a problem of a slightly different type.

It is a well-known theorem by Hille that if the Abel-Poisson means $u(r, x)$ of a function $f \in C(\mathbf{T})$ satisfy

$$\lim_{r\to 1-0} \frac{u(r, x) - f(x)}{1 - r} = 0 \tag{7.1}$$

uniformly in $\mathbf{T}$ then f equals a constant [3, p. 122]. For an account of further results in this direction see e.g. Sunouchi [10].

We shall here prove a theorem which shows that the above result remains true even under the much weaker assumption that (7.1) holds pointwise. Andrienko [1] has recently obtained a similar result for (C,1)-summability.

Let N be the distribution on $\mathbf{R}$ whose Fourier transform is $\hat{N}(t) = - i \cdot \operatorname{sign} t$, define $\tilde{f} = f * N$ (where the convolution is defined by means of the Fourier transform) and let P be the Poisson kernel, i.e. $P(x) = c_n(1 + |x|^2)^{-\frac{n+1}{2}}$ where the constants c_n are chosen so that $\hat{P}(0) = 1$.

The main results of this section is the following theorem.

THEOREM 7.1. *Let* $n = 1$ *and let* X *be the space* $L^1(\mathbf{R})$ *or* $L^1(\mathbf{T})$. *Suppose that* $f \in X$ *is finite everywhere and let* $-\infty \leq a < b \leq +\infty$. *If*

$$\lim_{R\to\infty} R(f * P_R(x) - f(x)) = g(x) \quad for\ all \quad x \in [a, b]\,, \tag{7.2}$$

where g *is a locally integrable and finite function, then*

$$\frac{d\tilde{f}}{dx} = -g \quad in \quad [a, b]\,.$$

Proof. We start by noting that the proof of theorem 4.2 actually gives a stronger result than stated in that theorem. We have, in fact, only used the assumptions (ii) and (iii) to prove that the integral

$$\int_1^\infty (f * H_t)(x) \frac{dt}{t^3}$$

is convergent and that

$$\varphi^*(x) = \overline{\lim_{R\to\infty}}\, 2R^2 \int_R^\infty (f * H_t)(x) \frac{dt}{t^3}$$

and

$$\varphi_*(x) = \underline{\lim_{R\to\infty}}\, 2R^2 \int_R^\infty (f * H_t)(x) \frac{dt}{t^3}$$

both are finite for all x and that $\varphi_* \geq \chi$ for some locally integrable function χ.

The assumptions on f_* and f^* in theorem 5.2 and 5.4 could therefore be replaced by, for instance, the assumption that for all $x \in [a, b]$

$$\int_1^\infty (f * H_t)(x) \frac{dt}{t^3} < \infty$$

and

$$\lim 2R^2 \int_R^\infty (f * H_t)(x) \frac{dt}{t^3} = g(x)\,,$$

where g is a finite and locally integrable function.

Using this observation it is now an easy matter to deduce theorem 7.1.

If we set $y = R^{-1}$ we can write

$$R(f * P_R - f) = \frac{1}{y} \int_0^y \frac{d}{dt} (f * P_{t^{-1}}) dt$$

Let φ be the tempered distribution whose Fourier transform is $-|y|\hat{f}(y)$. Then the Fourier transform of $d/dt(f * P_{t^{-1}})$ equals

$$\frac{d}{dt}(\hat{f}\hat{P}_{t^{-1}}) = \frac{d}{dt}(\hat{f} \cdot e^{-t|y|}) = \hat{\varphi} \cdot \hat{P}_{t^{-1}} = (\varphi * P_{t^{-1}})^{\wedge} .$$

Using this we can write (7.2) as

$$\lim_{y \to 0+} \frac{1}{y} \int_0^y (\varphi * P_{t^{-1}})(x) dt = g(x)$$

which, after a change of variable, gives

$$\lim_{R \to \infty} R \int_R^\infty (\varphi * P_t)(x) \frac{dt}{t^2} = g(x) .$$

By means of a partial integration we finally obtain

$$\lim_{R \to \infty} 2R^2 \int_R^\infty (\varphi * P_t)(x) \frac{dt}{t^3} = g(x) \quad \text{for} \quad x \in [a, b] .$$

But, since $f \in X$, we know that

$$\int_N^{2N} |\varphi| dx = o(N^2) \quad (\text{or } \sum_N^{2N} |\varphi(k)| = o(N^2))$$

and hence we can by the observation in the beginning of this proof, use theorem 5.2 or 5.4 to conclude that

$$\varphi = g \quad \text{a.e. in} \quad [a, b] .$$

On the other hand,

$$(\tilde{f})^{\wedge}(t) = -i \hat{f}(t) \operatorname{sign} t$$

and thus

$$\left(\frac{d\tilde{f}}{dx}\right)^{\wedge}(t) = t \hat{f}(t) \operatorname{sign} t = -\hat{\varphi}$$

which proves the theorem.

Remark 7.1. It might be worth pointing out that the particular property of the Poisson kernel that makes the above proof work is that the function $(-x P)^{\sim}$ belongs to $\mathcal{H}$.

By an analogous method we can, using theorem 5.3, obtain the following result.

THEOREM 7.2. *Let* $n \geq 2$ *and let* $B \subset \mathbf{R}^n$ *be an arbitrary open ball. Assume that* $f \in L^\infty(\mathbf{R}^n) \cap L^1(\mathbf{R}^n)$ *is finite-valued. If*

$$\lim_{R\to\infty} R(f * P_R(x) - f(x)) = 0 \quad \textit{for all} \quad x \in B\,,$$

then $f = 0$ *in* B.

As a corollary to theorem 7.1 we can get the result obtained by Andrienko [1] in the case when $g = 0$.

COROLLARY 7.1. *Let* X, f *and* g *be as in theorem 7.1 and denote the Fejér kernel by* D. *If*

$$\lim_{R\to\infty} R(f * D_R(x) - f(x)) = g(x) \quad \textit{for all} \quad x \in [a, b] \tag{7.3}$$

then $d\tilde{f}/dx = -g$ *in* $[a, b]$.

Proof. It is sufficient to prove that (7.3) implies (7.2).

Fix $x \in \mathbf{R}$ and assume that $f(x) = 0$ which can be done without loss of generality. Set $\psi(t) = \hat{f}(t)e^{ixt} + \hat{f}(-t)e^{-ixt}$.

We can then write

$$f * P_R(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(t)e^{-Rt}e^{ixt}\,dt = \frac{1}{2\pi}\int_0^\infty \psi(t)dt\int_{t/R}^\infty \left(u - \frac{t}{R}\right)e^{-u}\,du =$$

$$= \frac{1}{2\pi}\int_0^\infty ue^{-u}\,du\int_0^{uR}\psi(t)\left(1 - \frac{t}{uR}\right)dt = \int_0^\infty ue^{-u}\,Q(uR)du\,,$$

where, by assumption, $Q(u) = g(x)\cdot u^{-1} + o(u^{-1})$ as $u \to \infty$ and where $|Q|$ is bounded. By means of a simple change of variable we now immediately obtain our corollary.

We finish this section by remarking that lemma 3.2 could be formulated as a pointwise saturation theorem. The method of proof of the lemma does in fact give the following result, which is similar to a theorem proved by H. S. Shapiro [6, p. 27] (in the case when $n = 1$) under stronger assumptions on f.

THEOREM 7.3. *Suppose that* H *is a positive radial function on* $\mathbf{R}^n$ *which satisfies*

(i) $\displaystyle\int_{\mathbf{R}^n} H\,dx = 1\,,$

(ii) $\displaystyle\int_{|x|\geq R} H\,dx = o(R^{-2})$ *as* $R \to \infty$.

If f is a bounded continuous function on $\mathbf{R}^n$ for which

$$\lim_{R\to\infty} R^2(f * H_R(x) - f(x)) = g(x) \tag{7.4}$$

at each point $x \in \mathbf{R}^n$, where g is finite and locally integrable, then $\Delta f = g$.

Proof. From condition (ii) it follows that it is sufficient to prove the theorem in the case when f (and hence also g) has compact support.

By solving the equation

$$\Delta K = \frac{\partial^2 K}{\partial r^2} + \frac{n-1}{r}\frac{\partial K}{\partial r} = H$$

we get a function

$$K(x) = \int_{|x|}^{\infty} r^{1-n}\, dr \int_{|x|\geq r} H(y)dy$$

satisfying $\Delta K = H - \delta$ and (7.4) can be written

$$\lim_{R\to\infty} (f * \Delta K_R)(x) = g(x)\,.$$

The argument used in the proof of lemma 3.2 now gives the result.

Note. After the completion of the manuscript the author was informed that prof. H. Berens recently has obtained theorem 7.1 by a different but related argument [2].

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