On the local and global non-characteristic Cauchy problem when the solutions are holomorphic functions or analytic functionals in the space variables

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1. Introduction

In [5], p. 116, L. Hörmander proves the linear Cauchy-Kovalevsky theorem by a method of successive approximations. The main lemma used by Hörmander in that proof is also found in M. Nagumo [6]. Nagumo uses the lemma and Schauder's fixed point theorem to prove the Cauchy-Kovalevsky theorem. L. V. Ovsjannikov [7] has used the ideas in the lemma to prove a theorem that could be called an abstract Cauchy-Kovalevsky theorem. We shall call it the Ovsjannikov theorem. The theorem treats a Cauchy problem for an ordinary differential equation for functions with values in certain Banach spaces that form a scale of Banach spaces. See also F. Treves [22] and [23]. The Ovsjannikov theorem can be used to prove the linear Cauchy-Kovalevsky theorem as is done in [7], [22], and [23].

In [23], p. 24, Treves proves the dual Ovsjannikov theorem taking as scale of Banach spaces the duals of the spaces in the original scale. Then on pp. 53—58 [23] Treves takes the dual of the scale used by Ovsjannikov to prove the linear Cauchy-Kovalevsky theorem and applies the dual Ovsjannikov theorem. This gives the dual Cauchy-Kovalevsky theorem. Here the coefficients are analytic functions just as in the ordinary theorem but the solution is a function of the time variable with values in the space of analytic functionals on the space of analytic functions of the space variables.

The purpose of this paper is the following. We shall give another proof of the dual Cauchy-Kovalevsky theorem, Theorem 1. We shall also prove a global version of that theorem, Theorem 2, together with a global version of the ordinary Cauchy-Kovalevsky theorem, Theorem 3. In the proofs of the dual theorems we shall use the Fourier-Borel transformation of analytic functionals. Then the dual theorems are transformed into theorems for partial differential equations of infinite order in the space variables. The solutions of the transformed problems are holomorphic functions that are entire functions of exponential order one in the space variables. The coefficients are polynomials in the space variables. We have treated problems of this kind for equations of finite order in [11] and [14]. See also S. Steinberg and F. Treves [19]. In [14] we use a higher order version of the Ovsjannikov theorem which was proved in [13]. The methods in [14] are applied directly in the proof of the dual Cauchy-Kovalevsky theorem here called Theorem 1.

Looking at Hörmander's proof in [5], p. 116, one notes that the radius of convergence of the solution is proportional to the sum of the absolute values of the coefficients. This is very disturbing when one wants to prove global theorems for equations with variable coefficients. We have earlier used other methods to overcome this difficulty, see [8], [9], and [10]. In the proofs of Theorem 2 and Theorem 3 we shall *modify the Ovsjannikov technique in such a way that the radius of* convergence of the solution has a lower bound that is proportional to the sum of the absolute values of the coefficients in the principal part.

A comparison with the results in [8], [9], and [10] specialized to the situation in Theorem 3 and this theorem shows that Theorem 3 is more general. But it should be stressed that the Ovsjannikov theorem is not applicable to general Goursat problems in its present form.

We use the higher order version of the Ovsjannikov theorem [13] in the present paper. This is simple. But some information is lost. For technical reasons the dual theorems Theorem 1 and Theorem 2 are given as the Fourier-Borel transformation of the dual problem. The original dual problem can be rewritten as a problem for a first order system. Then the original first order Ovsjannikov theorem can be applied to the Fourier Borel transform of the system. In this way one proves that in Theorem 1 $\hat{v}(t)$ and $\hat{u}(t)$ belong to $HG(0, s, \varepsilon(s' - s)/2a, 0)$ and not only to $HG(0, s, \varepsilon((s' - s)/2a)^m, 0)$. We have used this fact in [15] to prove local and global uniqueness theorems of Holmgren type.

It would also be interesting to study other dual problems suggested by the results in [11] and [14]. S. Steinberg and F. Treves have already studied a special case in [19].

Below we use various ideas from papers by F. Treves, S. Steinberg and F. Treves, L. V. Ovsjannikov, and by ourselves. See the reference list. As to the content we give the preliminaries in Section 2. In Section 3 we prove our theorems on the Fourier-Borel side of the problem. Section 4 contains a discussion on the dual Cauchy-Kovalevsky theorem. The ordinary global theorem is treated in Section 5.

2. Preliminaries

By $x = (x_1, \ldots, x_n)$ we denote a point in $\mathbf{C}^n$ and by $y = (y_1, \ldots, y_n)$ we denote a point in the dual of $\mathbf{C}^n$. The scalar product is denoted by $xy = x_1y_1 + \ldots + x_ny_n$. We use standard multi-index notation. So $\alpha = (\alpha_1, \ldots, \alpha_n)$

is a multi-index with nonnegative integers as components. We write $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $\alpha! = \alpha_1! \ldots \alpha_n!$, $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. If $\alpha_j \leq \beta_j$, $1 \leq j \leq n$, then we denote this by $\alpha \leq \beta$. We also write $D_t = \partial/\partial t$, $D_x = (D_1, \ldots, D_n) = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$ and $D_x^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$. Differentiation may denote real or complex differentiation or differentiation in the natural way in formal power series. A formal power series u is also written as

$$u = \sum_\alpha D_x^\alpha u(0)(\alpha!)^{-1} x^\alpha .$$

Let $s > 0$. If

$$\|u\|_{0,s} = \sup_\alpha |D_x^\alpha u(0)| s^{|\alpha|} < \infty , \tag{2.1}$$

then u is said to belong to the space $G(0, s)$. See [14]. If u is in $G(0, s)$ and if we look at u as a holomorphic function then

$$u(x) = \sum_\alpha D_x^\alpha u(0)(\alpha!)^{-1} x^\alpha ,$$

is an entire function of exponential type such that for some $C > 0$

$$|u(x)| \leq C \exp(s^{-1}(|x_1| + \ldots |x_n|)), \quad x \in \mathbf{C}^n . \tag{2.2}$$

This follows directly from (2.1). If (2.2) is true then it follows from the Cauchy formula that $u \in G(0, s'')$ for every s'', $0 < s'' < s$.

Let H be the space of entire functions in $\mathbf{C}^n$ with the topology of convergence on compact sets in $\mathbf{C}^n$. Let H' be the topological dual of H. If $u \in H'$ and $h \in H$ we define differentiation on H' by

$$(D_j u)(h) = u(-D_j h) .$$

The Fourier-Borel transformation of u is defined by

$$\hat{u}(y) = u(e^{xy}) .$$

It is well known that

$$H' \ni u \to \hat{u} \in \bigcup_{s>0} G(0, s) ,$$

is a bijection, see [20], p. 474, and the argument above. As usual we have

$$\widehat{D_x u}(y) = -y\hat{u}(y) , \quad \text{and} \quad \widehat{xu}(y) = D_y \hat{u}(y) .$$

The following lemma is crucial. Its proof is contained in a careful reading of the proof of Lemma 4.1 in [14].

LEMMA 1. *Let $0 < a < s < s' < 2a$ and let $u \in G(0, s')$. Let further β and γ be multi-indices. Then $y^\beta D_y^\gamma u$ belongs to $G(0, s)$ and there exists a constant $C' > 0$ independent of a, s, s', u, β, and γ such that*

$$\|y^\beta D_y^\gamma u\|_{0,s} \leq C'(|\beta|/e)^{|\beta|} s'^{|\beta|-|\gamma|}((s'-s)/2a)^{-|\beta|} \|u\|_{0,s'} . \tag{2.3}$$

The function $\hat{g}(t)$ is a function of the complex variable t with values in $G(0, s)$. If for some $\varrho > 0$ and for some complex number t' it is holomorphic in some neighbourhood of $|t - t'| \leq \varrho$ then we say that $\hat{g}(t)$ belongs to the function class $HG(0, s, \varrho, t')$. See also [14].

3. The Fourier-Borel transformation of the Cauchy problem

We start by formulating the local version of the Fourier-Borel transform of the dual Cauchy-Kovalevsky theorem. Here local means local in the time direction.

THEOREM 1. *Let* $a_{j\beta}(t, x) = \sum_{\gamma} a_{j\beta\gamma}(t)x^{\gamma}$, $|\beta| \leq j$, $1 \leq j \leq m$, *be complex valued functions that are analytic in some neighbourhood of the origin in* $\mathbf{C}^{n+1}$. *Here* m *is a fixed integer and* β *and* γ *are multi-indices. There exist further numbers* a, s', $0 < a < s' < 2a$, $\varrho > 0$, *and a function* $\hat{g}(t) \in HG(0, s', \varrho, 0)$. *There also exists a constant* $C > 0$ *such that*

$$\sum_{|\beta| \leq j} \sum_{\gamma} |a_{j\beta\gamma}(t)| a^{-|\gamma|} \leq C, \quad 1 \leq j \leq m, \quad |t| \leq \varrho\,. \tag{3.1}$$

The operator D_t^{-j} *denotes iterated integration* j *times radially from the origin in the complex plane. It follows that there exists an* ε, $0 < \varepsilon \leq \varrho$ *such that to every* s *in* $a < s < s'$ *there exists a unique function* $\hat{v}(t)$ *in* $HG(0, s, \varepsilon((s' - s)/2a)^m, 0)$ *that satisfies*

$$\hat{v}(t) = \sum_{j=1}^{m} \sum_{|\beta| \leq j} y^{\beta} \sum_{\gamma} a_{j\beta\gamma}(t) D_y^{\gamma} D_t^{-j} \hat{v} + \hat{g}(t)\,. \tag{3.2}$$

Let

$$\hat{u}(t) = D_t^{-m} \hat{v}(t)\,.$$

Then $\hat{u}(t) \in HG(0, s, \varepsilon((s' - s)/2a)^m, 0)$. *From* (3.2) *it follows that* $\hat{u}$ *is the unique solution of this kind of the Cauchy problem*

$$D_t^m \hat{u}(t) = \sum_{j=1}^{m} \sum_{|\beta| \leq j} y^{\beta} \sum_{\gamma} a_{j\beta\gamma} D_y^{\gamma} D_t^{m-j} \hat{u} + \hat{g}(t), \quad D_t^j \hat{u}(0) = 0, 0 \leq j < m\,. \tag{3.3}$$

Proof. Let $u \in G(0, s'')$, $a < s < s'' < s' < 2a$. It follows from Lemma 1 and from (3.1) that for some new $C > 0$ depending on a, C', j and the old C

$$\Big\| \sum_{|\beta| \leq j} y^{\beta} \sum_{\gamma} a_{j\beta\gamma}(t) D_y^{\gamma} u \Big\|_{0, s} \leq C'(j/e)^j ((s'' - s)/2a)^{-j} \sum_{|\beta| \leq j} \sum_{\gamma} |a_{j\beta\gamma}(t)| a^{-|\gamma|} \|u\|_{0, s''} \leq$$
$$\leq C((s'' - s)/2a)^{-j} \|u\|_{0, s''}\,.$$

With this estimate we apply Theorem 1′ in [14]. The proof is completed.

We now prove a global version of Theorem 1. Here we use the modified Ovsjannikov technique mentioned in the introduction.

THEOREM 2. *Let* $a_{j\beta}(t, x) = \sum_\gamma a_{j\beta\gamma}(t)x^\gamma$, $|\beta| \le j$, $1 \le j \le m$, *be entire functions in* $\mathbf{C}^{n+1}$. *They are restricted by*

$$|\beta| = j \Rightarrow a_{j\beta}(t, x) = \sum_{|\gamma| \le j} a_{j\beta\gamma}(t)x^\gamma\,. \tag{3.4}$$

The real valued function $s''(r)$ *is decreasing for* $r > 0$ *and* $0 < s''(r) < 1$. *The function* $\hat{g}(t)$ *belongs to* $HG(0, s''(r), r, 0)$ *for all* $r > 0$. *The operator* D_t^{-j} *denotes iterated integration* j *times radially from the origin in the complex plane. It follows that there exist a decreasing function* $s(r), 0 < s(r) \le s''(r)$, $r > 0$, *and a unique function* $\hat{v}(t) \in HG(0, s(r), r, 0)$, $r > 0$, *that satisfies* (3.2). *Let*

$$\hat{u}(t) = D_t^{-m}\hat{v}(t)\,.$$

Then $\hat{u}(t) \in HG(0, s(r), r, 0)$, $r > 0$, *and* $\hat{u}(t)$ *satisfies* (3.3) *for all* t *in* $\mathbf{C}^1$.

Proof. We take an arbitrary $r > 0$. We want to show that a solution of (3.2) exists for $|t| \le r$ and that it belongs to $HG(0, s'(r), r, 0)$ for some number $s'(r)$. We note that there exists a constant C such that

$$\sum_{j=1}^m \sum_{|\beta|=j} \sum_\gamma |a_{j\beta\gamma}(t)| \le C, \quad |t| \le r + 1\,. \tag{3.5}$$

Lemma 1 then says that for $u \in G(0, s')$, $0 < a < s < s' \le 2a < 1$,

$$\|\sum_{|\beta|=j} \sum_\gamma y^\beta a_{j\beta\gamma}(t)D_y^\gamma u\|_{0,s} \le CC'(j/e)^j((s'-s)/2a)^{-j}\|u\|_{0,s'}, \quad |t| \le r, \quad 1 \le j \le m\,. \tag{3.6}$$

We also have that for some $C(a)$, $0 < a < 2^{-1}$,

$$\sum_{|\beta|<j} \sum_\gamma |a_{j\beta\gamma}(t)|a^{-|\gamma|} < C(a), \quad |t| \le r, \quad 1 \le j \le m\,. \tag{3.7}$$

From Lemma 1 it follows that

$$\|\sum_{|\beta|<j} \sum_\gamma y^\beta a_{j\beta\gamma}(t)D_y^\gamma u\|_{0,s} \le C(a)C'(j/e)^j((s'-s)/2a)^{1-j}\|u\|_{0,s'}, \quad |t| \le r, \quad 1 \le j \le m\,. \tag{3.8}$$

We now define $\hat{v}_0(t) = 0$ and $\hat{v}_k(t)$, $k \ge 1$, by

$$\hat{v}_{k+1}(t) = \sum_{j=1}^m \sum_{|\beta| \le j} y^\beta(\sum_\gamma a_{j\beta\gamma}(t)D_y^\gamma D_t^{-j}\hat{v}_k(t)) + \hat{g}(t), \quad k = 0, 1, 2, \ldots \tag{3.9}$$

We shall show that there exist constants M and K such that for $2a \le s''(r)$ and $a \le s < s' \le 2a$ with

$$d = (s'-s)/2a$$

$$\|\hat{v}_{k+1}(t) - \hat{v}_k(t)\|_{0,s} \le M(K|t|)^k d^{-mk}, \quad |t| \le 1, \quad k = 0, 1, 2, \ldots. \tag{3.10}$$

For $k = 0$ there exists an M since we may choose

$$M = \sup \|\hat{g}(t)\|_{0,2a}, \quad |t| \le r\,.$$

We now assume that (3.10) is true for all $k \leq k'$. We choose $s'' = s + 2ad/(k'+1)$. Then we have $(s''-s)/2a = d/(k'+1)$ and $(s'-s'')/2a = k'd/(k'+1)$. From (3.6)—(3.10) we get with s' replaced by s'' in (3.6) and (3.8)

$$A = \|\hat{v}_{k'+2}(t) - \hat{v}_{k'+1}(t)\|_{0,s} \leq$$
$$\leq \sum_{j=1}^{m} MK^{k'}(dk'/(k'+1))^{-mk'}(d/(k'+1))^{-j}(D_t^{-j}|t|^{k'})C'(j/e)^j(C + C(a)(d/(k'+1)))\,.$$

For a fixed a it is obvious that there exists a k'' such that

$$C + C(a)d/(k'+1) \leq 2C\,, \quad k' \geq k''\,.$$

So we let

$$M = \sup_{|t| \leq r} \|\hat{g}(t)\|_{0,2a} \prod_{k=1}^{k''} (1 + C(a)(C(k+1))^{-1})$$

and

$$K = 2C'm^{m+1}Ce^m\,.$$

Then we have for $k' \geq k''$

$$A \leq \sum_{j=1}^{m} MK^{k'}d^{-mk'-j}(me^m)^{-1}K|t|^{k'+j}e^m(k'+1)^j((k'+1)\ldots(k'+j))^{-1} \leq$$
$$\leq M(K|t|)^{k'+1}d^{-m(k'+1)}\,.$$

For $k' < k''$ in the definition of M we replace k'' by k' and call this new number $M_{k'}$. Then it is obvious from above that for $0 \leq k \leq k''$ (3.10) is true with M replaced by M_k. Since $M_k \leq M$ we have now shown that (3.10) is true for all k. The method of proof is the modified Osvjannikov technique mentioned in the introduction.

We now note that the successive approximations converge in $HG(0, a, K^{-1}, 0)$. We also note that K only depends on C and is independent of a. If we now let D_t^{-1} denote integration radially from a fixed arbitrary t' in the complex plane, $|t'| \leq r$, in (3.9) we get a solution of (3.2) with this D_t^{-1}. The solution is in $HG(0, a', K^{-1}, t')$ for some $a' < 1$. We note that K^{-1} is independent of a' and t'.

We now assume that we have a solution of (3.3) in $HG(0, s(r''), r'', 0)$ for $r'' \leq r' \leq r$, i.e. there exists a function $s(r'')$, $r'' \leq r'$ such that this is true. If now r' is chosen maximal and if $r' < r$ then take a t', $|t'| = r'$. Let

$$u_0 = \sum_{j=0}^{m-1} (t-t')^j (j!)^{-1} D_t^j \hat{u}(t')\,,$$

and look at the equation

$$D_t^m u(t) = \sum_{j=1}^{m} \sum_{|\beta| \leq j} y^\beta \sum_{\gamma} a_{j\beta\gamma}(t) D_y^\gamma D_t^{m-j}(u(t) + u_0(t)) + \hat{g}(t),\ \ D_t^j u(t') = 0, \quad (3.11)$$
$$0 \leq j < m\,.$$

We realize that $u = \hat{u} - u_0$ solves this equation in $\{t\,;\, |t| \leq r',\ \ |t - t'| < K^{-1}\}$. We now choose $2a' = s(r')$ and we find that $u(t)$ exists and is in $HG(0, a', K^{-1}, t')$. But t' is arbitrary so we can choose $s(r'') = 2^{-1}s(r')$, $r' < r'' \leq r' + K^{-1}$. Therefore r' is not maximal and we have a contradiction. Note that $u_0(t) \in HG(0, s(r'), r, 0)$. Theorem 2 is proved.

Remark 1. It is obvious from Theorem 1 in [14] that we can give a »continuous» version of each of the »holomorphic» theorems above. The proofs are even simpler.

Remark 2. There is also an »integrable» version of the theorems above. A look at (3.2) shows that integrability of $\hat{v}(t)$ in t is the weakest and also most natural condition we can put on $\hat{v}(t)$. This is a general fact for all Cauchy problems in the (3.2) form. But we have not worked out the details here so the precise formulation is open. See [13] and [14].

4. The dual Cauchy-Kovalevsky theorem

We assume now that the hypothesis of Theorem 1 is satisfied. Then we apply the inverse Fourier-Borel transformation to the equation in (3.3). We get

$$D_t^m u(t) = \sum_{j=1}^{m} \sum_{|\beta| \leq j} (-1)^{|\beta|} D_x^\beta (a_{j\beta}(t, x) D_t^{m-j} u(t)) + g(t),\ D_t^j u(0) = 0,\ 0 \leq j < m\,. \quad (4.1)$$

Here $g(t)$ and $u(t)$ are functions of the complex variable t with values in H'. The regularity in t of these functions is given via the Fourier-Borel transform and Theorem 1. We note that $u(t)$ is carried in $\{x\,;\, \max_j |x_j| \leq s^{-1}\}$ if $\hat{u}(t) \in G(0, s)$. See Section 2 and [21], p. 474. See also the introduction where it is pointed out that the conclusion of Theorem 1 can be given in a stronger form. Note that if (3.1) is true for a certain a then we can choose the same ε in Theorem 1 for all bigger a. This fact is also used in [15].

In some way the results here are more special than those given by F. Treves in [23], pp. 53—58. But the computing character of our result is very attractive for some applications. See [15].

We now use the inverse Fourier-Borel transformation on (3.3) under the hypothesis of Theorem 2. Then $u(t)$ is carried in $\max_j |x_j| \leq (s(|t|))^{-1}$ for all t. This is the global dual Cauchy-Kovalevsky theorem.

There are some indications that (3.4) is essentially necessary in the hypothesis of Theorem 2. The analytic solution of

$$D_t \hat{u} = y D_y^2 \hat{u},\ \ \hat{u}(0, y) = e^y\,,$$

is

$$\hat{u}(t, y) = \sum_{j=0}^{\infty} t^j (j!)^{-1} Q_j(y)\,.$$

Here the functions Q_j are defined recursively by $Q_0(y) = e^y$, $Q_{j+1} = yD_y^2Q_j(y)$, $j \geq 0$. It follows that $Q_j(y) = (yD_y^2)^j e^y$. We now assert that

$$Q_j(1) \geq e(j!), \quad j = 0, 1, \ldots . \tag{4.2}$$

The following proof of (4.2) is due to N. O. Wallin. We get

$$Q_j(y) = \sum_{k=0}^{\infty} (k!)^{-1}(yD_y^2)^j y^k = \sum_{k=j+1}^{\infty} (k!)^{-1}k(k-1)^2 \ldots (k-j+1)^2(k-j)y^{k-j} .$$

It follows that (4.2) is true. From this we conclude that our solution of the Cauchy problem is divergent in $(t, x) = (2, 1)$. That it converges for sufficiently small t follows from Theorem 3 in [14].

5. The global Cauchy-Kovalevsky theorem

We shall now prove the global Cauchy-Kovalevsky theorem for entire functions. A formal power series u is said to belong to the class $G(1, s)$ if

$$\|u\|_{1,s} = \sup_{\alpha} |D_x^{\alpha} u(0)|(\alpha!)^{-1}s^{|\alpha|} < \infty .$$

A careful reading of the proof of Lemma 4.1 in [14] gives the following lemma.

LEMMA 2. *Let* $0 < a < s < s' < 2a$ *and let* $u \in G(1, s')$. *Let further* β *and* γ *be multi-indices such that* $|\gamma| \leq m$ *for some fixed integer* m. *It follows that there exists a constant* $C > 0$ *independent of* a, s, s', u, β *and* γ *such that*

$$\|x^{\beta} D_x^{\gamma} u\|_{1,s} \leq C s'^{|\beta|-|\gamma|}(e^{-1}|\gamma|)^{|\gamma|}((s'-s)/2a)^{-|\gamma|}\|u\|_{1,s'} .$$

Let $u(x)$ be an entire function in $\mathbf{C}^n$. If we look upon it as a formal power series then it is obvious that $u \in \bigcap_{s>0} G(1, s) = G$. It is also obvious that the converse is true. If $f(t, x)$ is entire in $\mathbf{C}^{n+1}$ then $g(t) = f(t, x)$ is a function of t with values in G. It is easy to show from the Cauchy inequality that for every $s > 0$, $g(t)$ is holomorphic in t for all t with values in $G(1, s)$. The converse is still simpler. Now we state the theorem.

THEOREM 3. *Let the functions* $a_{j\gamma}(t, x) = \sum_{\beta} a_{j\gamma\beta}(t)x^{\beta}$, $|\gamma| \leq j$, $1 \leq j \leq m$, *and* $f(t, x)$ *be entire functions in* $\mathbf{C}^{n+1}$. *We have the following restrictions.*

$$|\gamma| = j \Rightarrow a_{j\gamma}(t, x) = \sum_{|\beta| \leq j} a_{j\gamma\beta}(t)x^{\beta}, \quad 1 \leq j \leq m , \tag{5.1}$$

It follows that there exists a unique entire function $u(t, x)$ *that satisfies*

$$D_t^m u = \sum_{j=1}^{m} \sum_{|\gamma| \leq j} a_{j\gamma}(t, x)D_x^{\gamma}D_t^{m-j}u(t, x) + f(t, x), \quad D_t^j u(0, x) = 0, \quad 0 \leq j < m . \tag{5.2}$$

Proof. We shall only sketch the proof since it is not so different from the proof of Theorem 2. Let $r > 0$ be a fixed arbitrary number and let t', $|t'| \leq r$, be a complex number. For $a > 1$ with an obvious analogue to the notation in Section 2 we note that $f(t, x) \in HG(1, 2a, 1, t')$. If $u \in G(1, s')$ and if $a < s < s' \leq 2a$ then it follows from Lemma 2 that

$$A = \| \sum_{|\gamma|<j} \sum_{\beta} a_{j\gamma\beta}(t) x^\beta D_x^\gamma u \|_{1,s} \leq Cm^m (\sum_{|\gamma|<j} \sum_{\beta} |a_{j\gamma\beta}(t)| (2a)^{|\beta|}) ((s'-s)/2a)^{-j+1} \|u\|_{1,s'},$$
$$1 \leq j \leq m .$$

We also get

$$B = \| \sum_{|\gamma|=j} \sum_{|\beta| \leq j} a_{j\gamma\beta}(t) x^\beta D_x^\gamma u \|_{1,s} \leq Cm^m ((s'-s)/2a)^{-j} \|u\|_{1,s'} \sum_{|\beta| \leq |\gamma| = j} |a_{j\gamma\beta}(t)|,$$
$$1 \leq j \leq m .$$

So there is a C_1 depending on a and r and a C_2 depending on r only such that

$$A \leq C_1((s'-s)/2a)^{-j+1} \|u\|_{1,s'}, \quad |t| \leq r ,$$

and

$$B \leq C_2((s'-s)/2a)^{-j} \|u\|_{1,s'}, \quad |t| \leq r .$$

We now compare A and B with (3.8) and (3.6). Then we find that the radius of convergence in t for the successive approximations is independent of a. So we find as in the proof of Theorem 2 that the solution of (5.2) exists for $|t| \leq r$. The solution is also entire in x. But r is arbitrary. The theorem is proved.

Since the equation $D_t u = x^2 D_x u + tx^2$ has no entire solution, see [12], we conclude that (5.1) is essentially necessary.

We note that Theorem 3 is more general than Theorem 2 in [10]. In the reference list the reader will find other aspects on global and local Cauchy problems and on the related Goursat problems.

Added in proof: The »Ovsjannikov theorem» was proved already in 1960 by T. Yamanaka, *Comment. Math. Univ. St. Paul.*, 9—10 (1960—1961), 7—10.

References

1. Asadullin, E. A., *An investigation of normal partial differential equations.* Thesis, Kazan 1950.
2. —»— Existence of analytic integrals of partial differential equations of non-normal type. *Kazah. Gos. Ped. Inst. Učen. Zap.* 10 (1955), 169—179.
3. Fridlender, V. R. & Salehov, G. S., On questions about the Cauchy-Kowalevski theorem and its inverse. *Uspehi Mat. Nauk.* 7 (1952), 169—192.
4. Friedman, A., A new proof and generalizations of the Cauchy-Kowalevski theorem to non-analytic and to non-normal systems. *Proceedings of Symposia in pure mathematics*, 4, Am. Math. Soc. Providence, R. I., 1961, pp. 115—119.

5. Hörmander, L., *Linear partial differential operators*. Berlin, Springer 1963.
6. Nagumo, M., Über das Anfangswertproblem partieller Differentialgleichungen. *Japan. J. Math.* 18 (1942), 41–47.
7. Ovsjannikov, L. V., A singular operator in a scale of Banach spaces. *Dokl. Akad. Nauk SSSR* 163 (1965), 819–822. (English translation Soviet Math. Dokl. 6 (1965), 1025–1028.)
8. Persson, J., Linear Goursat problems for entire functions when the coefficients are variable. *Ann. Scuola Norm. Sup. Pisa Cl. Sc.*, s. 3, 23 (1969), 87–98.
9. —»— Global linear Goursat problems for functions of Gevrey-Lednev type. *Ann. Scuola Norm. Sup. Pisa. Cl. Sc.*, s. 3, 23 (1969), 387–412.
10. —»— The function class $\gamma_H(\beta, \delta, d)$ and global Goursat problems. *Ann. Scuola Norm. Sup. Pisa Cl. Sc.*, s. 3, 24 (1970), 633–639.
11. —»— Some results on classical solutions of the equation $D_t^m u = x^q D_x^n u$, $m < n$. *Boll. Un. Mat. Ital.*, s. 4, 3 (1970), 426–440.
12. —»— The equation $D_t u = x^2 D_x u + tx^2$ has no entire solution. *Norske Vid. Selsk. Forh. (Trondheim)*. To appear.
13. —»— The Fourier transform on L^1_{loc} and its application to Cauchy problems for linear partial differential equations. *Rend. Sem. Mat. Univ. Padova*, 45. To appear.
14. —»— Linear characteristic Cauchy problems for partial differential operators with variable not only time dependent coefficients. *Boll. Un. Mat. Ital.*, s. 4, 4 (1971), 91–102.
15. —»— On the supports of solutions of linear partial differential equations with analytic coefficients. *Ann. Mat. Pura Appl.* To appear.
16. Pucci, C., Teoremi di esistenza e di unicità per il problema di Cauchy nella teoria delle equazioni lineari a derivate parziali. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* (8) 13 (1952), 18–23 and 111–116.
17. —»— Nuove ricerche sul problema di Cauchy. *Mem. Accad. Sci. Torino* Parte 1, s. 3, 1 (1954), 45–67.
18. Riquier, M. Sur l'application de la méthode des fonctions majorantes à certains systèmes differentiels. *C. R. Acad. Sci. Paris Sér. A–B*, 75 (1897) 1018–1019.
19. Steinberg, S. & Treves, F., Pseudo Fokker-Planck equations and hyperdifferential operators. *J. Differential Equations* 8 (1970), 333–366.
20. Talenti, G., Un problema di Cauchy. *Ann. Scuola Norm. Sup. Pisa Cl. Sc.*, s. 3, 18 (1964), 165–186.
21. Treves, F., *Linear partial differential equations with constant coefficients*. Gordon and Breach, New York, 1966.
22. —»— On the theory of linear partial differential operators with analytic coefficients. *Trans. Am. Math. Soc.* 137 (1969), 1–20.
23. —»— *Ovcyannikov theorem and hyperdifferential operators*. Instituto de Matematica pura e aplicada. Rio de Janeiro 1968.
24. Žitomirskiĭ, Ja. I., Uniqueness classes of the Cauchy problem for linear equations with increasing coefficients. *Izv. Akad. Nauk SSSR* 31, 7 (1967), 763–782. (English translation *Math. USSR Izvestija*, 1 (1967), 737–756.)

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