Some remarks on the value distribution of meromorphic functions

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1. Introduction

1. Let E be a closed set in the complex plane and f a meromorphic function outside E omitting a set F. We shall consider the following problem: If E is thin, under what conditions is F thin, too? In Chapter 2 we consider the case when E and F are of Hausdorff dimension less than one. In Chapter 3 E and F are countable sets with one limit point, and in Chapter 4 E is a countable set whose points converge to infinity, f is entire, and F is allowed to contain at most one finite value.

2. Sets of dimension less than one

2. Let f be meromorphic and non-constant outside a closed set E in the complex plane. It is known that if the logarithmic capacity of E is zero then f cannot omit a set of positive capacity, and if E has linear measure zero then f cannot omit a set of positive $(1+\varepsilon)$-dimensional measure. If the dimension of E is greater than one then there exists a non-constant function f which is regular and bounded outside E. Carleson [1] has proved that there exists a set E of positive capacity such that if f omits 4 values outside E then f is rational. We consider the following problem: Let E be of dimension less than one. Can f omit a set whose dimension is greater than the dimension of E?

We denote by $\mathrm{Dim}\,(A)$ the Hausdorff dimension of a set A, and let $\dim\,(A)$ be the dimension of A obtained by using coverings consisting of discs with equal radii. For example for usual Cantor sets these dimensions are equal. We have the following answer to our question:

THEOREM 1. *Let E be a closed set with $\dim\,(E) < 1$. If f is meromorphic and non-constant outside E and omits F then* $\mathrm{Dim}\,(F) \leq \dim\,(E)$.

The proof will be given in 3 and 4.

[1] This research was done at the Institut Mittag-Leffler. The author takes pleasure in thanking Professor Lennart Carleson for helpful suggestions.

3. It does not mean any essential restriction to assume that $\infty \in F$, $E \subset \{z : |z| < 1\}$, and that f is non-rational. In order to prove Theorem 1 it is sufficient to prove that for any α, $\dim (E) < \alpha < 1$, and any R, $0 < R < \infty$, we have $\operatorname{Dim} (B) \leq \alpha$ where $B = F \cap \{w : |w| \leq R\}$. Let these α and R be chosen. We define

$$U(a, r) = \{z : |z - a| < r\} .$$

Then we can choose a sequence r_n with $\lim_{n\to\infty} r_n = 0$ and coverings

$$\bigcup_{\nu=1}^{N_n} U(a_\nu, r_n) \supset E$$

such that

$$\lim_{n\to\infty} N_n r_n^\alpha = 0 . \tag{1}$$

For any $a \in B$ we define

$$f_a(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=4} \frac{d\zeta}{(f(\zeta) - a)(\zeta - z)} .$$

f_a is regular in $|z| < 4$ and therefore $f_a(z) \not\equiv 1/(f(z) - a)$. We set $G = \{z : 3 < |z| < 4\}$. Because $f_a(z)$ and $1/(f(z) - a)$ are continuous functions of a $(a \in B)$ for any fixed $z \in G$ and B is compact, there exists $\beta_1 > 0$ such that

$$\sup_{z\in G} \left| f_a(z) - \frac{1}{f(z) - a} \right| > \beta_1 \tag{2}$$

for any $a \in B$.

We assume that $r_n < 1/2$ for any n. Let n be fixed in the following considerations. Let D be the component of the complement of

$$\bigcup_{\nu=1}^{N_n} U(a_\nu, 2r_n)$$

which contains the point at infinity. The boundary of D consists of simple closed curves. These can be divided into continuous curves γ_ν, $\nu = 1, 2, \ldots, N_n$, such that the length of any γ_ν is at most $4\pi r_n$. Then we get for $z \in G$, $a \in B$,

$$\left| f_a(z) - \frac{1}{f(z) - a} \right| = \left| \frac{1}{2\pi i} \sum_{\nu=1}^{N_n} \int\limits_{\gamma\nu} \frac{d\zeta}{(f(\zeta) - a)(\zeta - z)} \right| \leq 2r_n \sum_{\nu=1}^{N_n} \frac{1}{\omega_\nu(a)}$$

where $\omega_\nu(a) = \min_{z\in\gamma_\nu} |f(z) - a|$. It follows from (2) that there exists a constant $\beta > 0$ not depending on the choice of n such that

$$\sum_{\nu=1}^{N_n} 1/\omega_\nu(a) > \beta/r_n \tag{a}$$

for any $a \in B$.

We need the following

LEMMA. *There exists an absolute constant* $K > 4$ *such that if* $\omega_\nu(a) \leq \varrho$ $(\varrho > 0)$ *for some* $a \in B$ *then* $\omega_\nu(b) > K\varrho$ *for any* $b \in B - U(a, 2K\varrho)$.

Proof. The length of γ_ν is at most $4\pi r_n$ and $U(\zeta, r_n) \cap E = \emptyset$ for any $\zeta \in \gamma_\nu$. The lemma follows from Schottky's theorem.

4. Let λ, $\alpha < \lambda < 1$, be chosen. We choose a positive integer k such that

$$\lambda > \alpha(1 + \lambda^k) . \tag{b}$$

Let $q_0 = N_n$. For $m = 1, 2, \ldots, k$ we set

$$\varrho_m = 2^m r_n q_{m-1} \beta^{-1} , \tag{c}$$

$$q_m = (\varrho_m / r_n)^\lambda , \tag{d}$$

and let p_m be the integer defined by $q_m \leq p_m < q_m + 1$. Let ϱ_{k+1} be defined by (c) and $q_{k+1} = p_{k+1} = 1$. It follows from these definitions that for $1 \leq m \leq k + 1$

$$\varrho_m = M_m r_n N_n^{\lambda^{m-1}} \tag{e}$$

where M_m is a positive constant depending only on β. It does not mean any essential restriction to assume that $N_n \to \infty$ as $n \to \infty$ because otherwise E consists of a finite number of points. Therefore it follows from (e) that we can assume that $\varrho_{m+1} < \varrho_m$ for any m.

Let m, $1 \leq m \leq k + 1$, be fixed. If possible, we choose $b_{m,1} \in B$ such that the inequality $\omega_\nu(b_{m,1}) \leq \varrho_m$ is satisfied at least for p_m different values of ν. We set $C_{m,1} = U(b_{m,1}, 2K\varrho_m)$ where K is the constant of the lemma. In the same manner, starting with the set

$$B - \bigcup_{p=1}^{s-1} C_{m,p}$$

$(s > 1)$ we define the disc $C_{m,s}$. Let this method yield the discs $C_{m,s}$, $s = 1, 2, \ldots, S_m$. Then it follows from the lemma that $S_m \leq N_n / q_m$.

Let us suppose that there exists

$$b \in B - \bigcup_{m=1}^{k+1} \bigcup_{p=1}^{S_m} C_{m,p} .$$

Then $\omega_\nu(b) > \varrho_{k+1}$ for each ν and $\varrho_{m+1} < \omega_\nu(b) \leq \varrho_m$ $(1 \leq m \leq k)$ is satisfied at most for $p_m - 1$ different values of ν. Therefore it follows from (c) that

$$\sum_{\nu=1}^{N_n} 1/\omega(b_\nu) \leq \frac{N_n}{\varrho_1} + \frac{p_1 - 1}{\varrho_2} + \ldots + \frac{p_k - 1}{\varrho_{k+1}} \leq \sum_{m=1}^{k+1} \frac{\beta}{2^m r_n} < \beta / r_n .$$

This is a contradiction to (a) and so

$$B \subset \bigcup_{m=1}^{k+1} \bigcup_{p=1}^{S_m} C_{m,p} .$$

It follows from (d) and (e) that the radii of $C_{m,p}$ satisfy the inequality

$$\sum_{m=1}^{k+1} S_m(2K\varrho_m)^\lambda \leq A_1 \sum_{m=1}^{k+1} \frac{N_n \varrho_m^\lambda}{q_m} \leq A_2 N_n^{1+\lambda^k} r_n^\lambda$$

where A_1 and A_2 are positive constants not depending on the choice of n. From (1) and (b) it follows that

$$N_n^{1+\lambda^k} r_n^\lambda < (N_n r_n^\alpha)^{1+\lambda^k} \to 0$$

as $n \to \infty$. Therefore $\operatorname{Dim}(B) \leq \lambda$. This is true for any $\lambda > \alpha$ and so $\operatorname{Dim}(B) \leq \alpha$. This completes the proof of Theorem 1.

3. Countable sets

5. Let A and B be two countable sets whose points converge to infinity. If B is given then it is always possible to construct A such that there exists a meromorphic function omitting B outside A. In fact, we take an entire function f and set $A = f^{-1}(B)$. In this manner, it is easy to construct A such that there exists a countable family of entire functions omitting B outside A. Then the following question arises: Is the family of transcendental entire functions omitting B outside A always at most countable? Theorem 2 gives a negative answer.

THEOREM 2. *Given any countable set* $B = \{b_n\}$ *with* $\lim b_n = \infty$ *then we can construct a countable set* $A = \{a_n\}$ *with* $\lim a_n = \infty$ *such that there exists a non-countable family of transcendental entire functions omitting* B *outside* A.

Proof. Let $a_0 \neq 0$ and $a_0 \notin B$. We shall choose inductively a sequence $\{f_n\}$ of polynomials such that the product

$$a_0 \prod_{n=1}^{\infty} (f_n(z))^{\varepsilon_n} \tag{1}$$

$(\varepsilon_n(1-\varepsilon_n) = 0)$ converges uniformly in bounded domains for any choice of the sequence $\{\varepsilon_n\}$.

Let $f_1(z) = z$ and $r_1 = 1 + |b_1|$. Let f_ν and r_ν be defined for $\nu = 1, 2, \ldots, n-1$ $(n > 1)$. We denote by G_n the family of the polynomials

$$g(z) = a_0 \prod_{\nu=1}^{n-1} (f_\nu(z))^{\varepsilon_\nu}$$

where $\varepsilon_\nu(1-\varepsilon_\nu) = 0$ for any ν. We choose $r_n > 2r_{n-1}$ such that $g(z) \notin B$ on $|z| = r_n$ for any $g \in G_n$. Then there exists $\delta_n > 0$ such that $|g(z) - b| \geq \delta_n$ on $|z| = r_n$ for all $b \in B$ and $g \in G_n$. We set

$$M_n = \max_{g \in G_n} \{\max_{|z| = r_n} |g(z)|\}$$

and
$$A_n = \bigcup_{g \in G_n} \{z : g(z) \in B \text{ and } |z| \leq r_n\}.$$

Because $\lim b_\nu = \infty$ and G_n consists of finitely many polynomials, A_n contains a finite number of points. We define

$$h_n(z) = \prod_{a \in A_n} (z - a)^{t_a}$$

where t_a is the largest multiplicity of the root a of the equations $g(z) = b$ for all $g \in G_n$ and $b \in B$. We set $f_n(z) = 1 - \varrho_n h_n(z)$ where $\varrho_n > 0$. Let $b \in B$ and $g \in G_n$. On $|z| = r_n$ we have

$$\left| \frac{g(z) f_n(z) - g(z)}{g(z) - b} \right| \leq \frac{M_n \varrho_n |h_n(z)|}{\delta_n}.$$

Now we see that we can choose the sequence $\{\varrho_\nu\}$ such that

$$\left| \frac{g(z) \prod_{\nu=n}^{\infty} (f_\nu(z))^{\varepsilon_\nu} - g(z)}{g(z) - b} \right| < 1 \tag{2}$$

on $|z| = r_n$ for all $b \in B$ and $g \in G_n$, and any sequence $\{\varepsilon_\nu\}$ $(n \geq 2)$.

Let F be the family of entire functions defined by (1). We define

$$A = \bigcup_{f \in F} \{z : f(z) \in B\}.$$

Let $f \in F$ and $b \in B$. We choose $n \geq 2$. We write

$$f(z) = g(z) \prod_{\nu=n}^{\infty} (f_\nu(z))^{\varepsilon_\nu} \tag{3}$$

where $g \in G_n$. It follows from (2) and Rouché's theorem that the functions f and g have the same number of b-points in $|z| < r_n$. It follows from the construction of the sequence $\{f_\nu\}$ that the b-points of g lying in $|z| < r_n$ are b-points of f, and not of smaller multiplicity. Therefore in $|z| < r_n$, f can take a value of B only at the points of A_n and we see that

$$A \subset \bigcup_{n=2}^{\infty} A_n.$$

If we choose f such that $f \in G_{n+1}$ we get

$$(A_{n+1} - A_n) \cap \{z : |z| < r_n\} = \emptyset$$

and we see that ∞ is the only limit point of A. Clearly F contains a non-countable set of transcendental entire functions. Theorem 2 is proved.

4. Picard sets for entire functions

6. Following Lehto, we call a set E in the complex plane a Picard set for entire functions if every non-rational entire function omits at most one finite value outside E. Lehto [3] has proved that a countable set $E = \{a_n\}$ whose points converge to infinity is a Picard set for entire functions if the points a_n satisfy the condition

$$|a_n/a_{n+1}| = O(n^{-2}) .$$

Matsumoto [5] has proved the same assertion under the condition

$$\limsup_{n\to\infty} \frac{\exp (K/\log |a_{n+1}/a_n|)}{\log |a_{n+1}|} < \infty$$

(K a positive constant) when $\{|a_n|\}_{n=1,2,\dots}$ is strictly increasing. Winkler [6] has proved this assertion in the case that E is a finite union of sets whose points satisfy the condition $|a_{n+1}/a_n| \geq q > 1$ and

$$\{z : e^{-|a|^\varepsilon} < |z - a| < |a|^{-p}\} \cap E = \emptyset$$

($\varepsilon > 0$, $p > 0$) for all sufficiently large $|a|$, $a \in E$. We shall give an essentially best possible density condition under which a countable set is a Picard set for entire functions.

7. We shall need the following results in our considerations. We define

$$\varrho(f(z)) = \frac{|f'(z)|}{1 + |f(z)|^2} ,$$

and by $h(r)$ we denote an arbitrary positive function of the positive variable r, with the property $h(r) = O(r)$ as $r \to \infty$. Lehto [4] has proved the following

THEOREM A. *Let f be meromorphic in a neighbourhood of the singularity $z = \infty$. If for a sequence $\{z_n\}$, $\lim_{n\to\infty} z_n = \infty$ and*

$$\lim_{n\to\infty} h(|z_n|)\, \varrho(f(z_n)) = \infty$$

then Picard's theorem holds for f in the union of any infinite subsequence of the discs

$$C_n = \{z : |z - z_n| < \varepsilon h(|z_n|)\}$$

for each $\varepsilon > 0$.

Clunie and Hayman [2] have proved the following

THEOREM B. *If f is an entire non-rational function then*

$$\limsup_{z\to\infty} \frac{|z|\varrho(f(z))}{\log |z|} = \infty .$$

8. Now we prove the following

THEOREM 3. *A countable set* $E = \{a_n\}$ *whose points converge to infinity is a Picard set for entire functions if there exists* $\varepsilon > 0$ *such that*

$$\left\{z : 0 < |z - a_n| < \frac{\varepsilon |a_n|}{\log |a_n|}\right\} \cap E = \emptyset \tag{1}$$

for all sufficiently large n.

Proof. Contrary to our assertion, let us suppose that there exists an entire non-rational function f omitting two finite values outside E. Then we can assume that f omits the values 0 and 1 in the complement of E.

From Theorem A and Theorem B it follows that we can choose a sequence $\{z_n\}$ such that $\lim z_n = \infty$ and Picard's theorem holds for f in the union of any infinite subsequence of the discs $C_n = U(z_n, r_n)$ where

$$r_n = \frac{\varepsilon |z_n|}{8 \log |z_n|} .$$

Then C_n contains at least one zero or 1-point of f for sufficiently large n, say for $n > n_1$. It follows from (1) that $U(z_n, 4r_n)$ contains at most one point of E if n is large enough, say if $n > n_2 > n_1$. Let $n > n_2$. Let us suppose that C_n contains a 1-point of f. Because f has no zeros in $U(z_n, 2r_n)$, it follows from the maximum principle that there exists a point ζ on $|z - z_n| = 2r_n$ such that $|f(\zeta)| \leq 1$. Because f has neither zeros nor 1-points in the ring domain $r_n < |z - z_n| < 4r_n$, it follows from Schottky's theorem that $|f(z)| < M$ on $|z - z_n| = 2r_n$ where M is an absolute constant. Then $|f(z)| < M$ in C_n. If C_n contains a zero of f we consider the function $1 - f(z)$, and we see that $|f(z)| < M + 1$ in C_n for $n > n_2$. This is a contradiction because f omits at most one finite value in the union of these discs. Theorem 3 is proved.

9. We now prove that the condition (1) is best possible.

THEOREM 4. *Corresponding to each real-valued function* $h(r)$ *satisfying the condition* $h(r) \to \infty$ *as* $r \to \infty$, *there exists a countable set* $E = \{a_n\}$ *whose points converge to infinity, which is not a Picard set for entire functions, and which satisfies the condition*

$$\left\{z : 0 < |z - a_n| < \frac{|a_n|}{h(|a_n|) \log |a_n|}\right\} \cap E = \emptyset \tag{2}$$

for all sufficiently large n.

Proof. In order to prove our assertion, we shall show that the set of the zeros and 1-points of the entire function

$$f(z) = \prod_{n=1}^{\infty} (1 - z/e^{t_n})^{t_n}$$

(t_n being positive integers, $t_{n+1} > 4t_n$) satisfies the condition (2) if t_n tends to infinity with a sufficient rapidity.

Let $n \geq 3$. We define $D_n = \{z : |z - e^{t_n}| \leq \exp(t_n - \frac{1}{2}t_{n-1})\}$ and

$$g(z) = (1 - z/e^{t_n})^{t_n} \prod_{\nu=1}^{n-1} (1 - e^{t_n}/e^{t_\nu})^{t_\nu} .$$

It is easy to see that if t_ν tends to infinity sufficiently rapidly then

$$|f(z) - g(z)| \leq \tfrac{1}{4}\,|g(z)| \tag{3}$$

in D_n. On the boundary of D_n we have $|f(z)| \geq \frac{1}{2}|g(z)| > 2$. Therefore f has only a finite number of 1-points outside the union of the discs D_ν, $\nu \geq 3$.

Let ζ_ν, $\nu = 1, 2, \ldots, t_n$, be the 1-points of g. We set

$$C_\nu = \left\{z : \left|\arg \frac{z - e^{t_n}}{\zeta_\nu - e^{t_n}}\right| < \frac{\pi}{2t_n}\,,\quad 1/2 < |g(z)| < 2\right\} .$$

Then $C_\nu \subset D_n$. On the boundary rays of C_ν we have $\operatorname{Re} g(z) = 0$. Then it is easy to see that $|g(z)| \leq 2|1 - g(z)|$ at the boundary points of D_n and C_ν, $\nu = 1, 2, \ldots, t_n$. Now it follows from (3) and Rouché's theorem that f has exactly t_n 1-points in D_n, each C_ν containing one 1-point of f. Then the distance between two different 1-points of f in D_n is at least $r = t_n^{-1} \exp(t_n - \sum_{\nu=1}^{n-1} t_\nu)$. Because the term $\sum_{\nu=1}^{n-1} t_\nu$ does not depend on t_n, we can assume that t_n is chosen so large that

$$r > \frac{|z|}{h(|z|)\log|z|}$$

for any $z \in D_n$. Therefore if $E = \{a_n\}$ is the set of the zeros and 1-points of f then E satisfies the condition (2) for all sufficiently large n. Theorem 4 is proved.

References

1. Carleson, L., A remark on Picard's theorem. *Bull. Amer. Math. Soc.*, 67 (1961), 142–144.
2. Clunie, J. & Hayman, W. K., The spherical derivative of integral and meromorphic functions. *Comment. Math. Helv.*, 40 (1966), 117–148.
3. Lehto, O., A generalization of Picard's theorem. *Ark. Mat.*, 3 (1958), 495–500.
4. —»— The spherical derivative of functions meromorphic in the neighbourhood of an isolated singularity. *Comment. Math. Helv.*, 33 (1959), 196–205.
5. Matsumoto, K., Remark on Lehto's paper »A generalization of Picard's theorem». *Proc. Japan Acad.*, 38 (1962), 636–640.
6. Winkler, J., Über Picardmengen ganzer Funktionen. *Manuscripta Math.*, 1 (1969), 191–199.

Received May 15, 1970

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