Ark. Mat., **43** (2005), 181–200

Continuity of pluricomplex Green functions with poles along a hypersurface

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I. Introduction

Let Ω be a bounded domain in $\mathbf{C}^n$ and A be a complex hypersurface of Ω. Following [LS2] we define

$$\mathcal{F}_{A,\Omega}=\{u\in \mathrm{PSH}(\Omega): u\le 0 \text{ and } \nu_u(a)\ge \nu_A(a) \text{ for all } a\in A\},$$

and

$$G_{A,\Omega}(z)=\sup_{u\in\mathcal{F}_{A,\Omega}} u(z),$$

where $\mathrm{PSH}(\Omega)$ is the class of plurisubharmonic functions on Ω (including the constant function $-\infty$), $\nu_u(a)$ denotes the Lelong number of u at a and $\nu_A(a)$ denotes the multiplicity of A at a. Recall also that, if u is plurisubharmonic in a neighborhood of $a\in\mathbf{C}^n$ then

$$\nu_u(a)=\lim_{r\to 0}\frac{\sup_{|z-a|=r} u(z)}{\log r}.$$

The pluricomplex Green function $G_{A,\Omega}$ was first introduced in a more general setting by F. Lárusson and R. Sigurdsson in [LS2]. In the same paper, the authors studied boundary behavior and uniqueness of $G_{A,\Omega}$. They also discussed the relationship between $G_{A,\Omega}$ and disc functionals and their envelopes (see also [LS1]). In particular, the following result about the "almost" continuity of $G_{A,\Omega}$ was claimed in Theorem 3.9 of [LS2]: *Let X be a relatively compact domain in a Stein manifold with a strong plurisubharmonic barrier at every boundary point. Let A be the divisor of a holomorphic function f on X which extends to a continuous function on $\overline{X}$. Then the set of points in X where $G_{A,\Omega}$ is discontinuous is pluripolar.*

Unfortunately, it turns out as reported in [LS3], that there is a gap in the proof of this statement. It is not known whether this "theorem" is valid even in the case where X is a ball in $\mathbf{C}^n$ and A is an (arbitrary) divisor (see also the remark after

Proposition 3.1). The major difficulty in their method (and ours) lies in the fact that $G_{A,\Omega}$ is uncontrollable near $A\cap\partial\Omega$.

The primary goal of this note is to find special cases where the continuity of $G_{A,\Omega}$ can be verified. We now briefly outline the content of our work. In Theorem 2.2, we prove that $G_{A,\Omega}$ is continuous if A is a finite union of hyperplanes passing through $0\in\Omega$ and if Ω satisfies some seemingly natural assumptions together with a more restrictive one: there exists a neighborhood V of $A\cap\partial\Omega$ such that V can be pushed inside Ω by linear maps $g_t(z)=tz$ for all t sufficiently close to 1. This technical condition is essential in our method. However, it is possible to derive from Theorem 2.2 a couple of consequences which are easier to appreciate. For instance, it follows rather quickly from this result that $G_{A,\Omega}$ is continuous provided that Ω is a bounded convex domain in $\mathbf{C}^n$, $0\in\Omega$ and A is a finite union of hyperplanes going through 0 (see the remark after Corollary 2.9). We also show in Corollary 2.11 that $G_{A,\Omega}$ is continuous provided that Ω is a smoothly bounded B-regular pseudoconvex domain in $\mathbf{C}^2$, and A is the coordinate line $\{(z_1,z_2):z_2=0\}$ such that $A\cap\Omega$ is a smoothly bounded simply connected domain (viewed as a subset of $\mathbf{C}$) and that Ω projects onto $A\cap\Omega$.

It should be stressed that not so many explicit examples of $G_{A,\Omega}$ are known up to present. In the last section we provide explicit computations of $G_{A,\Omega}$ in special cases. In particular, using the results obtained in Section 2 we are able to derive a specific formula in the case where Ω is the unit ball in $\mathbf{C}^2$ and A is the union of coordinate lines. Finally in Proposition 3.4 we discuss the continuity at the *boundary* of $G_{A,\Omega}$ in the case Ω is the unit bidisk. This result is strongly motivated by Example 3.4 in [LS2].

The main idea in our work is to connect $G_{A,\Omega}$ with the holomorphic convex hull of compact sets. This is inspired from the classical paper [Br].

Acknowledgment. The present work, an expanded version of [Ng], was completed during my stay at the Department of Mathematics and Physics, Mid Sweden University, Sundsvall, as a postdoctoral fellowship. I would like to express my hearty thanks to Professor Urban Cegrell for the kind invitation. Next my thanks go to Ragnar Sigurdsson for his warm hospitality during my first visit to Sundsvall in October 2000, and for explaining to me the papers [LS1] and [LS2], in particular for drawing my attention to Example 3.3. I am grateful also to Alexander Rashkovskii for fruitful conversations we had during my stay in Sundsvall in October 2000. Finally I am indebted to the referees for their constructive comments and suggestions on earlier versions of this paper.

This work was supported in part by the National Basic Research Program in Natural Sciences.

II. Continuity of Green functions

We first fix some notation. Throughout this paper, by U we mean a pseudoconvex domain in $\mathbf{C}^n$. If K is a compact subset of U then by $\widehat{K}_U$ we denote the holomorphic hull of K in U, i.e.,

$$\widehat{K}_U=\{z\in U:|f(z)|\leq\|f\|_K \text{ for all } f \text{ holomorphic on } U\}.$$

If $U=\mathbf{C}^n$ then $\widehat{K}_U$ becomes the usual polynomial convex hull of K and we will drop the subscript U in this case. The solution to the Levi problem (see [Hö]) implies that $\widehat{K}_U=\widehat{K}_{\mathrm{PSH}(U)}$, where $\widehat{K}_{\mathrm{PSH}(U)}$ denotes the plurisubharmonic hull of K, i.e.,

$$\widehat{K}_{\mathrm{PSH}(U)}=\Big\{z\in U:u(z)\leq\sup_{x\in K}u(x) \text{ for all } u\in\mathrm{PSH}(U)\Big\}.$$

If Ω is a pseudoconvex domain contained in U then we say that Ω is Runge in U if every holomorphic function on Ω can be approximated uniformly on compact subsets by holomorphic functions on U. We will frequently appeal to the following well-known fact: If U' is a Runge pseudoconvex domain in U and K is a compact subset of U' then the holomorphic hulls of K in U and U' are the same. The following result due to Bishop [Bi] about representing measure is also quite useful: For every point a in the holomorphic hull $\widehat{K}_U$ of K there exists a probability measure μ supported on K which satisfies

$$\log|f(a)|\leq\int_K\log|f|\,d\mu\quad\text{for all } f \text{ holomorphic on } U.$$

By an approximation result of Bremermann (see Théorème 9 in [Si1]), the above inequality in fact holds not only for functions of the form $\log|f|$ but also for all plurisubharmonic functions on U and hence for all plurisubharmonic functions on neighborhoods of $\widehat{K}_U$.

We next recall some elements of pluripotential theory pertaining to our work. Let Ω be a domain in $\mathbf{C}^n$. A function $u\in\mathrm{PSH}(\Omega)$ is called *maximal* if for every subdomain $\Omega'\Subset\Omega$ and for every $u'\in\mathrm{PSH}(\Omega)$ such that $u'\leq u$ on $\partial\Omega'$ we have $u'\leq u$ on Ω'. The following concept is quite important in solving the (complex) Dirichlet problem in higher dimension. A boundary point $\xi\in\partial\Omega$ is said to have a strong plurisubharmonic barrier if there exists a plurisubharmonic function u on Ω, $u<0$, such that $\lim_{z\to\xi}u(z)=0$ and such that for every neighborhood V of ξ we have $\sup_{\Omega\setminus V}u<0$. Now, according to Sibony (see [Si2]) Ω is called *B-regular* if every (real-valued) continuous function on $\partial\Omega$ can be extended to a plurisubharmonic function on Ω which is continuous on $\overline{\Omega}$. A basic result of Sibony (Théorème 2.1

in [Si2], see also [Wa] and [LS2]) states that Ω is B-regular if and only if every point in $\partial\Omega$ admits a strong plurisubharmonic barrier. Using this criterion, it is easy to check that every bounded strictly pseudoconvex domain Ω (possibly with non-smooth boundary), i.e. $\Omega=\{z:\varrho(z)<0\}$, where ϱ is a strictly plurisubharmonic function on a neighborhood of $\bar{\Omega}$, is B-regular.

In order to formulate the main result of this section (Theorem 2.2), we need a piece of terminology. A point $\xi\in\partial\Omega$ is said to be *good* if there exists a strong plurisubharmonic barrier u at ξ and a sequence of plurisubharmonic functions $\{u_j\}_{j=1}^{\infty}$ defined on neighborhoods of $\bar{\Omega}$ such that $u^*(z)=\lim_{j\to\infty} u_j(z)$, $z\in\bar{\Omega}$, where u^* is the upper regularization of u on $\bar{\Omega}$. It should be remarked at this point that, if Ω is a B-regular domain with $\mathcal{C}^1$ boundary then every $\xi\in\partial\Omega$ is a good boundary point. To see this, we pick a strong plurisubharmonic barrier u at ξ, according to Theorem 4.1 in [Wi] we can find a sequence of continuous plurisubharmonic functions on $\bar{\Omega}$ that decreases to u^* on $\bar{\Omega}$. As Ω has $\mathcal{C}^1$ boundary, by Theorem 1 in [FW], every continuous plurisubharmonic function on $\bar{\Omega}$ can be approximated uniformly on $\bar{\Omega}$ by plurisubharmonic functions on neighborhoods of $\bar{\Omega}$. It follows easily from these observations that ξ is a good boundary point (we do not know if the hypothesis about smoothness of $\partial\Omega$ can be dropped).

Next we recall some results in [LS2] adapted to our special situation.

Proposition 2.1. *Let Ω be a bounded domain in $\mathbf{C}^n$, A be a complex hypersurface which is defined by a holomorphic function f on a neighborhood of $\bar{\Omega}$. Assume further that f generates the ideal sheaf of A. Then $G_{A,\Omega}$ is maximal plurisubharmonic on $\Omega\setminus A$, $G_{A,\Omega}-\log|f|$ is locally bounded near $A\cap\Omega$. Moreover, if there exists a strong plurisubharmonic barrier at $\xi\in(\partial\Omega)\setminus A$ then*

$$\lim_{x\to\xi} G_{A,\Omega}(x)=0.$$

Proof. See Propositions 2.4 and 3.2 in [LS2]. □

Note. From now on, we always assume that the hypersurface A is defined by a holomorphic function f on a neighborhood of $\bar{\Omega}$ such that f generates the ideal sheaf of A, in other words $df\neq 0$ on a dense subset of A. By the preceding result $G_{A,\Omega}-\log|f|$ is real-valued plurisubharmonic on $\Omega\setminus A$ and locally bounded near $A\cap\Omega$. Thus it extends through A to a locally bounded plurisubharmonic function $\widetilde{G}_{A,\Omega}$ on Ω. Thus the continuity of $\widetilde{G}_{A,\Omega}$ on Ω implies that of $G_{A,\Omega}$ on Ω.

We are now able to formulate the main result of this paper.

Theorem 2.2. *Let Ω be a relatively compact pseudoconvex domain in U and A be a finite union of complex hyperplanes passing through $0\in\Omega$. Assume further* *that the following conditions are satisfied*:

(i) $\overline{\Omega}$ *is holomorphically convex in* U *and* $\operatorname{Int}(\overline{\Omega})=\Omega$.

(ii) *There exist* $\alpha<1$ *and a (relatively) open subset* V *of* $A\cap\partial\Omega$ *on* $\partial\Omega$ *such that* $tV\Subset\Omega$, $\alpha<t<1$.

(iii) *Every* $\xi\in(\partial\Omega)\setminus V$ *is a good boundary point.*

Then $\widetilde{G}_{A,\Omega}$ *is continuous on* Ω.

It seems that the simplest situation where Theorem 2.2 is applicable is the case where $\Omega=B$, the unit ball and $A=\{(z_1,z'):z_1=0\}$. In this special case, an explicit computation of $G_{A,\Omega}$ is available (see [LS2], p. 1535). More precisely, by writing $z=(z_1,z')$ we have

$$G_{A,B}(z)=\log\frac{|z_1|}{(1-|z'|^2)^{1/2}}.$$

The proof of the theorem relies on some lemmas. We first need the following simple fact.

Lemma 2.3. *Let* X *and* Y *be two bounded domains in* $\mathbf{C}^n$ *such that* $X\subset Y\Subset U$. *Let* f *be a holomorphic function continuous up to the boundary of* Y. *We define*

$$\widetilde{X}=\{(z,w):|w|\le|f(z)| \text{ for } z\in\partial X\},$$
$$\widetilde{Y}=\{(z,w):|w|\le|f(z)| \text{ for } z\in\partial Y\}.$$

Then $\widehat{\widetilde{X}}_{U\times\mathbf{C}}\subset\widehat{\widetilde{Y}}_{U\times\mathbf{C}}$.

Proof. It suffices to show that $\widetilde{X}\subset\widehat{\widetilde{Y}}_{U\times\mathbf{C}}$. For this, we pick $(z_0,w_0)\in\widetilde{X}$. Write $w_0=\lambda f(z_0)$, where $|\lambda|\le 1$. By the maximum modulus principle, for every g holomorphic on $U\times\mathbf{C}$ we have

$$|g(z_0,\lambda f(z_0))|\le\max_{z\in\partial Y}|g(z,\lambda f(z))|.$$

It follows that $(z_0,w_0)\in\widehat{\widetilde{Y}}_{U\times\mathbf{C}}$. $\square$

The next lemma is almost standard, we record it here for the readers convenience.

Lemma 2.4. *Let* Ω *be a relatively compact domain in* U. *Assume that* $\overline{\Omega}$ *is holomorphically convex in* U. *Then* $\overline{\Omega}$ *has a neighborhood basis of Runge strictly pseudoconvex domains with* $\mathcal{C}^\infty$ *boundaries.*

Proof. Since $\overline{\Omega}$ is holomorphically convex in U, by a result of Catlin (see Proposition 1.3 in [Si3]) we can find a smooth non-negative plurisubharmonic exhaustion function ϱ on U such that ϱ vanishes precisely on $\bar{\Omega}$ and that ϱ is strictly plurisubharmonic off $\bar{\Omega}$. By the Sard theorem we can find a sequence $\{\varepsilon_j\}_{j=1}^{\infty}$ of positive numbers decreasing to 0 such that $\{z : \varrho(z) = \varepsilon_j\}$ is smooth for every j. Now for each j we let Ω_j be the connected component of $\{z : \varrho(z) < \varepsilon_j\}$ that contains Ω. The lemma now follows. □

Lemma 2.5. *Let Ω be a relatively compact domain in U. Let φ be a real-valued continuous function on $\partial\Omega$. Set*

$$K = \{(z, w) : \log|w| + \varphi(z) \le 0, \ z \in \partial\Omega\}.$$

Then

$$\{(z, w) : \log|w| + \Phi(z) \le 0, \ z \in \Omega\} \subset \widehat{K}_{U \times \mathbf{C}},$$

where

$$\Phi(z) := \sup\{u(z) : u \in \mathrm{PSH}(\Omega) \text{ and } u^*(\xi) \le \varphi(\xi), \ \xi \in \partial\Omega\}. \tag{1}$$

This lemma is probably well known, but due to the lack of a precise reference we offer a short proof.

Proof. Fix $(z_0, w_0) \in \Omega \times \mathbf{C}$ such that $|w_0| < e^{-\Phi(z_0)}$. Let h be holomorphic on $U \times \mathbf{C}$ and satisfying $\|h\|_K \le 1$. Expanding h into power series (in w) one gets

$$h(z, w) = \sum_{j=0}^{\infty} w^j h_j(z).$$

Using the Cauchy inequalities we obtain $(1/j) \log|h_j| \le \varphi$ on $\partial\Omega$. Since $(1/j) \log|h_j|$ is plurisubharmonic on U we infer that $(1/j) \log|h_j| \le \Phi$ on Ω. This implies that

$$|h(z_0, w_0)| \le \sum_{j=0}^{\infty} |w_0 e^{\Phi(z_0)}|^j = \frac{1}{1 - |w_0| e^{\Phi(z_0)}}.$$

Applying this inequality to powers of h, it follows that $(z_0, w_0) \in \widehat{K}_{U \times \mathbf{C}}$. Since the latter set is closed, it includes all points $(z, w) \in \Omega \times \mathbf{C}$ such that $|w| \le e^{-\Phi(z)}$. This completes the proof. □

Remark. If we assume in addition that Ω is regular in the real sense, i.e. every continuous function on $\partial\Omega$ is the boundary values of some harmonic function on Ω, then Φ is plurisubharmonic on Ω (see Lemma 3.7 in [LS1]).

Retaining the notation used in Lemma 2.3, we move to the next lemma, which is the key in our argument.

Lemma 2.6. *Assume further that $\bar{\Omega}$ is holomorphically convex in U and that $\operatorname{Int}(\bar{\Omega})=\Omega$. Let f be a holomorphic function on a neighborhood of $\bar{\Omega}$. We set*

$$K=\{(z,w): z\in\partial\Omega \text{ and } |w|\leq|f(z)|\}.$$

Then we have

$$\widehat{K}_{U\times\mathbf{C}}\cap(\Omega\times\mathbf{C})=\{(z,w): z\in\Omega \text{ and } \log|w|+F(z)\leq 0\},$$

where F is a lower semicontinuous function in Ω such that F^*, its upper regularization, is locally bounded, maximal plurisubharmonic in Ω and $F^*\leq\widetilde{G}_{A,\Omega}$ on Ω.

Proof. Since $\bar{\Omega}$ is holomorphically convex in U, by shrinking U we may assume that f is holomorphic on U. We split the proof into two steps.

Step 1. In this step, we suppose in addition that Ω is a strictly pseudoconvex domain (possibly with non-smooth boundary), i.e. $\Omega=\{z\in U:\varrho(z)<0\}$, where ϱ is a strictly plurisubharmonic function on a neighborhood of $\bar{\Omega}$. Shrinking U again we may assume that ϱ is strictly plurisubharmonic on a neighborhood of $\bar{U}$.

Choose a sequence of real-valued, $\mathcal{C}^2$ functions $\{f_j\}_{j=1}^{\infty}$ defined on U such that $f_j\uparrow-\log|f|$ (we may e.g. take $f_j=-\log(|f|+1/j)$). Fix $j\geq 1$, let F_j be defined as in (1) (with φ replaced by f_j). Since Ω is strictly pseudoconvex we have that F_j is a continuous maximal plurisubharmonic function on Ω with boundary values f_j. Further, according to Lemma 2.5 we have

$$\{(z,w): z\in\Omega \text{ and } \log|w|+F_j(z)\leq 0\}\subset\widehat{(K_j)}_{U\times\mathbf{C}}, \tag{2}$$

where

$$K_j=\{(z,w): z\in\partial\Omega \text{ and } \log|w|+f_j(z)\leq 0\}.$$

Next we claim that F_j can be continued to a continuous plurisubharmonic function on U. Indeed, since ϱ is strictly plurisubharmonic on a neighborhood of $\bar{U}$ and F_j is continuous and maximal on Ω, the following function will be continuous and plurisubharmonic on U provided that λ_j is large enough

$$\widetilde{F}_j(z)=\begin{cases} F_j(z), & z\in\Omega,\\ \lambda_j\varrho(z)+f_j(z), & z\in U\setminus\Omega.\end{cases}$$

The claim follows. Since $\bar{\Omega}$ is holomorphically convex in U, by Theorem 4.3.4 in [Hö] we get

$$\widehat{(K_j)}_{U\times\mathbf{C}}\subset\{(z,w): z\in\bar{\Omega} \text{ and } \log|w|+F_j(z)\leq 0\}. \tag{3}$$

Combining (2) and (3) and using the fact that $F_j \equiv f_j$ on $\partial\Omega$, we obtain

$$\widehat{(K_j)}_{U\times\mathbf{C}} = \{(z,w) : z \in \overline{\Omega} \text{ and } \log|w| + F_j(z) \leq 0\}.$$

Notice that $K_j \downarrow K$, hence $\widehat{(K_j)}_{U\times\mathbf{C}} \downarrow \widehat{K}_{U\times\mathbf{C}}$. This implies that

$$\widehat{K}_{U\times\mathbf{C}} = \{(z,w) : z \in \overline{\Omega} \text{ and } \log|w| + F(z) \leq 0\},$$

where F is the increasing pointwise limit of F_j. Therefore F is lower semicontinuous, furthermore its upper regularization F^* is maximal plurisubharmonic and locally bounded in $\Omega \setminus A$. Since $\log|f| + F_j \leq 0$ on the boundary of Ω for every $j \geq 1$, by the maximum principle we have $\log|f| + F_j \leq 0$ on Ω. This implies that $\log|f| + F_j \in \mathcal{F}_{A,\Omega}$ and hence $\log|f| + F_j \leq G_{A,\Omega}$ in Ω. Thus $F_j \leq \widetilde{G}_{A,\Omega}$ on Ω. Consequently $F^* \leq \widetilde{G}_{A,\Omega}$ on Ω. Therefore F^* is maximal plurisubharmonic and locally bounded in Ω. Thus the case where Ω is strictly pseudoconvex has been settled.

Step 2. The general case. By Lemma 2.4 we can find a sequence $\{\Omega_j\}_{j=1}^{\infty}$ of smoothly bounded, strictly pseudoconvex domains such that $\overline{\Omega}_j$ is holomorphically convex in U and $\Omega_j \downarrow \overline{\Omega}$. Now for each j we define

$$L_j = \{(z,w) : z \in \partial\Omega_j \text{ and } |w| \leq |f(z)|\}.$$

It follows from the result obtained in Step 1 that

$$\widehat{(L_j)}_{U\times\mathbf{C}} = \{(z,w) : z \in \overline{\Omega}_j \text{ and } \log|w| + F_j(z) \leq 0\},$$

where F_j is lower semicontinuous on Ω_j such that F_j^* is locally bounded, maximal plurisubharmonic on Ω_j and $\log|f| + F_j^* \leq 0$ on Ω_j. As $\mathrm{Int}(\overline{\Omega}) = \Omega$ we see that the sets L_j converges to K in the Hausdorff metric. By Lemma 2.3 we have $\widehat{(L_j)}_{U\times\mathbf{C}} \downarrow \widehat{K}_{U\times\mathbf{C}}$. On the other hand, the sequence $\{F_j\}_{j=1}^{\infty}$ increases to a lower semicontinuous function F on Ω. So F^* is maximal plurisubharmonic on Ω and $F^* \leq \widetilde{G}_{A,\Omega}$ on Ω. The desired conclusion follows from these observations. $\square$

Remark. If $\xi \in (\partial\Omega) \setminus A$ is a good boundary point then $\lim_{z\to\xi} F(z) = -\log|f(\xi)|$. To see this, we claim first that $\widehat{K}_{U\times\mathbf{C}} \cap (\{\xi\} \times \mathbf{C}) = \{(\xi,w) : |w| \leq |f(\xi)|\}$. Assuming this, then by Lemma 2.6 clearly we have $\liminf_{z\to\xi} F(z) = -\log|f(\xi)|$. On the other hand, as $\log|f| + F^* \in \mathcal{F}_{A,\Omega}$ we infer that $\limsup_{z\to\xi} F(z) \leq -\log|f(\xi)|$. Thus we are done. It remains to prove the claim. Take a point $(\xi, w_0) \in \widehat{K}_{U\times\mathbf{C}}$. Then we can find a representing measure μ such that for all plurisubharmonic u on a neighborhood of $\widehat{K}_{U\times\mathbf{C}}$ the following holds $u(\xi, w_0) \leq \int_K u\, d\mu$. Let φ be a strong plurisubharmonic barrier at ξ, and $\{\varphi_j\}_{j=1}^{\infty}$ be a sequence of plurisubharmonic functions on neighborhoods of $\overline{\Omega}$ that decreases to φ^* on $\overline{\Omega}$. Since $\overline{\Omega}$ is holomorphically convex in U, we may consider φ_j as a plurisubharmonic function on a neighborhood of $\widehat{K}_{U\times\mathbf{C}}$ for each $j\geq 1$. Applying the above inequality to φ_j and then passing to the limit we see that μ must be supported on $\{\xi\}\times\mathbf{C}$. The claim now follows.

The next lemma is quite similar to Lemma A9 in [AW].

Lemma 2.7. *Let Ω be a relatively compact pseudoconvex domain in U. Assume that* $\operatorname{Int}(\overline{\Omega})=\Omega$ *and that $\overline{\Omega}$ is holomorphically convex in U. Then Ω is Runge in U, i.e. every holomorphic function on Ω can be approximated locally uniformly by holomorphic functions on U.*

Proof. Let K be a compact subset of Ω. In view of Theorem 4.3.2 in [Hö], it suffices to check that $\widehat{K}_{\mathrm{PSH}(U)}\Subset\Omega$. Let $\{\Omega_j\}$ be a neighborhood basis of Runge pseudoconvex domains of $\overline{\Omega}$. Since $-\log\operatorname{dist}(\,\cdot\,,\partial\Omega_j)\in\mathrm{PSH}(\Omega_j)$ we infer $\widehat{K}_{\mathrm{PSH}(U)}\Subset\Omega$. $\square$

The final preparatory lemma is the following easy fact whose proof is left to the readers.

Lemma 2.8. *Let Ω be a bounded domain in $\mathbf{C}^n$ and $\{\varphi_j\}_{j=1}^{\infty}$ be a sequence of lower semicontinuous functions on Ω that increases to a lower semicontinuous function φ on Ω. Then for every sequence $\{a_j\}_{j=1}^{\infty}$ that converges to $a\in\overline{\Omega}$ we have*

$$\varphi_*(a)\leq\liminf_{j\to\infty}\varphi_j(a_j),$$

where $\varphi_*(z)$ is the lower regularization of φ, which is defined on $\overline{\Omega}$. Moreover, if φ_* is continuous at a then $\lim_{j\to\infty}\varphi_j(a_j)=\varphi_*(a)$.

Proof of Theorem 2.2. Let f be a homogeneous polynomial that defines A. Define

$$K=\{(z,w):z\in\partial\Omega \text{ and } |w|\leq|f(z)|\}.$$

Then by Lemma 2.6 we get

$$\widehat{K}_{U\times\mathbf{C}}\cap(\Omega\times\mathbf{C})=\{(z,w):z\in\Omega \text{ and } \log|w|+F(z)\leq 0\},$$

where F is a lower semicontinuous function on Ω satisfying $F^*\leq\widetilde{G}_{A,\Omega}$ on Ω. Thus in view of the lower semicontinuity of F, it suffices to prove that $\widetilde{G}_{A,\Omega}\leq F$. Indeed, otherwise we could find $z_0\in\Omega$ such that $F(z_0)<\widetilde{G}_{A,\Omega}(z_0)$. Since the (real) interval $[1,t_0]$ as a subset of $\mathbf{C}^n$ is not pluri-thin at 1 we can find a sequence $\{t_j\}_{j=1}^{\infty}\subset(1,t_0)$ such that $t_j\downarrow 1$ and

$$\widetilde{G}_{A,\Omega}(z_0/t_j)>F(z_0)+\varepsilon \quad \text{for all } j \text{ and for some } \varepsilon>0. \tag{4}$$

We may further assume that $t_j z_0 \in \Omega$ for all j. Let ϱ be a $\mathcal{C}^\infty$ strictly plurisubharmonic exhaustion function on Ω, then we choose a sequence $\alpha_j \uparrow \infty$ such that for each $j \geq 1$ there exists a connected component Ω_j of the level set $\{z : \varrho(z) < \alpha_j\}$ that contains $\{0, z_0/t_j, V/t_j\}$ (this is possible in view of (ii)). Further, by the Sard theorem we may also achieve that $\{z : \varrho(z) = \alpha_j\}$ is a smooth real hypersurface (possibly disconnected), in particular its connected components have disjoint closures. It follows that $\overline{\Omega}_j$ is holomorphically convex in Ω. As Ω is Runge in U by Lemma 2.7, we conclude that $\overline{\Omega}_j$ is holomorphically convex in U.

Fix $j \geq 1$, let U_j be the connected component of $t_j \Omega_j \cap \Omega$ that includes 0. Set

$$V_j := \{(z, w) : z \in \partial U_j \text{ and } |w| \leq |f(z)|\},$$
$$W_j := \{(z, w) : z \in \partial \Omega_j \text{ and } |w| \leq |f(z)|\}.$$

By Lemmas 2.5 and 2.6 we obtain lower semicontinuous functions G_j (resp. H_j) on U_j (resp. Ω_j) as constructed in (1) such that

$$\widehat{(V_j)}_{U \times \mathbf{C}} \cap (U_j \times \mathbf{C}) \supset \{(z, w) : z \in U_j \text{ and } \log |w| + G_j(z) \leq 0\}, \tag{5}$$

$$\widehat{(W_j)}_{U \times \mathbf{C}} \cap (\Omega_j \times \mathbf{C}) = \{(z, w) : z \in \Omega_j \text{ and } \log |w| + H_j(z) \leq 0\}. \tag{6}$$

It should be observed that, as the components of $t_j \Omega_j \cap \Omega$ may not have disjoint closures, we cannot assert that $\overline{U}_j$ is holomorphically convex in U, so we have to use Lemma 2.5 in (5). Assume that $\deg f = d$, then under the linear map $\lambda_j(z, w) = (z/t_j, w/t_j^d)$ the set V_j transforms into

$$V_j' = \left\{(z, w) : z \in \frac{1}{t_j}(\partial U_j) \text{ and } |w| \leq |f(z)|\right\}.$$

Clearly $U_j/t_j \subset \Omega_j$, so by Lemma 2.3 we have $\widehat{(V_j')}_{U \times \mathbf{C}} \subset \widehat{(W_j)}_{U \times \mathbf{C}}$. Applying the inverse of λ_j and combining (5) and (6) we arrive at

$$H_j(z) \leq G_j(t_j z) + d \log t_j, \quad z \in U_j/t_j.$$

Since the point z_0/t_j lies in U_j/t_j we obtain

$$H_j(z_0/t_j) \leq G_j(z_0) + d \log t_j. \tag{7}$$

On the other hand, we have $\log |w| + \widetilde{G}_{A,\Omega}(z) \leq 0$ on W_j. It follows from Theorem 4.3.4 in [Hö] that

$$\widehat{(W_j)}_{U \times \mathbf{C}} = \widehat{(W_j)}_{\Omega \times \mathbf{C}} \subset \{(z, w) : z \in \overline{\Omega}_j \text{ and } \log |w| + \widetilde{G}_{A,\Omega}(z) \leq 0\}.$$

It implies that $\widetilde{G}_{A,\Omega}\le H_j$ on Ω_j. Putting this and (7) together one gets

$$\widetilde{G}_{A,\Omega}(z_0/t_j)\le G_j(z_0)+d\log|t_j|. \tag{8}$$

Combining this with (4) we obtain

$$F(z_0)-G_j(z_0)\le d\log|t_j|-\varepsilon.$$

Thus

$$\limsup_{j\to\infty}(F(z_0)-G_j(z_0))\le-\varepsilon.$$

In order to derive a contradiction from the above inequality, it is useful to recall the construction of F and G_j given in Lemma 2.6. Let $\{X_k\}_{k=1}^{\infty}$ be a sequence of smoothly bounded strictly pseudoconvex domains decreasing to $\overline{\Omega}$ such that $\overline{X}_k$ is holomorphically convex in U. Let $Y_{k,j}$ be the connected component of $X_k\cap t_j\Omega_j$ that includes U_j. Let $\{f_l\}_{l=1}^{\infty}$ be a sequence of real-valued continuous functions increasing to $-\log|f|$ on U. For each $l\ge1$ we have continuous functions as constructed in (1) (with f_l instead of f) $G_{k,j,l}$ on $Y_{k,j}$ and $F_{k,l}$ on X_k. Observe that $Y_{k,j}$ is a strictly pseudoconvex domain (possibly with non-smooth boundary), so from the proof of Lemma 2.6 we obtain

$$\begin{aligned} G_{k,j}&=\lim_{l\to\infty}G_{k,j,l}, & F_k&=\lim_{l\to\infty}F_{k,l},\\ G_j&=\lim_{k\to\infty}G_{k,j}, & F&=\lim_{k\to\infty}F_k. \end{aligned}$$

Notice that the constructions of $G_{k,j,l}$ and $F_{k,l}$ imply

$$|G_{k,j,l}(z_0)-F_{k,l}(z_0)|\le\sup_{\partial Y_{k,j}}|f_l-F_{k,l}|. \tag{9}$$

Let $\xi_{k,j,l}$ be a point on $\partial Y_{k,j}$ that realizes the supremum on the right-hand side. Since $F_{k,l}\equiv f_l$ on ∂X_k, we may achieve that $\xi_{k,j,l}\in(\partial Y_{k,j})\cap X_k$. By passing to a subsequence we may assume further in view of (ii) that $\xi_{k,j,l}$ tends to $\xi^*_{k,j}\in\overline{\Omega}\setminus V$ when l tends to ∞. Letting l tend to ∞ in (9) and using Lemma 2.8 we obtain

$$\begin{aligned}|G_{k,j}(z_0)-F_k(z_0)|&\le\max\{-\log|f(\xi^*_{k,j})|-(F_k)_*(\xi^*_{k,j}),\log|f(\xi^*_{k,j})|+(F_k)^*(\xi^*_{k,j})\}\\ &=-\log|f(\xi^*_{k,j})|-(F_k)_*(\xi^*_{k,j}).\end{aligned}$$

Here in the second line we have used the fact that $(F_k)^*\le-\log|f|$ on $\overline{\Omega}\setminus V$. We may assume that $\xi^*_{k,j}$ converges to $\xi^{**}_j\in\overline{\Omega}$ when k tends to ∞. It follows that

$$|G_j(z_0)-F(z_0)|\le-\log|f(\xi^{**}_j)|-F_*(\xi^{**}_j).$$

Again switching to a subsequence we may assume that ξ_j^{**} converges to $\xi^{***}\in(\partial\Omega)\setminus V$. Now the remark after Lemma 2.6 implies $F_*(\xi^{***})=-\log|f(\xi^{***})|$. This yields a contradiction. The proof is thereby concluded. □

Remarks. (a) In the proof of Theorem 2.2 we have established a connection between the holomorphic hull of K and the Green function $G_{A,\Omega}$. In fact, we have actually shown that $F^*\equiv\widetilde{G}_{A,\Omega}$ on Ω. This is closely related to a previous work of Bremermann (see [Br]).

(b) Theorem 2.2 depends heavily on the geometric nature of $\mathbf{C}^n$, so it seems hard to find an analogue of this result in the manifold setting.

(c) The hypothesis that A is a finite union of hyperplanes is not so restrictive as we will see later. On the other hand, the assumption on the existence of the neighborhood V of $A\cap\partial\Omega$ is essential for our method. It is desirable to know if a weaker hypothesis is sufficient.

Corollary 2.9. *Let Ω be a bounded pseudoconvex domain which is* (*strongly*) *starshaped, i.e. $t\Omega\Subset\Omega$, $t\in[0,1)$. Let A be a finite union of complex hyperplanes passing through $0\in\Omega$. Then $\widetilde{G}_{A,\Omega}$ is continuous on Ω.*

Proof. By Proposition 5 in [Ka] we have that $\overline{\Omega}$ is polynomially convex. Thus all assumptions given in Theorem 2.2 are satisfied (with $U=\mathbf{C}^n$ and $V=\partial\Omega$), so $\widetilde{G}_{A,\Omega}$ is continuous. □

Remark. The assumptions on Ω are obviously satisfied in the case where Ω is a bounded convex domain containing the origin.

The next result treats a special case where there exist many "good" boundary points on $\partial\Omega$. We first fix the notation, let Ω be a bounded pseudoconvex domain in $\mathbf{C}^n$ and ϱ be a $\mathcal{C}^1$ plurisubharmonic function on a pseudoconvex neighborhood U of $\overline{\Omega}$ such that $\Omega:=\{z\in U:\varrho(z)<0\}$. Clearly for t_0 sufficiently close to 1 we have $t\Omega\Subset U$, $t\in(1,t_0)$. Set

$$\varrho_\xi(t)=\varrho(\xi t),\quad \xi\in\partial\Omega,\ t\in(1,t_0).$$

Let A be a finite union of hyperplanes passing through $0\in\Omega$. Then we have the following result.

Corollary 2.10. *With the same notation as above, assume that for every $\xi\in A\cap\partial\Omega$ we have $\varrho'_\xi(1)>0$, and that every $\xi\in(\partial\Omega)\setminus A$ is a good boundary point. Then $\widetilde{G}_{A,\Omega}$ is continuous on Ω.*

Proof. It is well known that $\overline{\Omega}$ is holomorphically convex in U. Moreover the maximum principle implies that $\operatorname{Int}(\overline{\Omega})=\Omega$. In order to apply Theorem 2.2, it suffices to check the condition (ii) given there. We argue by contradiction, assume that there exists no such neighborhood V. By passing to a subsequence we may assume that there exist a sequence $t_j\uparrow 1$ and a sequence $\{\xi_j\}_{j=1}^{\infty}=\{(\xi_j^{(1)},\dots,\xi_j^{(n)})\}_{j=1}^{\infty}$ belonging to $\partial\Omega$ and a point $\xi^*\in A\cap\partial\Omega$ such that $\lim_{j\to\infty}\xi_j=\xi^*$, and that $\varrho(t_j\xi_j)\geq 0$. As $\varrho(\xi_j)=0$, we infer that $\varrho'_{\xi_j}(\lambda_j)\leq 0$ for some $\lambda_j\in(t_j,1)$. By letting j tend to ∞ we get $\varrho'_{\xi^*}(1)\leq 0$, a contradiction. □

Corollary 2.11. *Let Ω be a bounded B-regular pseudoconvex domain with $\mathcal{C}^3$ boundary. Let $A=\{(z_1,z_2):z_2=0\}$. Assume that $A\cap\Omega$ is a simply connected domain with $\mathcal{C}^\infty$ boundary (viewed as a subset of $\mathbf{C}$) and that $\pi(\Omega)\times\{0\}=A\cap\Omega$, where π is the projection $(z_1,z_2)\mapsto z_1$. Then $\widetilde{G}_{A,\Omega}$ is continuous on Ω.*

Proof. Since $A\cap\Omega$ is a smoothly bounded simply connected domain (viewed as a subset of $\mathbf{C}$) we can find a conformal map ψ from $A\cap\Omega$ onto the unit disk D. By Theorem 4.8.17 in [BG], ψ extends to a $\mathcal{C}^\infty$ diffeomorphism from a neighborhood of $\overline{A\cap\Omega}$ to a neighborhood of $\overline{D}$. Furthermore, as Ω projects onto $A\cap\Omega$ we infer that the map $\Psi(z_1,z_2)=(z_1,\psi(z_2))$ sends Ω biholomorphically onto Ω', a bounded B-regular pseudoconvex domain with $\mathcal{C}^3$ boundary such that $A\cap\Omega'$ is the unit disk. It suffices to check that $\widetilde{G}_{A,\Omega'}$ is continuous on Ω'. According to Theorem 4.1 in [Si2], we can find a function $\varrho\in\mathrm{PSH}(U)\cap\mathcal{C}^3(U)$, where U is a pseudoconvex neighborhood of $\overline{\Omega}'$ such that $\Omega'=\{z\in U:\varrho(z)<0\}$. By shrinking U we may assume that $\overline{\Omega}'$ is holomorphically convex in U. By the remark made before Proposition 2.1 we see that every $\xi\in\partial\Omega$ is a good boundary point. Now let $\tilde\varrho(z_1):=\varrho(z_1,0)$. Since $\tilde\varrho<0$ on D and $\tilde\varrho$ vanishes on ∂D, by the Hopf lemma, the outward normal derivative of $\tilde\varrho$ is positive at every point on the circle $|z_1|=1$. Thus using Corollary 2.10 we conclude that $\widetilde{G}_{A,\Omega'}$ is continuous on Ω'. The desired conclusion follows. □

So far, we have dealt only with cases where the pole set A is in some sense "linear". For the general case, a natural idea is to find a "good" holomorphic mapping that transforms Ω and A to a new setup such that one of the above results is applicable. The next result illustrates this idea.

Proposition 2.12. *Let Ω be the unit ball $\{(z,w):|z|^2+|w|^2<1\}$ in $\mathbf{C}^2$ and A be the complex curve $\{(z,w):w=z^m\}$, where m is a positive integer. Then $\widetilde{G}_{A,\Omega}$ is continuous on Ω.*

Proof. Consider the holomorphic map $g(z,w)=(z^m,w)$. It is easy to check that under g the ball Ω is mapped properly onto the Thullen domain

$$\Omega'=\{(z',w'):|z'|^{2/m}+|w'|^2<1\}$$

and A is transformed to the complex line $A'=\{(z',w'):w'=z'\}$. Now we claim that $\widetilde{G}_{A',\Omega'}\circ g\equiv\widetilde{G}_{A,\Omega}$. Indeed, from the definition we have $\widetilde{G}_{A',\Omega'}\circ g\leq\widetilde{G}_{A,\Omega}$. The reverse inequality is obtained by pulling back the plurisubharmonic function $\widetilde{G}_{A',\Omega'}$. Thus the claim is verified. It is straightforward to see that Ω' and A' verify all assumptions given in Corollary 2.9, thus $\widetilde{G}_{A',\Omega'}$ is continuous on Ω'. The continuity of $\widetilde{G}_{A,\Omega}$ now follows from this and the claim above. $\square$

III. Some examples

The first explicit examples of the Green function $G_{A,\Omega}$ appeared in [LS2], where the authors compute $G_{A,\Omega}$ in the cases where Ω is a ball or a polydisk and A is a coordinate hyperplane. Motivated from these examples, we deal with the case where Ω is a Hartogs domain in $\mathbf{C}^2$.

Proposition 3.1. *Let X be a bounded domain in $\mathbf{C}$ and φ be a real-valued subharmonic function on X. Let $A=\{(z,w):w=0\}$ and*

$$\Omega=\{(z,w)\in X\times\mathbf{C}:\log|w|+\varphi(z)<0\}.$$

Then $G_{A,\Omega}(z,w)=\log|w|+\varphi(z)$.

Proof. It is obvious that $\log|w|+\varphi(z)\in\mathcal{F}_{A,\Omega}$. Hence $\widetilde{G}_{A,\Omega}(z,w)\geq\varphi(z)$, $(z,w)\in\Omega$. Fix $z\in X$, as $G_{A,\Omega}(z,w)\leq 0$, $(z,w)\in\Omega$, by applying the maximum principle to the disk $\{w:|w|<e^{-\varphi(z)}\}$ we obtain

$$\widetilde{G}_{A,\Omega}(z,w)\leq\varphi(z),\quad (z,w)\in\Omega.$$

The desired conclusion follows. $\square$

Remarks. (a) Let

$$\Omega=\{(z,w):|z|^2+|w|^2<1 \text{ and } \log|z|+|w|^2<0\}.$$

Then Ω is a bounded hyperconvex domain in $\mathbf{C}^2$, which is not B-regular as $\partial\Omega$ contains (non-trivial) analytic disks. On the other hand, using Proposition 3.1 we can verify easily that $G_{A,\Omega}$ is a continuous plurisubharmonic function on $\overline{\Omega}\setminus A$ with boundary values 0, where $A=\{(z,w):z=0\}$. This example shows that $G_{A,\Omega}$ may still behave nicely (in some sense) on non-B-regular domains.

(b) Let φ be a bounded subharmonic function on the unit disk D such that φ is discontinuous at some point $a\in D$. Set

$$\Omega:=\{(z,w)\in D\times\mathbf{C}:\log|w|+\varphi(z)<0\}.$$

Then by Proposition 3.1 $G_{A,\Omega}(z,w)=\log|w|+\varphi(z)$ where $A=\{(z,w):w=0\}$. Thus $G_{A,\Omega}$ is discontinuous at every point of $\{a\}\times\{w:|w|<e^{-\varphi(a)}\}$. We will show that there exists no (weak) plurisubharmonic barrier at any point of $\{a\}\times\{w:|w|=e^{-\varphi(a)}\}$. Assume contrary that there exists such a barrier. Then we could find a negative plurisubharmonic function u on Ω, a point (a,w_0) such that $|w_0|=e^{-\varphi(a)}$ and that $\lim_{\xi\to(a,w_0)}u(\xi)=0$. Since φ is discontinuous at a, there exists a sequence $\{z_j\}_{j=1}^{\infty}$ converging to a such that $\varphi(z_j)<\varphi(a)-\varepsilon$ for some $\varepsilon>0$. Now for each j we set $\varphi_j(t):=u(z_j,te^{-\varphi(a)+\varepsilon})$. Then each φ_j is a negative subharmonic function on the unit disk D. Define $\widetilde{\varphi}(t)=(\limsup_{j\to\infty}\varphi_j(t))^*$. It is clear that $\widetilde{\varphi}$ is subharmonic on D and $\widetilde{\varphi}\le 0$. Notice also that $\widetilde{\varphi}(0)<0$ so $\widetilde{\varphi}\not\equiv 0$. On the other hand, as u is a strong barrier at (a,w_0) we infer that $\widetilde{\varphi}(w_0e^{\varphi(a)-\varepsilon})=0$. This is a contradiction to the maximum principle. This example indicates that the assumption on the existence of a strong plurisubharmonic barrier at every boundary point in Theorem 3.9 in [LS2] cannot be entirely omitted. $\square$

Before stating the next result, recall that by an *analytic disk* in $\mathbf{C}^n$ we mean a holomorphic mapping from the unit disk D in $\mathbf{C}$ to $\mathbf{C}^n$ which is continuous up to ∂D.

Proposition 3.2. *Let Ω be a bounded domain in $\mathbf{C}^n$ such that $\overline{\Omega}$ is polynomially convex. Let A be a complex hypersurface defined by a holomorphic f on a neighborhood of $\overline{\Omega}$. Set $M=\|f\|_{\Omega}$ and*

$$S=\{z\in\partial\Omega:|f(z)|=M\}.$$

Assume that there exists an analytic disk $\Phi: D\to\Omega$ such that $\Phi(\partial D)\subset S$ and that Ω has a strong plurisubharmonic barrier at every point along $\Phi(\partial D)$. Then for every $\xi\in D$ we have

$$(\widetilde{G}_{A,\Omega}\circ\Phi)(\xi)=-\log M.$$

Proof. Let

$$K=\{(z,w):z\in\partial\Omega \text{ and } |w|\le|f(z)|\}.$$

By Lemma 2.6 we can express

$$\widehat{K}\cap(\Omega\times\mathbf{C})=\{(z,w):z\in\Omega \text{ and } \log|w|+F(z)\le 0\}, \tag{10}$$

where F is a lower semicontinuous function on Ω. First we claim that

$$\widehat{K}\cap\{(z,w):w=M\}=\widehat{S}\times\{(z,w):w=M\}.$$

It is obvious that the set on the right-hand side is contained in the left-hand side one. To show the reverse inclusion, let $(z_0, M)\in\widehat{K}$, then we can find a probability measure μ supported on K such that

$$\log|p(z_0, M)| \leq \int_K \log|p(z,w)|\, d\mu \quad \text{for all polynomials } p. \tag{11}$$

For each $m\geq 1$ we set $p_m(w)=(Mw+1)^m/2^m$. It is easy to see that p_m peaks at M on the closed disk $\{(z,w):|w|\leq M\}$. Apply (11) to p_m and let m tend to ∞ we obtain via the Lebesgue convergence theorem that $\operatorname{supp}\mu\subset K\cap\{(z,w):w=M\}$. It implies that $z_0\in\widehat{S}$. The claim is proved. It now follows from (10) that $F(z)=-\log M$ for every point $z\in\Omega$ satisfying $(z,M)\in\widehat{K}$. Combining this and the claim above one obtains

$$F(z) = -\log M, \quad z\in\widehat{S}\cap\Omega.$$

Next, observe that $\widetilde{G}_{A,\Omega}\geq F$ on Ω, so we have $\widetilde{G}_{A,\Omega}(z)\geq -\log M$, $z\in\widehat{S}\cap\Omega$. Now the maximum principle implies that $\Phi(D)\subset\widehat{S}\cap\Omega$. From this and the last inequality we deduce that

$$g(\xi) := (\widetilde{G}_{A,\Omega}\circ\Phi)(\xi)+\log M \geq 0, \quad \xi\in D.$$

As Ω has a strong plurisubharmonic barrier at every point along $\Phi(\partial D)$, in view of Proposition 2.1 we have $g\equiv 0$ on ∂D. Because g is subharmonic on D, applying again the maximum principle we get $g\equiv 0$ on D. This concludes the proof. $\square$

As an application of the preceding result, we give an explicit formula of $G_{A,\Omega}$ in the case Ω is the unit ball in $\mathbf{C}^2$ and A is the union of coordinate lines.

Example 3.3. *Let Ω be the unit ball in $\mathbf{C}^2$, A be the (singular) complex curve $\{(z,w):zw=0\}$. We set*

$$\Omega_1 = \left\{(z,w)\in\Omega : \frac{1}{\sqrt{2}} \leq |w|\right\},$$
$$\Omega_2 = \left\{(z,w)\in\Omega : \max\{|z|, |w|\} < \frac{1}{\sqrt{2}}\right\},$$
$$\Omega_3 = \left\{(z,w)\in\Omega : \frac{1}{\sqrt{2}} \leq |z|\right\}.$$

Then we have

$$G_{A,\Omega}(z,w) = \begin{cases} \log|z| - \frac{1}{2}\log(1-|w|^2), & (z,w)\in\Omega_1, \\ \log|zw| - \log\frac{1}{2}, & (z,w)\in\Omega_2, \\ \log|w| - \frac{1}{2}\log(1-|z|^2), & (z,w)\in\Omega_3. \end{cases}$$

Proof. First it is immediate to check that the function on the right-hand side belongs to the class $\mathcal{F}_{A,\Omega}$. Second, it is clear that the function $f(z,w)=zw$ attains its maximum modulus precisely on the torus

$$S=\left\{(z,w):|z|=|w|=\frac{1}{\sqrt{2}}\right\}.$$

Notice further that, to each point x in Ω_2, there is an analytic disk in $\mathbf{C}^2$ passing through x whose boundary is contained in S. Applying Proposition 3.2 we obtain

$$G_{A,\Omega}(z,w)=\log|zw|-\log\tfrac{1}{2},\quad (z,w)\in\Omega_2.$$

On the other hand, by the definition of $G_{A,\Omega}$ we have

$$G_{A,\Omega}(z,w)\le G_{A_1,\Omega}(z,w)=\log|z|-\tfrac{1}{2}\log(1-|w|^2),\quad (z,w)\in\Omega,$$

and

$$G_{A,\Omega}(z,w)\le G_{A_2,\Omega}(z,w)=\log|w|-\tfrac{1}{2}\log(1-|z|^2),\quad (z,w)\in\Omega,$$

where $A_1=\{(z,w):z=0\}$ and $A_2=\{(z,w):w=0\}$. $\square$

Remarks. (a) It is of interest to notice that in Ω_1 (resp. Ω_3) the function $G_{A,\Omega}$ coincides with $G_{A_1,\Omega}$ (resp. $G_{A_2,\Omega}$). This might be considered as an analogue to the complex cones Γ_p and Γ_q discovered independently by D. Coman and F. Wikström for the Green function with two poles p and q of equal weights inside the ball (see [Co], p. 260).

(b) It is proved in [LS2], p. 1528, that if $G_{A,\Omega}(x)=\inf_{a\in A}G_a(x)$, where G_a denotes the Green function of Ω with single pole a, then the complex geodesic realizing the hyperbolic distance from x to A hits A in only one point. Using our Example 3.3 we will show that the converse statement is false. Indeed, it is elementary to see that for every point $x\in\Omega_2\setminus A$ and close enough to A_1 (resp. A_2), the complex geodesic realizing the hyperbolic distance from x to A is the complex line passing through x and orthogonal to A_1 (resp. A_2). Therefore it meets A only at one point. However we have

$$G_{A,\Omega}(x)<\min\{G_{A_1,\Omega}(x),G_{A_2,\Omega}(x)\}=\inf_{a\in A}G_a(x).$$

(c) After completing this paper, the author has learned from Professor Peter Pflug that a generalization of Example 3.3 to complex ellipsoids has been obtained in [Ja].

We finish this paper by proving the following result which generalizes in part Example 3.4 in [LS2] about the boundary behavior of $G_{A,\Omega}$ when Ω is the unit bidisk and A is the line $\{(z,w):z=0\}$.

Proposition 3.4. *Let Ω be the unit bidisk $D\times D$ in $\mathbf{C}^2$. Let A be a complex hypersurface defined by a holomorphic function f on a neighborhood of $\overline{\Omega}$. Assume that f does not vanish on a neighborhood of $\{(z,w):|z|=1$ and $|w|\leq 1\}$. Then for every $\xi=(z_0,w_0)$ with $|z_0|=1$ and $|w_0|\leq 1$ we have*

$$\lim_{x\to\xi} G_{A,\Omega}(x)=0.$$

Remark. The above mentioned example in [LS2] shows that the conclusion in Proposition 3.4 may fail if $|z_0|<1$ and $|w_0|=1$.

Proof. We set

$$K=\{(z,w,u):(z,w)\in\partial\Omega \text{ and } |u|\leq |f(z,w)|\}.$$

By Lemma 2.6 there exists F lower semicontinuous on Ω such that $F^*\leq \widetilde{G}_{A,\Omega}$ and

$$\widehat{K}\cap(\Omega\times\mathbf{C})=\{(z,w,u):\log|u|+F(z,w)\leq 0\}. \tag{12}$$

On the other hand, by repeating the argument used in the proof of Proposition 3.2 we obtain

$$\widehat{K}\cap(\{z_0\}\times\mathbf{C}^2)=(K\cap(\{z_0\}\times\mathbf{C}^2))\widehat{\ }. \tag{13}$$

Notice that

$$K\cap(\{z_0\}\times\mathbf{C}^2)=\{(z_0,w,u):|w|\leq 1 \text{ and } \log|u|-\log|f(z_0,w)|\leq 0\}.$$

Since $f(z_0,\cdot\,)$ does not vanish on a neighborhood of $\{w:|w|\leq 1\}$ we infer that $\log|u|-\log|f(z_0,w)|$ is plurisubharmonic on $\mathbf{C}\times D'$, where D' is a neighborhood of $\overline{D}$. It follows that $K\cap(\{z_0\}\times\mathbf{C}^2)$ is polynomially convex. Combining this and (13) we get

$$\widehat{K}\cap(\{z_0\}\times\mathbf{C}^2)=K\cap(\{z_0\}\times\mathbf{C}^2).$$

It follows that

$$\widehat{K}\cap(\{\xi\}\times\mathbf{C})=K\cap(\{\xi\}\times\mathbf{C}).$$

Let

$$\alpha=\liminf_{\substack{x\to\xi\\ x\in\Omega}} F(x).$$

As $\widehat{K}$ is closed, it follows from (12) that

$$\{(\xi,u):|u|\leq e^{-\alpha}\}\subset\widehat{K}\cap(\{\xi\}\times\mathbf{C}).$$

Thus $-\log|f(\xi)|\leq\alpha$. So

$$-\log|f(\xi)|\leq\liminf_{\substack{x\to\xi\\ x\in\Omega}}\widetilde{G}_{A,\Omega}(x).$$

The desired conclusion now follows. □

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Received August 18, 2003
in revised form February 14, 2004

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