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Tensor products of direct sums

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Abstract. A similar formula to the one established by Ansemil and Floret for symmetric tensor products of direct sums is proved for alternating and Jacobian tensor products. It is then applied to stable spaces where a number of isomorphisms between spaces of tensors or multilinear forms are unveiled. A connection between these problems and irreducible group representations is made.

Preliminaries

The n-fold tensor product of n vector spaces $\{E_i\}_{i=1}^n$ is defined recursively as

$$E_1\otimes\ldots\otimes E_{n-1}\otimes E_n=(E_1\otimes\ldots\otimes E_{n-1})\otimes E_n.$$

Let E and F be normed spaces. A norm μ on $E\otimes F$ is said to be a reasonable crossnorm if

(1) $\mu(x\otimes y)\leq\|x\|\,\|y\|$ for every $x\in E$ and $y\in F$;

(2) the linear functional $\varphi\otimes\psi$ on $E\otimes_\mu F$ is bounded and $\|\varphi\otimes\psi\|\leq\|\varphi\|\,\|\psi\|$ for every $\varphi\in E^*$ and $\psi\in F^*$.

The projective and the injective norms π and ε satisfy these conditions and it can be shown that a norm μ on $E\otimes F$ is a reasonable crossnorm if and only if $\varepsilon(u)\leq\mu(u)\leq\pi(u)$ for every $u\in E\otimes F$ (see [2] and [6] for details).

A uniform crossnorm is an assignment to each pair E and F of Banach spaces of a reasonable crossnorm on $E\otimes F$ which behaves well with respect to the formation of tensor product of operators, in the sense that if $S: E\to X$ and $T: F\to Y$ are bounded linear operators and the spaces $E\otimes F$ and $X\otimes Y$ are endowed with the assigned norms, then $S\otimes T: E\otimes F\to X\otimes Y$, defined by $S\otimes T(x\otimes y)=Sx\otimes Ty$, is bounded and $\|S\otimes T\|\leq\|S\|\,\|T\|$.

If for all normed spaces $\{E_i\}_{i=1}^n$ and all k with $1\leq k\leq n$, the norm μ satisfies

$$(E_1\otimes_\mu\ldots\otimes_\mu E_k)\otimes_\mu(E_{k+1}\otimes_\mu\ldots\otimes_\mu E_n)=E_1\otimes_\mu\ldots\otimes_\mu E_n$$

then we say that μ induces a tensor topology. The projective and the injective norms π and ε satisfy this property. Tensors topologies are defined in the wider context of locally convex spaces, as explained in [1].

We say that a tensor topology τ is symmetric if the mapping

$$x_1\otimes\ldots\otimes x_n \longmapsto (x_1\otimes\ldots\otimes x_n)^\sigma = x_{\sigma(1)}\otimes\ldots\otimes x_{\sigma(n)}$$

extended by linearity to the whole of $\bigotimes_\tau^n E$, is continuous for every σ in S_n, where S_n is the group of permutations of the set $\{1,2,\ldots,n\}$. In particular, if μ is a symmetric uniform crossnorm, i.e. $\mu(x_{\sigma(1)}\otimes\ldots\otimes x_{\sigma(n)})=\mu(x_1\otimes\ldots\otimes x_n)$ for every σ in S_n, then the topology induced by μ is a symmetric tensor topology.

A locally convex space E is stable if E is topologically isomorphic to its square $E\oplus E=E^2$. Díaz and Dineen [3], working with a stable locally convex space E, proved that there exists an isomorphism between the spaces of continuous n-linear forms $\mathcal{L}(^nE)$ and symmetric n-linear forms $\mathcal{L}_s(^nE)$ (see also [4]). Later, Ansemil and Floret [1] dealt with the predual problem. First they establish a general formula for the symmetric n-fold tensor product of a direct sum of spaces:

$$\bigotimes_s^n(F_1\oplus F_2)\simeq\bigoplus_{k=0}^n\Big(\bigotimes_s^k F_1\Big)\otimes\Big(\bigotimes_s^{n-k} F_2\Big)$$

and then they use it for stable spaces to show that for all tensor topologies, the full n-fold tensor product of a stable space E is isomorphic to its symmetric n-fold tensor product.

In this note we prove similar formulas for the alternating n-fold tensor product, analyse in detail the 3-fold tensor product and deduce a formula for the Jacobian tensor product. We apply these results to stable spaces, obtaining a number of isomorphisms. Finally we make a connection between these problems and irreducible group representations.

1. Alternating tensors

The antisymmetrisation operator $A\colon\bigotimes^n E\to\bigotimes^n E$ is defined as

$$A(x_1\otimes\ldots\otimes x_n)=x_1\otimes_a\ldots\otimes_a x_n=\frac{1}{n!}\sum_{\alpha\in S_n}\chi(\alpha)x_{\alpha(1)}\otimes\ldots\otimes x_{\alpha(n)}$$

for elementary tensors and extended by linearity to the whole of $\bigotimes^n E$, with $\chi(\alpha)$ denoting the sign of a permutation α. The range of A, that we denote by $\bigotimes_a^n E$, is called the alternating n-fold tensor product of E. We want to show that a similar result to that of Ansemil and Floret holds for the alternating tensor product, that is if the vector space E is the direct sum of two subspaces F_1 and F_2 then

$$\bigotimes_a^n E \simeq \bigoplus_{k=0}^{n} \Big(\bigotimes_a^k F_1\Big) \otimes \Big(\bigotimes_a^{n-k} F_2\Big).$$

The proof in [1], which uses the fact that the n-fold symmetric tensor product is the linear span of the vectors $\bigotimes^n x = \bigotimes_s^n x$, with x in E, is no longer valid, since $\bigotimes_a^n x = 0$ for all x.

Clearly, as k ranges from 0 to n, all of $(\bigotimes_a^k F_1) \otimes (\bigotimes_a^{n-k} F_2)$ are subspaces of $\bigotimes^n E$ and their sum is direct in $\bigotimes^n E$.

The elements of $(\bigotimes_a^k F_1) \otimes (\bigotimes_a^{n-k} F_2)$ are not alternating tensors, but can be antisymmetrised. Thus, if $x_1 \otimes_a \dots \otimes_a x_k$ and $x_{k+1} \otimes_a \dots \otimes_a x_n$ are elements of $\bigotimes_a^k F_1$ and $\bigotimes_a^{n-k} F_2$, respectively, then

$$\begin{aligned} &A((x_1 \otimes_a \dots \otimes_a x_k) \otimes (x_{k+1} \otimes_a \dots \otimes_a x_n)) \\ &\qquad = A\bigg(\frac{1}{k!(n-k)!} \sum_{\tau,\varrho} \chi(\tau)\chi(\varrho) x_{\tau(1)} \otimes \dots \otimes x_{\tau(k)} \otimes x_{\varrho(k+1)} \otimes \dots \otimes x_{\varrho(n)}\bigg), \end{aligned}$$

where τ ranges over S_k and ϱ over the set of all permutations of $\{k+1, \dots, n\}$. Let $(\tau\varrho)$ be the element of S_n defined by

$$(\tau\varrho)(i) = \begin{cases} \tau(i), & 1 \le i \le k, \\ \varrho(i), & k+1 \le i \le n. \end{cases}$$

It is easy to see that $\chi(\tau\varrho) = \chi(\tau)\chi(\varrho)$. Fix now τ and ϱ. As σ ranges over S_n, so does $\sigma(\tau\varrho)$. Therefore

$$\begin{aligned} &A((x_1 \otimes_a \dots \otimes_a x_k) \otimes (x_{k+1} \otimes_a \dots \otimes_a x_n)) \\ &\qquad = \frac{1}{n!} \frac{1}{k!(n-k)!} \sum_{\sigma \in S_n} \sum_{\tau,\varrho} \chi(\sigma)\chi(\tau\varrho) x_{\sigma(\tau\varrho)(1)} \otimes \dots \otimes x_{\sigma(\tau\varrho)(k)} \\ &\qquad\quad \otimes x_{\sigma(\tau\varrho)(k+1)} \otimes \dots \otimes x_{\sigma(\tau\varrho)(n)} \\ &\qquad = \frac{1}{k!(n-k)!} \sum_{\tau,\varrho} \frac{1}{n!} \sum_{\alpha \in S_n} \chi(\alpha) x_{\alpha(1)} \otimes \dots \otimes x_{\alpha(k)} \otimes x_{\alpha(k+1)} \otimes \dots \otimes x_{\alpha(n)} \\ &\qquad = \frac{1}{k!(n-k)!} \sum_{\tau,\varrho} x_1 \otimes_a \dots \otimes_a x_n \\ &\qquad = x_1 \otimes_a \dots \otimes_a x_n. \end{aligned}$$

Now let us try to "decompose" elementary tensors in $\bigotimes_a^n E$. We use the notation of [1]:

$$S_n^k = \{\eta \in S_n : \eta|_{\{1,\dots,k\}} \text{ and } \eta|_{\{k+1,\dots,n\}} \text{ are increasing}\}$$

and

$$\begin{aligned} T_n &= \{f : f\colon \{1,2,\dots,n\} \to \{1,2\}\}, \\ T_n^k &= \{f : f \in T_n \text{ and } \operatorname{card} f^{-1}(1) = k\}. \end{aligned}$$

If P_1 and P_2 are the projections of E onto F_1 and F_2 we can write

$$\begin{aligned} x_1 \otimes_a \dots \otimes_a x_n &= (P_1 x_1 + P_2 x_1) \otimes_a \dots \otimes_a (P_1 x_n + P_2 x_n) \\ &= \sum_{f \in T_n} P_{f(1)} x_1 \otimes_a \dots \otimes_a P_{f(n)} x_n = \sum_{k=0}^n \sum_{f \in T_n^k} P_{f(1)} x_1 \otimes_a \dots \otimes_a P_{f(n)} x_n. \end{aligned}$$

For f in T_n^k let $f^{-1}(1) = \{i_1, \dots, i_k\}$ with $i_1 \le i_2 \le \dots \le i_k$ and $f^{-1}(2) = \{i_{k+1}, \dots, i_n\}$ with $i_{k+1} \le \dots \le i_n$. Then the permutation η with $\eta(l) = i_l$ for $1 \le l \le n$ is an element of S_n^k. On the other hand, for every η in S_n^k, we can define a function $f\colon \{1,\dots,n\} \to \{1,2\}$ such that $f^{-1}(1) = \{\eta(1), \dots, \eta(k)\}$ and $f^{-1}(2) = \{\eta(k+1), \dots, \eta(n)\}$, so there is a one-to-one correspondence between S_n^k and T_n^k. Thus

$$\begin{aligned} x_1 \otimes_a \dots \otimes_a x_n &= \sum_{k=0}^n \sum_{\eta \in S_n^k} \chi(\eta) P_1 x_{\eta(1)} \otimes_a \dots \otimes_a P_1 x_{\eta(k)} \otimes_a P_2 x_{\eta(k+1)} \otimes_a \dots \otimes_a P_2 x_{\eta(n)} \\ &= \sum_{k=0}^n \sum_{\eta \in S_n^k} \chi(\eta) A\big((P_1 x_{\eta(1)} \otimes_a \dots \otimes_a P_1 x_{\eta(k)}) \\ &\quad \otimes (P_2 x_{\eta(k+1)} \otimes_a \dots \otimes_a P_2 x_{\eta(n)})\big) \\ &= A\Bigg(\sum_{k=0}^n \sum_{\eta \in S_n^k} \chi(\eta) \frac{1}{k!(n-k)!} \sum_{\tau,\varrho} \chi(\tau)\chi(\varrho) P_1 x_{\tau\eta(1)} \otimes \dots \otimes P_1 x_{\tau\eta(k)} \\ &\quad \otimes P_2 x_{\varrho\eta(k+1)} \otimes \dots \otimes P_2 x_{\varrho\eta(n)}\Bigg), \end{aligned}$$

where τ and ϱ range over the set of all permutations of $\{\eta(1), \dots, \eta(k)\}$ and of $\{\eta(k+1), \dots, \eta(n)\}$, respectively, for every fixed η in S_n^k. Certainly $(\tau\varrho)\eta$ is an element of S_n and

$$\chi((\tau\varrho)\eta) = \chi(\tau)\chi(\varrho)\chi(\eta).$$

Conversely, if σ is an element of S_n, let η be the permutation in S_n^k such that $\{\eta(1),\dots,\eta(k)\}=\{\sigma(1),\dots,\sigma(k)\}$ and $\{\eta(k+1),\dots,\eta(n)\}=\{\sigma(k+1),\dots,\sigma(n)\}$. If we put $\tau(\eta(l))=\sigma(l)$ for $1\le l\le k$ and $\varrho(\eta(j))=\sigma(j)$ for $k+1\le j\le n$ then τ and ϱ are permutations of $\{\eta(1),\dots,\eta(k)\}$ and $\{\eta(k+1),\dots,\eta(n)\}$, respectively, and $\sigma=(\tau\varrho)\eta$. We can then write

$$\begin{aligned} x_1\otimes_a\dots\otimes_a x_n &= A\Big(\sum_{k=0}^{n}\binom{n}{k}\frac{1}{n!}\sum_{\sigma\in S_n}\chi(\sigma)P_1x_{\sigma(1)}\otimes\dots\otimes P_1x_{\sigma(k)} \\ &\qquad \otimes P_2x_{\sigma(k+1)}\otimes\dots\otimes P_2x_{\sigma(n)}\Big) \\ &= A\Big(\sum_{k=0}^{n}\binom{n}{k}\Big(\bigotimes^k P_1\Big)\otimes\Big(\bigotimes^{n-k}P_2\Big)(x_1\otimes_a\dots\otimes_a x_n)\Big). \end{aligned}$$

Now define $Q_k\colon \bigotimes^n E\to(\bigotimes^k F_1)\otimes(\bigotimes^{n-k}F_2)$ by $Q_k=\binom{n}{k}(\bigotimes^k P_1)\otimes(\bigotimes^{n-k}P_2)$. As is easily seen from the calculations above

$$\begin{aligned} Q_k(x_1\otimes_a\dots\otimes_a x_n) = \sum_{\eta\in S_n^k}&\chi(\eta)(P_1x_{\eta(1)}\otimes_a\dots\otimes_a P_1x_{\eta(k)}) \\ &\otimes(P_2x_{\eta(k+1)}\otimes_a\dots\otimes_a P_2x_{\eta(n)}), \end{aligned}$$

therefore Q_k maps $\bigotimes_a^n E$ into $(\bigotimes_a^k F_1)\otimes(\bigotimes_a^{n-k}F_2)$. We are now ready to prove the announced result.

Theorem 1. *Let E be a vector space such that $E=F_1\oplus F_2$. Then the linear mapping*

$$Q\colon \bigotimes_a^n E\longrightarrow\bigoplus_{k=0}^{n}\Big(\bigotimes_a^k F_1\Big)\otimes\Big(\bigotimes_a^{n-k}F_2\Big)$$

defined by $Q(u)=\bigoplus_{k=0}^n Q_k(u)$ for all u in $\bigotimes_a^n E$ is an isomorphism, its inverse being the restriction of the antisymmetrisation operator A to $\bigoplus_{k=0}^n(\bigotimes_a^k F_1)\otimes(\bigotimes_a^{n-k}F_2)$.

Proof. Since for elementary tensors we have

$$x_1\otimes_a\dots\otimes_a x_n = A\Big(\sum_{k=0}^{n}Q_k(x_1\otimes_a\dots\otimes_a x_n)\Big),$$

we obtain

$$u=A(Q(u))$$

for every u in $\bigotimes_a^n E$.

To prove that QA is the identity on $\bigoplus_{k=0}^{n}(\bigotimes_a^k F_1)\otimes(\bigotimes_a^{n-k} F_2)$, let w_k be elements of $(\bigotimes_a^k F_1)\otimes(\bigotimes_a^{n-k} F_2)$. Because of the linearity, it is enough to work with elementary tensors w_k. Fix a value between 0 and n and call it l. Let $w_l = (x_1\otimes_a\dots\otimes_a x_l)\otimes(x_{l+1}\otimes_a\dots\otimes_a x_n)$. Then

$$\begin{aligned}
Q(A(w_l)) &= \sum_{k=0}^{n} Q_k(A(w_l))\\
&= \sum_{k=0}^{n}\binom{n}{k}\Big(\bigotimes^k P_1\Big)\otimes\Big(\bigotimes^{n-k} P_2\Big)(x_1\otimes_a\dots\otimes_a x_l\otimes_a x_{l+1}\otimes_a\dots\otimes_a x_n)\\
&= \sum_{k=0}^{n}\binom{n}{k}\frac{1}{n!}\sum_{\sigma\in S_n}\chi(\sigma)P_1x_{\sigma(1)}\otimes\dots\otimes P_1x_{\sigma(k)}\otimes P_2x_{\sigma(k+1)}\otimes\dots\otimes P_2x_{\sigma(n)}.
\end{aligned}$$

If $k<l$ then $\{\sigma(k+1),\dots,\sigma(n)\}$ contains at least one natural number j with $1\le j\le l$ and thus $P_1x_{\sigma(1)}\otimes\dots\otimes P_1x_{\sigma(k)}\otimes P_2x_{\sigma(k+1)}\otimes\dots\otimes P_2x_{\sigma(n)}=0$ since $P_2x_j=0$. In the same way, all the terms corresponding to $k>l$ will vanish and so

$$Q(A(w_l)) = \binom{n}{l}\frac{1}{n!}\sum_{\sigma\in S_n}\chi(\sigma)P_1x_{\sigma(1)}\otimes\dots\otimes P_1x_{\sigma(l)}\otimes P_2x_{\sigma(l+1)}\otimes\dots\otimes P_2x_{\sigma(n)}.$$

It is clear now that the terms of the sum will be 0 unless $\{\sigma(1),\dots,\sigma(l)\}=\{1,\dots,l\}$ and $\{\sigma(l+1),\dots,\sigma(n)\}=\{l+1,\dots,n\}$, in which case $\sigma=(\tau\varrho)$ with τ and ϱ permutations of $\{1,\dots,l\}$ and $\{l+1,\dots,n\}$ respectively. Thus

$$\begin{aligned}
Q(A(w_l)) &= \binom{n}{l}\frac{1}{n!}\sum_{\tau,\varrho}\chi(\tau)\chi(\varrho)x_{\tau(1)}\otimes\dots\otimes x_{\tau(l)}\otimes x_{\varrho(l+1)}\otimes\dots\otimes x_{\varrho(n)}\\
&= \binom{n}{l}\frac{1}{n!}\Big(\sum_{\tau}\chi(\tau)x_{\tau(1)}\otimes\dots\otimes x_{\tau(l)}\Big)\otimes\Big(\sum_{\varrho}\chi(\varrho)x_{\varrho(l+1)}\otimes\dots\otimes x_{\varrho(n)}\Big)\\
&= \binom{n}{l}\frac{1}{n!}l!(n-l)!(x_1\otimes_a\dots\otimes_a x_l)\otimes(x_{l+1}\otimes_a\dots\otimes_a x_n)\\
&= w_l
\end{aligned}$$

and so, by linearity

$$Q\Big(A\Big(\sum_{l=0}^{n} w_l\Big)\Big) = \sum_{l=0}^{n} w_l$$

for all w_l in $(\bigotimes_a^l F_1)\otimes(\bigotimes_a^{n-l} F_2)$ and all $0\le l\le n$. Therefore Q is an isomorphism. $\square$

Remark. By induction the result can be extended for a finite direct sum of subspaces:

$$\bigotimes_a^n \Big(\bigoplus_{j=1}^m F_j\Big) = \bigoplus_{\substack{k_1+\ldots+k_m=n\\ 0\le k_j\le n}} \bigotimes_{j=1}^m \Big(\bigotimes_a^{k_j} F_j\Big).$$

2. Jacobian tensors

The space $E\otimes E$ is the direct sum of $E\otimes_s E$ and $E\otimes_a E$, an element $x\otimes y$ of $E\otimes E$ being expressed uniquely as a sum of elements of $E\otimes_s E$ and $E\otimes_a E$:

$$x\otimes y = x\otimes_s y + x\otimes_a y.$$

When the order of the tensor product increases, more components of an elementary tensor will come into play. When $n=3$ we have to deal with a third component, $J(x\otimes y\otimes z)$, such that

$$x\otimes y\otimes z = x\otimes_s y\otimes_s z + x\otimes_a y\otimes_a z + J(x\otimes y\otimes z).$$

Thus

$$J(x\otimes y\otimes z) = \tfrac{1}{3}(2x\otimes y\otimes z - y\otimes z\otimes x - z\otimes x\otimes y)$$

and it is easy to see that J is a projection which satisfies the identity

$$J(x\otimes y\otimes z) + J(y\otimes z\otimes x) + J(z\otimes x\otimes y) = 0$$

for which reason we call the element $J(x\otimes y\otimes z)$ the Jacobian component of $x\otimes y\otimes z$ and the space $J(\bigotimes^3 E)$ the subspace of Jacobian tensors. We will occasionally denote $J(x\otimes y\otimes z)$ by $x\otimes_J y\otimes_J z$ and $J(\bigotimes^3 E)$ by $\bigotimes_J^3 E$. We can then write

$$\bigotimes^3 E = \Big(\bigotimes_s^3 E\Big) \oplus \Big(\bigotimes_a^3 E\Big) \oplus \Big(\bigotimes_J^3 E\Big).$$

Our goal is to find a formula for $\bigotimes_J^3 (F_1\oplus F_2)$. Unlike the case for the symmetric and antisymmetric tensors, if σ is a permutation of $\{x,y,z\}$ there is no way of expressing the relation between $x\otimes_J y\otimes_J z$ and $\sigma(x)\otimes_J \sigma(y)\otimes_J \sigma(z)$ without using other terms involving at least one more permutation. It is then expected that we will encounter some difficulties and that the formula will not be as “uniform” as for the symmetric or alternating case.

Let us write $x_1 = P_1 x$ and $x_2 = P_2 x$ for an element x in E. We have

$$\begin{aligned} x \otimes_J y \otimes_J z = {} & x_1 \otimes_J y_1 \otimes_J z_1 \\ & + x_1 \otimes_J y_1 \otimes_J z_2 + x_1 \otimes_J y_2 \otimes_J z_1 + x_2 \otimes_J y_1 \otimes_J z_1 \\ & + x_1 \otimes_J y_2 \otimes_J z_2 + x_2 \otimes_J y_1 \otimes_J z_2 + x_2 \otimes_J y_2 \otimes_J z_1 \\ & + x_2 \otimes_J y_2 \otimes_J z_2. \end{aligned}$$

Now we notice that each of the terms on the second line in the formula above is the Jacobian tensor product of two elements of F_1 and one element of F_2 and, using the Jacobian formula, their sum can be expressed as

$$J(x_1 \otimes y_1 \otimes z_2 - z_1 \otimes x_1 \otimes y_2 + x_2 \otimes y_1 \otimes z_1 - y_2 \otimes z_1 \otimes x_1).$$

Consider the projection $Q_1^a \colon \bigotimes^3 E \to (F_1 \otimes F_1) \otimes F_2$ defined by

$$Q_1^a(x \otimes y \otimes z) = x_1 \otimes y_1 \otimes z_2 - z_1 \otimes x_1 \otimes y_2.$$

It is easy to see that

$$Q_1^a(x \otimes_J y \otimes_J z) = x_1 \otimes y_1 \otimes z_2 - z_1 \otimes x_1 \otimes y_2.$$

The same thing is true for $Q_1^b \colon \bigotimes^3 E \to (F_1 \otimes F_1) \otimes F_2$ defined by

$$Q_1^b(x \otimes y \otimes z) = y_1 \otimes z_1 \otimes x_2 - z_1 \otimes x_1 \otimes y_2$$

and thus

$$\begin{aligned} & J(x_1 \otimes y_1 \otimes z_2 - z_1 \otimes x_1 \otimes y_2 + x_2 \otimes y_1 \otimes z_1 - y_2 \otimes z_1 \otimes x_1) \\ & \qquad = J(Q_1^a(x \otimes_J y \otimes_J z) + [Q_1^b(x \otimes_J y \otimes_J z)]^{(3,1,2)}), \end{aligned}$$

where $(x \otimes y \otimes z)^{(3,1,2)} = z \otimes x \otimes y$.

In the same way, working with the projections Q_2^a and Q_2^b of $\bigotimes^3 E$ onto $F_1 \otimes (F_2 \otimes F_2)$, defined by

$$\begin{aligned} Q_2^a(x \otimes_J y \otimes_J z) &= x_1 \otimes y_2 \otimes z_2 - y_1 \otimes z_2 \otimes x_2, \\ Q_2^b(x \otimes_J y \otimes_J z) &= z_1 \otimes x_2 \otimes y_2 - y_1 \otimes z_2 \otimes x_2, \end{aligned}$$

we obtain

$$\begin{aligned} & x_1 \otimes_J y_2 \otimes_J z_2 + x_2 \otimes_J y_1 \otimes_J z_2 + x_2 \otimes_J y_2 \otimes_J z_1 \\ & \qquad = J(x_1 \otimes y_2 \otimes z_2 - y_1 \otimes z_2 \otimes x_2 + x_2 \otimes y_2 \otimes z_1 - z_2 \otimes x_2 \otimes y_1) \\ & \qquad = J(Q_2^a(x \otimes_J y \otimes_J z) + [Q_2^b(x \otimes_J y \otimes_J z)]^{(2,3,1)}) \end{aligned}$$

with $(x\otimes y\otimes z)^{(2,3,1)}=y\otimes z\otimes x$.

Defining $Q_0(x\otimes y\otimes z)=x_1\otimes y_1\otimes z_1$ and $Q_3(x\otimes y\otimes z)=x_2\otimes y_2\otimes z_2$ and noticing that

$$Q_0(x\otimes_J y\otimes_J z)=x_1\otimes_J y_1\otimes_J z_1,$$
$$Q_3(x\otimes_J y\otimes_J z)=x_2\otimes_J y_2\otimes_J z_2,$$

we have

$$\begin{aligned} x\otimes_J y\otimes_J z = {} & Q_0(x\otimes_J y\otimes_J z)\\ & +J(Q_1^a(x\otimes_J y\otimes_J z)+[Q_1^b(x\otimes_J y\otimes_J z)]^{(3,1,2)})\\ & +J(Q_2^a(x\otimes_J y\otimes_J z)+[Q_2^b(x\otimes_J y\otimes_J z)]^{(2,3,1)})\\ & +Q_3(x\otimes_J y\otimes_J z). \end{aligned}$$

Let $Q_1=Q_1^a+(Q_1^b)^{(3,1,2)}$ and $Q_2=Q_2^a+(Q_2^b)^{(2,3,1)}$.

Theorem 2. *Let E be a vector space such that $E=F_1\oplus F_2$. Then the linear mapping*

$$Q\colon \bigotimes\nolimits_J^3 E \longrightarrow \Big(\bigotimes\nolimits_J^3 F_1\Big)\oplus((F_1\otimes F_1)\otimes F_2)^2\oplus(F_1\otimes(F_2\otimes F_2))^2\oplus\Big(\bigotimes\nolimits_J^3 F_2\Big),$$

defined by $Q(u)=\bigoplus_{k=0}^3 Q_k(u)$ for all u in $\bigotimes_J^3 E$, is an isomorphism, its inverse being the restriction of the projection J to

$$\Big(\bigotimes\nolimits_J^3 F_1\Big)\oplus((F_1\otimes F_1)\otimes F_2)^2\oplus(F_1\otimes(F_2\otimes F_2))^2\oplus\Big(\bigotimes\nolimits_J^3 F_2\Big).$$

Proof. Since we have $JQ_0(x\otimes_J y\otimes_J z)=Q_0(x\otimes_J y\otimes_J z)$ and $JQ_3(x\otimes_J y\otimes_J z)=Q_3(x\otimes_J y\otimes_J z)$, it follows that

$$JQ(u)=u$$

for all u in $\bigotimes_J^3 E$.

It remains to show that QJ is the identity on

$$\Big(\bigotimes\nolimits_J^3 F_1\Big)\oplus((F_1\otimes F_1)\otimes F_2)^2\oplus(F_1\otimes(F_2\otimes F_2))^2\oplus\Big(\bigotimes\nolimits_J^3 F_2\Big).$$

Because of the linearity, it is enough to work with elementary tensors.

Pick an element $w_0=x\otimes_J y\otimes_J z$ in $\bigotimes_J^3 F_1$. Clearly $Jw_0=w_0$ and $Q_0(Jw_0)=w_0$. Since the projection of E on F_2 appears in the formulas for Q_1, Q_2 and Q_3, we obtain $Q_1(Jw_0)=Q_2(Jw_0)=Q_3(Jw_0)=0$ which means that $Q(Jw_0)=w_0$.

Take now $w_1=x\otimes y\otimes z$ in $(F_1\otimes F_1)\otimes F_2$. Clearly $Q_0(Jw_1)=0$ since $z_1=0$. The expressions of Q_2 and Q_3 contain at least two projections on F_2 and so $Q_2(Jw_1)=Q_3(Jw_1)=0$. Then

$$\begin{aligned}Q_1(J(x\otimes y\otimes z)) &= Q_1^a(x\otimes_J y\otimes_J z)+[Q_1^b(x\otimes_J y\otimes_J z)]^{(3,1,2)}\\ &= x_1\otimes y_1\otimes z_2-z_1\otimes x_1\otimes y_2+x_2\otimes y_1\otimes z_1-y_2\otimes z_1\otimes x_1=x\otimes y\otimes z=w_1\end{aligned}$$

and so $Q(J(w_1))=w_1$.

In the same way it can be shown that $Q(J(w_2))=w_2$ and $Q(J(w_3))=w_3$ for all elements w_2 in $F_1\otimes(F_2\otimes F_2)$ and w_3 in $\bigotimes_J^3 F_2$. Therefore QJ is the identity. $\square$

3. Topological results

Both of the theorems proved so far state the existence of algebraic isomorphisms. When working with a topological structure on E we would like these isomorphisms to be topological as well.

Suppose E is a normed space. Let us analyse first the antisymmetric tensors. The isomorphism Q is the sum of Q_k's, where

$$Q_k(x_1\otimes_a\ldots\otimes_a x_n)=\binom{n}{k}\Big(\Big(\bigotimes^k P_1\Big)\otimes\Big(\bigotimes^{n-k} P_2\Big)\circ A\Big)(x_1\otimes\ldots\otimes x_n).$$

In order for the antisymmetrisation operator A to be continuous we need to work with a norm μ on $\bigotimes^n E$ for which the mapping $u\mapsto u^\sigma$ is continuous for every σ in S_n, a condition satisfied by symmetric tensor norms. We also need the continuity of $\bigotimes^k P_1$ and $\bigotimes^{n-k} P_2$, so we would like tensor products of continuous operators to remain continuous. Thus μ should be a uniform cross-norm. Finally, we want to be able to associate the factors of an n-fold tensor product in any way we want, therefore we require that the norm μ induces a tensor topology. It is clear now that once these conditions are satisfied, all of the spaces $(\bigotimes_{\mu,a}^k F_1)\otimes_\mu(\bigotimes_{\mu,a}^{n-k} F_2)$ are continuously embedded into $\otimes_\mu^n E$ and so the inverse of Q, which is the restriction of A to the direct sum of these spaces, is also continuous. The same remarks remain valid when working with locally convex spaces.

Theorem 3. *Let E be a locally convex space such that $E=F_1\oplus F_2$. Then*

$$\bigotimes_{\tau,a}^n E\cong\bigoplus_{k=0}^n\Big(\bigotimes_{\tau,a}^k F_1\Big)\otimes_\tau\Big(\bigotimes_{\tau,a}^{n-k} F_2\Big)$$

for every symmetric tensor topology τ.

Let us apply now this result to stable spaces. Díaz and Dineen [3] showed that the spaces of continuous n-linear forms and symmetric continuous n-linear forms on a stable space E are isomorphic. Using their ideas, Ansemil and Floret [1] extended this result to the predual of those two spaces, respectively the n-fold and the symmetric n-fold tensor products endowed with the projective topology. They also showed that the isomorphism holds for all the symmetric tensor topologies. The proof of the next corollary follows that in [1], but we give it for the sake of completeness.

Corollary 1. *Let E be a stable locally convex space. Then, for all symmetric tensor topologies τ and all positive integers n, the spaces $\bigotimes_{\tau,a}^n E$ and $\bigotimes_{\tau}^n E$ are isomorphic.*

Proof. We are going to prove it by induction. The result is clear for $n=1$ since both spaces are, in this case, equal to E. Let us write $E=F_1\oplus F_2$ with both F_1 and F_2 isomorphic to E and denote $\bigotimes_{\tau,a}^k E$ by G_k and $\bigotimes_{\tau}^k E$ by H_k. Assume $G_k\cong H_k$ for all $k<n$.

Since we are working with a tensor topology,

$$\begin{aligned} H_k &= \Big(\bigotimes_{\tau}^{k-1} E\Big)\otimes_\tau (F_1\oplus F_2) = \Big(\Big(\bigotimes_{\tau}^{k-1} E\Big)\otimes_\tau F_1\Big)\oplus\Big(\Big(\bigotimes_{\tau}^{k-1} E\Big)\otimes_\tau F_2\Big) \\ &\cong \Big(\Big(\bigotimes_{\tau}^{k-1} E\Big)\otimes_\tau E\Big)^2 = H_k^2. \end{aligned}$$

By the previous theorem,

$$\begin{aligned} G_n &= \Big(\bigotimes_{\tau,a}^{n} F_1\Big)\oplus\bigoplus_{k=1}^{n-1}\Big(\Big(\bigotimes_{\tau,a}^{k} F_1\Big)\otimes_\tau\Big(\bigotimes_{\tau,a}^{n-k} F_2\Big)\Big)\oplus\Big(\bigotimes_{\tau,a}^{n} F_2\Big) \\ &\cong G_n^2\oplus\left(\bigoplus_{k=1}^{n-1} H_k\otimes_\tau H_{n-k}\right) = G_n^2\oplus H_n^{n-1}\cong G_n^2\oplus H_n. \end{aligned}$$

Let V be the topological complement of G_n in H_n. Then

$$H_n = G_n\oplus V\cong G_n^2\oplus H_n\oplus V = G_n\oplus H_n^2\cong G_n^2\oplus H_n^3\cong G_n^2\oplus H_n\cong G_n. \quad \square$$

Let us denote by $\mathcal{L}_a({}^nE)$ the space of alternating n-linear forms, namely those forms B that satisfy

$$B(x_{\sigma(1)},\dots,x_{\sigma(n)}) = \chi(\sigma)B(x_1,\dots,x_n)$$

for all σ in S_n, and endow it with the topology of uniform convergence on bounded subsets of E. Let us note that for a normed space E we deal with the usual "sup" norm in both $\mathcal{L}_a({}^nE)$ and $\mathcal{L}({}^nE)$. When working with the projective topology π, the dual of $\bigotimes_{\pi,a}^n E$ is $\mathcal{L}_a({}^nE)$ and by duality we have the following result.

Corollary 2. *For a stable locally convex space E and all positive integers n, the spaces $\mathcal{L}_a({}^nE)$ and $\mathcal{L}({}^nE)$ are isomorphic.*

Moving on to Jacobian tensors and working with normed or locally convex spaces, it is easy to see, by analysing the expression of the algebraic isomorphism in Theorem 2 that, in order to obtain a topological isomorphism, the topology we work with must meet the same requirements as in the case of the alternating tensors that we have just dealt with. In other words, we need to work with a symmetric tensor topology, in which case the following statement holds.

Theorem 4. *Let E be a locally convex space such that $E = F_1 \oplus F_2$. Then*

$$\bigotimes_{\tau,J}^{3} E \cong \Big(\bigotimes_{\tau,J}^{3} F_1\Big) \oplus ((F_1 \otimes_\tau F_1) \otimes_\tau F_2)^2 \oplus (F_1 \otimes_\tau (F_2 \otimes_\tau F_2))^2 \oplus \Big(\bigotimes_{\tau,J}^{3} F_2\Big)$$

for every symmetric tensor topology τ.

If E is a stable locally convex space then so is $\bigotimes_\tau^3 E$, where τ is a tensor topology. Writing $E = F_1 \oplus F_2$ with F_1 and F_2 isomorphic to E, from the preceding corollary we obtain

$$\bigotimes_{\tau,J}^{3} E \cong \Big(\bigotimes_{\tau,J}^{3} E\Big) \oplus \Big(\bigotimes_{\tau}^{3} E\Big)^4 \oplus \Big(\bigotimes_{\tau,J}^{3} E\Big) \cong \Big(\bigotimes_{\tau,J}^{3} E\Big)^2 \oplus \Big(\bigotimes_{\tau}^{3} E\Big)$$

and so the preceding result and the proof of Corollary 1 give the following corollary.

Corollary 3. *Let E be a stable locally convex space. Then, for all symmetric tensor topologies τ, the spaces $\bigotimes_{\tau,J}^3 E$ and $\bigotimes_\tau^3 E$ are isomorphic.*

The dual result will involve the space $\mathcal{L}_J(^3E)$ of Jacobian 3-linear forms, namely those which satisfy

$$B(x,y,z) + B(y,z,x) + B(z,x,y) = 0$$

for all x, y and z in E, endowed with the topology of uniform convergence on bounded subsets of E. Since $\mathcal{L}_J(^3E)$ is the dual of $\bigotimes_{\pi,J}^3 E$, by duality we obtain the next result.

Corollary 4. *For a stable locally convex space E the spaces $\mathcal{L}_J(^3E)$ and $\mathcal{L}(^3E)$ are isomorphic.*

4. Further ideas

Let E be a vector space. As has been mentioned before

$$E \otimes E = (E \otimes_s E) \oplus (E \otimes_a E).$$

Having a formula for both symmetric and alternating tensor products of direct sums, it has been proved that when E is a stable locally convex space and we work with a symmetric tensor topology, both $E\otimes_s E$ and $E\otimes_a E$ are isomorphic to $E\otimes E$.

The same thing remains true for the 3-fold tensor product of a locally convex space E, again, the result coming from the fact that

$$\bigotimes^3 E=\Big(\bigotimes_s^3 E\Big)\oplus\Big(\bigotimes_a^3 E\Big)\oplus\Big(\bigotimes_J^3 E\Big)$$

and that we have a formula for symmetric, antisymmetric and Jacobian tensor products of direct sums.

It is interesting now to notice a connection between these facts and the irreducible representations of the symmetric groups of different orders (see [5] for details on representations). When $n=2$, the symmetric group S_2 has just two such representations, the trivial one, corresponding to symmetric tensors and the alternating one (corresponding to alternating tensors). Moving on to $n=3$, a new irreducible representation comes into play, the standard one, corresponding to the Jacobian tensors. Analysing the character table of S_3, where, on the top row, 1 represents the identity permutation, (12) the transpositions and (123) the 2-cycles,

S_3	1	(12)	(123)
trivial	1	1	1
alternating	1	-1	1
standard	2	0	-1

we see the way the correspondence is given; for instance $x\otimes_J y\otimes_J z$ will contain twice the identity $x\otimes y\otimes z$, none of the transpositions $y\otimes x\otimes z$, $x\otimes z\otimes y$ or $z\otimes y\otimes x$ and the negatives of the 2-cycles $y\otimes z\otimes x$ and $z\otimes x\otimes y$.

Now, the greater the value of n, the more irreducible representations (in fact the number of partitions of n) S_n will have (5 for $n=4$, 7 for $n=5$, 11 for $n=6$, etc.) and to each such representation a subspace of $\bigotimes^n E$ will be associated. Since there exist formulas for symmetric and alternating n-tensors of direct sums for any degree n, it would be interesting to investigate whether such formulas can be found for other types of tensors corresponding to other representations than the trivial and the alternating one, as is the case with the standard representation when $n=3$. Could these formulas be so "uniform" so that for a stable space E, each of these subspaces of $\bigotimes^n E$ are isomorphic to $\bigotimes^n E$ itself, as is the case for $n=2$ and $n=3$?

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