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MEASURE-PRESERVING DIFFEOMORPHISMS WITH AN ARBITRARY SPECTRAL MULTIPLICITY

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Dedicated to the memory of Karol Borsuk

1. Introduction

One of the most important problems (still open) in ergodic theory is to decide whether or not any ergodic finite entropy transformation has a smooth model (e.g. [24], p. 186, [4], Problem 2.16, p. 56; see also [17]). In this paper we continue a program initiated in [14] and [15] to present smooth versions of some complicated measure-theoretic constructions in ergodic theory. This time we focus on the construction of analytic diffeomorphisms whose maximal spectral multiplicity is equal to a given natural number n.

Assume that $\tau : (Y, \mathcal{C}, \nu) \to (Y, \mathcal{C}, \nu)$ is an ergodic measure-preserving automorphism of a standard Borel space. It induces a unitary operator $U_\tau : L^2(Y, \nu) \to L^2(Y, \nu)$, where $U_\tau f = f \circ \tau$. The spectral properties of U_τ are called the *spectral properties of* τ (see Appendix in [23]). Denote $Z(f) = \text{span}\{f\tau^k : k \in \mathbf{Z}\}$ for $f \in L^2(Y, \nu)$. We say that a number $m \in \mathbf{N} \cup \{\infty\}$ belongs to the set of *essential values* $E(\tau)$ if there exist $f_1, \ldots, f_m \in L^2(Y, v)$ with $Z(f_i) \perp Z(f_j)$ for

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$i \neq j, i, j = 1, ..., m$ and such that the measures σ_{f_i} (σ_{f_i} is called the *spectral measure* of f_i), $i = 1, ..., m$ on $\mathbf{T}$ determined by

$$\widehat{\sigma}_{f_i}(p) = \int_T z^p d\sigma_{f_i}(z) = (U_\tau^p f_i, f_i), \quad p \in \mathbf{Z}$$

are all equivalent and moreover there is no $f \in L^2(Y, \nu)$ whose spectral measure is equivalent to σ_1 and $Z(f) \perp Z(f_i), i = 1, ..., m$. The greatest element in $E(\tau)$ is then called the *maximal spectral multiplicity (msm)* of τ.

Constructions of τ with the msm equal to n are already known and presented in [7], [25], [26], [27]. The history of the spectral multiplicity problem in ergodic theory till 1983 is described in [25] (see also [1], [5], [6], [10], [13], [16], [18], [19], [21]). However, none of the constructions of an ergodic automorphism with a given msm is smooth, that is a construction in which τ is a diffeomorphism on a finite dimensional compact manifold and μ is a smooth measure. Such examples have necessarily zero entropy and we recall that the horocycle flow on a two-dimensional orientable surface of constant negative curvature which has zero entropy has infinite msm ([22]). In the present paper we construct analytic diffeomorphisms on finite dimensional tori, ergodic with respect to Lebesgue measure, having given msm. For $n = 1$ a simple example is delivered by any irrational rotation; if we require τ not to have pure discrete spectrum an appropriate example is constructed in [15]. In [2] certain C^∞-constructions of τ with continuous spectrum and msm equal to 1 are delivered. Now, fix $n \geq 2$. Our examples will be of the following form:

$$\tau = T_\varphi, \qquad T_\varphi : \mathbf{T}^{n+1} \to \mathbf{T}^{n+1},$$

where $T : \mathbf{T} \to \mathbf{T}$, $Tz = z \cdot e^{2\pi i\alpha}$, with α irrational, $\varphi : \mathbf{T} \to \mathbf{T}^n$ is an analytic cocycle and

$$T_\varphi(z, (w_1, ..., w_n)) = (Tz, \varphi(z) \cdot (w_1, ..., w_n)).$$

Consequently, $\varphi = (\varphi_1, ..., \varphi_n)$, where $\varphi_i : \mathbf{T} \to \mathbf{T}$ is analytic and $T_{\varphi_i} : \mathbf{T}^2 \to \mathbf{T}^2$ is a measure-theoretic factor of T_φ. By a result of [12], if there exists $1 \leq i \leq n$ such that the topological degree of φ_i is not zero then T_{φ_i} and hence T_φ has infinite msm. Therefore, in all our constructions, for each $j = 1, \ldots, n$

$$\varphi_j(e^{2\pi ix}) = e^{2\pi i\widetilde{\varphi}_j(x)},$$

where $\widetilde{\varphi}_j : \mathbf{R} \to \mathbf{R}$ is a 1-periodic real analytic map and also $\int_0^1 \widetilde{\varphi}_j(x)dx = 0$. Now, by a result of M. Herman [11], p. 189, for each $j = 1, ..., n$

$$\varphi_j^{(q_r)}(e^{2\pi ix}) = e^{2\pi i(\widetilde{\varphi}_j(x)+\widetilde{\varphi}_j(x+\alpha)+\cdots+\widetilde{\varphi}_j(x+(q_r-1)\alpha))}$$

tends uniformly to one when r goes to infinity ($\{q_r\}$ is the sequence of denominators of α). Hence, for each $F \in L^2(\mathbf{T}^{n+1})$ we have

$$F \circ (T_\varphi)^{q_r} \to F \quad \text{in} \quad L^2(\mathbf{T}^{n+1})$$

and consequently the maximal spectral type of U_{T_φ} is a Dirichlet measure, in particular it is singular. Therefore, all diffeomorphisms under consideration will have singular spectrum. The question of whether it is possible to construct an analytic diffeomorphism on a finite dimensional torus with finite msm and nonsingular spectrum remains open.

In [26], E.A. Robinson has shown that for a given subset $E \subset \mathbf{N}$ satisfying

(i) $1 \in E$,

(ii) if $m, n \in E$ then $\mathrm{lcm}(n, m) \in E$,

there exists an ergodic τ such that $E(\tau) = E$. In the last section we explain how the same result can be achieved with τ an analytic diffeomorphism on a finite dimensional torus, ergodic with respect to Lebesgue measure. We recall that it is still an open problem whether a finite subset of natural numbers can be realized as the set of essential values of an ergodic automorphism (see also [13]).

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2. Definitions and notation

Let $(Y, \mathcal{C}, \nu)$ be a standard Borel probability space with a normalized measure ν. Assume that $\tau : (Y, \mathcal{C}, \nu) \to (Y, \mathcal{C}, \nu)$ is an automorphism. Let $C(\tau)$ denote the *centralizer* of τ, i.e. the set of all not necessarily invertible measure-preserving transformations commuting with τ. An automorphism τ is said to be *rigid* if there exists an increasing sequence $\{n_t\}$ such that

$$(1) \qquad (\forall A \in \mathcal{C}) \quad \nu(\tau^{n_t} A \triangle A) \to 0.$$

Any sequence $\{n_t\}$ for which (1) holds will be called a *rigidity time* for τ. Equivalently, $\{n_t\}$ is a rigidity time for τ if and only if

$$(\forall f \in L^2(Y, \nu)) \quad (U_\tau)^{n_t} f \to f \quad \text{in} \quad L^2(Y, \nu).$$

The automorphisms τ considered in the paper will be generally a special kind of skew products. Assume that $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is an ergodic automorphism of a standard Borel space. Let G be a compact abelian metric group with Haar measure m. A measurable function $\varphi : X \to G$ will be called a *cocycle*. For $n > 0$ we put

$$\varphi^{(n)}(x) = \varphi(x)\varphi(Tx) \ldots \varphi(T^{n-1}x).$$

A cocycle φ determines an automorphism T_{φ} (called a *G-extension of* T) on $(X \times G, \widetilde{\mathcal{B}}, \widetilde{\mu})$ by

$$T_{\varphi}(x,g) = (Tx, g \cdot \varphi(x)), \tag{2}$$

where $\widetilde{\mathcal{B}}$ is the product σ -algebra and $\widetilde{\mu} = \mu \times m$. A cocycle φ is said to be a *coboundary* (or a *G-coboundary*) if it is of the form

$$\varphi(x) = f(Tx)/f(x)$$

for a measurable function $f : X \to G$. We say that two cocycles $\varphi, \psi : X \to G$ are *cohomologous* if φ/ψ is a coboundary. We will say that φ is *ergodic* if the automorphism T_{φ} is ergodic. The following is classical [3].

$$\begin{cases} \varphi \text{ is ergodic if and only if for no character } \chi \in \widehat{G},\ \chi \neq 1, \\ \text{the cocycle } \chi \circ \varphi \text{ is a } \mathbf{T}-\text{coboundary.} \end{cases} \tag{3}$$

Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be an ergodic rotation on a compact monothetic group X with Haar measure μ (i.e. T is assumed to have discrete spectrum). Suppose that $\varphi : X \to G$ is an ergodic cocycle.

PROPOSITION 1. (cf. [20]) *Every element $\widehat{S}$ of the centralizer of T_{φ} is of the form*

$$\widehat{S}(x,g) = S_{f,v}(x,g) = (Sx, f(x)v(g)),$$

where $S \in C(T)$, $f : X \to G$ is measurable and $v : G \to G$ is a continuous group automorphism; equivalently f, v satisfy the following functional equation

$$\varphi \circ S / v(\varphi) = f \circ T / f.$$

□

REMARK 1. We will also consider maps $\widetilde{\varphi} : X \to \mathbf{R}^n$ for $n \geq 1$. As before we will call such a function a *cocycle* whenever it is measurable. We denote

$$\widetilde{\varphi}^{(n)}(x) = \widetilde{\varphi}(x) + \widetilde{\varphi}(Tx) + \cdots + \widetilde{\varphi}(T^{n-1}x)$$

for $n > 0$. □

We will identify the circle $\mathbf{T}$ with $X = [0, 1)$. Therefore, real functions defined on the circle will be identified with one-periodic functions defined on $\mathbf{R}$. Let μ denote Lebesgue measure on X. Assume that $T : X \to X$ is an irrational rotation, $Tx = x + \alpha \pmod 1$, $x \in X$. Let

$$\alpha = [0; a_1, a_2, \dots]$$

be the continued fraction expansion of α. The positive integers a_n are said to be the *partial quotients* of α. Put

$$q_0 = 1, q_1 = a_1, q_{n+1} = a_{n+1}q_n + q_{n-1}; \quad p_0 = 0, p_1 = 1, p_{n+1} = a_{n+1}p_n + p_{n-1}.$$

The rationals p_n/q_n are called the *convergents* of α and the inequality

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}$$

holds. The following formula

$$q_{n+1}\|q_n\alpha\| + q_n\|q_{n+1}\alpha\| = 1$$

is satisfied. Here $\|t\|$ is the distance of a real number t from the set of integers. By $\{t\}$ we denote the fractional part of t.

Hence, from the continued fraction expansion of α we obtain, for each n, two Rokhlin towers $\xi_n, \overline{\xi}_n$ whose union coincides with the whole circle. For n even

$$\xi_n = \{[0, \{q_n\alpha\}), T[0, \{q_n\alpha\}), \ldots, T^{(a_{n+1}q_n+q_{n-1})-1}[0, \{q_n\alpha\})\},$$

$$\overline{\xi}_n = \{[\{q_{n+1}\alpha\}, 1), \ldots, T^{q_n-1}[\{q_{n+1}\alpha\}, 1)\}.$$

Given a subsequence $\{n_k\}$ of natural numbers we will denote

$$I_k = [0, \{a_{2n_k+1}\, q_{2n_k}\alpha\}), \qquad J_t^k = T^{(t-1)q_{2n_k}}(0, \{q_{2n_k}\alpha\}],$$

$t = 1, \ldots, a_{2n_k+1}$. Notice that

$$I_k = \bigcup_{t=1}^{a_{2n_k+1}} J_t^k,$$

and

$$|J_1^k| < \frac{1}{a_{2n_k+1}\, q_{2n_k}}. \tag{4}$$

3. Coboundaries with values in $\mathbf{R}^n$

Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be an ergodic automorphism of a standard Borel space. Let $\widetilde{\varphi} : X \to \mathbf{R}$ be a cocycle.

DEFINITION 1. *A set $F \subset X$ of positive measure is called a fixing set for $\widetilde{\varphi}$ if for each natural number $n \geq 1$*

$$|\widetilde{\varphi}^{(n)}(x)| \leq 1 \quad \textit{whenever} \quad x, T^n x \in F. \tag{5}$$

In [14] it has been proved that $\widetilde{\varphi}$ is an $\mathbf{R}$-coboundary if and only if $\widetilde{\varphi}$ has a fixing set. Note that if $\widetilde{\varphi}$ is given as $\sum_{k\geq 1}\widetilde{\varphi}(k)$, $\widetilde{\varphi}(k): X \to \mathbf{R}$ and each of $\widetilde{\varphi}(k)$ has a fixing set F_k such that if $x, T^N x \in F_k$ then $\widetilde{\varphi}(k)^{(N)}(x) = 0$ and besides if $\sum_{k\geq 1}\mu(F_k^c) < 1$ then $F = \bigcap_{k\geq 1} F_k$ is a fixing set for $\widetilde{\varphi}$, hence such a $\widetilde{\varphi}$ is a coboundary.

COROLLARY 1. *Let $\widetilde{\varphi} : X \to \mathbf{R}^n$ be an $\mathbf{R}^n$-cocycle, $\widetilde{\varphi} = (\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n)$, where each $\widetilde{\varphi}_i : X \to \mathbf{R}$ is an $\mathbf{R}$-cocycle. Then $\widetilde{\varphi}$ is an $\mathbf{R}^n$-coboundary if and only if each $\widetilde{\varphi}_i$ has a fixing set.* □

4. Algebraic automorphisms of $\mathbf{T}^n$

We will be interested in periodic continuous algebraic automorphisms of $\mathbf{T}^n$, $n \geq 1$. Note that given $n \geq 2$ there is $v : \mathbf{T}^n \to \mathbf{T}^n$ such that the period of v is n and moreover for certain character $\chi : \mathbf{T}^n \to \mathbf{T}$ all the characters

$$\chi, \chi \circ v, \ldots, \chi \circ v^{n-1}$$

are different. Indeed, it is enough to take

$$v(z_1, z_2, \ldots, z_n) = (z_2, z_3, \ldots, z_n, z_1) \tag{6}$$

and for instance $\chi(z_1, z_2, \ldots, z_n) = z_1 z_2^2 \ldots z_n^n$. By $v : \mathbf{R}^n \to \mathbf{R}^n$ we will denote the corresponding lifting of v to $\mathbf{R}^n$. From now on (till Section 9) by v we denote the automorphism defined by (6).

For some more information about periodic algebraic automorphisms of $\mathbf{T}^n$ we refer to the last section.

5. A class of $\mathbf{R}^n$-cocycles having an analytic coboundary modification

We will recall here a notion of an $\mathcal{AACCP}$ (almost analytic cocycle construction procedure) from [15] which is to construct a real 1-periodic cocycle $\widetilde{\varphi} : \mathbf{R} \to \mathbf{R}$ such that in its $\mathbf{R}$-cohomology class (for certain α) there is an analytic cocycle.

An $\mathcal{AACCP}$ is given by a collection of parameters, say $\mathcal{PAR}$, as follows. We are given a sequence $\{M_k\}$ of natural numbers and an array $\{(d_{k,1}, \ldots, d_{k,M_k})\}$, $d_{k,i} \in \mathbf{R}$ satisfying for each k

$$\sum_{i=1}^{M_k} d_{k,i} = 0. \tag{7}$$

Denote $D_k = \max_{1\le i\le M_k} |d_{k,i}|$. Choose a sequence $\{\varepsilon_k\}$ of positive real numbers satisfying

$$\sum_{k=1}^{\infty} \sqrt{\varepsilon_k} M_k < +\infty, \tag{8}$$

$$\sum_{k=1}^{\infty} \varepsilon_k < 1, \tag{9}$$

$$\varepsilon_k < \frac{1}{D_k^2}, \quad k = 1, 2, \ldots . \tag{10}$$

Finally, we are given $A > 1$ completing the parameters of the $\mathcal{AACCP}$.

We say that this $\mathcal{AACCP}$ *is realized over an irrational number* α with continued fraction expansion $[0; a_1, a_2, \ldots]$ and convergents p_n/q_n, $n \ge 1$ if there exists a strictly increasing sequence $\{n_k\}$ of natural numbers such that

$$A^{N_k} \frac{D_k M_k \|P_k\|_{\mathcal{F}}}{a_{2n_k+1} q_{2n_k}} < \frac{1}{2^k} \tag{11}$$

and

$$\frac{D_k \|P_k'\|_\infty}{a_{2n_k+1} q_{2n_k}} < \sqrt{\varepsilon_k} \tag{12}$$

where $\{P_k\}$ is a sequence of "bump" real trigonometric polynomials, i.e.

$$\begin{cases} \text{(i)} & \int_0^1 P_k(t)dt = 1, \\ \text{(ii)} & P_k \ge 0, \\ \text{(iii)} & P_k(t) < \varepsilon_k \text{ for each } t \in (\eta_k/2, 1), \end{cases} \tag{13}$$

where the η_k's are chosen in such a way that

$$4 M_k \eta_k < \frac{\varepsilon_k}{q_{2n_k}} \tag{14}$$

and N_k is the degree of P_k. Finally, $a_{2n_k+1} > 1$ and

$$\frac{1}{a_{2n_k+1} q_{2n_k}} < \frac{1}{2}\eta_k. \tag{15}$$

Using the above parameters define a cocycle

$$\widetilde{\varphi} = \sum_{k=1}^{\infty} \widetilde{\varphi}(k)$$

as follows. In view of (14), (15) (and (4)), in the interval $I_k = [0, \{a_{2n_k+1} q_{2n_k} \alpha\})$ we can choose $w_{k,1}, \ldots, w_{k,M_k}$ to be consecutive pairwise disjoint intervals of the same length contained between η_k and $2\eta_k$ such that each $w_{k,i}$ consists of say e_k

consecutive subintervals J_t^k, where e_k is an odd number. Let $J_{s_{k,i}}^k$ be the central subinterval in $w_{k,i}$ and now define

$$\widetilde{\varphi}(k)(x) = \begin{cases} d_{k,i} & \text{if } x \in J_{s_{k,i}}^k, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the $\widetilde{\varphi}(k)$'s have disjoint supports so $\widetilde{\varphi}$ is well defined. We will also write $\widetilde{\varphi} = \widetilde{\varphi}(\mathcal{PAR}, \alpha)$. Let $m_{k,i}$ be determined by $J_{s_{k,i}}^k = T^{m_{k,i}} J_1^k$.

As proved in [15]

(A) The set of α's over which an $\mathcal{AACCP}$ is realized is a G_δ and dense subset of the circle.

(B) If an $\mathcal{AACCP}$ is realized over α then there exists an analytic cocycle $f : \mathbf{T} \to \mathbf{R}$ which is α-cohomologous to $\widetilde{\varphi}$.

Suppose that for each $j = 1, \ldots, n$ we have a collection (depending on j) of parameters $\mathcal{PAR}_j$ with the same sequences $\{\varepsilon_k\}, \{M_k\}$ and $A > 1$. Suppose that moreover we have an α for which for every $j = 1, \ldots, n$ we have a realization of the $\mathcal{AACCP}$ with $\mathcal{PAR}_j$ that is a cocycle $\widetilde{\varphi}_j = \widetilde{\varphi}_j(\mathcal{PAR}_j, \alpha)$ in such a way that the intervals $w_{k,1}, \ldots, w_{k,M_k}$ and $J_{s_{k,i}}^k$ are common for all j. We will refer to the arising cocycle $\widetilde{\varphi} = (\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n) : \mathbf{T} \to \mathbf{R}^n$ as an n-dimensional $\mathcal{AACCP}$ realized over α. Thus in an n-dimensional $\mathcal{AACCP}$ the only parameters that vary with j are

$$\{(d_{j,k,1}, \ldots, d_{j,k,M_k}\}, \quad j = 1, \ldots, n, \ k \geq 1,$$

where for each j, k

$$\sum_{i=1}^{M_k} d_{j,k,i} = 0.$$

COROLLARY 2. *Assume that we are given an n-dimensional $\mathcal{AACCP}$ with $\mathcal{PAR}_j$, $j = 1, \ldots, n$. Then there exists a dense G_δ set of irrational numbers such that for each α from this set the the corresponding cocycle $\widetilde{\varphi} = (\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n) : \mathbf{T} \to \mathbf{R}^n$, where $\widetilde{\varphi}_j = \widetilde{\varphi}_j(\mathcal{PAR}_j, \alpha)$ $(j = 1, \ldots, n)$ has an analytic $\mathbf{R}^n$-coboundary modification, that is there are $f_1, \ldots, f_n : \mathbf{T} \to \mathbf{R}$ analytic such that for each $j = 1, \ldots, n$ the cocycle $\widetilde{\varphi}_j$ is α-cohomologous to f_j.* □

6. Multiplicity function for group extensions

Suppose that $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is an ergodic rotation on a compact monothetic group and G is a compact metric abelian group with Haar measure m. Let $\varphi : X \to G$ be an ergodic cocycle. Now, the space $L^2(X \times G, \widetilde{\mu})$ with $\widetilde{\mu} = \mu \times m$

can be decomposed as

$$L^2(X \times G, \widetilde{\mu}) = \bigoplus_{\chi \in \widehat{G}} H_\chi,$$

where $H_\chi = \{f \otimes \chi : f \in L^2(X, \mu)\}$. Note that H_χ is a U_{T_φ}-invariant closed subspace and that $U_{T_\varphi} : H_\chi \to H_\chi$ is unitarily equivalent to

$$V_{\varphi,T,\chi} : L^2(X, \mu) \to L^2(X, \mu)$$

$$V_{\varphi,T,\chi}(f)(x) = \chi(\varphi(x))f(Tx), \quad x \in Y.$$

Suppose now that for each χ the operator $V_{\varphi,T,\chi}$ has a simple spectrum. Denote by σ_χ the maximal spectral type of $V_{\varphi,T,\chi}$.

The following facts are proved in [7]:

FACT 1. *If there exists $S_{f,v} \in C(T_\varphi)$ that is if $\varphi S/v\varphi = fT/f$ then $V_{\varphi,T,\chi}$ and $V_{\varphi,T,\chi\circ v}$ are unitarily equivalent.* □

FACT 2. *Let $\chi, \gamma \in \widehat{G}$ be nontrivial and suppose that there exists a rigidity time $m_t = m_t(\chi, \gamma)$ for T such that*

$$\lim_{t\to\infty} \int_X w(\varphi^{(m_t)}(x))d\mu(x) = \delta_w$$

(where $w = \chi$ or γ) with $0 < |\delta_w| < 1$ and $\delta_\chi \neq \delta_\gamma$. Then σ_χ and σ_γ are mutually singular. □

The following result is standard.

COROLLARY 3. *For each n-dimensional $\mathcal{AACCP}$ and every $\chi \in \widehat{\mathbf{T}}^n$ the corresponding unitary operator $V_{\chi,T,\varphi} : L^2(\mathbf{T}) \to L^2(\mathbf{T})$ has simple spectrum, i.e. its msm is equal to* 1. □

These two facts and the above corollary give us a plan to construct an n-dimensional $\mathcal{AACCP}$ with the msm equal to n: we will have to build an n-dimensional $\mathcal{AACCP}$ for which we will be able to solve the functional equation

$$\widetilde{\varphi}S - \widetilde{v}\widetilde{\varphi} = gT - g$$

and then we will have to prove that if $\chi \neq \gamma \circ v^r$ for each r then for χ and γ we can find a "good" (in the sense of Fact 2) rigidity time $\{m_t\}$. This will put some further restrictions on the set of parameters as we will see in the next sections.

REMARK 2. Note also that if we construct φ in the above way, then in particular for each non-trivial character $\chi \in \widehat{\mathbf{T}}^n$ we will have a rigidity time $\{m_t\}$ for T such that

$$\int_{\mathbf{T}} \chi(\varphi^{(m_t)}(x))d\mu(x) \to \delta_\chi$$

with $|\delta_\chi| < 1$. This excludes the possibility of $\chi\varphi$ being a coboundary and it follows from (3) that the cocycle φ is ergodic.

7. Solving the functional equation $\widetilde{\varphi}S - \widetilde{v}\widetilde{\varphi} = gT - g$

We will work with an n-dimensional $\mathcal{AACCP}$ by putting more and more assumptions on the subsequence $\{a_{2n_k+1}\}$ (this will not change the residuality of the set of possible realizations). In this section we require $\{a_{2n_k+1}\}$ to satisfy additionally

$$a_{2n_k+1} = r_k t_k, \tag{16}$$

where $r_k \nearrow \infty$ and

$$\frac{1}{t_k} < \frac{1}{16 \cdot 2^k}. \tag{17}$$

Denote $[b_{i,s}^{(k)}, c_{i,s}^{(k)}) = T^i(J_s^k)$.

We will inductively choose non-negative integers k_l, w_l and positive numbers δ_l, $l \geq 1$, so that $k_l < k_{l+1}$ and

$$\delta_l \leq |J_1^{k_l}| \frac{1}{16 \cdot 2^l}, \tag{18}$$

$$\frac{w_l + 1}{q_{2n_{k_l}}} < \frac{1}{16 \cdot 2^l}, \tag{19}$$

$$q_{2n_{k_l}} |I_{k_l}| \geq 1 - \frac{1}{4 \cdot 2^l}, \tag{20}$$

$$\begin{cases} \text{the closed intervals } B_l = [v_l - \delta_l/2, v_l + \delta_l/2], \\ v_l = c_{w_l, r_{k_l}}^{(k_l)} \text{ form a decreasing sequence.} \end{cases} \tag{21}$$

Let us start with $w_1 = 0$, $0 < \delta_1 < \frac{1}{32}|J_1^{k_1}|$, where k_1 is chosen so that (19) and (20) hold true. Suppose that we have defined $k_1, \ldots, k_l, w_1, \ldots, w_l$, and $\delta_1, \ldots, \delta_l$, so that (18), (19), (20) and (21) are satisfied. Since T is strictly ergodic, there exists a positive integer m such that for every $x \in [0, 1)$

$$\{x, Tx, \ldots, T^{m-1}x\} \cap \operatorname{Int} B_l \neq \emptyset. \tag{22}$$

Select k_{l+1} so that $k_{l+1} > k_l$ and

$$m/q_{2n_{k_{l+1}}} < \frac{1}{16 \cdot 2^{l+1}}$$

and

$$q_{2n_{k_{l+1}}} |I_{k_{l+1}}| \geq 1 - \frac{1}{4 \cdot 2^{l+1}}.$$

Now, take $x = c_{0,r_{k_{l+1}}}^{(k_{l+1})}$ and according to (22) choose $v_{l+1} = T^{w_{l+1}}x \in \mathrm{Int}(B_l)$. Then, we choose δ_{l+1} satisfying (18) so that $B_{l+1} = [v_{l+1}-\delta_{l+1}/2, v_{l+1}+\delta_{l+1}/2] \subset \mathrm{Int}(B_l)$. Notice that $|B_l| \le \delta_l$ so from (21) there exists a unique $\beta \in [0,1)$ such that

$$\beta \in \bigcap_{l=1}^{\infty} B_l. \tag{23}$$

Denote $S : [0,1) \to [0,1)$, $Sx = x + \beta$. Put

$$Z_{1,l} = \bigcup_{i=q_{2n_{k_l}}-w_l}^{q_{2n_{k_l}}-1} T^i(I_{k_l}), \qquad Z_{2,l} = \bigcup_{i=0}^{q_{2n_{k_l}}-1} \bigcup_{s=1}^{a_{2n_{k_l}}+1} T^i[b_{i,s}^{(k_l)} - \delta_l/2, b_{i,s}^{(k_l)} + \delta_l/2],$$

$$Z_{3,l} = I_{k_l}, \quad Z_{4,l} = \bigcup_{i=0}^{q_{2n_{k_l}}-1} T^i \left(\bigcup_{s=a_{2n_{k_l}}+1-r_{k_l}}^{a_{2n_{k_l}}+1} J_s^{k_l} \right).$$

Denote

$$F_l = \bigcup_{i=0}^{q_{2n_{k_l}}-1} T^i(I_{k_l}) \setminus (Z_{1,l} \cup Z_{2,l} \cup Z_{3,l} \cup Z_{4,l}).$$

It follows from (19) that

$$\mu(Z_{1,l}) \le \frac{w_l}{q_{2n_{k_l}}} \le \frac{1}{16 \cdot 2^l}.$$

In view of (18)

$$\mu(Z_{2,l}) \le \frac{\delta_l}{|J_1^{k_l}|} \le \frac{1}{16 \cdot 2^l}$$

and (by (19))

$$\mu(Z_{3,l}) \le \frac{1}{q_{2n_{k_l}}} \le \frac{1}{16 \cdot 2^l}.$$

Finally, by (17)

$$\mu(Z_{4,l}) < \frac{1}{16 \cdot 2^{k_l}} \le \frac{1}{16 \cdot 2^l}.$$

Therefore, by (20)

$$\mu(F_l) \ge 1 - \frac{1}{4 \cdot 2^l} - \frac{1}{4 \cdot 2^l} = 1 - \frac{1}{2 \cdot 2^l}, \quad l = 1, 2, \ldots \tag{24}$$

Assume that an n-dimensional $\mathcal{AACCP}$ is given with

$$M_k = \widetilde{r}_k t_k, \qquad r_k = \widetilde{r}_k \widetilde{\widetilde{r}}_k, \quad \widetilde{r}_k \nearrow \infty,$$

$k \ge 1$ and realized over an α satisfying (16). We will not use all parameters of the $\mathcal{AACCP}$ but only those given by the subsequence $\{k_l\}$. Equivalently, for $k \ne k_l$ we put

$$d_{j,k,i} = 0, \quad j = 1, \ldots, n, \quad i = 1, \ldots, M_k.$$

Moreover, we require that for each k

$$\tag{25} \widetilde{v}(d_{1,k,s}, d_{2,k,s}, \ldots, d_{n,k,s}) = (d_{1,k,s+\widetilde{r}_k}, d_{2,k,s+\widetilde{r}_k}, \ldots, d_{n,k,s+\widetilde{r}_k})$$

$s = 1, 2, \ldots, M_k - \widetilde{r}_k - 1$ and that in the definition of $\widetilde{\varphi}(k)$ we choose the intervals $w_{k,i}$ so that

$$\tag{26} J^k_{s_{k,i}} = J^k_{\widetilde{\widetilde{r}}_k i}, \quad i = 1, \ldots, M_k.$$

In view of (25) and the definition of the cocycle $\widetilde{\varphi}(k)$ we will have

$$\tag{27} (\widetilde{v}\widetilde{\varphi}(k))|J^k_p = \widetilde{\varphi}(k)|J^k_{p+r_k}$$

(i.e. the equality of the constant values of the corresponding functions), $p = 1, 2, \ldots, a_{2n_k+1} - r_k - 1$.

THEOREM 1. *For the $\mathcal{AACCP}$ described above and $Sx = x + \beta$ with β defined by* (23) *the functional equation*

$$\widetilde{\varphi}S - \widetilde{v}\widetilde{\varphi} = gT - g$$

has a measurable solution $g : \mathbf{T} \to \mathbf{R}^n$.

PROOF. Since the proof of this theorem is almost the same as the one of Theorem 2 in [14], we will only sketch it. Denote

$$\widetilde{\psi} = \widetilde{\varphi}S - \widetilde{v}\widetilde{\varphi} \quad \text{and} \quad \widetilde{\psi}(k) = \widetilde{\varphi}(k)S - \widetilde{v}\widetilde{\varphi}(k)$$

and notice that $\widetilde{\psi} = \sum_{k \geq 1} \widetilde{\psi}(k)$. Put $F = \bigcap_{k \geq 1} F_k$ which has positive measure (for $k \neq k_l$ the set F_k is the whole circle) in view of (24). Therefore all we need to prove is that

$$\text{if} \quad x, T^N x \in F_k \quad \text{then} \quad \widetilde{\psi}(k)^{(N)}(x) = 0.$$

This statement is clear if x and $T^N x$ belong to the same interval $T^p J^k_m$ since the sum of constant values of $\widetilde{\varphi}(k)$ on I_k is equal to zero and J^k_1 is the base of a Rokhlin tower. Therefore the general case can be reduced to the case where $x, T^N x \in S_k$ and $Tx, \ldots, T^{N-1}x \in \bigcup_{i=0}^{q_{2n_k}-1} T^i I_k$. Now notice that $\widetilde{\psi}(k)$ takes possible non-zero values only on I_k and $T^{q_{2n_k} - w_k} I_k$ and moreover in view of (27) and (23)

$$\widetilde{v}\widetilde{\psi}(k)|J^k_{p+1} \cap F_k = \widetilde{\psi}(k)|T^{q_{2n_k} - w_k} J^k_p \cap F_k$$

(the equality of the constant values of the corresponding functions) which completes the proof. □

As a consequence of Fact 1 and (6) we obtain the following

COROLLARY 4. *For the $\mathcal{AACCP}$ defined in Theorem* 1 *the msm is at least n.* □

8. An n-dimensional $\mathcal{AACCP}$ with the msm equal to n

We will put some more restrictions on the $\mathcal{AACCP}$'s described in Section 7 (and their realizations). Recall that

$$a_{2n_k+1} = r_k t_k, \qquad r_k = \widetilde{r}_k \widetilde{\widetilde{r}}_k, \quad \widetilde{r}_k \nearrow \infty.$$

It is assumed that the only nonzero values of $\widetilde{\varphi}(k)$ are on the intervals $J^k_{\widetilde{\widetilde{r}}_k i}$, $i = 1, \ldots, M_k$ and (27) holds true. Denote

$$m_k = q_{2n_k} \widetilde{\widetilde{r}}_k. \tag{28}$$

Then $m_k\alpha \to 0 \bmod 1$, so $\{m_k\}$ is a rigidity time for T. Set

$$\overline{d}_{k,j} = (e^{2\pi i d_{1,k,j}}, \ldots, e^{2\pi i d_{n,k,j}}), \quad j = 1, \ldots, \widetilde{r}_k.$$

PROPOSITION 2. *For each character* $\chi \in \widehat{\mathbf{T}}^n$

$$\left| \int_{\mathbf{T}} \chi(\varphi^{(m_k)}(x))\, d\mu(x) - \frac{1}{\widetilde{r}_k} \sum_{j=1}^{\widetilde{r}_k} \frac{1}{n} \sum_{t=0}^{n-1} \chi v^t(\overline{d}_{k,j}) \right| \to 0.$$

PROOF. Let us start with the obvious observation that

$$\left| \int_{\mathbf{T}} \chi(\varphi^{(m_k)}(x))\, d\mu(x) - \int_{\mathbf{T}} \chi({}_k\varphi^{(m_k)}(x))\, d\mu(x) \right| \to 0,$$

where ${}_k\varphi(x) = \varphi(1)(x)\varphi(2)(x) \ldots \varphi(k)(x)$ and $\varphi(k)(x) = e^{2\pi i \widetilde{\varphi}(k)(x)}$. Denote by $\overline{e}_{j,p}$ the constant value of $\varphi(j)$ on J^j_p for $j \geq 1$, $p = 1, \ldots, a_{2n_j+1}$. Consequently, by (25), $\overline{e}_{j,r_j+p} = v(\overline{e}_{j,p})$ and $\overline{e}_{j,p} = (1, \ldots, 1)$ except for those p which are multiples of $\widetilde{\widetilde{r}}_j$. The cocycle ${}_k\varphi$ is constant on the levels $TI_k, T^2I_k, \ldots, T^{q_{2n_k}-1}I_k$ taking the corresponding values, say, $\overline{b}_{k,2}, \ldots, \overline{b}_{k,q_{2n_k}}$, where

$$\prod_{s=2}^{q_{2n_k}} \overline{b}_{k,s} = (1, \ldots, 1) \tag{29}$$

(this statement follows from Lemma 3 [14]). We also have

$$\left| \int_{\mathbf{T}} \chi({}_k\varphi^{(m_k)}(x))\, d\mu(x) - \int_{G_k} \chi({}_k\varphi^{(m_k)}(x))\, d\mu(x) \right| \to 0,$$

where $G_k = \bigcup_{s=0}^{q_{2n_k}-1} T^s I_k$. We claim that

$$\left| \int_{\mathbf{T}} \chi({}_k\varphi^{(m_k)}(x))\, d\mu(x) - \frac{1}{r_k} \sum_{i=1}^{r_k-1} \frac{1}{n} \sum_{t=0}^{n-1} \chi v^t(\overline{e}_{k,i} \cdot \overline{e}_{k,i+1} \cdot \ldots \cdot \overline{e}_{k,i+\widetilde{\widetilde{r}}_k-1}) \right| \to 0. \tag{30}$$

Indeed, in view of (29), denoting $l_k = q_{2n_k}|J_1^k|$,

$$\int_{G_k} \chi({}_k\varphi^{(m_k)}(x))\,d\mu(x) = l_k \sum_{i=1}^{a_{2n_k+1}-\tilde{\tilde{r}}_k} \chi(\bar{e}_{k,i}\cdot \bar{e}_{k,i+1}\cdot\ldots\cdot \bar{e}_{k,i,\tilde{\tilde{r}}_k-1}) + o(1)$$

$$=l_k(\sum_{i=1}^{r_k} \chi(\bar{e}_{k,i}\cdot \bar{e}_{k,i+1}\cdot\ldots\cdot \bar{e}_{k,i+\tilde{\tilde{r}}_k-1})$$

$$+\sum_{i=1}^{r_k} \chi(\bar{e}_{k,i+r_k}\cdot \bar{e}_{k,i+r_k+1}\cdot\ldots\cdot \bar{e}_{k,i+r_k+\tilde{\tilde{r}}_k-1}) + \ldots) + o(1)$$

$$=l_k \sum_{i=1}^{r_k}\sum_{p=0}^{t_k-1} \chi v^p(\bar{e}_{k,i}\cdot \bar{e}_{k,i+1}\cdot\ldots\cdot \bar{e}_{k,i+\tilde{\tilde{r}}_k-1}) + o(1)$$

$$=l_k \sum_{i=1}^{r_k}\frac{t_k}{n}\sum_{p=0}^{n-1} \chi v^p(\bar{e}_{k,i}\cdot \bar{e}_{k,i+1}\cdot\ldots\cdot e_{k,i+\tilde{\tilde{r}}_k-1}) + o(1)$$

$$=t_k l_k \sum_{i=1}^{r_k}\frac{1}{n}\sum_{p=0}^{n-1} \chi v^p(\bar{e}_{k,i}\cdot \bar{e}_{k,i+1}\cdot\ldots\cdot \bar{e}_{k,i+\tilde{\tilde{r}}_k-1}) + o(1).$$

But $a_{2n_k+1}l_k \to 1$, so $|t_k l_k - \frac{1}{r_k}|r_k \to 0$ and consequently

$$\Bigg| t_k l_k \sum_{i=1}^{r_k}\frac{1}{n}\sum_{p=0}^{n-1} \chi v^p(\bar{e}_{k,i}\cdot \bar{e}_{k,i+1}\cdot\ldots\cdot \bar{e}_{k,i+\tilde{\tilde{r}}_k-1})$$

$$-t_k \frac{1}{a_{2n_k+1}} \sum_{i=1}^{r_k}\frac{1}{n}\sum_{p=0}^{n-1} \chi v^p(\bar{e}_{k,i}\cdot \bar{e}_{k,i+1}\cdot\ldots\cdot \bar{e}_{k,i+\tilde{\tilde{r}}_k-1})\Bigg| \to 0.$$

Thus

$$\int_{G_k} \chi({}_k\varphi^{(m_k)}(x))\,d\mu(x) = \frac{1}{r_k}\sum_{i=1}^{r_k-1}\frac{1}{n}\sum_{p=0}^{n-1} \chi v^p(\bar{e}_{k,i}\cdot \bar{e}_{k,i+1}\cdot\ldots\cdot \bar{e}_{k,i+\tilde{\tilde{r}}_k-1}) + o(1).$$

Since the expression $\bar{e}_{k,i}\cdot\bar{e}_{k,i+1}\cdot\ldots\cdot\bar{e}_{k,i+\tilde{\tilde{r}}_k-1}$ takes the same value for $\tilde{\tilde{r}}_k$ consecutive i's, we obtain that

$$\int_{G_k} \chi({}_k\varphi^{(m_k)}(x))\,d\mu(x) = \frac{1}{r_k}\sum_{j=1}^{r_k/\tilde{\tilde{r}}_k} \tilde{\tilde{r}}_k \frac{1}{n}\sum_{p=0}^{n-1} \chi v^p(\bar{d}_{k,j}) + o(1)$$

$$= \frac{1}{\tilde{r}_k}\sum_{j=1}^{\tilde{r}_k}\frac{1}{n}\sum_{p=0}^{n-1} \chi v^p(\bar{d}_{k,j}) + o(1)$$

and the result follows. □

On $\widetilde{\mathbf{T}}^n$ consider the orbit equivalence relation $\sim$:

$$\chi \sim \gamma \quad \text{if there exists } s\in\mathbf{Z} \text{ such that } \chi v^s = \gamma.$$

THEOREM 2. *There exists an n-dimensional AACCP giving rise to $\varphi : \mathbf{T} \to \mathbf{T}^n$ and its realization over T an irrational rotation by α such that*

(i) $S_{\exp 2\pi i g, v} \in C(T_\varphi)$ *for certain* $Sx = x + \beta$ *and a measurable* $g : \mathbf{T} \to \mathbf{R}^n$,

(ii) *for any pair* $\chi, \gamma \in \widehat{\mathbf{T}}^n$ *of nontrivial characters which are not equivalent there exists a rigidity time* $\widetilde{m}_t = \widetilde{m}_t(\chi, \gamma)$ *for* T *satisfying*

$$\int_{\mathbf{T}} \chi(\varphi^{(\widetilde{m}_t)})\, d\mu \to \delta_{\chi,\gamma},$$

$$\int_{\mathbf{T}} \gamma(\varphi^{(\widetilde{m}_t)})\, d\mu \to \delta_{\gamma,\chi},$$

where $|\delta_{\chi,\gamma}|, |\delta_{\gamma,\chi}| < 1$ *and* $\delta_{\chi,\gamma} \neq \delta_{\gamma,\chi}$. *In particular* T_φ *is ergodic and the msm of* T_φ *is equal to* n.

PROOF. For each nontrivial $\chi \in \widehat{\mathbf{T}}^n$ put

$$f_\chi(\cdot) = \frac{1}{n} \sum_{t=0}^{n-1} \chi v^t(\cdot) \, : \, \mathbf{T}^n \to \mathbf{C}.$$

We have that f_χ is continuous, $\|f_\chi\|_\infty \leq 1$ with $\int_{\mathbf{T}^n} f_\chi = 0$ and if χ and γ are not equivalent then f_χ and f_γ are orthogonal. Partition

$$\mathbf{N} = \bigcup_{\chi,\gamma \text{ nontrivial}, \, \chi \not\sim \gamma} \mathbf{N}_{\chi,\gamma}$$

in such a way that $\mathbf{N}_{\chi,\gamma} = \mathbf{N}_{\gamma,\chi}$ is infinite for each pair (χ, γ). For each $y = (y_1, \ldots, y_n) \in \mathbf{R}^n$ we put $\overline{y} = (e^{2\pi i y_1}, \ldots, e^{2\pi i y_n})$.

Now, fix χ and $\tilde{\gamma}$ what are not equivalent. In view of Proposition 2, all we have to prove is to show that there exists a choice of parameters $d_{k,j} \in \mathbf{R}^n$, $j = 1, \ldots, \widetilde{r}_k$ for $k \in \mathbf{N}_{\chi,\gamma}$ in such a way that

$$(31) \qquad \begin{cases} \displaystyle\lim_{k\to\infty, k \in \mathbf{N}_{\chi,\gamma}} \frac{1}{\widetilde{r}_k} \sum_{j=1}^{\widetilde{r}_k} f_\chi(\overline{d}_{k,j}) & = \delta_{\chi,\gamma}, \\ \displaystyle\lim_{k\to\infty, k \in \mathbf{N}_{\chi,\gamma}} \frac{1}{\widetilde{r}_k} \sum_{j=1}^{\widetilde{r}_k} f_\gamma(\overline{d}_{k,j}) & = \delta_{\gamma,\chi} \end{cases}$$

with

$$(32) \qquad |\delta_{\chi,\gamma}|, |\delta_{\gamma,\chi}| < 1 \quad \text{and} \quad \delta_{\chi,\gamma} \neq \delta_{\gamma,\chi}.$$

To this end, first, find $y_1, \ldots, y_q \in \mathbf{R}^n$ such that if we denote

$$\delta_{\chi,\gamma} = \delta_{\chi,\gamma}(\overline{y}_1, \ldots, \overline{y}_q) = \frac{1}{q} \sum_{j=1}^{q} f_\chi(\overline{y}_j),$$

$$\delta_{\gamma,\chi} = \delta_{\gamma,\chi}(\overline{y}_1, \ldots, \overline{y}_q) = \frac{1}{q} \sum_{j=1}^{q} f_\gamma(\overline{y}_j)$$

then (32) holds (indeed, such points do exist since if q is large enough and $(\overline{y}_j)$ are sufficiently "well-distributed" then $\delta_{\gamma,\chi}, \delta_{\chi,\gamma}$ are close to the corresponding integrals, hence close to zero; and if for $y_1, \dots, y_q$ we have $\delta_{\gamma,\chi}(\overline{y}_1, \dots, \overline{y}_q) = \delta_{\chi,\gamma}(\overline{y}_1, \dots, \overline{y}_q)$ then by changing one term, say y_1, and replacing it by u_1 we find that $f_\chi(\overline{y}_1) - f_\gamma(\overline{y}_1) \neq f_\chi(\overline{u}_1) - f_\gamma(\overline{u}_1)$, so $\delta_{\gamma,\chi}(\overline{u}_1, \dots, \overline{y}_q) \neq \delta_{\chi,\gamma}(\overline{u}_1, \dots, \overline{y}_q)$, without changing the closeness to zero of the latter two numbers because the functions under consideration are bounded by 1).

For $k \in \mathbf{N}_{\chi,\gamma}$ large enough we put

$$\begin{aligned} &d_{k,j} = y_j \quad \text{for} \quad j = 1, \dots, q, \\ &d_{k,j+q} = d_{k,j} \quad \text{for} \quad j = sq, s = 0, \dots, [\widetilde{r}_k/q] - 1, \\ &d_{k,j} \quad \text{arbitrary for} \quad j = [\widetilde{r}_k/q]q + 1, \dots, \widetilde{r}_k - 1, \\ &d_{k,\widetilde{r}_k} = -\sum_{j=1}^{\widetilde{r}_k - 1} d_{k,j}. \end{aligned}$$

Then clearly for the corresponding parameters $\overline{d}_{k,j}$ we obtain (31) and (32). □

COROLLARY 5. *For each natural number $n \geq 1$, on the $(n+1)$-dimensional torus there exists an analytic diffeomorphism τ, ergodic with respect to Lebesgue measure, and whose msm is equal to n.* □

9. Essential values of the multiplicity function for analytic diffeomorphisms

Let $E \subset \mathbf{N}$ be a finite set satisfying

(i) $1 \in E$,

(ii) if $m, n \in E$ then $\operatorname{lcm}(n, m) \in E$.

We will now describe how to define a d-dimensional (the number d depends on E) $\mathcal{AACCP}$ such that for the arising T_φ we have

$$E(T_\varphi) = E.$$

Let $w : \mathbf{Z}^d \to \mathbf{Z}^d$ be an algebraic periodic automorphism. Denote by

$$E(w) = \{k \in \mathbf{N} : (\exists x \in \mathbf{Z}^d) \operatorname{card}\{x, w(x), w^2(x), \dots\} = k\}.$$

PROPOSITION 3. *The set $E(w)$ enjoys both* (i) *and* (ii).

PROOF. Let A denote the integer-valued matrix associated to w with $\det(A) = \pm 1$. Suppose that N is the smallest positive integer such that $w^N = \mathrm{id}$. Denote by $\widehat{w} : \mathbf{C}^d \to \mathbf{C}^d$ the linear isomorphism given by A. Since A is a root of the identity matrix, its Jordan form must be diagonal. Thus $\mathbf{C}^d = \bigoplus_{i=1}^d \mathcal{P}_i$, where $\dim \mathcal{P}_i = 1$, say $\mathcal{P}_i = \mathbf{C}_{p_i}$, $\mathcal{P}_i$ is $\widehat{w}$-invariant and the action of $\widehat{w}$ on $\mathcal{P}_i$ is the multiplication by a primitive root of unity of order N_i for certain N_i dividing N.

It follows from this that if $\chi = \sum_{i=1}^d a_i p_i \in \mathbf{Z}^d \setminus \{0\}$ then the smallest positive integer N' such that $w^{N'}\chi = \chi$ is equal to $\operatorname{lcm}\{N_i : a_i \neq 0\}$. Therefore all we have to prove is that given another $\gamma \in \mathbf{Z}^d \setminus \{0\}$ with $\gamma = \sum_{i=1}^d b_i p_i$ we can find two nonzero integers c, d so that if we denote $c\chi + d\gamma = \sum_{i=1}^d e_i p_i$ then $e_i = 0$ if and only if $a_i = b_i = 0$. This last statement is however clear; c big enough and $d = 1$ will do. □

Given $n \geq 1$ by $R(n)$ we denote the set of all primitive roots of unity of order n. Set

$$P(n)(x) = \prod_{\rho \in R(n)} (x - \rho).$$

Call class $\mathcal{C}$ the family of unitary polynomials with coefficients in $\mathbf{Z}$, having constant coefficient equal to ± 1. By $\phi(n)$ we denote the Euler function, i.e. the cardinality of the set of positive integers less than n which are coprime with n. Note that the degree of $P(n)$ is equal to $\phi(n)$.

PROPOSITION 4. ([28], section 8.4) *$P(n)$ belongs to class $\mathcal{C}$; furthermore it is the minimal polynomial for each $\rho \in R(n)$.* □

We will also need the following proposition.

PROPOSITION 5. *If a finite set $E \subset \mathbf{N}$ satisfies* (i) *and* (ii) *then there exist $d \geq 1$ and an algebraic automorphism $w : \mathbf{Z}^d \to \mathbf{Z}^d$ such that $E = E(w)$.*

PROOF. Let m be the maximal element of E that is the lcm of all the elements of E. Put $P_E = \prod_{k \in E} P(k)$. We have $P_E(t) = t^d + c_{d-1}t^{d-1} + \ldots + c_0$. By Proposition 4, P_E belongs to class $\mathcal{C}$. Denote by $A = [a_{ij}]_{1 \leq i,j \leq d}$ its companion matrix, that is all entries of A are equal to zero except for $a_{i+1,i} = 1, a_{id} = c_{i-1}$ for $i = 1, \ldots, d$. Consequently the entries of A are integers, $\det(A) = \pm 1$ and let w denote the associated automorphism of $\mathbf{Z}^d$. As P_E is the characteristic polynomial of A and P_E has d different roots (all of them being m-th roots of 1), we have $A^m = I$ and all eigenvalues of A have multiplicity 1, so there exists a base of $\mathbf{C}^d$ consisting of eigenvectors. The eigenvector corresponding to an eigenvalue $\rho \in R(k)$, $k \in E$ is of course periodic with least period k. Considered as vectors in $\mathbf{C}^d$, the $\mathbf{Z}^d$-vectors (representing characters) are linear combinations of the vectors of the base. This and the condition (ii) imply that all nontrivial characters have the least periods belonging to E.

There remains to prove that whenever $k \in E$ we can find a character χ such that $\operatorname{card}\{\chi, w(\chi), \dots\} = k$. Consider the matrix $P(k)(A) = \prod_{\rho \in R(k)}(A - \rho I)$. Obviously, a base of eigenvectors of A is also a base of eigenvectors of $P(k)(A)$; moreover if p is an eigenvector from the base then its eigenvalue for $P(k)(A)$ is 0 if and only if its eigenvalue for A belongs to $R(k)$. Therefore, $\ker P(k)(A)$ is exactly the subspace of $\mathbf{C}^d$ generated by the eigenvectors corresponding to the eigenvalues in $R(k)$. Since $P(k)(A)$ has coefficients in $\mathbf{Z}$, its kernel contains a nonzero vector χ with integer coordinates and, as in the proof of Proposition 3, the least period of χ is equal to k. □

Notice that by small modifications of the proofs of Theorem 1, Proposition 2 and Theorem 2 we can get the conclusions (i) and (ii) of Theorem 2 with v being an arbitrary periodic automorphism. Now, by (ii) of Theorem 2 we always have that $E(T_\varphi) = E(w)$, where w denotes the corresponding dual automorphism on $\mathbf{Z}^n$. Therefore, by Proposition 5, we have proved the following

COROLLARY 6. *If a finite subset E satisfies* (i) *and* (ii) *then there exist $d = d(E) \geq 2$, an irrational rotation T and an analytic cocycle $\varphi : \mathbf{T} \to \mathbf{T}^d$ such that the set of essential values of the multiplicity function of the diffeomorphism T_φ is exactly E.* □

REMARK 3. By the method presented in [7] it follows that we can drop the finitness assumption E. If E is infinite then we will produce maps $\tau = T_\varphi$ preserving Lebesgue measure (and ergodic with respect to it) with $E(\tau) = E$ where $\varphi : \mathbf{T} \to \mathbf{T}^\infty$ has the property that all its projections $\varphi \circ \operatorname{proj}_i : \mathbf{T} \to \mathbf{T}$ are analytic maps of $\mathbf{T}$.

However, if we fix a finite $d \geq 2$ then using our method there is an upper bound on the cardinality of the sets E we can obtain. This will be seen if we prove that for each $d > 1$ there exists N such that for each periodic algebraic automorphism v of $\mathbf{Z}^d$ with least period n we have $n \leq N$.

Indeed, an automorphism v is determined by a $d \times d$ matrix A on $\mathbf{Z}$ whose characteristic polynomial P_v is in class $\mathcal{C}$. Since $A^n = I$, all eigenvalues of A are n-th roots of unity, hence belonging to some $R(k)$ with $k|n$. Moreover there is a base of $\mathbf{C}^d$ consisting of eigenvectors of A (see the proof of Proposition 3). By Proposition 4,

(33) all elements of $R(k)$ are eigenvalues of A whenever some $\rho \in R(k)$ is.

Let $N = \max\{\operatorname{lcm}(n_i) : \sum_{i=1}^{u} \phi(n_i) = d,\ 1 \leq u \leq d\}$. (Note that the maximum is well-defined since given s there are only finitely many r such that $\phi(r) = s$.) The matrix A has d eigenvalues and if $\{n_1, \dots, n_u\}$ $(n_i|n)$ denotes the set of degrees of the eigenvalues of A then by (33) and Proposition 4 the characteristic polynomial of

A is equal to $\prod_{i=1}^{u} P(n_i)$, so $\phi(n_1) + \cdots + \phi(n_u) = d$. Now, if $x_{i,s}$ is an eigenvector corresponding to an n_i-root then $x_{i,s}$ has period n_i. Consequently, each $x \in \mathbf{C}^d$ is periodic with the least period not bigger than $\operatorname{lcm}(n_i)$. □

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