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MULTIPLICITY OF POSITIVE SOLUTIONS FOR THE EQUATION $\Delta u + \lambda u + u^{2^*-1} = 0$ IN NONCONTRACTIBLE DOMAINS

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Dedicated to the memory of Juliusz Schauder

1. Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^n$ with $n \geq 3$, $2^* = 2n/(n-2)$ the critical exponent for the Sobolev embedding of $H_0^{1,2}(\Omega)$ in $L^p(\Omega)$, and λ a real parameter. In this paper we study the following problem:

$$P_\lambda(\Omega) \qquad \begin{cases} \Delta u + \lambda u + u^{2^*-1} = 0 & \text{in } \Omega, \\ u \in H_0^{1,2}(\Omega), \ u > 0, & \text{in } \Omega. \end{cases}$$

It is easy to verify (see [5]) that Problem $P_\lambda(\Omega)$ has no solution for $\lambda \geq \lambda_1$, where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^{1,2}(\Omega)$.

If $\lambda \leq 0$, the well known Pokhozhaev identity (see [24], [5]) implies that there is no solution of $P_\lambda(\Omega)$ when Ω is starshaped.

In [5] Brézis and Nirenberg proved that, if $n \geq 4$, Problem $P_\lambda(\Omega)$ has a solution for every $\lambda \in]0, \lambda_1[$; the situation is more complex for $n = 3$ (see [5]) and a complete answer has been given only if Ω is a sphere: in this case $P_\lambda(\Omega)$ has a solution if and only if $\lambda \in]\lambda_1/4, \lambda_1[$. In [25] Rey proved that, for $\lambda > 0$ small enough, the number of solutions of $P_\lambda(\Omega)$ is related to the properties of the Green function of Ω; in this way he shows that, for $\lambda > 0$ small enough and for $n \geq 5$, $P_\lambda(\Omega)$ has at least $\operatorname{cat}\Omega$ solutions, where $\operatorname{cat}\Omega$ denotes the Lyusternik-Schnirelman category of Ω in itself.

Through a different approach, based on some ideas introduced by Benci and Cerami in [2], it is possible to obtain, for $\lambda > 0$ small enough, the same number of solutions as Rey in [25], but under the weaker assumption $n \geq 4$ (see Lazzo [15]).

In this paper we will prove the following result (see Theorem 3.2 for a more precise statement):

THEOREM 1.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ with $n \geq 4$. Suppose that Ω is not contractible in itself (i.e. $\operatorname{cat}\Omega > 1$). Then there exists $\overline{\lambda} \in]0, \lambda_1[$ such that for every $\lambda \in]0, \overline{\lambda}[$ Problem $P_\lambda(\Omega)$ has at least $\operatorname{cat}\Omega + 1$ distinct solutions. More precisely, if we set $m = \operatorname{cat}\Omega$ and denote by S the best Sobolev constant (see Definition 2.2), then for every $\lambda \in]0, \overline{\lambda}[$:*

(a) *there exist m solutions $u_{1,\lambda}, \dots, u_{m,\lambda}$ such that*

$$\int_\Omega (u_{i,\lambda})^{2^*}\, dx < S^{n/2} \qquad \forall i = 1, \dots, m,$$

(b) *there exists at least one solution $\widehat{u}_\lambda$ such that*

$$S^{n/2} < \int_\Omega (\widehat{u}_\lambda)^{2^*}\, dx < 2S^{n/2}.$$

Notice that the solutions $u_{1,\lambda}, \dots, u_{m,\lambda}$ correspond to those found in [25] and [15]; on the contrary, the solution $\widehat{u}_\lambda$ does not correspond to the one obtained by Bahri and Coron in [1] when $\lambda = 0$ and Ω is a bounded domain with nontrivial topology (in a suitable sense). In fact (unless Ω has little holes as in [11] and [27]) the solution u_0 given in [1] corresponds, presumably, to higher energy values (i.e. $\int_\Omega u_0^{2^*}\, dx > 2S^{n/2}$); hence the solution $\widehat{u}_\lambda$ cannot converge to u_0 as $\lambda \to 0$.

So the existence of the solution $\widehat{u}_\lambda$ points out a new phenomenon, and it is natural to deem that in domains Ω nontrivial in the sense of [1] (hence noncontractible), for $\lambda > 0$ small enough, there also exists a solution u_λ, distinct from those given by Theorem 1.1, which converges as $\lambda \to 0$ to the solution obtained in [1].

Notice that the solutions $u_{1,\lambda}, \dots, u_{m,\lambda}$ converge weakly to zero in $H_0^{1,2}(\Omega)$ for $\lambda \to 0$ and concentrate near some points of Ω (see [6], [16], [26]); on the contrary, the solution $\widehat{u}_\lambda$ can converge, under suitable assumptions, to a function $\widehat{u}_0 > 0$, which is a solution of the limit problem $P_0(\Omega)$: Theorem 5.2 gives a sufficient condition which guarantees this convergence.

Moreover, if the solution $\widehat{u}_\lambda$ does not converge as $\lambda \to 0$ to a solution of $P_0(\Omega)$, then it converges weakly to zero in $H_0^{1,2}(\Omega)$ and concentrates near *two* points of Ω, as one can deduce from Lemma 5.1.

The results in 4 show that the number of solutions of $P_\lambda(\Omega)$ is related not to the topology of Ω, but to the topology of a domain $\widetilde{\Omega}$ which differs from Ω by a set of small capacity (see Definition 4.1): for instance, if $\widetilde{\Omega}$ is a noncontractible domain, we can modify $\widetilde{\Omega}$ by a closed subset K with sufficiently small capacity, in such a way that Problem $P_\lambda(\Omega)$ with $\Omega = \widetilde{\Omega} \setminus K$ has, for $\lambda > 0$ small enough, at least $\operatorname{cat} \widetilde{\Omega} + 1$ distinct solutions, even if the domain Ω is contractible in itself.

Finally, let us point out that, if Ω has particular symmetry properties, then the number of solutions may increase considerably (see 3.8 and 4.3): for instance, if Ω is a domain homotopically equivalent to the $(k-1)$-dimensional sphere S_{k-1} and is symmetric with respect to a point $x_0 \notin \widetilde{\Omega}$, then Problem $P_\lambda(\Omega)$ has, for $\lambda > 0$ small enough, at least $2k+1$ solutions, even if the category of Ω is only 2.

2. Preliminaries and notations

Let Ω be a smooth bounded domain in $\mathbb{R}^n$ with $n \geq 3$ and set $2^* = 2n/(n-2)$.

Throughout this paper we shall denote by $\|\cdot\|_p$ the norm in $L^p(\Omega)$ and by $\|u\|_{H_0^{1,2}(\Omega)} = \|Du\|_2$ the Dirichlet norm in the Sobolev space $H_0^{1,2}(\Omega)$. By λ_1 we shall denote the first eigenvalue of the Laplace operator $-\Delta$ in $H_0^{1,2}(\Omega)$:

$$\lambda_1 = \min\{\|Du\|_2 \,:\, u \in H_0^{1,2}(\Omega),\ \|u\|_2 = 1\}.$$

Every function u in $H_0^{1,2}(\Omega)$ will be extended by 0 outside Ω. Moreover, we set $u^+ = \max(u, 0)$, $u^- = \max(-u, 0)$.

Let $f_\lambda : H_0^{1,2}(\Omega) \to \mathbb{R}$ be the functional defined by

$$f_\lambda(u) = \int_\Omega |Du|^2\, dx - \lambda \int_\Omega u^2\, dx.$$

We shall consider f_λ restricted to

$$V = \left\{ u \in H_0^{1,2}(\Omega) \,:\, \int_\Omega (u^+)^{2^*}\, dx = 1 \right\}.$$

It is easy to verify that V is a differentiable C^2 submanifold of $H_0^{1,2}(\Omega)$ with codimension 1.

The following lemma shows that the solutions of Problem $P_\lambda(\Omega)$ correspond to the critical points of the functional f_λ restricted to V.

LEMMA 2.1. *Let $\lambda < \lambda_1$ and Ω be a bounded domain in $\mathbb{R}^n$ with $n \geq 3$. Then the following properties are equivalent:*

(a) *u is a solution of Problem $P_\lambda(\Omega)$.*

(b) *$u/\|u\|_{2^*}$ is a critical point for the functional f_λ on V and*

$$u = \left[f_\lambda\left(\frac{u}{\|u\|_{2^*}} \right) \right]^{(n-2)/4} \frac{u}{\|u\|_{2^*}}.$$

PROOF. It is easy to verify that (a) $\Rightarrow$ (b) (notice that $u/\|u\|_2^* \in V$ because $u > 0$).

Now we show that (b) $\Rightarrow$ (a): let $\overline{u} = u/\|u\|_{2^*}$ be a critical point for f_λ on V. Then we have

$$\Delta \overline{u} + \lambda \overline{u} + \mu (\overline{u}^+)^{2^*-1} = 0 \qquad \text{with } \mu \in \mathbb{R}.$$

It follows that $\overline{u} \geq 0$: in fact, multiplying by $\overline{u}^-$ and integrating, we get

$$\int_\Omega |D\overline{u}^-|^2\, dx - \lambda \int_\Omega |\overline{u}^-|^2\, dx = 0.$$

Hence, since $\int_\Omega |D\overline{u}^-|^2\, dx \geq \lambda_1 \int_\Omega (\overline{u}^-)^2\, dx$, it follows that $(\lambda_1 - \lambda) \int_\Omega (\overline{u}^-)^2\, dx \leq 0$ with $\lambda_1 - \lambda > 0$, so $\overline{u}^- = 0$. Therefore $\overline{u} = \overline{u}^+$ solves the equation

$$\Delta \overline{u} + \lambda \overline{u} + \mu (\overline{u})^{2^*-1} = 0 \qquad \text{with } \mu = f_\lambda(\overline{u}) > 0$$

(because $\lambda < \lambda_1$); hence it is easy to prove that $u = [f_\lambda(\overline{u})]^{(n-2)/4} \overline{u}$ is a solution of Problem $P_\lambda(\Omega)$. □

Of course, the lack of compactness for the Sobolev embedding of $H_0^{1,2}(\Omega)$ in $L^{2^*}(\Omega)$ causes some difficulties in finding critical points of f_λ on V.

DEFINITION 2.2. Let S be the best Sobolev constant for the embedding of $H_0^{1,2}(\Omega)$ in $L^{2^*}(\Omega)$:

$$S = \inf\{ \|Du\|_2^2 \, : \, u \in H_0^{1,2}(\Omega); \ \|u\|_{2^*} = 1 \}.$$

It is well known (see [5]) that S is independent of Ω and depends only on the dimension n. Moreover, the infimum is never achieved when Ω is a bounded domain, while if $\Omega = \mathbb{R}^n$ the infimum is achieved only by the functions

$$\overline{U}_{\mu, x_0}(x) = \frac{c_\mu}{(\mu + |x - x_0|^2)^{(n-2)/2}} \qquad \text{with } \mu > 0 \text{ and } x_0 \in \mathbb{R}^n,$$

(where c_μ are normalization constants).

LEMMA 2.3. *Let Ω be a bounded domain in $\mathbb{R}^n$ with $n \geq 3$ and λ_1 be the first eigenvalue of $-\Delta$ in $H_0^{1,2}(\Omega)$. Then we have:*

(a) *If $\lambda \leq 0$, the minimum $\min_V f_\lambda$ does not exist and $\inf_V f_\lambda = S$.*

(b) *For all $\lambda \in \mathbb{R}$ we have $\inf_V f_\lambda \leq S$; if $\lambda \in]0, \lambda_1]$ and $\inf_V f_\lambda < S$, then the minimum $\min_V f_\lambda$ exists (notice that $\inf_V f_\lambda = -\infty$ for $\lambda > \lambda_1$).*

(c) *If $n \geq 4$, then $\inf_V f_\lambda < S$ for $\lambda > 0$ (so $\min_V f_\lambda$ exists if and only if $\lambda \in]0, \lambda_1]$).*

(d) *If $n = 3$, then there exists $\lambda^* \in]0, \lambda_1[$ such that $\inf_V f_\lambda < S$ if and only if $\lambda > \lambda^*$.*

The proof can be easily deduced from well known results obtained in [5] (it suffices to observe that for $\lambda \leq \lambda_1$ we have $f_\lambda(u^+) \leq f_\lambda(u)$ for all $u \in H_0^{1,2}(\Omega)$ and that, obviously, $u \in V$ if and only if $u^+ \in V$).

LEMMA 2.4. *Let Ω be a bounded domain in $\mathbb{R}^n$ with $n \geq 3$, $\lambda \in]0, \lambda_1[$, $S_\lambda = \inf_V f_\lambda$ (notice that $S_\lambda > 0$ for $\lambda < \lambda_1$). Then, for every $c < (S_\lambda^{n/2} + S^{n/2})^{2/n}$, $c \neq S$, the functional f_λ restricted to V has the following compactness property (Palais-Smale condition): if the sequence $(u_i)_i$ in V satisfies:*

$$f_\lambda(u_i) \to c, \qquad f'_{\lambda|V}(u_i) \to 0 \qquad \text{in } H^{-1}(\Omega),$$

then $(u_i)_i$ is relatively compact in $H_0^{1,2}(\Omega)$ (it is well known that this does not hold for $c = S$).

The proof makes use of the following proposition:

PROPOSITION 2.5. *For every $\lambda \in \mathbb{R}$ let $F_\lambda : H_0^{1,2}(\Omega) \to \mathbb{R}$ be the functional defined by*

$$F_\lambda(u) = \frac{1}{2}\int_\Omega |Du|^2\, dx - \frac{\lambda}{2}\int_\Omega u^2\, dx - \frac{1}{2^*}\int_\Omega (u^+)^{2^*}\, dx.$$

Suppose that $(u_i)_i$ is a sequence in $H_0^{1,2}(\Omega)$ such that $(F_\lambda(u_i))_i$ is a bounded sequence and $F'_\lambda(u_i) \to 0$ in $H^{-1}(\Omega)$. Then there exists a subsequence of $(u_i)_i$ (which we shall denote again by $(u_i)_i$), a function u_0 in $H_0^{1,2}(\Omega)$, which is a critical point for the functional F_λ, and an integer $k \geq 0$ such that $u_i \rightharpoonup u_0$ weakly in $H_0^{1,2}(\Omega)$ and

$$\|Du_i\|_2^2 \to \|Du_0\|_2^2 + kS^{n/2},$$

$$F_\lambda(u_i) \to F_\lambda(u_0) + \frac{k}{n} S^{n/2}.$$

For the proof it suffices to argue as in [28]. Moreover, in our case we can observe that the solutions of the problem:

$$\begin{cases} \Delta u + (u^+)^{2^*-1} = 0 & \text{in } \mathbb{R}^n, \\ u \in L^{2^*}(\mathbb{R}^n), \ |Du| \in L^2(\mathbb{R}^n), \end{cases}$$

are nonnegative; so all nontrivial solutions are given by the functions (see [17], [29], [14])

$$U_{\mu,x_0} = \frac{[n(n-2)\mu]^{(n-2)/4}}{[\mu + |x - x_0|^2]^{(n-2)/2}} \qquad \text{with } \mu > 0 \text{ and } x_0 \in \mathbb{R}^n,$$

which yields

$$\int_{\mathbb{R}^n} |DU_{\mu,x_0}|^2 \, dx = \int_{\mathbb{R}^n} U_{\mu,x_0}^{2^*} \, dx = S^{n/2}$$

(see [29]).

PROOF OF LEMMA 2.4. The sequence $(u_i)_i$ is bounded in $H_0^{1,2}(\Omega)$, because $\lambda < \lambda_1$ and the sequence $(f_\lambda(u_i))_i$ is bounded.

Since $f'_{\lambda|V}(u_i) \to 0$ in $H^{-1}(\Omega)$, it follows that there exists a sequence $(\mu_i)_i$ in $\mathbb{R}$ such that

$$\Delta u_i + \lambda u_i + \mu_i (u_i^+)^{2^*-1} \to 0 \qquad \text{in } H^{-1}(\Omega).$$

Therefore $\mu_i - f_\lambda(u_i) \to 0$ and so $\mu_i \to c$ (notice that $c \geq S_\lambda$ and $S_\lambda > 0$ for $\lambda < \lambda_1$).

If we set $U_i = \mu_i^{(n-2)/4} u_i$, the sequence $(U_i)_i$ satisfies

$$F_\lambda(U_i) \to \frac{1}{n} c^{n/2}; \qquad F'_\lambda(U_i) \to 0 \qquad \text{in } H^{-1}(\Omega).$$

By Proposition 2.5, it suffices to prove that the integer k in that proposition is 0.

We have $(1/n)c^{n/2} = F_\lambda(U_0) + (k/n)S^{n/2}$ where $U_0 \in H_0^{1,2}(\Omega)$ solves the equation $\Delta U_0 + \lambda U_0 + (U_0^+)^{2^*-1} = 0$.

We claim that $U_0 \not\equiv 0$: otherwise we would have $c = k^{2/n} S$ with $k \neq 0$ because $c > 0$, $k \neq 1$ since $c \neq S$, and $k < 2$ because $c < (S_\lambda^{n/2} + S^{n/2})^{2/n} \leq 2^{2/n} S$.

On the other hand, $U_0^- = 0$ because $\lambda < \lambda_1$ and so $U_0 > 0$; moreover, it follows that $F_\lambda(U_0) \geq (1/n) S_\lambda^{n/2}$ because $f_\lambda(U_0/\|U_0\|_{2^*}) \geq S_\lambda$.

Therefore we have $c \geq (S_\lambda^{n/2} + kS^{n/2})^{2/n}$, which implies $k < 1$, since $c < (S_\lambda^{n/2} + S^{n/2})^{2/n}$ according to our assumptions.

Hence $k = 0$ and so $(u_i)_i$ (or a subsequence) converges to $U_0/\|U_0\|_{2^*}$ in $H_0^{1,2}(\Omega)$.

DEFINITION 2.6. Let Ω be a smooth bounded domain in $\mathbb{R}^n$ with $n \geq 3$. Choose $r > 0$ small enough that

$$\Omega_r^+ = \{x \in \mathbb{R}^n : \operatorname{dist}(x, \Omega) \leq r\}, \qquad \Omega_r^- = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq r\}$$

are homotopically equivalent to Ω.

Let $\beta : V \to \mathbb{R}^n$ be the map defined by

$$\beta(u) = \int_\Omega x(u^+(x))^{2^*} \, dx, \qquad \forall u \in V.$$

LEMMA 2.7. *Under the notations introduced in Definition* 2.6,

$$\lim_{\lambda \to 0} \inf\{f_\lambda(u) : u \in V, \ \beta(u) \notin \Omega_r^+\} > S$$

(*we set* $\inf \emptyset = +\infty$ *if* $\{u \in V : \beta(u) \notin \Omega_r^+\} = \emptyset$).

PROOF. By contradiction, suppose that there exist two sequences $(\varepsilon_i)_i$ in $\mathbb{R}$ and $(u_i)_i$ in V, with $\lim_{i\to\infty} \varepsilon_i = 0$ and $\beta(u_i) \notin \Omega_r^+$ for $i \in \mathbb{N}$, such that $\lim_{i\to\infty} f_{\varepsilon_i}(u_i) \leq S$. It follows that $\lim_{i\to\infty} f_{\varepsilon_i}(u_i^+) \leq S$ because $f_{\varepsilon_i}(u_i^+) \leq f_{\varepsilon_i}(u_i)$ whenever $\varepsilon_i \leq \lambda_1$. Since $\int_\Omega |Du_i^+|^2 \, dx \geq S$ for $i \in \mathbb{N}$ and

$$\int_\Omega (u_i^+)^2 \, dx \leq (\operatorname{meas} \Omega)^{2/n} \left(\int_\Omega (u_i^+)^{2^*} \, dx \right)^{2/2^*} = (\operatorname{meas} \Omega)^{2/n},$$

we obtain $\lim_{i\to\infty} \int_\Omega |Du_i^+|^2 \, dx = S$. Therefore, by a well known result of P. L. Lions (see [16]), we get $\lim_{i\to\infty} \operatorname{dist}(\beta(u_i), \Omega) = 0$, which is impossible because $\beta(u_i) \notin \Omega_r^+$ for $i \in \mathbb{N}$. □

Lastly, we prove a simple algebraic relation which will be useful in the sequel.

LEMMA 2.8. *For every* $p \geq 2$ *and all* $a, b \in \mathbb{R}^+$,

$$(a+b)^p \geq a^p + b^p + pa^{p-1}b.$$

PROOF. We have

$$\begin{aligned}(a+b)^p - a^p - b^p &= p(p-1) \int_0^a dt \int_0^b (\tau + t)^{p-2} \, d\tau \\ &\geq bp(p-1) \int_0^a t^{p-2} \, dt = pa^{p-1}b.\end{aligned}$$

3. Multiplicity of positive solutions

DEFINITION 3.1. Let Y be a topological space and $X \subset Y$. We say that the *Lyusternik-Schnirelman category* of X in Y is m (and we write $\operatorname{cat}(X, Y) = m$) if m is the least nonnegative integer such that X can be covered by m closed subsets of Y, each contractible in Y.

By the category of Z we mean the category of Z in itself: $\operatorname{cat} Z = \operatorname{cat}(Z, Z)$.

THEOREM 3.2. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ with $n \geq 4$. Assume that Ω is not contractible in itself* ($\operatorname{cat}\Omega > 1$). *Then there exists $\overline{\lambda} \in]0, \lambda_1[$ such that for every $\lambda \in]0, \overline{\lambda}[$ Problem $P_\lambda(\Omega)$ has at least* $\operatorname{cat}\Omega + 1$ *solutions. More precisely, if we set $m = \operatorname{cat}\Omega$ and $S_\lambda = \inf_V f_\lambda$, then for each $\lambda \in]0, \overline{\lambda}[$ there exist at least m solutions $u_{1,\lambda}, \dots, u_{m,\lambda}$ such that*

$$S_\lambda \leq f_\lambda\left(\frac{u_{i,\lambda}}{\|u_{i,\lambda}\|_{2^*}}\right) < S \qquad \forall i = 1, \dots, m,$$

and there exists at least one solution $\widehat{u}_\lambda$ such that

$$S < f_\lambda\left(\frac{\widehat{u}_\lambda}{\|\widehat{u}_\lambda\|_{2^*}}\right) < (S_\lambda^{n/2} + S^{n/2})^{2/n}$$

(*notice that $S_\lambda > 0$ for $\lambda < \lambda_1$*). *Moreover,*

$$\inf\left\{f_\lambda\left(\frac{\widehat{u}_\lambda}{\|\widehat{u}_\lambda\|_{2^*}}\right) : \lambda \in]0, \overline{\lambda}[\right\} > S.$$

For the proof, see 3.5 and 3.7.

DEFINITION 3.3. Let $\varphi \in C_0^\infty(B(0, r))$ be a function with radial symmetry such that $0 \leq \varphi \leq 1$ and $\varphi = 1$ in $B(0, r/2)$.

For every $\mu > 0$ set $\psi_\mu(x) = \varphi(x) U_\mu(x)$, where

$$U_\mu(x) = \frac{[n(n-2)\mu]^{(n-2)/4}}{[\mu + |x|^2]^{(n-2)/2}}.$$

Let $\Phi_\mu : \Omega_r^- \to V$ be the map defined by

$$\Phi_\mu(y)[x] = \frac{\psi_\mu(x - y)}{\|\psi_\mu\|_{2^*}}$$

(notice that $\beta \circ \Phi_\mu(y) = y$ for $y \in \Omega_r^-$).

We shall use the following lemma:

LEMMA 3.4 (see [5], [7], [10]). *Under the notations introduced in Definition 3.3:*

(a) $f_\lambda(\psi_\mu/\|\psi_\mu\|_{2^*}) = \begin{cases} S - \lambda c_1\mu + O(\mu^{(n-2)/2}) & \text{if } n \geq 5, \\ S - \lambda c_2\mu|\log\mu| + O(\mu) & \text{if } n = 4, \end{cases}$

(b) $\|\psi_\mu\|_1 \leq c_3\mu^{(n-2)/4}$,

(c) $\|\psi_\mu\|_{2^*}^{2^*} = S^{n/2} + O(\mu^{n/2})$,

where c_1, c_2, c_3 are suitable positive constants.

3.5. PROOF OF THEOREM 3.2. By Lemma 2.7, there exists $\overline{\lambda} \in]0, \lambda_1[$ such that

$$\inf\{f_{\overline{\lambda}}(u) : u \in V,\ \beta(u) \notin \Omega_r^+\} > S.$$

Let $\lambda \in]0, \overline{\lambda}[$; then we have

$$\inf\{f_\lambda(u) : u \in V,\ \beta(u) \notin \Omega_r^+\} \geq \inf\{f_{\overline{\lambda}}(u) : u \in V,\ \beta(u) \notin \Omega_r^+\} > S.$$

Moreover, from Lemma 3.4 it follows that $f_\lambda(\psi_\mu/\|\psi_\mu\|_{2^*}) < S$ for μ small enough.

For every $c \in \mathbb{R}$ we set $f_\lambda^c = \{u \in V : f_\lambda(u) \leq c\}$; let $c_{\lambda,\mu} = f_\lambda(\psi_\mu/\|\psi_\mu\|_{2^*})$. Then clearly $\phi_\mu(\Omega_r^-) \subset f_\lambda^{c_{\lambda,\mu}}$. As in [2] and [15], one can prove that for every c such that

$$c_{\lambda,\mu} \leq c < \inf\{f_\lambda(u) : u \in V,\ \beta(u) \notin \Omega_r^+\},$$

we have

$$\operatorname{cat}(\Phi_\mu(\Omega_r^-), f_\lambda^c) \geq \operatorname{cat}(\Omega_r^-, \Omega_r^+) = \operatorname{cat}\Omega.$$

Since $c_{\lambda,\mu} < S$, from Lemma 2.4 we deduce that in the sublevel $f_\lambda^{c_{\lambda,\mu}}$ there exist at least $\operatorname{cat}\Omega$ critical points for f_λ on V. Moreover, since Ω is not contractible in itself (i.e. $\operatorname{cat}\Omega > 1$), we also have $\operatorname{cat}(\Phi_\mu(\Omega_r^-), f_\lambda^c) > 1$, that is, $\Phi_\mu(\Omega_r^-)$ is not contractible in f_λ^c for any c such that

$$c_{\lambda,\mu} \leq c < \inf\{f_\lambda(u) : u \in V,\ \beta(u) \notin \Omega_r^+\}.$$

Set

$$\widehat{c}_{\lambda,\mu} = \inf\{c \in \mathbb{R} : \Phi_\mu(\Omega_r^-) \text{ is contractible in } f_\lambda^c\}.$$

Hence

$$\widehat{c}_{\lambda,\mu} \geq \inf\{f_\lambda(u) : u \in V,\ \beta(u) \notin \Omega_r^+\} > S.$$

Notice that for $\varepsilon \in]0, \widehat{c}_{\lambda,\mu} - S[$ we must have

$$\inf\{\|f'_{\lambda|V}(u)\|_{H^{-1}(\Omega)} : u \in V,\ |f_\lambda(u) - \widehat{c}_{\lambda,\mu}| < \varepsilon\} = 0;$$

otherwise there would exist $\overline{\varepsilon} \in]0, \widehat{c}_{\lambda,\mu} - S[$ such that $f_\lambda^{\widehat{c}_{\lambda,\mu}-\overline{\varepsilon}}$ is a deformation retract of $f_\lambda^{\widehat{c}_{\lambda,\mu}+\overline{\varepsilon}}$; but this is impossible: the set $\Phi_\mu(\Omega_r^-)$, which is contained in $f_\lambda^{\widehat{c}_{\lambda,\mu}-\overline{\varepsilon}}$, is contractible in $f_\lambda^{\widehat{c}_{\lambda,\mu}+\overline{\varepsilon}}$ but not in $f_\lambda^{\widehat{c}_{\lambda,\mu}-\overline{\varepsilon}}$, by definition of $\widehat{c}_{\lambda,\mu}$. Therefore, there exists a sequence $(u_i)_i$ in V such that $f_\lambda(u_i) \to \widehat{c}_{\lambda,\mu}$ and $f'_{\lambda|V}(u_i) \to 0$ to $H^{-1}(\Omega)$. Then, in order to prove that $\widehat{c}_{\lambda,\mu}$ is a critical value for f_λ on V, by Lemma 2.4, it suffices to prove that $\widehat{c}_{\lambda,\mu} < (S_\lambda^{n/2} + S^{n/2})^{2/n}$ for μ small enough. This fact will be deduced (see Remark 3.7) from the following lemma.

LEMMA 3.6. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ with $n \geq 4$ and $\lambda \in]0, \lambda_1[$. Let $S_\lambda = \min_V f_\lambda$ and $u_\lambda \in V$ be such that $f_\lambda(u_\lambda) = S_\lambda$ (this minimum exists by Lemma 2.3 and clearly $u_\lambda > 0$ in Ω, since $\lambda < \lambda_1$).*

For every $y \in \Omega_r^-$ and $t \in [0,1]$ let us set

$$\widehat{u}_{\lambda,\mu}(y,t) = \frac{tu_\lambda + (1-t)\Phi_\mu(y)}{\|tu_\lambda + (1-t)\Phi_\mu(y)\|_{2^*}}$$

where $\Phi_\mu : \Omega_r^- \to V$ is the function defined in 3.3. Then there exists $\widehat{\mu} > 0$ such that for every $\mu \in]0, \widehat{\mu}[$,

$$\max\{f_\lambda(\widehat{u}_{\lambda,\mu}(y,t)) : t \in [0,1],\ y \in \Omega_r^-\} < (S_\lambda^{n/2} + S^{n/2})^{2/n}.$$

PROOF. Note that for every $u \in H_0^{1,2}(\Omega)$ with $u \not\equiv 0$ we have

$$\max\left\{\frac{\tau^2}{2}\int_\Omega (|Du|^2 - \lambda u^2)\,dx - \frac{\tau^{2^*}}{2^*}\int_\Omega |u|^{2^*}\,dx : \tau \geq 0\right\} = \frac{1}{n}\left[f_\lambda\left(\frac{u}{\|u\|_{2^*}}\right)\right]^{n/2};$$

so it is equivalent to prove that, if we set

$$F_\lambda(u) = \frac{1}{2}\int_\Omega (|Du|^2 - \lambda u^2)\,dx - \frac{1}{2^*}\int_\Omega |u|^{2^*}\,dx \qquad \forall u \in H_0^{1,2}(\Omega),$$

we have, for μ small enough,

$$\max\{F_\lambda(\alpha u_\lambda + \beta\Phi_\mu(y)) \,:\, y \in \Omega_r^-,\ \alpha \geq 0,\ \beta \geq 0\} < \frac{1}{n}(S_\lambda^{n/2} + S^{n/2}).$$

Since

$$F_\lambda(\alpha u_\lambda + \beta\Phi_\mu(y)) = \frac{1}{2}f_\lambda(\alpha u_\lambda + \beta\Phi_\mu(y)) - \frac{1}{2^*}\int_\Omega (\alpha u_\lambda + \beta\Phi_\mu(y))^{2^*}\,dx,$$

we estimate separately the two terms: we have

$$\begin{aligned} f_\lambda(\alpha u_\lambda + \beta\Phi_\mu(y)) =& \alpha^2 S_\lambda + \beta^2 f_\lambda(\Phi_\mu(y)) \\ &+ 2\alpha\beta\int_\Omega [Du_\lambda D\Phi_\mu(y) - \lambda u_\lambda\Phi_\mu(y)]\,dx, \end{aligned}$$

where

$$\int_\Omega [Du_\lambda D\Phi_\mu(y) - \lambda u_\lambda \Phi_\mu(y)]\,dx = S_\lambda \int_\Omega u_\lambda^{2^*-1}\Phi_\mu(y)\,dx$$

because $\Delta u_\lambda + \lambda u_\lambda + S_\lambda u_\lambda^{2^*-1} = 0$. Using Lemma 2.8 we obtain

$$\int_\Omega (\lambda u_\lambda + \beta\Phi_\mu(y))^{2^*}\,dx \geq \alpha^{2^*} + \beta^{2^*} + 2^*\alpha^{2^*-1}\beta \int_\Omega u_\lambda^{2^*-1}\Phi_\mu(y)\,dx.$$

Hence, if we set $\varepsilon_\mu(y) = \int_\Omega u_\lambda^{2^*-1}\Phi_\mu(y)\,dx$, we have

$$\begin{aligned} F_\lambda(\alpha u_\lambda + \beta\Phi_\mu(y)) \leq &\frac{\alpha^2}{2}S_\lambda - \frac{\alpha^{2^*}}{2^*} + \frac{\beta^2}{2} f_\lambda\left(\frac{\psi_\mu}{\|\psi_\mu\|_{2^*}}\right) \\ &- \frac{\beta^{2^*}}{2^*} + \alpha\beta(S_\lambda - \alpha^{2^*-2})\varepsilon_\mu(y). \end{aligned}$$

By Lemma 3.4, $\lim_{\mu\to 0^+} f_\lambda(\psi_\mu/\|\psi_\mu\|_{2^*}) = S$, and moreover,

$$|\varepsilon_\mu(y)| \leq \|u_\lambda\|_{L^\infty(\Omega)}^{2^*-1} \frac{\|\Phi_\mu\|_1}{\|\Phi_\mu\|_{2^*}} = O(\mu^{(n-2)/4}).$$

Therefore, if we set

$$\begin{aligned} \Gamma(\alpha,\beta,\mu,y) = &\frac{\alpha^2}{2}S_\lambda - \frac{\alpha^{2^*}}{2^*} + \frac{\beta^2}{2} f_\lambda\left(\frac{\psi_\mu}{\|\psi_\mu\|_{2^*}}\right) \\ &- \frac{\beta^{2^*}}{2^*} + \alpha\beta(S_\lambda - \alpha^{2^*-2})\varepsilon_\mu(y), \end{aligned}$$

we have

$$\lim_{\mu\to 0^+} \Gamma(\alpha,\beta,\mu,y) = \frac{\alpha^2}{2}S_\lambda - \frac{\alpha^{2^*}}{2^*} + \frac{\beta^2}{2}S - \frac{\beta^{2^*}}{2^*} \qquad \forall\alpha,\beta\in\mathbb{R}^+ \text{ and } \forall y\in\Omega_r^-;$$

moreover, there exist $\overline{\mu} > 0$ and two positive constants $\overline{c}_1$ and $\overline{c}_2$ such that

$$\Gamma(\alpha,\beta,\mu,y) \leq \overline{c}_1 - \overline{c}_2(\alpha^{2^*} + \beta^{2^*}) \qquad \forall\mu\in]0,\overline{\mu}[,\ \forall\alpha,\beta\in\mathbb{R}^+,\ \forall y\in\Omega_r^-.$$

It follows that for all $y\in\Omega_r^-$ and $\mu\in]0,\overline{\mu}[$ there exists a pair $M_{\mu,y} = (\alpha_{\mu,y},\beta_{\mu,y})$ with $\alpha_{\mu,y}\geq 0$ and $\beta_{\mu,y}\geq 0$ such that

$$\Gamma(\alpha_{\mu,y},\beta_{\mu,y},\mu,y) = \max\{\Gamma(\alpha,\beta,\mu,y)\ :\ \alpha\geq 0,\ \beta\geq 0\};$$

moreover, the set $\{M_{\mu,y} : y\in\Omega_r^-, \mu\in]0,\overline{\mu}[\}$ is bounded in $\mathbb{R}^2$.

If we set $\overline{M} = (S_\lambda^{(n-2)/4}, S^{(n-2)/4})$, we can prove that

$$\lim_{\mu\to 0^+} \sup\{|M_{\mu,y} - \overline{M}|_{\mathbb{R}^2}\ :\ y\in\Omega_r^-\} = 0.$$

Indeed, suppose by contradiction that there exist two sequences $(\mu_i)_i \to 0$ and $(y_i)_i \in \Omega_r^-$ such that

$$\lim_{i\to\infty}(\alpha_{\mu_i,y_i}, \beta_{\mu_i,y_i}) = (\alpha_0, \beta_0) \neq \overline{M}.$$

Then we would have

$$\lim_{i\to\infty}\Gamma(\alpha_{\mu_i,y_i}, \beta_{\mu_i,y_i}, \mu_i, y_i) = \frac{\alpha_0^2}{2}S_\lambda - \frac{\alpha_0^{2^*}}{2^*} + \frac{\beta_0^2}{2}S - \frac{\beta_0^{2^*}}{2^*} < \frac{1}{n}(S_\lambda^{n/2} + S^{n/2}),$$

while

$$\lim_{i\to\infty}\Gamma(S_\lambda^{(n-2)/4}, S^{(n-2)/4}, \mu_i, y_i) = \frac{1}{n}(S_\lambda^{n/2} + S^{n/2}),$$

in contradiction with the fact that

$$\Gamma(\alpha_{\mu_i,y_i}, \beta_{\mu_i,y_i}, \mu_i, y_i) \geq \Gamma(S_\lambda^{(n-2)/4}, S^{(n-2)/4}, \mu_i, y_i) \qquad \forall i \in \mathbb{N}.$$

Hence the pair $(\alpha_{\mu,y}, \beta_{\mu,y})$ is in the interior of $\mathbb{R}^+ \times \mathbb{R}^+$ for μ small enough and so necessarily $\frac{\partial\Gamma}{\partial\alpha}(\alpha_{\mu,y}, \beta_{\mu,y}, \mu, y) = 0$, that is,

$$\alpha_{\mu,y}(S_\lambda - \alpha_{\mu,y}^{2^*-2}) + \beta_{\mu,y}(S_\lambda - (2^*-1)\alpha_{\mu,y}^{2^*-2})\varepsilon_\mu(y) = 0,$$

which implies

$$|\lambda_{\mu,y}(S_\lambda - \lambda_{\mu,y}^{2^*-2})| \leq O(\mu^{(n-2)/4}).$$

Therefore we have

$$\Gamma(\alpha_{\mu,y}, \beta_{\mu,y}, \mu, y) \leq \frac{1}{n}S_\lambda^{n/2} + \frac{1}{n}\left[f_\lambda\left(\frac{\psi_\mu}{\|\psi_\mu\|_{2^*}}\right)\right]^{n/2} + O(\mu^{(n-2)/2}).$$

Since $f_\lambda(\psi_\mu/\|\psi_\mu\|_{2^*}) < S$ for μ small enough, by Lemma 3.4 we deduce that $\Gamma(\alpha_{\mu,y}, \beta_{\mu,y}, \mu, y)$ is less than

$$\frac{1}{n}(S_\lambda^{n/2} + S^{n/2}) + \frac{1}{n}S^{(n-2)/2}[-\lambda c_1\mu + O(\mu^{(n-2)/2})] + O(\mu^{(n-2)/2}) \qquad \text{for } n \geq 5$$

and less than

$$\frac{1}{n}(S_\lambda^{n/2} + S^{n/2}) + \frac{1}{n}S^{(n-2)/2}[-\lambda c_2\mu|\log\mu| + O(\mu)] + O(\mu) \qquad \text{for } n = 4.$$

In any case

$$\Gamma(\alpha_{\mu,y}, \beta_{\mu,y}, \mu, y) < \frac{1}{n}(S_\lambda^{n/2} + S^{n/2})$$

for μ small enough and so

$$\max\{F_\lambda(\alpha u_\lambda + \beta\Phi_\mu(y)) : y \in \Omega_r^-, \ \alpha \geq 0, \ \beta \geq 0\} < \frac{1}{n}(S_\lambda^{n/2} + S^{n/2}).$$

REMARK 3.7. In order to conclude the proof of Theorem 3.2, it suffices to observe that $\Phi_\mu(\Omega_r^-)$ is contractible in each sublevel which contains the set $\{\widehat{u}_{\lambda,\mu}(y,t) : y \in \Omega_r^-, t \in [0,1]\}$; hence we have

$$\widehat{c}_{\lambda,\mu} \leq \max\{f_\lambda(\widehat{u}_{\lambda,\mu}(y,t)) \,:\, y \in \Omega_r^-,\ t \in [0,1]\}.$$

So we get $\widehat{c}_{\lambda,\mu} < (S_\lambda^{n/2} + S^{n/2})^{2/n}$ for μ small enough.

3.8. Multiplicity of solutions in symmetrical domains. Notice that the number of solutions of Problem $P_\lambda(\Omega)$ for λ small enough can be even considerably greater than $\operatorname{cat}\Omega + 1$ if the noncontractible bounded domain Ω has some symmetry properties. Suppose, for instance, that Ω is a smooth bounded noncontractible domain which is symmetric with respect to 0 (i.e. $-\Omega = \Omega$) with $0 \notin \overline{\Omega}$. Let the *genus* of Ω (denoted by $\gamma(\Omega)$) be the least nonnegative integer m such that there exist m closed subsets $F_1, \dots, F_m$ in $\mathbb{R}^n \setminus \{0\}$ satisfying the following conditions:

$$F_i \cap (-F_i) = \emptyset \quad \forall i = 1, \dots, m \qquad \text{and} \qquad \bigcup_{i=1}^{m} (F_i \cup (-F_i)) \supset \Omega.$$

Then one can prove (see [18]) that, if $\lambda \in]0, \lambda_1[$ is small enough, then Problem $P_\lambda(\Omega)$ has at least $\gamma(\Omega)$ pairs of solutions $(u, \widetilde{u})$, with $\widetilde{u}(x) = u(-x)$, such that $f_\lambda(u) < S$. Notice that for λ small enough, since $0 \notin \overline{\Omega}$,

$$\inf\{f_\lambda(u) \,:\, u \in V,\ \beta(u) = 0\} > S$$

(see Lemma 2.7).

It follows that, if $f_\lambda(u) < S$, then $\widetilde{u} \neq u$ because $\beta(\widetilde{u}) = -\beta(u) \neq 0$. Hence Problem $P_\lambda(\Omega)$ has at least $2\gamma(\Omega) + 1$ distinct solutions.

For instance, if Ω is a symmetrical bounded domain homotopically equivalent to the $(k-1)$-dimensional sphere S_{k-1}, then $\gamma(\Omega) = k$ and so we have at least $2k + 1$ distinct solutions, even if the category of Ω is only 2.

4. Persistence of the solutions with respect to perturbations of small capacity

The results of this section show that the number of solutions of Problem $P_\lambda(\Omega)$ for λ small enough is related not just to the topology of Ω, but to the topology of a domain $\widetilde{\Omega}$ which differs from Ω by a set with sufficiently small capacity: if $\widetilde{\Omega}$

is a noncontractible domain, we can choose a closed set K with sufficiently small capacity in such a way that Problem $P_\lambda(\widetilde{\Omega}\backslash K)$ has the same number of solutions as $P_\lambda(\widetilde{\Omega})$, even if the domain $\Omega = \widetilde{\Omega}\backslash K$ is contractible.

DEFINITION 4.1. Let B be a smooth bounded domain in $\mathbb{R}^n$ with $n \geq 3$, $u \in H^{1,2}(B)$, $K \subset B$. We say that $u \geq 1$ *on* K *in the sense of* $H^{1,2}(B)$ if there exists a sequence $(u_i)_i$ in $C^1(\overline{B})$ such that $u_i \geq 1$ on K for all $i \in \mathbb{N}$ and $u_i \to u$ in $H^{1,2}(B)$. If the set

$$\{u \in H_0^{1,2}(B) \,:\, u \geq 1 \quad \text{on } K \text{ in the sense of } H^{1,2}(B)\}$$

is nonempty, the *capacity* of K with respect to B is the number

$$\operatorname{cap}_B K = \inf\left\{ \int_B |Du|^2\, dx \,: u \in H_0^{1,2}(B), \right.$$
$$\left. u \geq 1 \text{ on } K \text{ in the sense of } H^{1,2}(B) \right\}.$$

Moreover, we set $\operatorname{cap}_B \emptyset = 0$.

It is well known that there exists a unique function $\chi_K \in H_0^{1,2}(B)$ such that $\int_B |D\chi_K|^2\, dx = \operatorname{cap}_B K$ and $\chi_K \geq 1$ on K in the sense of $H^{1,2}(B)$. Furthermore, $0 \leq \chi_K \leq 1$ in B.

THEOREM 4.2. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ with $n \geq 4$. Suppose that Ω is noncontractible in itself and set $B = B(0,R)$ with $R > \sup\{|x|_{\mathbb{R}^n} : x \in \Omega\}$. Let $(\Omega_i)_i$ be a sequence of smooth domains such that $\Omega_i \subset \Omega$ for all $i \in \mathbb{N}$ and suppose that $\lim_{i\to\infty} \operatorname{cap}_B K_i = 0$, where $K_i = \Omega \setminus \Omega_i$. Then there exist $\overline{\lambda} \in]0,\lambda_1[$ and a map $j :]0,\overline{\lambda}[\to \mathbb{N}$ such that, if $\lambda \in]0,\overline{\lambda}[$ and $i > j(\lambda)$, then Problem $P_\lambda(\Omega_i)$ has at least $\operatorname{cat}\Omega + 1$ solutions (and one can give, for the corresponding critical values, some estimates analogous to the ones of Theorem 3.2).*

PROOF. We argue as in the proof of Theorem 3.2. Set

$$V_i = V \cap H_0^{(1,2)}(\Omega_i); \qquad f_{i,\lambda} = f_{\lambda|V_i}; \qquad S_{i,\lambda} = \inf_{V_i} f_{i,\lambda}.$$

We have clearly $S_{i,\lambda} \geq S_\lambda$ and

$$\inf\{f_{i,\lambda}(u) \,:\, u \in V_i,\ \beta(u) \notin \Omega_r^+\} \geq \inf\{f_\lambda(u) \,:\, u \in V,\ \beta(u) \notin \Omega_r^+\}.$$

As in Theorem 3.2, choose $\overline{\lambda} \in]0,\lambda_1[$ in such a way that

$$\inf\{f_\lambda(u) \,:\, u \in V,\ \beta(u) \notin \Omega_r^+\} > S \qquad \forall \lambda \in]0,\overline{\lambda}[.$$

Then fix $\mu > 0$ such that

$$\begin{aligned} f_\lambda\left(\frac{\psi_\mu}{\|\psi_\mu\|_{2^*}}\right) &< S < \inf\{f_\lambda(u) \,:\, u \in V,\ \beta(u) \notin \Omega_r^+\} \\ &\le \max\{f_\lambda(\widehat{u}_{\lambda,\mu}(y,t)) \,:\, y \in \Omega_r^-,\ t \in [0,1]\} \\ &< (S_\lambda^{n/2} + S^{n/2})^{2/n}. \end{aligned}$$

Since $\lim_{i\to\infty} \mathrm{cap}_B K_i = 0$, there exists a sequence $(\chi_i)_i$ in $H_0^{1,2}(B)$ such that $\chi_i \to 0$ in $H_0^{1,2}(B)$, $0 \le \chi_i \le 1$, and $\chi_i \ge 1$ on K_i in the sense of $H^{1,2}(B)$ for every $i \in \mathbb{N}$.

Let $\Phi_\mu^i : \Omega_r^- \to V_i$ be the map defined by

$$\Phi_\mu^i(y) = \frac{(1-\chi_i)\Phi_\mu(y)}{\|(1-\chi_i)\Phi_\mu(y)\|_{2^*}} \qquad \forall y \in \Omega_r^-.$$

Moreover, set

$$\widehat{u}^i_{\lambda,\mu}(y,t) = \frac{(1-\chi_i)\widehat{u}_{\lambda,\mu}(y,t)}{\|(1-\chi_i)\widehat{u}_{\lambda,\mu}(y,t)\|_{2^*}} \qquad \forall t \in [0,1] \text{ and } \forall y \in \Omega_r^-.$$

One can prove (as in the proof of Theorem A.1 of [3]) that $\Phi_\mu^i(y)$ and $\widehat{u}^i_{\lambda,\mu}(y,t)$ are well defined because, for i large enough,

$$\begin{aligned} \|(1-\chi_i)\Phi_\mu(y)\|_{2^*} &\neq 0 \quad \text{and} \\ \|(1-\chi_i)\widehat{u}_{\lambda,\mu}(y,t)\|_{2^*} &\neq 0 \qquad \forall y \in \Omega_r^-,\ \forall t \in [0,1]. \end{aligned}$$

Moreover, one can prove as in [3] that

$$\lim_{i\to\infty} \max\{f_{i,\lambda}(\Phi_\mu^i(y)) \,:\, y \in \Omega_r^-\} = \max\{f_\lambda(\Phi_\mu(y)) \,:\, y \in \Omega_r^-\}$$

and

$$\begin{aligned} \lim_{i\to\infty} \max\{f_{i,\lambda}(\widehat{u}^i_r(y,t)) \,:\, y \in \Omega_r^-,\ t \in [0,1]\} \\ = \max\{f_\lambda(\widehat{u}_{\lambda,\mu}(y,t)) \,:\, y \in \Omega_r^-,\ t \in [0,1]\}. \end{aligned}$$

Hence, by the estimates obtained in the proof of Theorem 3.2, there exists $j(\lambda) \in \mathbb{N}$ such that we have, for all $i > j(\lambda)$,

$$\begin{aligned} \max\{f_{i,\lambda}(\Phi_\mu^i(y) \,:\, y \in \Omega_r^-\} &< S < \inf\{f_{i,\lambda}(u) \,:\, u \in V_i,\ \beta(u) \notin \Omega_r^+\} \\ &\le \max\{f_{i,\lambda}(\widehat{u}^i_{\lambda,\mu}(y,t)) \,:\, y \in \Omega_r^-,\ t \in [0,1]\} \\ &< (S_\lambda^{n/2} + S^{n/2})^{2/n} \le (S_{i,\lambda}^{n/2} + S^{n/2})^{2/n}. \end{aligned}$$

Since $\beta \circ \Phi_\mu^i$ is homotopically equivalent to the identity in Ω_r^+ because $\beta \circ \Phi_\mu^i(y) \in B(y,r) \subset \Omega$ for all $y \in \Omega_r^-$, we deduce as in [3] that, if we set

$$c_{\lambda,\mu}^i = \max\{f_{i,\lambda}(\Phi_\mu^i(y)) \ : \ y \in \Omega_r^-\},$$

then we have $\operatorname{cat}(\Phi_\mu^i(\Omega_r^-), f_{i,\lambda}^c) \geq \operatorname{cat}\Omega$ for every c such that

$$c_{\lambda,\mu}^i \leq c < \inf\{f_{i,\lambda}(u) \ : \ u \in V_i, \ \beta(u) \notin \Omega_r^+\}.$$

This implies that $\Phi_\mu^i(\Omega_r^-)$ is not contractible in those sublevels $f_{i,\lambda}^c$ and that there exist at least $\operatorname{cat}\Omega$ critical points in the sublevel $f_{i,\lambda}^{c_{\lambda,\mu}^i}$. Moreover, if we set

$$\widehat{c}_{\lambda,\mu}^i = \inf\{c \ : \ \Phi_\mu^i(\Omega_r^-) \text{ is contractible in } f_{i,\lambda}^c\},$$

then we have

$$\begin{aligned} S &< \inf\{f_{i,\lambda}(u) \ : \ u \in V_i, \ \beta(u) \notin \Omega_r^+\} \leq \widehat{c}_{\lambda,\mu}^i \\ &\leq \max\{f_{i,\lambda}(\widehat{u}_{\lambda,\mu}^i(y,t) \ : \ y \in \Omega_r^-, \ t \in [0,1]\} \\ &< (S_{i,\lambda}^{n/2} + S^{n/2})^{2/n}. \end{aligned}$$

Hence, as in the proof of Theorem 3.2, from Lemma 2.4 we can deduce that $\widehat{c}_{\lambda,\mu}^i$ is a critical value for the functional $f_{i,\lambda}$ restricted to V_i.

REMARK 4.3. Simple examples show that suitable perturbations of a domain Ω by sets with small capacity can modify its topology.

For instance, in Theorem 4.2 it can happen that Ω is not contractible in itself but the nearby domains Ω_i are contractible. In this case the topology of the domains Ω_i would only guarantee the existence of one solution, while by Theorem 4.2 we obtain at least three solutions.

Moreover, if the domain Ω and the perturbed domains Ω_i have some symmetry properties, then the number of solutions increases even more. For instance, if we suppose that $0 \notin \overline{\Omega}$ and that $-\Omega = \Omega$, $-\Omega_i = \Omega_i$ for all $i \in \mathbb{N}$ (besides the assumptions of Theorem 4.2), then there exist $\overline{\lambda} \in]0, \lambda_1[$ and a map $j :]0, \overline{\lambda}[\to \mathbb{N}$ such that, if $\lambda \in]0, \overline{\lambda}[$ and $i > j(\lambda)$, then Problem $P_\lambda(\Omega_i)$ has at least $2\gamma(\Omega) + 1$ solutions, where $\gamma(\Omega)$ is the genus of Ω (see 3.8). More precisely, there exist at least $\gamma(\Omega)$ pairs of critical points $(u, \widetilde{u})$, with $\widetilde{u}(x) = u(-x)$, such that $f_\lambda(u) < S$ and $\widetilde{u} \neq u$; moreover, there exists at least a critical value between S and $(S_\lambda^{n/2} + S^{n/2})^{2/n}$.

5. Behaviour of the solutions as $\lambda \to 0$

The solutions u_λ of Problem $P_\lambda(\Omega)$ corresponding to critical values of f_λ between S_λ and S satisfy

$$\lim_{\lambda\to 0} \frac{\|Du_\lambda\|_2^2}{\|u_\lambda\|_{2^*}^2} = S.$$

In particular, this holds for the solutions $u_{1,\lambda}, \ldots, u_{m,\lambda}$ given by Theorem 3.2. Hence (see [5]) they converge weakly to zero in $H_0^{1,2}(\Omega)$ as $\lambda \to 0$.

One can also specify their asymptotic behaviour: each one concentrates near a point of Ω, as one can deduce from a well known result of P. L. Lions [16]; moreover the concentration point is interior to Ω and is a critical point for the regularization of the Green function of the domain Ω (see [6], [26]). On the contrary, the solution $\widehat{u}_\lambda$, which satisfies

$$S < f_\lambda\left(\frac{\widehat{u}_\lambda}{\|\widehat{u}_\lambda\|_{2^*}}\right) < (S_\lambda^{n/2} + S^{n/2})^{2/n},$$

under suitable conditions may converge to a function $\widehat{u}_0 > 0$, solution of the limit problem $P_0(\Omega)$; Theorem 5.2 gives a sufficient condition which guarantees this convergence.

Notice that, when the solution $\widehat{u}_\lambda$ does not converge to a solution of Problem $P_0(\Omega)$, then it concentrates near *two* points, as one can deduce from the following lemma.

LEMMA 5.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ with $n \geq 4$ and $(\lambda_k)_k$ be a sequence in $]0, \lambda_1[$ such that $\lambda_k \to 0$. For every $k \in \mathbb{N}$ let $\widehat{u}_{\lambda_k}$ be a solution of Problem $P_{\lambda_k}(\Omega)$ and suppose that*

$$\lim_{k\to\infty} f_{\lambda_k}\left(\frac{\widehat{u}_{\lambda_k}}{\|\widehat{u}_{\lambda_k}\|_{2^*}}\right) = c \in \mathbb{R}$$

(notice that $c \in]S, 2^{2/n}S[$ if $\widehat{u}_{\lambda_k}$ is the solution given by Theorem 3.2). Then we have:

(a) $\lim_{k\to\infty} f_0(\widehat{u}_{\lambda_k}/\|\widehat{u}_{\lambda_k}\|_{2^*}) = c$; $f'_{0|V}(\widehat{u}_{\lambda_k}/\|\widehat{u}_{\lambda_k}\|_{2^*}) \to 0$ *in* $H^{-1}(\Omega)$.

(b) *If $c \in]S, 2^{2/n}S[$, then there exists a subsequence of $(\widehat{u}_{\lambda_k})_k$ converging in $H_0^{1,2}(\Omega)$ to a solution $\widehat{u}_0$ of Problem $P_0(\Omega)$.*

(c) *If $c = 2^{2/n}S$, then at least one of the following cases happens:*

   $1°$ *there exists a subsequence of $(\widehat{u}_{\lambda_k})_k$ converging in $H_0^{1,2}(\Omega)$ to a solution $\widehat{u}_0$ of $P_0(\Omega)$ ($\widehat{u}_0 > 0$); or*

2° there exist two sequences $(x_{1,k})_k$, $(x_{2,k})_k$ *in* Ω *and two sequences of positive numbers* $(\varepsilon_{1,k})_k$, $(\varepsilon_{2,k})_k$ *converging to zero such that:*

$$\lim_{k\to\infty} \frac{1}{\varepsilon_{i,k}} \operatorname{dist}(x_{i,k}, \partial\Omega) = +\infty \qquad \text{for } i = 1, 2,$$

$$\lim_{k\to\infty} \max\left\{ \frac{\varepsilon_{2,k}}{\varepsilon_{1,k}}, \frac{\varepsilon_{1,k}}{\varepsilon_{2,k}}, \frac{|x_{1,k} - x_{2,k}|}{\varepsilon_{1,k} + \varepsilon_{2,k}} \right\} = +\infty,$$

$$\lim_{k\to\infty} \left\| \widehat{u}_{\lambda_k} - \sum_{i=1}^{2} \overline{U}_{\varepsilon_{i,k}}(x - x_{i,k}) \right\|_{H^{1,2}(\Omega)} = 0,$$

where $U_\varepsilon(x) = [n(n-2)\varepsilon]^{(n-2)/4} / [\varepsilon + |x|^2]^{(n-2)/2}$ *for* $\varepsilon > 0$.

PROOF. Since $\widehat{u}_{\lambda_k}/\|\widehat{u}_{\lambda_k}\|_{2^*}$ is bounded in $L^2(\Omega)$, we have

$$f_0\left(\frac{\widehat{u}_{\lambda_k}}{\|\widehat{u}_{\lambda_k}\|_{2^*}}\right) = \frac{\|D\widehat{u}_{\lambda_k}\|_2^2}{\|\widehat{u}_{\lambda_k}\|_{2^*}^2} = f_{\lambda_k}\left(\frac{\widehat{u}_{\lambda_k}}{\|\widehat{u}_{\lambda_k}\|_{2^*}}\right) + \lambda_k \frac{\|\widehat{u}_{\lambda_k}\|_2^2}{\|\widehat{u}_{\lambda_k}\|_{2^*}^2} \to c.$$

Moreover, since $f'_{\lambda_k|V}(\widehat{u}_{\lambda_k}/\|\widehat{u}_{\lambda_k}\|_{2^*}) = 0$, we have

$$\begin{aligned} \left\| f'_{0|V}\left(\frac{\widehat{u}_{\lambda_k}}{\|\widehat{u}_{\lambda_k}\|_{2^*}}\right) \right\|_{H^{-1}(\Omega)} &= \left\| f'_{0|V}\left(\frac{\widehat{u}_{\lambda_k}}{\|\widehat{u}_{\lambda_k}\|_{2^*}}\right) - f'_{\lambda_k|V}\left(\frac{\widehat{u}_{\lambda_k}}{\|\widehat{u}_{\lambda_k}\|_{2^*}}\right) \right\|_{H^{-1}(\Omega)} \\ &\le \left\| f'_0\left(\frac{\widehat{u}_{\lambda_k}}{\|\widehat{u}_{\lambda_k}\|_{2^*}}\right) - f'_{\lambda_k}\left(\frac{\widehat{u}_{\lambda_k}}{\|\widehat{u}_{\lambda_k}\|_{2^*}}\right) \right\|_{H^{-1}(\Omega)} \\ &\le \lambda_k c(\Omega) \frac{\|D\widehat{u}_{\lambda_k}\|_2}{\|\widehat{u}_{\lambda_k}\|_{2^*}} \end{aligned}$$

for a suitable positive constant $c(\Omega)$.

Since the sequence $(\widehat{u}_{\lambda_k}/\|\widehat{u}_{\lambda_k}\|_{2^*})_k$ is bounded in $H_0^{1,2}(\Omega)$, (a) is completely proved. The assertions (b) and (c) follow from (a), upon using well known results which describe the Palais-Smale sequences of the functional f_0 restricted to V (see [16], [28], [1], [4]). □

THEOREM 5.2. *Let* $\widetilde{\Omega}$ *be a contractible smooth bounded domain of* $\mathbb{R}^n$ *with* $n \ge 4$ *and* $\Omega \subset \widetilde{\Omega}$ *be a smooth bounded domain noncontractible in itself. Let* $(\Omega_i)_i$ *be a sequence of smooth bounded domains such that* $\Omega \subset \Omega_i \subset \widetilde{\Omega}$ *for all* $i \in \mathbb{N}$, *and* Ω *be a deformation retract of* Ω_i. *Set* $B = \widetilde{\Omega} \setminus \overline{\Omega}$ *and assume that* $\widetilde{\Omega} \setminus \Omega_i \subset B$ *for all* $i \in \mathbb{N}$ *and* $\lim_{i\to\infty} \operatorname{cap}_B(\widetilde{\Omega} \setminus \Omega_i) = 0$ *(see Definition* 4.1*). Then we have:*

(a) *There exists a sequence* $(\overline{\lambda}_i)_i$ *in* $\mathbb{R}^+$ *such that for all* $\lambda\in]0,\overline{\lambda}_i[$ *Problem* $P_\lambda(\Omega_i)$ *has at least* $\operatorname{cat}\Omega_i+1$ *solutions and one of them, which we shall denote by* $\widehat{u}_{i,\lambda}$, *satisfies*

$$S<f_\lambda\left(\frac{\widehat{u}_{i,\lambda}}{\|\widehat{u}_{i,\lambda}\|_{2^*}}\right)<2^{2/n}S.$$

(b) *Moreover, there exists a positive integer* j *such that, if* $i>j$, *then for every sequence* $(\lambda_k)_k$ *converging to zero in* $]0,\overline{\lambda}_i[$ *the sequence* $(\widehat{u}_{i,\lambda_k})_k$ *has a subsequence converging in* $H_0^{1,2}(\Omega_i)$ *to a function* $\widehat{u}_{i,0}>0$, *solution of the limit problem* $P_0(\Omega_i)$.

PROOF. (a) follows by Theorem 3.2 because Ω_i (as well as Ω) is noncontractible in itself. In order to obtain (b), it suffices to prove that

$$\begin{aligned}S&<\inf\left\{f_\lambda\left(\frac{\widehat{u}_{i,\lambda}}{\|\widehat{u}_{i,\lambda}\|_{2^*}}\right)\,:\,\lambda\in]0,\overline{\lambda}_i[\right\}\\&\le\sup\left\{f_\lambda\left(\frac{\widehat{u}_{i,\lambda}}{\|\widehat{u}_{i,\lambda}\|_{2^*}}\right)\,:\,\lambda\in]0,\overline{\lambda}_i[\right\}<2^{2/n}S\end{aligned}$$

for i large enough, and then to apply Lemma 5.1. Set $\widetilde{V}=\{u\in H_0^{1,2}(\widetilde{\Omega})\,:\,\int_{\widetilde{\Omega}}(u^+)^{2^*}=1\}$, $V_i=\widetilde{V}\cap H_0^{1,2}(\Omega_i)$ and $f_{i,\lambda}=f_{\lambda|V_i}$. Fix $r>0$ sufficiently small such that $\widetilde{\Omega}_r^-=\{x\in\widetilde{\Omega}\,:\,\operatorname{dist}(x,\partial\widetilde{\Omega})\ge r\}$ is homotopically equivalent to $\widetilde{\Omega}$ and also Ω_r^-,Ω_r^+ are homotopically equivalent to Ω. Let $\overline{\mu}>0$ be such that (see Definition 3.3)

$$f_0\left(\frac{\psi_\mu}{\|\psi_\mu\|_{2^*}}\right)=\frac{\|D\psi_\mu\|_2^2}{\|\psi_\mu\|_{2^*}^2}<\left(\frac{3}{2}\right)^{2/n}S\qquad\forall\mu\in]0,\overline{\mu}].$$

Define $\widetilde{\Phi}_{\overline{\mu}}\,:\,\widetilde{\Omega}_r^-\to\widetilde{V}$ by

$$\widetilde{\Phi}_{\overline{\mu}}(y)[x]=\frac{\psi_{\overline{\mu}}(x-y)}{\|\psi_{\overline{\mu}}\|_{2^*}}.$$

Since $\lim_{i\to\infty}\operatorname{cap}_B(\widetilde{\Omega}\setminus\Omega_i)=0$, there exists (see Definition 4.1) a sequence $(\chi_i)_i$ in $H_0^{1,2}(B)$ such that $0\le\chi_i\le1$, $\chi_i\ge1$ on $\widetilde{\Omega}\backslash\Omega_i$ in the sense of $H^{1,2}(B)$ for all $i\in\mathbb{N}$ and $\lim_{i\to\infty}\int_B|D\chi_i|^2\,dx=0$.

Let $\Phi_{\overline{\mu}}^i\,:\,\widetilde{\Omega}_r^-\to V_i$ be defined by

$$\widetilde{\Phi}_{\overline{\mu}}^i(y)=\frac{(1-\chi_i)\widetilde{\Phi}_{\overline{\mu}}(y)}{\|(1-\chi_i)\widetilde{\Phi}_{\overline{\mu}}(y)\|_{2^*}}\qquad\forall y\in\widetilde{\Omega}_r^-.$$

Arguing as in the proof of Theorem A.1 of [3] one can prove that the map $\widetilde{\Phi}_{\overline{\mu}}^i$ is well defined for i large enough, because $\|(1-\chi_i)\widetilde{\Phi}_{\overline{\mu}}(y)\|_{2^*}\neq0$ for $y\in\widetilde{\Omega}_r^-$, and moreover we have

$$\lim_{i\to\infty}\max\{f_0(\widetilde{\Phi}_{\overline{\mu}}^i(y))\,:\,y\in\widetilde{\Omega}_r^-\}=\max\{f_0(\widetilde{\Phi}_{\overline{\mu}}(y))\,:\,y\in\widetilde{\Omega}_r^-\}=f_0\left(\frac{\psi_{\overline{\mu}}}{\|\psi_{\overline{\mu}}\|_{2^*}}\right).$$

Since $f_0(\psi_{\overline{\mu}}/\|\psi_{\overline{\mu}}\|_{2^*}) < (3/2)^{2/n}S$, there exists $j \in \mathbb{N}$ such that

$$\max\{f_0(\widetilde{\Phi}^i_{\overline{\mu}}(y)) \,:\, y \in \widetilde{\Omega}^-_r\} < \left(\frac{3}{2}\right)^{2/n} S \qquad \forall i > j.$$

For every $i \in \mathbb{N}$, choose $r_i > 0$ sufficiently small such that

$$\Omega_i^+ = \{x \in \mathbb{R}^n \,:\, \operatorname{dist}(x, \Omega_i) \le r_i\}$$

is homotopically equivalent to Ω_i. By Lemma 2.7 there exists $\overline{\lambda}_i > 0$ such that

$$\inf\{f_{\overline{\lambda}_i}(u) \,:\, u \in V_i,\ \beta(u) \notin \Omega_i^+\} > S.$$

Let $\Phi_\mu \,:\, \Omega_r^- \to \widetilde{V} \cap H_0^{1,2}(\Omega)$ be defined as in Definition 3.3. For every $\lambda \in]0, \overline{\lambda}_i[$ we can choose $\mu \in]0, \overline{\mu}]$ in such a way that

$$\begin{aligned} f_\lambda\left(\frac{\psi_\mu}{\|\psi_\mu\|_{2^*}}\right) &= \max\{\Phi_\mu(y) \,:\, y \in \Omega_r^-\} < S \\ &< \inf\{f_{\overline{\lambda}_i}(u) \,:\, u \in V_i,\ \beta(u) \notin \Omega_i^+\} \\ &\le \inf\{f_\lambda(u) \,:\, u \in V_i,\ \beta(u) \notin \Omega_i^+\}. \end{aligned}$$

It follows (as in the proof of Theorem 3.2) that for every c such that

$$f_\lambda\left(\frac{\psi_\mu}{\|\psi_\mu\|_{2^*}}\right) \le c < \inf\{f_{i,\lambda}(u) \,:\, u \in V_i,\ \beta(u) \notin \Omega_i^+\}$$

we have

$$\operatorname{cat}(\Phi_\mu(\Omega_r^-), f^c_{i,\lambda}) \ge \operatorname{cat}(\Omega_r^-, \Omega_i^+) = \operatorname{cat}\Omega_i = \operatorname{cat}\Omega > 1.$$

In particular, we deduce that $\Phi_\mu(\Omega_r^-)$ is not contractible in the sublevel $f^c_{i,\lambda}$ for every c such that

$$f_\lambda\left(\frac{\psi_\mu}{\|\psi_\mu\|_{2^*}}\right) \le c < \inf\{f_{i,\lambda}(u) \,:\, u \in V_i,\ \beta(u) \notin \Omega_i^+\}.$$

Set

$$\widehat{c}^{\,i}_{\lambda,\mu} = \inf\{c \,:\, \Phi_\mu(\Omega_r^-) \text{ is contractible in } f^c_{i,\lambda}\}.$$

As in the proof of Theorem 3.2 one can prove that $\widehat{c}^{\,i}_{\lambda,\mu}$ is a critical value for the functional $f_{i,\lambda}$ restricted to V_i.

Let us prove that for all $i > j$ and $\lambda \in]0, \overline{\lambda}_i[$ the set $\Phi_\mu(\Omega_r^-)$ is contractible in the sublevel

$$f_{i,\lambda}^{(3/2)^{2/n}S} = \left\{u \in V_i \,:\, f_{i,\lambda}(u) \le \left(\frac{3}{2}\right)^{2/n} S\right\}.$$

Indeed, let $\Theta : \Phi_\mu(\Omega_r^-) \times [0,1] \to V_1$ be defined by

$$\Theta(\Phi_\mu(y), t) = \begin{cases} \Phi_{\mu_t}(y) & \forall t \in [0, 1/2], \ \forall y \in \Omega_r^-, \\ \widetilde{\Phi}^i_{\overline{\mu}}(\vartheta(y,t)) & \forall t \in [1/2, 1], \ \forall y \in \Omega_r^-, \end{cases}$$

where $\mu_t = \mu + 2t(\overline{\mu} - \mu)$ and $\vartheta : \widetilde{\Omega}_r^- \times [0,1] \to \widetilde{\Omega}_r^-$ has the following properties: $\vartheta(y, 0) = y$ for all $y \in \widetilde{\Omega}_r^-$ and there exists $y_0 \in \widetilde{\Omega}_r^-$ such that $\vartheta(y, 1) = y_0$ for all $y \in \widetilde{\Omega}_r^-$ (such a ϑ exists because $\widetilde{\Omega}_r^-$, as well as $\widetilde{\Omega}$, is contractible in itself).

Let us note that the function Θ defined above is continuous because $\Phi_{\overline{\mu}}(y) = \widetilde{\Phi}^i_{\overline{\mu}}(y)$ for $y \in \Omega_r^-$ since $\chi_i(x) = 0$ for $x \in \Omega$ and $\Phi_{\overline{\mu}}(y) \in H_0^{1,2}(\Omega)$ for $y \in \Omega_r^-$; moreover, obviously

$$\Theta(\Phi_\mu(y), 0) = \Phi_\mu(y) \qquad \forall y \in \Omega_r^- \quad \text{and}$$
$$\Theta(\Phi_\mu(y), 1) = \widetilde{\Phi}^i_{\overline{\mu}}(y_0) \qquad \forall y \in \Omega_r^-.$$

It remains to prove that

$$\Theta(\Phi_\mu(y), t) \in f_{i,\lambda}^{(3/2)^{2/n} S} \qquad \forall y \in \Omega_r^- \text{ and } \forall t \in [0,1].$$

To this end it suffices to notice that for all $y \in \Omega_r^-$ and $t \in [0, 1/2]$ we have

$$f_\lambda(\Phi_{\mu_t}(y)) \le f_0(\Phi_{\mu_t}(y)) = f_0\left(\frac{\psi_{\mu_t}}{\|\psi_{\mu_t}\|_{2^*}}\right) < \left(\frac{3}{2}\right)^{2/n} S$$

(by the choice of $\overline{\mu}$); for all $y \in \Omega_r^-$ and $t \in [1/2, 1]$ we have

$$\begin{aligned} f_\lambda(\widetilde{\Phi}^i_{\overline{\mu}}(\vartheta(y,t))) &\le f_0(\widetilde{\Phi}^i_{\overline{\mu}}(\vartheta(y,t))) \\ &\le \max\{f_0(\widetilde{\Phi}^i_{\overline{\mu}}(y)) : y \in \widetilde{\Omega}_r^-\} \\ &< \left(\frac{3}{2}\right)^{2/n} S \qquad \forall i > j \end{aligned}$$

(by the choice of $j \in \mathbb{N}$). It follows that

$$\widetilde{c}^{\,i}_{\lambda,\mu} < \left(\frac{3}{2}\right)^{2/n} S \qquad \forall i > j \text{ and } \forall \lambda \in]0, \overline{\lambda}_i[.$$

On the other hand, since $\Phi_\mu(\Omega_r^-)$ is not contractible in the sublevel $f_{i,\lambda}^c$ for

$$c < \inf\{f_{i,\lambda}(u) : u \in V_i, \ \beta(u) \notin \Omega_i^+\},$$

we have, for all $\lambda \in]0, \overline{\lambda}_i[$,

$$\widetilde{c}^i_{\lambda,\mu} \geq \inf\{f_\lambda(u) : u \in V_i,\ \beta(u) \notin \Omega_i^+\} \\ \geq \inf\{f_{\overline{\lambda}_i}(u) : u \in V_i,\ \beta(u) \notin \Omega_i^+\} > S$$

(by the choice of $\overline{\lambda}_i$).

So, if we denote by $\widehat{u}_{i,\lambda}$ the solution of $P_\lambda(\Omega_i)$ corresponding to the critical value $\widetilde{c}^i_{\lambda,\mu}$, then we have

$$S < \inf\{f_{\overline{\lambda}_i}(u) : u \in V_i,\ \beta(u) \notin \Omega_i^+\} \\ \leq \inf\left\{f_\lambda\left(\frac{\widehat{u}_{i,\lambda}}{\|\widehat{u}_{i,\lambda}\|_{2^*}}\right) : \lambda \in]0, \overline{\lambda}_i[\right\} \\ \leq \sup\left\{f_\lambda\left(\frac{\widehat{u}_{i,\lambda}}{\|\widehat{u}_{i,\lambda}\|_{2^*}}\right) : \lambda \in]0, \overline{\lambda}_i[\right\} \leq \left(\frac{3}{2}\right)^{2/n} S < 2^{2/n} S.$$

Then it suffices to apply (b) of Lemma 5.1 to conclude the proof.

REMARK 5.3. Assumptions like those of Theorem 5.2, concerning perturbations of small capacity of the topology of Ω, have already been considered in [21] and [23]. However, in those papers we considered directly Problem $P_0(\Omega_i)$ and the aim was to study the multiplicity of the solutions of $P_0(\Omega_i)$ with respect to the shape of the domain Ω_i.

Notice that in [21] and [23] the assumptions on Ω_i and Ω were used to obtain existence and multiplicity results for Problem $P_0(\Omega_i)$; on the contrary, in Theorem 5.2 such assumptions have only been used to guarantee the convergence of the solution $\widehat{u}_{i,\lambda}$ to a solution $\widehat{u}_{i,0}$ of $P_0(\Omega_i)$; indeed, the existence of the solution $\widehat{u}_{i,\lambda}$ comes from Theorem 3.2 and so it is independent of such assumptions.

Notice that the conditions of Theorem 5.2 are satisfied by domains with "little holes" considered by Coron in [11] and by Rey in [27]; in such domains the solution $\widehat{u}_{i,0}$ corresponds to those obtained by Coron and Rey.

Finally, let us mention that under the same assumptions of Theorem 5.2 it is also possible to prove (as in [23]) that $\lim_{i\to\infty}(\widehat{u}_{i,0}/\|\widehat{u}_{i,0}\|_{2^*}) = S$ and so $\widehat{u}_{i,0}$ converges weakly to zero in $H_0^{1,2}(\widetilde{\Omega})$ as $i \to \infty$ and concentrates near a point of $\widetilde{\Omega}$ (see [16]).

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