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LERAY RESIDUE FORMULA AND ASYMPTOTICS OF SOLUTIONS TO CONSTANT COEFFICIENT PDES

BOGDAN ZIEMIAN

Dedicated to Professor Jean Leray

0. Introduction

The main objective of the paper is the derivation of new integral representations for fundamental solutions to a class of constant coefficient operators $P(D)$. They can be considered as a generalization of the classical formulas [11], [4] on representation of solutions in terms of exponential functions, to cover the behaviour of solutions at infinity. While the classical existence proofs for fundamental solutions (see e.g. [5]) consist in avoiding the (complex) characteristic set $\operatorname{char} P = \{z \in \mathbb{C}^n : P(z) = 0\}$, we proceed in the opposite direction. The formulas derived involve integration over certain $(n-1)$-dimensional sets L^e_σ, "isomorphic" to the positive orthant $\overline{\mathbb{R}}^{n-1}_+$, contained in $\operatorname{char} P$, and can be regarded as n-dimensional Laplace type integrals. The formulas generalize the classical Leray residue formula [8] to the case where the cycle intersects the singular set of the integrand. They exhibit new features of the fundamental solutions which cannot be read off from the classical formulas. The most important one, perhaps, is an explicit expression yielding the asymptotic behaviour of the fundamental solution at infinity which can be regarded as its n-dimensional Borel resummation at infinity. Another feature is the possibility of deforming the set L^e_σ within the characteristic set, which exhibits n-dimensional resurgence phenomena in the spirit of J. Ecalle (see [3], [15], [14]). This in turn leads to the

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expectation that the new formulas may be useful in the study of the corresponding non-linear equations, thus leading to a generalization to the case of PDEs of the recent results on non-linear ODE by Ecalle, Ramis and Braaksma [2].

In the case of two variables the situation is easier and the results are presented in [14] (the "smooth" version) and in [15] (the "analytic" version). Especially the book [14] is recommended as a "gentle" introduction to the subject.

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1. The problem and results

Let $P(z)$ be a complex polynomial in $\mathbb{C}^n$. Denote by $\widehat{E} = \operatorname{reg}(1/P(i\beta))$ the regularization of the function $i\mathbb{R}^n \ni i\beta \mapsto 1/P(i\beta)$ to a tempered distribution on $\mathbb{R}^n$. The problem consists essentially in establishing a real Laplace inversion formula for $\widehat{E}$. Recall that, while the imaginary (= Fourier) inversion formula consists in expansion in oscillatory functions $\mathbb{R}^n \ni s \mapsto e^{is\beta}$, $\beta \in \mathbb{R}^n$, the real inversion formula is an expansion in exponential terms $e^{-s\alpha}$, $\alpha \in \mathbb{R}^n_+$. The intermediate case of expansions in $e^{-s\theta}$ with $\operatorname{Re}\theta \geq 0$ will be called the *Taylor-Fourier representation* (see §7).

Intuitively the situation is very easy: given the Fourier representation of the fundamental solution E to P,

$$E = \operatorname{reg} \int_{i\mathbb{R}^n} e^{-s\theta} \frac{d\theta}{P(\theta)},$$

one completes $i\mathbb{R}^n$ to a cycle (by adding points at infinity) and then deforms it to a (finite) sum of integrals of the "residue form" of $d\theta/P(\theta)$ over the sets L^e_σ as close as possible to the positive orthant $\mathbb{R}^{n-1}_+$ (note that when taking residues the dimension drops by 1). Clearly in doing so the geometry of the complex characteristic set of P will enter in a crucial way. The program is facing, however, serious difficulties: one is that the classical Leray residue theory (recalled below) does not apply since the cycle $i\mathbb{R}^n$ intersects the singular set of the integrand, another is the lack of suitable estimates at infinity to carry out the deformation procedure (see §6).

The residue form of Leray can be regarded as a parameter version of the Cauchy residue formula in the case of a single complex variable. More precisely, given a regular complex hypersurface S in an open set $U \subset \mathbb{C}^n$ and an n-form ω on $U \setminus S$ (which may be singular on S) we define an $(n-1)$-form $\operatorname{res}\omega$ on S by the identity

$$\int_{\tilde{\gamma}} \omega = \int_\gamma \operatorname{res}\omega$$

where γ is an $(n-1)$-cycle on S and $\widetilde{\gamma}$ is an n-cycle in $U \setminus S$ homotopic to the normal sphere bundle over γ. Locally if S is given as the zero set of a holomorphic function F on U, $S = \{z \in U : F(z) = 0\}$, then

$$\operatorname{res}\omega = \frac{F\omega}{dF}.$$

The principal feature of the residue form $\operatorname{res}\omega$ is that, although defined locally, it is a (unique) global $(n-1)$-form on S.

In the following to underline the asymptotic expansion character of the formulas obtained we shall restrict our attention to the positive orthant $\mathbb{R}^n_+ = \mathbb{R}_+ \times \ldots \times \mathbb{R}_+$ (with $\mathbb{R}_+ = (0, \infty)$) and work in the logarithmic coordinates $s_1 = -\ln x_1, \ldots, s_n = -\ln x_n$.

The choice of the positive orthant is completely irrelevant (cf. Remark 1.1) but simplifies the presentation. Note that in x variables the partial derivatives $\partial/\partial s_j$, $j = 1, \ldots, n$, become $x_j\partial/\partial x_j$, $j = 1, \ldots, n$. We shall denote by $x\partial/\partial x$ the vector $x\partial/\partial x = (x_1\partial/\partial x_1, \ldots, x_n\partial/\partial x_n)$. Hence $P(x\partial/\partial x)$ stands for the operator $P(x_1\partial/\partial x_1, \ldots, x_n\partial/\partial x_n)$. We shall study solutions to the equation

$$P\left(x\frac{\partial}{\partial x}\right)u = f \tag{1.1}$$

belonging to the scale $\mathfrak{M}'_a$, $a \in \mathbb{R}^n$, of S'-type spaces defined as follows:

Denote by $\mu : \mathbb{R}^n \to \mathbb{R}^n_+$ the diffeomorphism

$$\mu(s) = e^{-s} \overset{\text{def}}{=} (e^{-s_1}, \ldots, e^{-s_n}).$$

Let $a \in \mathbb{R}^n$ and define (for details see [14], §4)

$$\mathfrak{M}_a = \mathfrak{M}_a(\mathbb{R}^n_+) = \{\sigma \in C^\infty(\mathbb{R}^n_+) : (x^{a+1}\sigma) \circ \mu \in S(\mathbb{R}^n)\}$$

where we use the notation $x^z = x_1^{z_1} \ldots x_n^{z_n}$, $z \in \mathbb{C}^n$, $\mathbf{1} = (1, \ldots, 1)$, and $S(\mathbb{R}^n)$ denotes the Schwartz space of rapidly decreasing functions. We equip $\mathfrak{M}_a$ with the natural topology induced from $S(\mathbb{R}^n)$ and define $\mathfrak{M}'_a$ as the topological dual of $\mathfrak{M}_a$. It follows from the S'-version of the Schwartz kernel theorem ([14], Th. 4.3) that $\mathfrak{M}'_a$ is isomorphic in a natural way to $\mathfrak{M}'_{a_1}(\mathbb{R}_+; \mathfrak{M}'_{a'}(\mathbb{R}^{n-1}_+))$ (here $a = (a_1, a')$), the space of linear continuous functionals on $\mathfrak{M}_{a_1}$ with values in $\mathfrak{M}'_{a'}(\mathbb{R}^{n-1}_+)$.

Observing that $u \in \mathfrak{M}'_a(\mathbb{R}^n_+)$ if and only if $e^{as}(u \circ \mu) \in S'(\mathbb{R}^n)$, we can define the $\mathcal{M}_a$-*Mellin transform* of u by the formula

$$\mathcal{M}_a u = (2\pi)^{n/2}\mathcal{F}^{-1}(e^{as}(u \circ \mu)) \tag{1.2}$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transformation in $S'(\mathbb{R}^n)$ defined on test functions $\sigma \in S(\mathbb{R}^n)$ by the formula

$$\mathcal{F}^{-1}\sigma(y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\xi y}\sigma(\xi)\, d\xi.$$

Note that if u has support in a bounded set (in $\mathbb{R}^n$) then

$$\mathcal{M}_a u(b) = u[x^{-a-ib-\mathbf{1}}],$$

i.e. coincides with the n-dimensional Mellin transform of u defined in §5.

Now, before proceeding to the formulation of the main theorem we impose assumptions on the polynomial $P(z)$:

Let $P(z) = \sum_{|\alpha|\le m} a_\alpha z^\alpha$ be a complex polynomial in $\mathbb{C}^n$ and let $\overset{*}{\alpha} \in \mathbb{R}^n$. Set $\operatorname{char} P = \{z \in \mathbb{C}^n : P(z) = 0\}$.

CONDITION A_1. We say that P satisfies condition A_1 relative to the point $\overset{*}{\alpha}$ if there exist a finite number of points $B^1, \dots, B^k \in \mathbb{R}$ such that

$$\operatorname{char} P \cap (\overset{*}{\alpha} + i\mathbb{R}^n) \subset \bigcup_{j=1}^{k} \{z \in \mathbb{C}^n : \operatorname{Im} z_1 = B^j\}.$$

CONDITION A_2.[1] The iterated discriminant roots $c^k(z_1, \theta^{(k)}), k = 2, \dots, n$, for P (see §2) grow at most linearly:

$$|c^k(z_1, \theta^{(k)})| \le C\|(z_1, \theta^{(k)})\| \qquad \text{for } \|(z_1, \theta^{(k)})\| \text{ large.}$$

CONDITION A_3.[2] The iterated discriminant roots satisfy the following global Łojasiewicz inequalities: there exist constants $C > 0$ and $\rho, \kappa > 0$ such that

$$|c_l^2(z_1, \theta'') - c_j^2(z_1, \theta'')| \ge \frac{C}{\|(z_1, \theta'')\|^\rho} (\operatorname{dist}((z_1, \theta''), \{\Delta_2(P)(z_1, \theta'') = 0\}))^\kappa$$

for $l, j = 1, \dots, m_2$, $l \ne j$, and

$$|c_j^k(z_1, \theta^{(k)})| \ge \frac{C}{\|(z_1, \theta^{(k)})\|^\rho} (\operatorname{dist}((z_1, \theta^{(k)}), \{c_j^k(z_1, \theta^{(k)}) = 0\}))^\kappa$$

for $k = 3, \dots, n-1$, $j = 1, \dots, m_k$; finally,

$$|a_{m_1}(z_1, \theta'')| \ge \frac{C}{\|(z_1, \theta'')\|^\rho} (\operatorname{dist}((z_1, \theta''), \{\widetilde{\Delta}_2(P)(z_1, \theta'') = 0\}))^\kappa.$$

CONDITION A_4. The multivalued mapping E (given by (2.3) in §2) is positive definite in the following sense: for any fixed branch $E_{i_2,\dots,i_n}$ of the function $E = (E^1, \dots, E^n)$ there exist $v = (v_1, v_3, \dots, v_n) \in \mathbb{C}^{n-1}$ with $\operatorname{Re} v \ge 0$, $\overset{\circ}{E} \in \mathbb{R}^n$ and $\widetilde{\kappa} > 0$ such that for any $\varepsilon \in \mathbb{R}_+^n$,

$$\sum y_j(\overset{\circ}{E}{}^j + \operatorname{Re} E^j_{i_2,\dots,i_n}(z_1, \theta'')) \ge c_\varepsilon \|(z_1, \theta'')\|^{\widetilde{\kappa}}$$

for $\mathbb{R}^n \ni y \ge \varepsilon$ and $\|(z_1, \theta'')\|$ large, with $(z_1, \theta'') \in v\mathbb{R}_+^{n-1} \overset{\text{def}}{=} (v_1\mathbb{R}_+, v_3\mathbb{R}_+, \dots$ $\dots, v_n\mathbb{R}_+)$.

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[1]The author thanks Prof. T. Winiarski for pointing out that, subject to a linear change of variables, this condition is satisfied for any complex polynomial.

[2]The author is grateful to Prof. S. Łojasiewicz for informing him that this condition is always satisfied.

REMARK 1.1. Conditions A_1–A_4 distinguish the first variable z_1. To give them a more symmetric formulation one may assume that A_1–A_4 hold for the polynomial $\mathcal{P}(\zeta) = P(A^{-1}\zeta)$ and $\overset{*}{a} = A\overset{*}{\alpha}$ in place of P and $\overset{*}{\alpha}$, where A is a suitable matrix in $GL(n, \mathbb{R})$.

REMARK 1.2. Since Conditions A_1 and A_4 involve the behaviour of the iterated discriminant roots at infinity one may replace the polynomial P by its principal part $P_m(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha$ and verify those conditions for the iterated discriminant roots of P_m, which may be easier.

REMARK 1.3. Condition A_2 can be formulated geometrically in terms of the position of the set $\operatorname{char} P$ and the iterated discriminant set $\{\widetilde{\Delta}_{2,\dots,k}(z_1, \theta^{(k)}) = 0\}$ with respect to the sets $\{(z_1, \|(z_1, \theta^{(k)})\|, \theta^{(k)}) : z_1 \in \mathbb{C},\ \theta^{(k)} \in \mathbb{C}^{n-k}\}$.

EXAMPLE 1.1. Condition A_1 is satisfied for the polynomials

$$P(z_1, z') = \sum_{l=1}^{m} a_l z_1^l + \widetilde{P}(z'),$$

where $\widetilde{P}(z')$ is a polynomial in $z' = (z_2, \dots, z_n)$ with real coefficients, relative to the point $(\overset{*}{a}_1, 0)$ for any $\overset{*}{a}_1 \in \mathbb{R}$.

MAIN THEOREM. *Let $\overset{*}{\alpha} \in \mathbb{R}^n$ be such that $P(z)$ satisfies Condition A_1 relative to $\overset{*}{\alpha} + i\mathbb{R}^n$. Assume further that P satisfies Conditions A_2, A_3 and A_4. Consider the distributional equation*

$$P\left(x\frac{\partial}{\partial x}\right)u = f \qquad \text{on } \mathbb{R}^n_+ \tag{1.3}$$

where f is supported by a polyinterval $\{\widetilde{t} \leq x < t\}$, $\widetilde{t}, t \in \mathbb{R}^n_+$.

Then any solution u of (1.3) in the space $\mathfrak{M}_{\overset{*}{\alpha}}$ can be expressed as a sum of multidimensional Laplace integrals:

$$u(x) = \sum_{e \in \Sigma_{\overset{*}{\alpha}_1}(P)} \ \sum_{\sigma \in \{+,-\}^{n-1}} T^e_\sigma[x^z] \qquad \text{for } 0 < x < \widetilde{t}, \tag{1.4}$$

where T^e_σ are certain Laplace $(n-1)$-currents (cf. §4) supported by $\overline{\mathbb{R}}^{n-1}_+$-type sets $L^e_\sigma \subset \operatorname{char} P \cap \{z : \operatorname{Re} z > \overset{*}{\alpha}\}$, whose vertices e lie in $\Sigma_{\overset{*}{\alpha}_1}(P)$, the vertex set of P relative to $\overset{*}{\alpha}$. For each $e \in \Sigma_{\overset{*}{\alpha}_1}(P)$ and $\sigma \in \{+,-\}^{n-1}$ the Laplace current T^e_σ can be so chosen that restricted to $\operatorname{Int} L^e_\sigma$ it coincides with a suitable branch of the (multivalued) residue form $\operatorname{res}\left(\frac{\mathcal{M}f(\theta)d\theta}{P(\theta)}\right)$ defined on the regular part of the set $\operatorname{char} P$.

The proof of the Main Theorem is given in §7.

2. Iterated discriminants

We recall first the algebraic definition of the discriminant of a polynomial. Let $Q(z) = \sum_{k=0}^{m} a_k z^k$ with $a_k \in \mathbb{C}$. The *discriminant* of Q is the determinant of the matrix

$$\begin{vmatrix} 1 & & & b_m & & \\ a_{m-1} & 1 & & b_{m-1} & b_m & \\ a_{m-2} & a_{m-1} & 1 & b_{m-2} & b_{m-1} & b_m \\ \vdots & \vdots & a_{m-1} & \vdots & \vdots & \vdots \\ a_0 & \vdots & \vdots & b_1 & \vdots & \vdots \\ & a_0 & \vdots & & b_1 & \\ & & a_0 & & & b_1 \end{vmatrix}$$

where $b_k = a_k k$ for $k = 1, \ldots, m$. The discriminant of Q (denoted by ΔQ) vanishes if and only if $Q(z)$ and $\frac{d}{dz}Q(z)$ have a common root, i.e. when Q has multiple roots. The discriminant can also be computed in terms of the roots of Q. Namely if

$$Q(z) = a_m(z - c_1)\ldots(z - c_m) \tag{2.1}$$

then $\Delta Q = \prod_{j<k}(c_j - c_k)^2$.

In the following we shall deal with polynomials in several complex variables, i.e. the coefficients a_k will themselves be polynomials (in some parameter variables) and consequently the discriminants will also be polynomials (in the parameters). To prepare for this situation we introduce the following notation:

$$\theta' = (\theta_2, \ldots, \theta_n) \in \mathbb{C}^{n-1}, \qquad \theta'' = (\theta_3, \ldots, \theta_n) \in \mathbb{C}^{n-2},$$
$$\theta^{(k)} = (\theta_{k+1}, \ldots, \theta_n) \in \mathbb{C}^{n-k}$$

for $k = 3, \ldots, n-1$. Now if $P(z_1, \theta')$ is a complex polynomial in $(z_1, \theta') \in \mathbb{C}^n$ we denote by $\Delta_2(P)(z_1, \theta'')$ the discriminant of P considered as a function of θ_2 with the remaining variables as parameters.

Analogously to (2.1), if

$$P(z_1, \theta') = a_{m_2}^2(z_1, \theta'')(\theta_2 - c_1^2(z_1, \theta''))\ldots(\theta_2 - c_{m_2}^2(z_1, \theta'')) \tag{2.2}$$

then

$$\Delta_2(P)(z_1, \theta'') = \prod_{j<k}(c_j^2(z_1, \theta'') - c_k^2(z_1, \theta''))^2.$$

We define

$$\widetilde{\Delta}_2(P)(z_1, \theta'') = (a_{m_2}^2(z_1, \theta''))^{2m-1}\Delta_2(P)(z_1, \theta'')$$

and call it the *complete discriminant* of P with respect to θ_2. It follows that $\widetilde{\Delta}_2(P)(z_1, \theta'')$ is again a polynomial and we can take its complete discriminant with respect to θ_3 to obtain a polynomial $\widetilde{\Delta}_{3,2}(P)(z_1, \theta^{(3)})$. Continuing this procedure we arrive at polynomials $\widetilde{\Delta}_{k,\ldots,2}(P)(z_1, \theta^{(k)})$ for $k = 2, \ldots, n$ which we call the *complete iterated discriminants* of P. It may happen that at some stage k, $\widetilde{\Delta}_{k,\dots,2}(P)$ is identically zero. This would mean that $\widetilde{\Delta}_{k-1,\dots,2}(P)$ had a multiple polynomial factor of the form $(Q(z_1,\theta^{(k-1)}))^l$ (for some $l>1$). Then instead of this factor we take $Q(z_1,\theta^{(k-1)})$ and repeat the operation of reducing multiple factors until the discriminant becomes non-trivial. In the following we shall assume that all discriminants are reduced in the above sense.

Now, given a (reduced) discriminant $\widetilde{\Delta}_{k,\dots,2}(P)(z_1,\theta^{(k)})$ we can write it in the form

$$\begin{aligned}&\widetilde{\Delta}_{k,\dots,2}(P)(z_1,\theta^{(k)})\\ &\quad= a^k_{m_k}(z_1,\theta^{(k+1)})(\theta_{k+1}-c^k_1(z_1,\theta^{(k+1)}))^2\dots(\theta_{k+1}-c^k_{m_k}(z_1,\theta^{(k+1)}))^2.\end{aligned}$$

The functions c^k_j, $j=1,\dots,m_k$, are multivalued functions[3] in the space $\mathcal{O}(\mathbb{C}^k\setminus\{\Delta_{k+1,\dots,2}(z_1,\theta^{(k+1)})=0\})$, and $a^k_{m_k}(z_1,\theta^{(k+1)})$ is clearly a polynomial. We call c^k_j the *k-th iterated discriminant roots* of P. The set of c^k_j for $k=2,\dots,n$, $j=1,\dots,m_k$ (with c^2_j given by (2.2)) is called the set of iterated discriminant roots of P.

Observe that the $(n-1)$-th iterated discriminant roots are functions of z_1 only. Therefore we introduce the following

DEFINITION 2.1. Let $P(z_1,\theta')$ be a polynomial in $\mathbb{C}^n$. The *partial vertex set* of P (with respect to θ'), denoted by $\widetilde{\Sigma}_{z_1}(P)$, is the set[4]

$$\begin{aligned}\widetilde{\Sigma}_{z_1}(P)=\{\theta'\in\mathbb{C}^{n-1}:\theta_2&=c^2(z_1,c^3(z_1,\dots,c^{n-1}(z_1,c^n(z_1))\dots)),\\ \theta_3&=c^3(z_1,\dots,c^{n-1}(z_1,c^n(z_1))\dots)),\dots,\theta_n=c^n(z_1)\}.\end{aligned}$$

DEFINITION 2.2. Suppose P satisfies Condition A_1 relative to a point $\overset{*}{\alpha}\in\mathbb{R}^n$. The *vertex set* of P relative to $\overset{*}{\alpha}_1$ is

$$\Sigma_{\overset{*}{\alpha}_1}(P)=\Sigma^0_{\overset{*}{\alpha}_1}(P)\cup\Sigma^+_{\overset{*}{\alpha}_1}(P)$$

where

$$\Sigma^0_{\overset{*}{\alpha}_1}(P)=\bigcup_{j=1}^{k}(\overset{*}{\alpha}_1+iB^j)\times\widetilde{\Sigma}_{\overset{*}{\alpha}_1+iB^j}(P),\qquad\text{the \textit{boundary vertex set},}$$

$$\Sigma^+_{\overset{*}{\alpha}_1}(P)=\bigcup_{\{z_1:\widetilde{\Delta}_{n,\dots,2}(P)(z_1)=0,\operatorname{Re}z_1\geq\overset{*}{\alpha}_1\}}(z_1\times\widetilde{\Sigma}_{z_1}(P)),\qquad\text{the \textit{inner vertex set}}$$

[3] To be more precise, all c^k_j for $j=1,\dots,m_k$ form a single multivalued function which we denote by c^k. We shall use the symbol c^k instead of c^k_j if we do not mean any specific branch of c^k.

[4] According to the preceding footnote the set $\widetilde{\Sigma}_{z_1}(P)$, for a fixed $z_1\in\mathbb{C}$, should be understood as the set of all points θ' obtained by taking all branches of the multivalued functions c^j, $j=2,\dots,n$.

($\widetilde{\Delta}_{n,\dots,2}(P)$ denotes the complete iterated discriminant of P with respect to all variables θ').

The term "vertex set" comes from the fact that the points $e \in \Sigma_{\overset{*}{\alpha}_1}(P)$ will be vertices of certain $(n-1)$-dimensional "orthants" L_e which we introduce below.

Fix $e = (e_1, \dots, e_n) \in \Sigma_{\overset{*}{\alpha}_1}(P)$. This corresponds to a choice $c^2_{i_2}, \dots, c^n_{i_n}$ of branches of the functions $c^2, \dots, c^n$.

Consider the mapping $E_{i_2,\dots,i_n}$ defined by

$$(2.3)\quad (z_1, \theta_3, \dots, \theta_n) \mapsto (z_1, c^2_{i_2}(z_1, c^3_{i_3}(z_1, \dots, c^{n-1}_{i_{n-1}}(z_1, c^n_{i_n}(z_n) + \theta_n) + \theta_{n-1}), \dots) + \theta_3)), \dots, c^n_{i_n}(z_1) + \theta_n)$$

and let $\widetilde{L}_e$ be the orthant

$$\widetilde{L}_e = (e_1 + \overline{\mathbb{R}}_+) \times \overline{\mathbb{R}}^{n-2}_+.$$

We define

$$L_e = E_{i_2,\dots,i_n}(\widetilde{L}_e)$$

and call it an $\overline{\mathbb{R}}^{n-1}_+$*-type set* with vertex e.

EXAMPLE 2.1. We compute the partial vertex set $\Sigma_{z_1}(P)$ for the Brieskorn-Pham polynomials $P(z_1, \theta') = z_1^{m_1} + \theta_2^{m_2} + \dots + \theta_n^{m_n}$.

We have

$$\Delta_2(P) = z_1^{m_1} + \theta_3^{m_3} + \dots + \theta_n^{m_n},$$
$$\Delta_{k,\dots,2}(P) = z_1^{m_1} + \theta_{k+1}^{m_{k+1}} + \dots + \theta_n^{m_n},$$
$$\Delta_{n-1,\dots,2}(P) = z_1^{m_1} + \theta_n^{m_n}.$$

Consequently, $c^n(z_1) = \sqrt[m_n]{-z_1^{m_1}}$. Next we find

$$\Delta_{n-2,\dots,2}(P)(z_1, \theta_{n-1}, c^n(z_1)) = \theta_{n-1}^{m_{n-1}}$$

and hence

$$c^{n-1}(z_1, c^n(z_1)) = 0.$$

Continuing this procedure we find[5] for $z_1 \in \mathbb{C}$,

$$\Sigma_{z_1}(P) = \{(0, \dots, 0, \sqrt[m_n]{-z_1^{m_1}}) \in \mathbb{C}^{n-1}\}.$$

Let $\overset{*}{\alpha} = (\alpha_1, 0, \dots, 0) \in \mathbb{R}^n$. We shall first check that P satisfies Condition A_1 relative to the point $\overset{*}{\alpha}$. Indeed, the equation

$$P(\overset{*}{\alpha}_1 + i\beta_1, i\gamma') = 0 \qquad \text{for all } \gamma' \in \mathbb{R}^n$$

implies in particular

$$\operatorname{Im}(\overset{*}{\alpha}_1 + i\beta_1)^{m_1} = 0,$$

[5]Recall that for a fixed ζ the symbol $\sqrt[m]{\zeta}$ denotes the set of all complex roots of ζ.

which has a finite number of roots $\beta_1 = B^j$, $j = 1, \dots, m_1$. Since $\Delta_{n,\dots,2}(P)(z_1) = z_1^{m_1}$ we see that

$$\Sigma^+_{\overset{*}{\alpha}_1}(P) = \begin{cases} \emptyset & \text{if } \overset{*}{\alpha}_1 > 0, \\ \{0\} & \text{if } \overset{*}{\alpha}_1 \leq 0. \end{cases}$$

Thus we have

$$\Sigma^0_{\overset{*}{\alpha}_1}(P) = \{(\overset{*}{\alpha}_1 + iB^j, 0, \dots, 0, \sqrt[m]{-(\alpha_1 + iB^j)^m}) : j = 1, \dots, m_1\}.$$

3. Nilsson type integrals

In this section we investigate analyticity properties with respect to parameters of certain integrals of ramified functions. The case of integrals over a bounded cycle is classical and can be found e.g. in [10, 8, 7, 1]. We extend these results to the case of an unbounded cycle and establish explicit formulas for the analytic continuation with respect to the parameter.

Let $H(z, \theta)$ be a function of $z = (z_1, \dots, z_k) \in \mathbb{C}^k$ and $\theta \in \mathbb{C}$ such that there exists an algebraic variety V_H in $\mathbb{C}^k$ of the form $V_H = \{(z, \theta) : P(z, \theta) = 0\}$, where P is a non-trivial complex polynomial such that H is holomorphic on the universal covering space $\widetilde{\mathbb{C}^{k+1} \setminus V_H}$ of $\mathbb{C}^{k+1} \setminus V_H$.

THEOREM 3.1. *Let H be as above and let $\overset{*}{a} \in \mathbb{R}$. Fix $\overset{\circ}{z} = (\overset{\circ}{z}_1, \dots, \overset{\circ}{z}_k)$ such that $P(\overset{\circ}{z}, \overset{*}{a} + i\gamma) \neq 0$ for $\gamma \in \mathbb{R}$. Consider the integral $\mathcal{J}(z)$ defined (formally) in a neighbourhood of $\overset{\circ}{z}$ by the formula*

$$\mathcal{J}(z) = \int_{\overset{*}{a} + i\mathbb{R}} H(z, \theta)\, d\theta.$$

Suppose that either

(i) *locally uniformly in $z \in \mathbb{C}^k$ we have*

$$|H(z, \theta)| \leq C(z) \frac{1}{|\operatorname{Im} \theta|^2} \qquad \text{for } |\operatorname{Im} \theta| \text{ large,}$$

or (ii)

$$H(\overset{\circ}{z}, \theta) = o(|\theta|) \qquad \text{for } \operatorname{Re} \theta \geq \overset{*}{a} \text{ with } |\theta| \text{ large,} \tag{3.1}$$

and locally uniformly in $z \in \mathbb{C}^k$,

$$|H(z, \theta)| \leq C(z) \frac{1}{|\operatorname{Re} \theta|^2} \qquad \text{for } \operatorname{Re} \theta \text{ large positive and } \operatorname{Im} \theta \text{ bounded.} \tag{3.1'}$$

Then the function $\mathcal{J}$ is well defined in a neighbourhood of $\overset{\circ}{z}$ and extends analytically[6] to a multivalued function on $\widetilde{\mathbb{C}^k \setminus V_{\mathcal{J}}}$ where $V_{\mathcal{J}} = \{z : \Delta P(z) = 0\}$.

[6]Note that the extensions may be different for different initial points $\overset{\circ}{z}$.

Moreover, in case (ii) *the analytic continuation of* $\mathcal{J}$ *to the set* $\mathbb{C}^k \setminus (V_{\mathcal{J}} + \overline{\mathbb{R}}^k_+)$ *is given by the integrals*

$$\mathcal{J}(z) = \int_{\Gamma_z} H(z,\theta)\, d\theta \tag{3.2}$$

where for fixed $z \in \mathbb{C}^k \setminus (V_{\mathcal{J}} + \overline{\mathbb{R}}^k_+)$, Γ_z *is a curve in* $\mathbb{C}$ *encircling the set* $\{P(z,\theta) = 0\} + \overline{\mathbb{R}}_+$ *and contained in a small tubular neighbourhood of this set.*

PROOF. In case (i) the proof is done by deformation of the integration contour as in [10]. In case (ii) the situation is essentially the same but we first note that

$$\mathcal{J}(\overset{\circ}{z}) = \int_{\Gamma_{\overset{\circ}{z}}} H(\overset{\circ}{z},\theta)\, d\theta$$

due to the estimate (3.1).

REMARK 3.1. The curve Γ_z in (3.2) can be replaced by Γ_z^φ obtained by rotating Γ_z by an angle $-\pi/2 \le \varphi \le \pi/2$. Then the set of z for which $\mathcal{J}(z)$ is defined should be changed accordingly.

COROLLARY 3.1. *Suppose* $H(z,\theta) = F(z,\theta)/P(z,\theta)$ *where* P *is a polynomial and* F *is entire and such that condition* (ii) *in Theorem* 3.1 *is satisfied. Fix* $\overset{\circ}{z} \in \mathbb{C}^k$ *and* $\overset{*}{a} \in \mathbb{R}$ *as in Theorem* 3.1. *Write* $P(z,\theta) = a_m(z)\prod_{j=1}^m(\theta - c_j(z))$ *and define*

$$I^+(\overset{\circ}{z}) = \{j : \operatorname{Re} c_j(\overset{\circ}{z}) > \overset{*}{a}\}.$$

Then

$$\mathcal{J}(z) = \sum_{j\in I^+(\overset{\circ}{z})} \frac{F(z,c_j(z))}{a_m(z)\prod_{q=1,q\neq j}^m (c_j(z) - c_q(z))}.$$

Theorem 3.1(ii) and Remark 4.1 yield

COROLLARY 3.2. *If* $H(z,\cdot) \in \mathcal{O}^k_{(a)}(\mathbb{C} \setminus \bigcup_{j=1}^m L_j(z))$ *for some* $a > 0$ *where* $L_j(z) = c_j(z) + \overline{\mathbb{R}}_+$ *for* $j = 1, \dots, m$, $c_j(z)$ *being all roots of* $P(z,\theta) = 0$, *then*

$$\mathcal{J}(z) = \sum_{j=1}^m T_j(z)[1]$$

with $T_j(z) = b_j(H(z,\cdot)) \in L'_{(a)}(L_j)$ *being the difference of the boundary values of* $H(z,\cdot)$ *(across* L_j*) in the sense analogous to* (4.2).[7] *In particular, if for every* $j = 1, \dots, m$ *the functions*

$$\overline{\mathbb{R}}_+ \ni \gamma \mapsto H(z, c_j(z) + \gamma + i\varepsilon)$$

[7] Observe that, since $\mathbb{C} \setminus \bigcup_{j=1}^m L_j(z)$ is simply connected, $H(z,\cdot)$ is uniquely defined.

are integrable uniformly in $|\varepsilon| < \overset{\circ}{\varepsilon}$ *and locally uniformly in* z *then*

$$T_j(z) = \int_{\mathbb{R}_+} H_+(z, c_j(z) + \gamma)\, d\gamma - \int_{\mathbb{R}_+} H_-(z, c_j(z) + \gamma)\, d\gamma$$

where $H_\pm(z, c_j(z) + \gamma) = \lim_{\varepsilon \to 0_\pm} H(z, c_j(z) + \gamma + i\varepsilon)$.

4. Laplace distributions and currents. Generalized analytic functions

The purpose of this section is to provide a short introduction to the theory of Laplace integrals in several variables. A detailed exposition of the one-dimensional case featuring links with the classical theory of Laplace integrals can be found in [15]. We begin with the standard case of Laplace distributions supported by the positive orthant $\overline{\mathbb{R}}^n_+$.

Fix $\omega \in \mathbb{R}^n$. We introduce the space of *Laplace test functions*

$$(4.1) \quad L_{(\omega)}(\overline{\mathbb{R}}^n_+) = \{\varphi \in C^\infty(\overline{\mathbb{R}}^n_+) : \sup_{\alpha \in \overline{\mathbb{R}}^n_+} |e^{-a\alpha}(\partial/\partial\alpha)^\gamma \varphi(\alpha)| < \infty \quad \text{for some } a < \omega \text{ and any } \gamma \in \mathbb{N}_0^n\}$$

equipped with the topology of the inductive limit over $a < \omega$ of the topologies defined by the sequences of seminorms

$$q_{a,\gamma}(\varphi) = \sup_{\alpha \in \mathbb{R}^n_+} |e^{-a\alpha}(\partial/\partial\alpha)^\gamma \varphi(\alpha)| \qquad \text{for } \gamma \in \mathbb{N}_0^n.$$

The space $L'_{(\omega)}(\overline{\mathbb{R}}^n_+)$ of *Laplace distributions* supported by $\overline{\mathbb{R}}^n_+$ is by definition the topological dual of $L_{(\omega)}(\overline{\mathbb{R}}^n_+)$.

An alternative definition of Laplace distributions refers to the techniques of hyperfunction theory. For the purpose of this paper we only need the case of $n = 1$ which we outline below (the general case is treated in [15]).

Let W be a tubular neighbourhood of $\overline{\mathbb{R}}_+$ in $\mathbb{C}$ (i.e. W is an open set in $\mathbb{C}$ containing $(\overline{\mathbb{R}}_+)_\varepsilon = \{z \in \mathbb{C} : \operatorname{dist}(z, \mathbb{R}_+) < \varepsilon\}$ for some $\varepsilon > 0$). Let $a \in \mathbb{R}$ and set

$$\widetilde{\mathcal{O}}_{(a)}(W) = \{H \in \mathcal{O}(W) : \sup_{\zeta \in K} |e^{(a-\delta)\zeta} H(\zeta)| < \infty \text{ for every } \delta > 0 \text{ and every closed (in } \mathbb{C}) \text{ tubular subset } K \text{ of } W\}.$$

Next for $k \in \mathbb{R}$ we define

$$\widetilde{\mathcal{O}}^k_{(a)}(W \setminus \overline{\mathbb{R}}_+) = \{\Psi \in \mathcal{O}(W \setminus \overline{\mathbb{R}}_+) : \sup_{\zeta \in K} |e^{(a-\delta)\zeta} \beta^k \Psi(\alpha + i\beta)| < \infty \text{ for every } \delta > 0 \text{ and every closed tubular subset } K \text{ of } W\}.$$

The spaces $\widetilde{\mathcal{O}}_{(a)}(W)$ and $\widetilde{\mathcal{O}}^k_{(a)}(W \setminus \overline{\mathbb{R}}_+)$ are equipped with natural topologies. Then by the three line theorem, $\widetilde{\mathcal{O}}_{(a)}(W)$ is closed in $\widetilde{\mathcal{O}}^k_{(a)}(W \setminus \overline{\mathbb{R}}_+)$ and hence the quotient space $\widetilde{\mathcal{O}}^k_{(a)}(W \setminus \overline{\mathbb{R}}_+)/\widetilde{\mathcal{O}}_{(a)}(W)$ is Hausdorff. We have the following result (see [15]).

THEOREM 4.1. *There exists a natural topological isomorphism*

$$\varprojlim_{a<\omega} \varinjlim_{k\in\mathbb{R}} \widetilde{\mathcal{O}}^k_{(a)}(W \setminus \overline{\mathbb{R}}_+)/\widetilde{\mathcal{O}}_{(a)}(W) \simeq L'_{(\omega)}(\overline{\mathbb{R}}_+).$$

The isomorphism is given as follows: if $\Psi \in \widetilde{\mathcal{O}}^k_{(\kappa)}(W \setminus \overline{\mathbb{R}}_+)$ for some $k \in \mathbb{R}$ and $\kappa < \omega$ we define, for some $\varepsilon > 0$,

$$(4.2)\quad T[\varphi] = \lim_{\beta\to 0+} \int_{-\varepsilon+\mathbb{R}_+} \Psi(\alpha + i\beta)\widetilde{\varphi}(\alpha)\, d\alpha - \lim_{\beta\to 0+} \int_{-\varepsilon+\mathbb{R}_+} \Psi(\alpha - i\beta)\widetilde{\varphi}(\alpha)\, d\alpha$$

for $\varphi \in L_{(\theta)}(\overline{\mathbb{R}}_+)$ with $\kappa < \theta < \omega$ where $\widetilde{\varphi}$ is a smooth extension of φ to $[-\varepsilon, \infty)$. The inverse mapping is given by the assignment $T \mapsto [\Psi]$, where $[\Psi]$ denotes the equivalence class modulo $\widetilde{\mathcal{O}}_{(a)}(\mathbb{C} \setminus \overline{\mathbb{R}}_+)$ of the function

$$\Psi(z) = \frac{1}{2\pi i} T\left[\frac{e^{a(z-w)}}{z-w}\right], \qquad z \in \mathbb{C} \setminus \overline{\mathbb{R}}_+$$

(this is understood as the value of T on the test function $\overline{\mathbb{R}}_+ \ni w \mapsto \frac{e^{a(z-w)}}{z-w}$ for fixed $z \notin \overline{\mathbb{R}}_+$).

Now let f be a smooth diffeomorphism of an open (in $\mathbb{R}^n$) neighbourhood $U \supset \overline{\mathbb{R}}^n_+$ into a smooth manifold N of dimension n. Let $L = f(\overline{\mathbb{R}}^n_+)$. We then say that L is an $\overline{\mathbb{R}}^n_+$*-type set*. In order to define Laplace distributions on N supported by L it will be convenient to regard Laplace distributions as n-currents. Recall [13] that an n-current on a smooth oriented manifold N is a continuous linear functional on the space $C_0^\infty(N)$ which in local (orientation preserving) coordinate systems $H_1 : U_1 \to N$, $H_2 : U_2 \to N$ transforms according to the rule

$$(4.3)\quad T^{H_1}[\varphi] = T^{H_2}[J(H_1^{-1} \circ H_2)\varphi \circ H_1^{-1} \circ H_2] \qquad \text{for } \varphi \in C_0^\infty(H_1^{-1}(H_1(U_1) \cap H_2(U_2))).$$

In particular, T is a regular n-current on an open set $V \subset N$ if there exists a locally integrable n-form θ on V such that

$$T[\varphi] = \int \varphi\theta \qquad \text{for } \varphi \in C_0^\infty(N).$$

Observe that the transformation rule (4.3) agrees then with that for an n-form.

Now suppose $N \subset \mathbb{C}^n$. The space of Laplace test functions (of type $\omega \in \mathbb{R}^n$) is defined in practically the same way as in (4.1):

$$L_{(\omega)}(L) = \{\varphi \in C^\infty(L) : \sup_{z\in L} |e^{-az} X_1 \circ \ldots \circ X_p\varphi(z)| < \infty \text{ for some } a < \omega \text{ and any collection of bounded smooth vector fields } X_1, \ldots, X_p\ (p \in \mathbb{N}) \text{ on } N\}.$$

The mapping f considered above is called a *Laplace mapping* of type $(\omega, \widetilde{\omega})$ if

$$f_* L_{(\omega)}(\overline{\mathbb{R}}^n_+) = \{\varphi \circ f : \varphi \in L_{(\omega)}(\overline{\mathbb{R}}^n_+)\} \subset L_{(\widetilde{\omega})}(L).$$

We shall also consider a slightly more general situation where instead of $\mathbb{R}^n_+$ we take its deformation $\Gamma_v = \Gamma_{v_1} \times \ldots \times \Gamma_{v_n}$ where Γ_{v_j} is a smooth curve in $\mathbb{C}$ starting at zero and coinciding with $\mathbb{R}_+ v_j$ for large values of the parameter.

REMARK 4.1. It can be proved (see [15]) that for φ in

$$\underset{\sim}{L}_{(\theta)}(\overline{\mathbb{R}}_+) \stackrel{\text{def}}{=} \{\varphi \in \mathcal{O}(V) : \exists \delta > 0 \text{ so that } \sup_{z \in K} |e^{(-\theta+\delta)z}\varphi(z)| < \infty$$
$$\text{on any proper tubular subset } K \text{ of an open tubular set } V \text{ with } \overline{\mathbb{R}}_+ \subset V \subset \mathbb{C}\}$$

the limit in (4.2) coincides with the contour integral $T[\varphi] = \int_{\Gamma_\varepsilon} \Psi(z)\varphi(z)\,dz$, where $\Gamma_\varepsilon = \{z \in \mathbb{C} : \operatorname{dist}(z, \overline{\mathbb{R}}_+) = \varepsilon\}$ for sufficiently small $\varepsilon > 0$, oriented clockwise.

The most important property of Laplace distributions is that they can be evaluated on functions $z \mapsto x^z$ (for fixed x small enough) thus giving rise to the so called generalized analytic functions (cf. [15]):

Denote by $\widetilde{\mathbb{C} \setminus \{0\}}$ the universal covering space of $\mathbb{C} \setminus \{0\}$ and let $(\widetilde{\mathbb{C} \setminus \{0\}})^n$ be the product of n copies of $\widetilde{\mathbb{C} \setminus \{0\}}$. Let Z be a closed subset of $\mathbb{C}^n$ such that there exist (distinct) $\overline{\mathbb{R}}^n_+$-type sets $L_0, \ldots, L_b$ contained in $\mathbb{C}^n$ such that

$$Z \subset \mathbb{L} \stackrel{\text{def}}{=} \bigcup_{k=0}^{b} L_k.$$

Define $Z_k = Z \cap L_k$ for $k = 0, \ldots, b$.

DEFINITION 4.1. A function f on $\{0 < x < \rho\}$ is called a *generalized analytic function* (GAF for short) *of type* (Z, m), $m \in \mathbb{N}_0$, *and of convergence multiradius not less than* $\rho \in \mathbb{R}^n_+$ if there exist Laplace distributions $T_k \in L'_{(\ln \rho)}(L_k)$ of order at most m with $\operatorname{supp} T_k \subset Z_k$ $(k = 0, \ldots, b)$ such that f extends to a function on $(\widetilde{\mathbb{C} \setminus \{0\}})^n$ of the form

$$f(w) = \sum_{k=0}^{b} T_k[w^z] \qquad \text{for } w \in (\widetilde{\mathbb{C} \setminus \{0\}})^n,\ |w| < \rho \tag{4.4}$$

(here for $k = 0, \ldots, b$, and fixed w with $|w| < \rho$, w^z denotes the Laplace test function $L_k \ni z \mapsto w^z \in \mathbb{C}$).

It follows easily from the properties of Laplace distributions that the GAF f of Definition 4.1 is well defined and holomorphic on $\widetilde{D}(\rho) = \{w \in (\widetilde{\mathbb{C} \setminus \{0\}})^n : |w| < \rho\}$. Further, since the space $C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$ (of restrictions to $\overline{\mathbb{R}}^n_+$ of functions in $C_0^\infty(\mathbb{R}^n)$) is dense in $L_{(\omega)}(\overline{\mathbb{R}}^n_+)$, generalized analytic functions possess the following "analyticity" property which justifies their name.

PROPERTY A. If the function f given by (4.4) is flat of order r for all $r \in \mathbb{R}^n_+$ (i.e. if $x^{-\widetilde{r}}f(x)$ is bounded near zero for any $\widetilde{r} < r$), then $f \equiv 0$ on $\widetilde{D}(\rho)$.

A further study of GAFs will be done in §5 by means of the Mellin transformation.

5. The Mellin and the Cauchy transformation

The analysis of GAF is carried conveniently by means of the Mellin transformation which we recall below together with its basic properties. The *Mellin kernel* is defined as $x^{-z-\mathbf{1}}$ for $x \in \mathbb{R}^n_+$, $z \in \mathbb{C}^n$ and $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$. Recall that according to the vector notation of Section 1 we have $x^{-z-\mathbf{1}} = x_1^{-z_1-1} \cdot \ldots \cdot x_n^{-z_n-1}$ if $x = (x_1, \dots, x_n)$, $z = (z_1, \dots, z_n)$. We also write $\langle \beta \rangle = \|\beta\| + 1$ for $\beta \in \mathbb{R}^n$.

The *Mellin transform* of a bounded function f on $\mathbb{R}^n_+$ supported by a cube $I = \{0 < x < r\}$ ($r = (r_1, \dots, r_n) \in \mathbb{R}^n_+$) is defined as

$$\mathcal{M}f(z) = \int_I f(x)x^{-z-\mathbf{1}}\,dx \qquad \text{for } \operatorname{Re} z < 0.$$

This definition readily extends to the class of distributions on $\mathbb{R}^n_+$ supported by I and extendible to distributions on $\mathbb{R}^n$ (the so called *Mellin distributions*, cf. [14]). For such a distribution u we put

$$\mathcal{M}u(z) = u[x^{-z-\mathbf{1}}],$$

which makes sense for $z \in \mathbb{C}^n$ with $\operatorname{Re} z$ sufficiently small and defines a holomorphic function on that set.

In particular, distributions in the space $\mathfrak{M}'_a$ (considered in §1) supported by I are of that type and their Mellin transforms are well defined (and holomorphic) on $\{\operatorname{Re} z < a\}$.

We have the following Paley-Wiener type results for the Mellin transformation.

THEOREM 5.1 (cf. [14], Corollary 9.1). *In order that a function $F(z)$ be the Mellin transform of a distribution u supported by the polyinterval $\{r_+ < r < r_-\}$ for some $r_+, r_- \in \mathbb{R}^n_+$, it is necessary and sufficient that F be an entire function on $\mathbb{C}^n$ and that for every $b \in \mathbb{R}^n$, $\varepsilon > 0$ there exist $C = C(b, \varepsilon)$ and $s = s(b) \in \mathbb{N}_0$ such that*

$$|F(\alpha + i\beta)| \leq \begin{cases} C\langle\beta\rangle^s(r_+e^\varepsilon)^{-\alpha} & \text{for } \alpha \leq b, \\ C\langle\beta\rangle^s(r_-e^{-\varepsilon})^{-\alpha} & \text{for } \alpha > b. \end{cases} \tag{5.1}$$

Then for any $\sigma \in \{+, -\}^n$ we also have

$$|F(\alpha + i\beta)| \leq C\langle\beta\rangle^s(r_\sigma e^\sigma)^{-\alpha} \qquad \text{for } \sigma\alpha > \sigma b.$$

Moreover, u is smooth if and only if (5.1) *holds for any* $s \in \mathbb{R}$.

Before passing to the study of the Mellin transforms of GAFs we must get acquainted with the Mellin transforms of cut-off functions.

The simplest cut-off function is the characteristic function of an interval $(0, r]$ where $r > 0$. We denote it by χ_r. Its Mellin transform is readily computed:

$$\mathcal{M}(\chi_r)(z) = \int_0^r x^{-z-1} dx = \frac{r^{-z}}{-z} \qquad \text{for } z \neq 0.$$

Sometimes it is convenient to use smooth cut-off functions. To define them, fix $r > 0$ and let $\widetilde{r} = r\delta$ for some $0 < \delta < 1$. Take $\chi \in C^\infty(\mathbb{R}_+)$ such that $\chi(x) \equiv 1$ for $0 < x < \delta$ and $\chi(x) = 0$ for $x \geq 1$ and let $0 \leq \chi \leq 1$. Define

$$\chi_{\widetilde{r},r}(x) = \chi(x/r) \qquad \text{for } x > 0.$$

Clearly $\chi_{\widetilde{r},r} \in C^\infty(\mathbb{R}_+)$, $0 \leq \chi_{\widetilde{r},r} \leq 1$, $\chi_{\widetilde{r},r}(x) \equiv 1$ for $0 < x \leq \widetilde{r}$ and $\chi_{\widetilde{r},r}(x) \equiv 0$ for $x \geq r$.

In a sense χ_r can be regarded as a limit case of $\chi_{\widetilde{r},r}$ as $\widetilde{r} \to r$. To underline this we write $\chi_{r,r} = \chi_r$.

The Mellin transform of $\chi_{\widetilde{r},r}$ resembles, in many respects, that of χ_r. Namely, we have

PROPOSITION 5.1 ([14]). *The Mellin transform of the cut-off function $\chi_{\widetilde{r},r} = \chi(x/r)$ has the following properties:*

(i) $\mathcal{M}\chi_{\widetilde{r},r}(z) = r^{-z}\mathcal{M}\chi(z)$,

(ii) $\mathcal{M}\chi_{\widetilde{r},r}(z) = \frac{r^{-z}}{-z}\widetilde{G}_{\widetilde{r},r}(z)$
where $\widetilde{G}_{\widetilde{r},r} \in \mathcal{O}(\mathbb{C})$ and for any $p \in \mathbb{N}$,

$$\left|\frac{d^p}{dz^p}\widetilde{G}_{\widetilde{r},r}(z)\right| \leq \begin{cases} C_p \langle \ln r\rangle^p r^{-\operatorname{Re} z}/\langle \operatorname{Im} z\rangle & \text{for } \operatorname{Re} z \leq 0, \\ C_p \langle \ln r\rangle^p \widetilde{r}^{-\operatorname{Re} z}/\langle \operatorname{Im} z\rangle & \text{for } \operatorname{Re} z > 0, \end{cases}$$

for some constant C_p independent of r,

(iii) *for every $\varepsilon > 0$ and $p, j \in \mathbb{N}$ there exists a constant $C_{\varepsilon,j,p}$ (depending on χ but independent of r) such that for $|z| \geq \varepsilon$,*

$$\left|\frac{d^p}{dz^p}\mathcal{M}\chi_{\widetilde{r},r}(z)\right| \leq \begin{cases} C_{\varepsilon,j,p} \langle \ln r\rangle^p r^{-\operatorname{Re} z}/\langle \operatorname{Im} z\rangle^j & \text{for } \operatorname{Re} z \leq 0, \\ C_{\varepsilon,j,p} \langle \ln r\rangle^p \widetilde{r}^{-\operatorname{Re} z}/\langle \operatorname{Im} z\rangle^j & \text{for } \operatorname{Re} z > 0. \end{cases}$$

Now we are ready to describe the Mellin transforms of GAFs. Since in applications in §6 we only need the case of dimension 1 (with parameters) we restrict our considerations to that dimension. Further, it will be convenient to assume that the half-lines L_k, $k = 0, \ldots, b$, in the definition of a GAF are in standard position, i.e. $L_k = \zeta_k + \overline{\mathbb{R}}_+$ for some $\zeta_0, \ldots, \zeta_b \in \mathbb{C}$.

We have the following result:

THEOREM 5.2. *Let u be a generalized analytic function of type (Z,m) and of convergence radius not less than $\rho > 0$. Define*

$$\mathbb{L}_\varepsilon = \{z \in \mathbb{C} : \operatorname{dist}(z, \mathbb{L}) < \varepsilon\} \qquad \text{for } \varepsilon > 0$$

where $\mathbb{L}$ is as in §4. Then for any cut-off function $\chi_{\tilde{r},r}$ with $r < \rho$,

(i) *$\mathcal{M}(\chi_{\tilde{r},r}u)$ extends to a holomorphic function on $\mathbb{C} \setminus Z$,*

(ii) *for every $\varepsilon > 0$ and $j \in \mathbb{N}$ there exists a constant $C_{\varepsilon,j}$ (depending on δ as $\delta \to 0$ and on $\rho - r$ as $r \to \rho$) such that for $z \notin \mathbb{L}_\varepsilon$ we have*

$$|\mathcal{M}(\chi_{\tilde{r},r}u)(z)| \le \begin{cases} C_{\varepsilon,j}\langle \ln r\rangle^m r^{-\operatorname{Re} z}/\langle \operatorname{Im} z\rangle^j & \text{for } \operatorname{Re} z \le 0, \\ C_{\varepsilon,j}\langle \ln r\rangle^m \tilde{r}^{-\operatorname{Re} z}/\langle \operatorname{Im} z\rangle^j & \text{for } \operatorname{Re} z > 0, \end{cases}$$

(iii) *for every $\varepsilon > 0$ and $z \in \mathbb{L}_\varepsilon$,*

$$|\mathcal{M}(\chi_{\tilde{r},r}u)(z)| \le C_\varepsilon \langle \ln r\rangle^m \frac{\tilde{r}^{-\operatorname{Re} z}}{\operatorname{dist}(\operatorname{Im} z, \{\operatorname{Im} \zeta_0, \dots, \operatorname{Im} \zeta_b\})^{m+1}}$$

(with C_ε independent of r as $r \to 0_+$).

Similarly, for every function χ_r with $r < \rho$,

(i′) *$\mathcal{M}(\chi_r u)$ extends to a holomorphic function on $\mathbb{C} \setminus Z$,*

(ii′) *for every $\varepsilon > 0$ there exists a constant C_ε (independent of r as $r \to 0_+$) such that for $z \notin \mathbb{L}_\varepsilon$ we have*

$$|\mathcal{M}(\chi_r u)(z)| \le C_\varepsilon \langle \ln r\rangle^m \frac{r^{-\operatorname{Re} z}}{\langle \operatorname{Im} z\rangle},$$

(iii′) *for every $\varepsilon > 0$ and $z \in \mathbb{L}_\varepsilon$,*

$$|\mathcal{M}(\chi_r u)(z)| \le C_\varepsilon \langle \ln r\rangle^m \frac{\tilde{r}^{-\operatorname{Re} z}}{\operatorname{dist}(\operatorname{Im} z, \{\operatorname{Im} \zeta_0, \dots, \operatorname{Im} \zeta_b\})^{m+1}}.$$

(with C_ε independent of r as $r \to 0_+$).

Moreover, for $j = 0, \dots, b$ the Laplace distribution T_j coincides with the difference of the boundary values of $\frac{1}{2\pi i}\mathcal{M}(\chi_{\tilde{r},r}u)$ (and of $\frac{1}{2\pi i}\mathcal{M}(\chi_r u)$) across the line L_j, i.e.

$$(5.2) \quad T_j[\sigma] = \frac{1}{2\pi i} \lim_{\substack{\beta \to \operatorname{Im} \zeta_j \\ \beta > \operatorname{Im} \zeta_j}} \left(\int_{\mathbb{R}} \mathcal{M}(\chi_{\tilde{r},r}u)(\alpha + i\beta)\sigma(\alpha)\, d\alpha - \int_{\mathbb{R}} \mathcal{M}(\chi_{\tilde{r},r}u)(\alpha - i\beta)\sigma(\alpha)\, d\alpha \right) \qquad \text{for } \sigma \in L'_{(\ln \rho)}(L_j).$$

The converse of Theorem 5.2 is also true; however, we present it separately since the estimates in (ii) are replaced by much weaker ones which are easier to verify in applications.

THEOREM 5.3. *A distribution $u \in D'((0,\rho))$ extendible to zero is a generalized analytic function of type Z and of convergence radius not less than ρ if for any cut-off function $\chi_{\tilde{r},r}$ with $r < \rho$,*

(i) *$\mathcal{M}(\chi_{\tilde{r},r}u)$ holomorphic in $\{\operatorname{Re} z < \omega\}$ (for some $\omega \in \mathbb{R}$) extends to a holomorphic function on $\mathbb{C} \setminus Z$,*

(ii) *there exist $s \in \mathbb{R}$ and $\theta \in \mathbb{R}_+$ such that for any $\varepsilon > 0$ and any $\kappa > 0$ there is a constant $C_{\varepsilon,\kappa}$ (depending on r) such that for $z = \alpha + i\beta \notin \mathbb{L}_\varepsilon$,*

$$|\mathcal{M}(\chi_{\tilde{r},r}u)(\alpha+i\beta)| \le \begin{cases} C_{\varepsilon,\kappa}\langle\beta\rangle^s (re^{\kappa})^{-\alpha} & \text{for } \alpha \le 0, \\ C_{\varepsilon,\kappa} e^{(\theta+\kappa)|\beta|}(\tilde{r}e^{-\kappa})^{-\alpha} & \text{for } \alpha > 0, \end{cases}$$

(iii) *for some $\varepsilon > 0$ there exists $p \in \mathbb{N}_0$ such that for any $\kappa > 0$,*

$$|\mathcal{M}(\chi_{\tilde{r},r}u)(\alpha+i\beta)| \le C_\kappa \frac{(\tilde{r}e^{-\kappa})^{-\alpha}}{(\operatorname{dist}(\beta, \{\operatorname{Im}\zeta_0, \dots, \operatorname{Im}\zeta_b\}))^p} \qquad \text{for } \alpha + i\beta \in \mathbb{L}_\varepsilon.$$

Both theorems are proved in [15]. The proof of the second one relies on the following Phragmén-Lindelöf type theorem which is also shown in [15].

THEOREM 5.4. *If $F \in \mathcal{O}(\mathbb{C})$ and there exist $0 < \overset{*}{r} < r$, $\theta > 0$ and $s \in \mathbb{R}$ such that for every $\varepsilon > 0$,*

$$|F(\alpha+i\beta)| \le \begin{cases} C_\varepsilon\langle\beta\rangle^s (re^{\varepsilon})^{-\alpha} & \text{for } \alpha \le 0, \\ C_\varepsilon e^{(\theta+\varepsilon)|\beta|}(\overset{*}{r}e^{-\varepsilon})^{-\alpha} & \text{for } \alpha \ge 0, \end{cases}$$

then there exists $\tilde{s} \in \mathbb{R}$ such that for every $\varepsilon > 0$,

$$|F(\alpha+i\beta)| \le \begin{cases} C_\varepsilon\langle\beta\rangle^{\tilde{s}} (re^{\varepsilon})^{-\alpha} & \text{for } \alpha \le 0, \\ C_\varepsilon \langle\beta\rangle^{\tilde{s}}(\overset{*}{r}e^{-\varepsilon})^{-\alpha} & \text{for } \alpha \ge 0. \end{cases}$$

REMARK 5.1. One expects $\tilde{s} = s$; however, by examining the proof given in [15] we only get $\tilde{s} = s + 3$ (also for s negative).

Now we pass to the Cauchy transformation. To this end we observe that the Mellin transform of the cut-off function χ_1 is the Cauchy kernel $(-z)^{-1}$. This kernel is not, however, convenient for us since it has poor growth properties at infinity. On the other hand, it is seen from Proposition 5.1 that the Mellin transform of the cut-off function $\chi_{\tilde{r},r}$ has similar analyticity properties as the Cauchy kernel, and at the same time it is rapidly decreasing as $|\beta| \to \infty$. Therefore we call $G(z) = \mathcal{M}(\chi_{\tilde{r},r})(z)$ the *modified Cauchy kernel* and consider the corresponding convolution transformation which we call the *modified Cauchy transformation*:

$$\widetilde{\mathcal{C}}^{\pm}T[z] = \frac{1}{2\pi}T[G(z - \overset{*}{\alpha} - i\gamma)] \qquad \text{for } \pm\operatorname{Re} z > \pm\overset{*}{\alpha}$$

where $T \in S'(\mathbb{R})$ and $\overset{*}{\alpha}$ is a fixed real number.

We shall also need the following result on the modified Cauchy transformation with parameters:

THEOREM 5.5. *Let* $T \in S'(\mathbb{R}^n)$ *and fix* $\overset{*}{a} \in \mathbb{R}^n$. *Fix cut-off functions* $\chi \in C^\infty_{(0)}(\mathbb{R}_+)$, $\chi \equiv 1$ *in a neighbourhood of zero and* $\sigma \in \mathfrak{M}_{-1}(\mathbb{R}^{n-1}_+)$ *and define*

$$K(\zeta) = \mathcal{M}(\chi(x_1)\sigma(x'))(\zeta) \qquad \text{for } \zeta \in (\mathbb{C} \setminus \{0\}) \times \mathbb{C}^{n-1},$$
$$K'(\zeta') = \mathcal{M}'\sigma(\zeta') \qquad \text{for } \zeta' \in \mathbb{C}^{n-1}.$$

Set

$$\widetilde{C}^\pm(\zeta) = \widetilde{C}^\pm T(\zeta) = \frac{1}{(2\pi)^n} T[K(\zeta - \overset{*}{a} - i\gamma)] \quad \text{for } \pm \operatorname{Re}\zeta_1 > \pm\overset{*}{a}_1,\ \zeta' \in \mathbb{C}^{n-1},$$

and

$$\text{(5.3)} \quad \widetilde{C}'_{\overset{*}{a}_1}(\zeta') = (\widetilde{C}'T)_{\overset{*}{a}_1}(\zeta') = \frac{1}{(2\pi)^{n-1}} T[K'(\zeta' - \overset{*}{a}' - i\gamma')] \in S'(\mathbb{R}) \qquad \text{for } \zeta' \in \mathbb{C}^{n-1}$$

(*in* (5.3), T *is regarded as an element of* $S'(\mathbb{R}^{n-1}; S'(\mathbb{R}))$ *under the canonical isomorphism* $S'(\mathbb{R}^n) \simeq S'(\mathbb{R}^{n-1}; S'(\mathbb{R}))$; *see* [14], *Th.* 4.3). *Then*

$$\widetilde{C}^\pm T \in \mathcal{O}(\{\pm \operatorname{Re}\zeta_1 > \pm\overset{*}{a}_1\} \times \mathbb{C}^{n-1}),$$
$$(\widetilde{C}'T)_{\overset{*}{a}_1} \in \mathcal{O}(\mathbb{C}^{n-1}; S'(\mathbb{R}))$$

and in the sense of convergence in $S'(\mathbb{R}^n)$,

$$\text{(5.4)} \qquad \lim_{\substack{a \to \overset{*}{a} \\ a_1 < \overset{*}{a}_1}} \widetilde{C}^- T(a + i\cdot) - \lim_{\substack{a \to \overset{*}{a} \\ a_1 > \overset{*}{a}_1}} \widetilde{C}^+ T(a + i\cdot) = (\widetilde{C}'T)_{\overset{*}{a}_1}(\overset{*}{a}' + i\cdot)$$

(*here* $(\widetilde{C}'T)_{\overset{*}{a}_1}(\overset{*}{a}' + i\cdot) \in S'(\mathbb{R}^{n-1}; S(\mathbb{R}))$ *is regarded as an element of* $S'(\mathbb{R}^n)$).

PROOF. By translation in ζ we may assume $\overset{*}{a} = 0$. In view of formula (1.2) we have with $\omega(s) = \chi(e^{-s_1})\sigma(e^{-s'})$, $\omega'(s') = \sigma(e^{-s'})$,

$$K(a+ib) = \begin{cases} (2\pi)^{n/2}\mathcal{F}^{-1}(e^{as}\omega(s))(b) & \text{for } a_1 < 0, \\ (2\pi)^{n/2}\mathcal{F}^{-1}(e^{as}(\omega(s) - \omega'(s')))(b) & \text{for } a_1 > 0. \end{cases}$$

Now by the formula for the Fourier transform of convolution,

$$\widetilde{C}^- T(a + i\cdot) = \mathcal{F}^{-1}(e^{as}\omega(s)\mathcal{F}T) \qquad \text{for } a_1 < 0,$$
$$\widetilde{C}^+ T(a + i\cdot) = \mathcal{F}^{-1}(e^{as}(\omega(s) - \omega'(s'))\mathcal{F}T) \qquad \text{for } a_1 > 0.$$

First we prove that the limits in (5.4) exist. Since $\mathcal{F}$ and $\mathcal{F}^{-1}$ are topological isomorphisms of S onto S, it is enough to show that for every $\psi \in S(\mathbb{R}^n)$,

$$\text{(5.5)} \quad \begin{aligned} &e^{as}\omega(s)\psi(s) \to \omega(s)\psi(s) \quad \text{in } S(\mathbb{R}^n) \text{ as } a \to 0 \text{ with } a_1 < 0, \\ &e^{as}(\omega(s) - \omega'(s'))\psi(s) \to (\omega(s) - \omega'(s'))\psi(s) \quad \text{in } S(\mathbb{R}^n) \\ &\qquad\qquad \text{as } a \to 0 \text{ with } a_1 > 0. \end{aligned}$$

But this is simple in view of the properties of the supports of $\omega(s)$ and $\omega(s) - \omega'(s')$ and the fact that all derivatives of ω and ω' are bounded on $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$, respectively.

From (5.5) we get

$$\lim_{\substack{a'\to 0\\ a_1\to 0_-}} \widetilde{C}^- T(a+i\cdot) - \lim_{\substack{a'\to 0\\ a_1\to 0_+}} \widetilde{C}^+ T(a+i\cdot) = \mathcal{F}^{-1}(\omega'(s')\mathcal{F}T) = T * (\delta_{(0)} \otimes \mathcal{F}'^{-1}\omega')$$

where $\delta_{(0)}$ is the Dirac delta at zero in the variable s_1 and $\mathcal{F}'^{-1}$ denotes the inverse Fourier transformation in the variables $s' = (s_2, \dots, s_n)$. Now from (1.2) we see that

$$K'(ib') = (2\pi)^{(n-1)/2}\mathcal{F}'^{-1}\omega'(b')$$

and therefore

$$T * (\delta_{(0)} \otimes \mathcal{F}'^{-1}\omega') = (\widetilde{C}'T)_0(i\cdot),$$

which proves (5.4). The proof of the analyticity of $\widetilde{C}^{\pm}T$ and $(\widetilde{C}'T)_{\overset{*}{a}_1}$ is left to the reader.

COROLLARY 5.1. *Let H be a function holomorphic on an open set $U \subset \mathbb{C}^n$. Fix $\overset{*}{a} \in \mathbb{R}^n$ and suppose that the function $b \mapsto H(\overset{*}{a} + ib)$, defined for $b \in \mathbb{R}^n$ such that $\overset{*}{a} + ib \in U$, extends to a distribution in $S'(\mathbb{R}^n)$ which we denote by $H_{\overset{*}{a}}$. Further, suppose that there exists an open set $U^1 \subset \mathbb{C}$ such that for every $\zeta_1 \in U^1$ the function $b' \mapsto H_{\zeta_1}(\overset{*}{a}' + ib')$, defined for $b' \in \mathbb{R}^{n-1}$ such that $(\zeta_1, \overset{*}{a}' + ib') \in U$, extends to a distribution $H_{\zeta_1, \overset{*}{a}'}$ in $S'(\mathbb{R}^{n-1})$ and the distribution-valued function*

$$U^1 \ni \zeta_1 \mapsto H_{\zeta_1, \overset{*}{a}'} \in S'(\mathbb{R}^{n-1}) \tag{5.6}$$

is holomorphic on U^1. Finally, assume that there exists a regularization $\widetilde{H}_{\overset{*}{a}_1, \overset{*}{a}'} \in S'(\mathbb{R}; S'(\mathbb{R}^{n-1}))$ of the function $b_1 \mapsto H_{\overset{*}{a}_1 + ib_1, \overset{*}{a}'} \in S'(\mathbb{R}^{n-1})$, defined for $b_1 \in \mathbb{R}$ with $\overset{*}{a}_1 + ib_1 \in U^1$, such that $\widetilde{H}_{\overset{*}{a}_1, \overset{*}{a}'} = H_{\overset{*}{a}}$ under the canonical isomorphism $S'(\mathbb{R}; S'(\mathbb{R}^{n-1})) \simeq S'(\mathbb{R}^n)$. Then the function

$$\widetilde{C}'_{\zeta_1}(\zeta') = \frac{1}{(2\pi)^{n-1}} H_{\zeta_1, \overset{*}{a}'}[K'(\zeta' - \overset{*}{a}' - i\gamma')], \quad (\zeta_1, \zeta') \in U^1 \times \mathbb{C}^{n-1},$$

is holomorphic on $U^1 \times \mathbb{C}^{n-1}$, and for every fixed $\zeta' \in \mathbb{C}^{n-1}$ the distribution $\widetilde{C}'_{\overset{*}{a}_1}(\zeta') \in S'(\mathbb{R})$ is a regularization of the function

$$b_1 \mapsto \widetilde{C}'_{\overset{*}{a}_1 + ib_1}(\zeta')$$

defined for $b_1 \in \mathbb{R}$ such that $\overset{*}{a}_1 + ib_1 \in U^1$. Moreover, the function

$$\widetilde{\psi}(\zeta) = \begin{cases} \widetilde{C}^-(\zeta) & \text{for } \operatorname{Re}\zeta_1 < \overset{*}{a}_1, \ \zeta' \in \mathbb{C}^{n-1}, \\ \widetilde{C}^+(\zeta) + \widetilde{C}'_{\zeta_1}(\zeta') & \text{for } \operatorname{Re}\zeta_1 > \overset{*}{a}_1, \ \zeta' \in U^1, \ \zeta' \in \mathbb{C}^{n-1} \end{cases}$$

extends to a holomorphic function on $(\{\operatorname{Re}\zeta_1 < \overset{*}{a}_1\} \cup U^1) \times \mathbb{C}^{n-1}$ *(here* $\widetilde{C}^{\pm}(\zeta) = (\widetilde{C}^{\pm} H_{\overset{*}{a}})(\zeta)$ *as in Theorem* 5.5*).*

We also need the following result on the classical Cauchy transformation.

THEOREM 5.6. *Let* $\mathbb{C}^{n-1} \ni \zeta' \mapsto T_{\zeta'} \in E(\mathbb{R})$ *(= the space of compactly supported distributions) be a distribution-valued holomorphic function which is rapidly decreasing as a function of* $\operatorname{Im}\zeta'$*, locally uniformly in* $\operatorname{Re}\zeta'$*, and such that the orders of* $T_{\zeta'}$ *are uniformly bounded. Suppose that* $T_{\zeta'}$ *restricted to an interval* $(0, \overset{*}{b})$, $\overset{*}{b} > 0$, *is a function* $T_{\zeta'}(\gamma_1)$ *for* $\zeta' \in \mathbb{C}^{n-1}$*, and for* $j = 0, 1$ *and some* $p, l \in \mathbb{N}_0$,

$$\left\|\frac{\partial^j}{\partial \gamma_1^j} T_{a'+i\cdot}(\gamma_1)\right\|_{S;l} \le \frac{c_l}{\gamma_1^p}, \qquad \gamma_1 \in (0, \overset{*}{b}),$$

locally uniformly with respect to $a' \in \mathbb{R}^{n-1}$*, where*

$$\|\sigma\|_{S;l} = \sup_{x \in \mathbb{R}^{n-1}} \langle x \rangle^l \Big(\sum_{|\alpha| \le l} \Big| \Big(\frac{\partial}{\partial x}\Big)^{\alpha} \sigma(x) \Big| \Big)$$

for $\sigma \in S(\mathbb{R}^{n-1})$*. Then for* $\overset{*}{a}_1 \le a_1 < 0$ *and small* $b_1 > 0$,

$$\|C^- T_{a'+i\cdot}(a_1 + ib_1)\|_{S;l} \le \frac{\widetilde{C}_l}{b_1^{\widehat{p}}}$$

locally uniformly in $a' \in \mathbb{R}^{n-1}$*, where* $\widehat{p} = \max(p, 1+\widetilde{p})$, $\widetilde{p} = \sup_{\zeta' \in \mathbb{C}^{n-1}} \operatorname{order} T_{\zeta'}$ *and*

$$C^- T_{\zeta'}(\zeta_1) = -\frac{1}{2\pi} T_{\zeta'}\left[\frac{1}{\zeta_1 - i\gamma_1}\right] \qquad \text{for } \zeta' \in \mathbb{C}^{n-1},\ \operatorname{Re}\zeta_1 < 0.$$

Finally, we shall make use of the modified Cauchy transformation to derive the following corollary of Theorem 5.4.

COROLLARY 5.2. *Suppose* F *is holomorphic in* $U = \{\operatorname{Re} z > \omega\} \setminus \mathbb{L}$ *for some* $\omega \in \mathbb{R}$ *with* $\mathbb{L} = \bigcup_{k=0}^{b} (\zeta_k + \overline{\mathbb{R}}_+)$ *where* $\operatorname{Re}\zeta_k > \omega$*. Further, suppose that there exists* $\overset{\circ}{\alpha} \in \mathbb{R}$ *with* $\omega < \overset{\circ}{\alpha} < \operatorname{Re}\zeta_k$ *for* $k = 0, \ldots, b$ *such that the following estimates hold: there exist* $0 < \overset{*}{r}$ *and* $\theta > 0$ *such that for any* $j \in \mathbb{N}$ *and any* $\varepsilon, \kappa > 0$,

$$|F(\overset{\circ}{\alpha} + i\beta)| \le C_j / \langle \beta \rangle^j, \tag{5.7}$$

$$|F(\alpha + i\beta)| \le C_{\varepsilon,\kappa} e^{(\theta+\varepsilon)|\beta|} (\overset{*}{r} e^{-\varepsilon})^{-\alpha} \qquad \text{for } \alpha \ge \overset{\circ}{\alpha},\ \alpha + i\beta \notin \mathbb{L}_\kappa. \tag{5.8}$$

Then for any $j \in \mathbb{N}$ *and any* $\varepsilon, \kappa > 0$,

$$|F(\alpha + i\beta)| \le C_{j,\varepsilon,\kappa} \langle \beta \rangle^{-j} (\overset{*}{r} e^{-\varepsilon})^{-\alpha} \qquad \text{for } \alpha \ge \overset{\circ}{\alpha},\ \alpha + i\beta \notin \mathbb{L}_\kappa.$$

PROOF. Take the modified Cauchy kernel $G(z) = \mathcal{M}\chi_{\widetilde{r},r}(z)$ with $\widetilde{r} = \overset{*}{r}$ and define the right and left modified Cauchy transformations

$$\mathcal{C}^{\pm}F_{\overset{\circ}{\alpha}}(z) = \frac{1}{2\pi}\int_{\mathbb{R}} F(\overset{\circ}{\alpha}+i\gamma)G(z-\overset{\circ}{\alpha}-i\gamma)\,d\gamma \qquad \text{for } \pm\operatorname{Re} z > \pm\overset{\circ}{\alpha}.$$

Then from (5.7) and Proposition 5.1(iii) we easily find that for any $j \in \mathbb{N}$,

$$|\mathcal{C}^{+}F_{\overset{\circ}{\alpha}}(\alpha+i\beta)| = C_j\langle\beta\rangle^{-j}\overset{*}{r}^{-\alpha} \qquad \text{for } \alpha > \overset{\circ}{\alpha},$$
$$|\mathcal{C}^{-}F_{\overset{\circ}{\alpha}}(\alpha+i\beta)| \le C_j\langle\beta\rangle^{-j}r^{-\alpha} \qquad \text{for } \alpha < \overset{\circ}{\alpha}.$$

Now fix $\kappa = \overset{\circ}{\kappa}$ and let $\Gamma = \operatorname{bd}\mathbb{L}_{\overset{\circ}{\kappa}}$ be the contour encircling the set $\mathbb{L}$ and oriented in the positive direction. Define

$$\Psi(z) = \frac{1}{2\pi}\int_{\Gamma} F(\theta)G(z-\theta)\,d\theta.$$

It is seen that Ψ is independent of $\overset{\circ}{\kappa}$ and by choosing $\overset{\circ}{\kappa}$ small Ψ can be extended to a function holomorphic on $\mathbb{C}\setminus\mathbb{L}$. Moreover, as above we find that, for any $j \in \mathbb{N}$ and $\kappa, \varepsilon > 0$,

$$|\Psi(\alpha+i\beta)| \le \begin{cases} C_j\langle\beta\rangle^{-j}r^{-\alpha} & \text{for } \alpha \le \overset{\circ}{\alpha}, \\ C_{j,\varepsilon,\kappa}\langle\beta\rangle^{-j}(\overset{*}{r}e^{-\varepsilon})^{-\alpha} & \text{for } \alpha \ge \overset{\circ}{\alpha},\ \alpha+i\beta \notin \mathbb{L}_{\kappa}. \end{cases}$$

Further, it can be verified by using the Cauchy integral formula and Proposition 5.1(ii) that $\overset{*}{F}(z) = F(z) - \Psi(z)$ extends holomorphically to $\{\operatorname{Re} z > \omega\}$ and by the three line theorem it satisfies the estimate

$$|\overset{*}{F}(\alpha+i\beta)| \le C_{j,\varepsilon,\kappa}\langle\beta\rangle^{-j}(\overset{*}{r}e^{-\varepsilon})^{-\alpha} \qquad \text{for } \alpha \ge \overset{\circ}{\alpha}$$

(actually, all these facts are standard in the theory of Fourier hyperfunctions, cf. e.g. [6], [15]). It follows from the above and Theorem 5.5 that the function $\widetilde{F} - \Psi$ with

$$\widetilde{F}(z) = \begin{cases} \mathcal{C}^{-}F_{\overset{\circ}{\alpha}}(z) & \text{for } \operatorname{Re} z < \overset{\circ}{\alpha}, \\ \mathcal{C}^{+}F_{\overset{\circ}{\alpha}}(z) + F(z) & \text{for } \operatorname{Re} z > \overset{\circ}{\alpha}, \end{cases}$$

satisfies the assumptions of Theorem 5.4. The desired assertion follows therefore in view of Remark 4.1.

6. The main lemma

This section contains a technical result which is fundamental for the proof of the Main Theorem given in §7. In the case of two variables the result coincides essentially with Theorem 17.2 of [15].

MAIN LEMMA. *Let $\overset{*}{\alpha} \in \mathbb{R}^n$ and let $P(z_1, \dots, z_n)$ be a complex polynomial in $\mathbb{C}^n$ $(n \geq 2)$ which satisfies Conditions A_1, A_2 and A_3. Consider the distributional equation*

$$P\left(x\frac{\partial}{\partial x}\right)u = f \qquad \text{on } \mathbb{R}^n_+$$

where f is supported by a compact subset of $\mathbb{R}^n_+$. Then any solution u in the space $\mathfrak{M}_{\overset{*}{\alpha}}(\mathbb{R}^n_+)$ can be expressed in the form[8]

$$(6.1) \quad u(x_1)[\sigma(x')] = \sum_{e_1 \in \Pi_1(\Sigma_{\overset{*}{\alpha}_1})} \widetilde{T}^{e_1}[\sigma(x')][x_1^{z_1}] \qquad \text{for } 0 < x_1 < \delta_1,\ \sigma \in C_0^\infty([\widetilde{\delta}', \delta'])$$

for $\delta_1 \in \mathbb{R}_+$ and $\widetilde{\delta}'$, $\delta' \in \mathbb{R}^{n-1}_+$ sufficiently small, where for any fixed σ, $\widetilde{T}^{e_1}[\sigma]$ is a Laplace distribution in the variable z_1, and

$$\Pi_1(\Sigma_{\overset{*}{\alpha}_1}) = \bigcup_{j=1}^{k}(\overset{*}{\alpha}_1 + iB^j) \cup \bigcup \{z_1 : \widetilde{\Delta}_{n,\dots,2}(P)(z_1) = 0 \text{ with } \operatorname{Re} z_1 \geq \overset{*}{\alpha}_1\}.$$

PROOF. Let $\operatorname{supp} f \subset (\widetilde{t}, t)$ for some $\widetilde{t}, t \in \mathbb{R}^n_+$. Let $\chi_{\tilde{r},r}$ be the cut-off function as in §5 with $r < t_1$ and let $\sigma \in C_0^\infty((0, t'))$ have support in $[\widetilde{\delta}', \delta']$ with some $0 < \widetilde{\delta}' < \delta' < t'$. Define

$$\begin{aligned} G(z_1) &= \mathcal{M}\chi_{\tilde{r},r}(z_1) && \text{for } z_1 \in \mathbb{C} \setminus \{0\}, \\ K'(z') &= \mathcal{M}\sigma(z') && \text{for } z' \in \mathbb{C}^{n-1}, \\ K(z_1, z') &= G(z_1)K'(z') && \text{for } z_1 \in \mathbb{C} \setminus \{0\},\ z' \in \mathbb{C}^{n-1}, \\ F(z) &= \mathcal{M}f(z) && \text{for } z \in \mathbb{C}^n. \end{aligned}$$

Let $(1/P)_{\overset{*}{\alpha}}$ be a distribution is $S'(\mathbb{R}^n)$ extending the function $\beta \mapsto 1/P(\overset{*}{\alpha} + i\beta)$ for β such that $P(\overset{*}{\alpha} + i\beta) \neq 0$, and such that

$$\mathcal{M}_{\overset{*}{\alpha}} u = F(\overset{*}{\alpha} + i\cdot)\left(\frac{1}{P}\right)_{\overset{*}{\alpha}}.$$

We have

$$\mathcal{M}(\chi_{\tilde{r},r}(x_1)\sigma(x')u)(z) = \widetilde{C}^-(z)$$

where

$$\widetilde{C}^\pm(z) = \frac{1}{(2\pi)^n}\left(\frac{1}{P}\right)_{\overset{*}{\alpha}}[F(\overset{*}{\alpha} + i\gamma)K(z - \overset{*}{\alpha} - i\gamma)] \quad \text{for } \pm\operatorname{Re} z_1 > \pm\overset{*}{\alpha}_1,\ z' \in \mathbb{C}^{n-1}.$$

Further, it follows from Corollary 5.1 that for fixed $z' \in \mathbb{C}^{n-1}$ the holomorphic extension of $\widetilde{C}^-(\cdot, z')$ to $\mathbb{C}$ coincides for $\operatorname{Re} z_1 > \overset{*}{\alpha}_1$ with $\widetilde{C}^+(z) + \widetilde{C}'_{z_1}(z')$ where

[8] Here we regard u as an element of $\mathfrak{M}'_{\overset{*}{\alpha}_1}(\mathbb{R}_+; \mathfrak{M}_{\overset{*}{\alpha}'}(\mathbb{R}^{n-1}_+))$; see §1.

$\widetilde{C}'_{z_1}(z_1)$ is the holomorphic extension of the function

$$\widetilde{C}'_{z_1}(z') = \frac{1}{(2\pi i)^{n-1}} \int_{\operatorname{Re}\theta' = \overset{*}{\alpha}'} \frac{K'(z'-\theta')F(z_1,\theta')}{P(z_1,\theta')}\, d\theta' \tag{6.2}$$

defined initially for $z_1 = \overset{*}{\alpha}_1 + i\beta_1$ with $\beta_1 \neq B^j$, $j = 1, \dots, k$, and $z' \in \mathbb{C}^{n-1}$.

In order to find the holomorphic extension of $\widetilde{C}'_{z_1}$ in the variable z_1 we shall make use of Theorem 3.1 consecutively with respect to the variables $\theta_2, \theta_3, \dots, \theta_n$.

We shall first estimate the function $\widetilde{H}(z,\theta') = K'(z'-\theta')F(z_1,\theta')$ in the variables θ', locally uniformly in $z = (z_1, z')$. By Theorem 5.1 we have

$$|F(z_1,\theta')| \le C(z_1)\langle \operatorname{Im}\theta'\rangle^s \max(t'^{-\operatorname{Re}\theta'}, \widetilde{t}'^{-\operatorname{Re}\theta'}) \qquad \text{for } \theta' \in \mathbb{C}^{n-1} \tag{6.3}$$

for some $s \in \mathbb{R}$, with $C(z_1) \le C\langle \operatorname{Im} z_1\rangle^s t_1^{-\operatorname{Re} z_1}$ for $\operatorname{Re} z_1 > \overset{*}{\alpha}_1$. Again by the same theorem applied to the smooth function σ we have, for any $j \in \mathbb{N}$,

$$|K'(\theta')| \le \frac{C_j}{\langle \operatorname{Im}\theta'\rangle^j} \max(\delta'^{-\operatorname{Re}\theta'}, \widetilde{\delta}'^{-\operatorname{Re}\theta'}) \qquad \text{for } \theta' \in \mathbb{C}^{n-1}. \tag{6.4}$$

Hence, in particular, for $\operatorname{Re}\theta_2 > \overset{*}{\alpha}_2$ we get

$$|\widetilde{H}(z,\theta_2,\theta'')| \le \frac{C_j(z,\theta'')}{\langle \operatorname{Im}\theta_2\rangle^j} \left(\frac{\delta_2}{t_2}\right)^{\operatorname{Re}\theta_2}$$

for any $j \in \mathbb{N}$. Since for $|\theta_2|$ large $P(z_1,\theta_2,\theta'')$ is bounded from below locally uniformly in (z_1,θ'') we hence get

$$\left|\frac{\widetilde{H}(z,\theta')}{P(z_1,\theta')}\right| \le \frac{\widetilde{C}_j(z,\theta'')}{\langle \operatorname{Im}\theta_2\rangle^j} \left(\frac{\delta_2}{t_2}\right)^{\operatorname{Re}\theta_2} \qquad \text{for } \operatorname{Re}\theta_2 > \overset{*}{\alpha}_2.$$

Since $\delta_2 < t_2$ we see that the assumptions (ii) in Theorem 3.1 are satisfied. Hence and from Corollary 3.1 we see that the function $J_2(z,\theta'')$ defined in a neighbourhood of a fixed $\overset{\circ}{z}$ with $\operatorname{Im}\overset{\circ}{z} \neq B^j$, $j = 1, \dots, k$, by

$$J_2(z,\theta'') = \int_{\overset{*}{\alpha}_2 + i\mathbb{R}} \frac{\widetilde{H}(z,\theta_2,\theta'')}{P(z_1,\theta_2,\theta'')}\, d\theta_2 \tag{6.5}$$

extends to a multivalued function outside the set $\{\widetilde{\Delta}_2(P)(z_1,\theta'') = 0\}$ and can be written explicitly as

$$J_2(z,\theta'') = \sum_{i_2 \in I_2^+(\overset{\circ}{z},\theta'')} Q_{i_2}(z,\theta'')$$

where

$$Q_{i_2}(z,\theta'') = \frac{K'(z' - (c_{i_2}^2(z_1,\theta''),\theta''))F(z_1, c_{i_2}^2(z_1,\theta''),\theta'')}{a_{m_1}(z_1,\theta'')\prod_{j\neq i_2}(c_{i_2}^2(z_1,\theta'') - c_j^2(z_1,\theta''))}$$

with

$$I_2^+(\overset{\circ}{z},\theta'') = \{j : \operatorname{Re} c_j^2(\overset{\circ}{z},\theta'') > \overset{*}{\alpha}_2\}.$$

Now we want to apply Theorem 3.1 to the function $J_2(z,\theta'')$ with respect to the variable θ_3. To this end we have to estimate Q_{i_2}. Denoting by $\widetilde{H}_{i_2}$ its numerator we find from (6.3) and (6.4) that for $\operatorname{Re}\theta_3 > \overset{*}{\alpha}_3$,

$$|\widetilde{H}_{i_2}(z,\theta'')| \le C(z,\theta^{(3)}) A_2^{|c_{i_2}^2(z_1,\theta'')|}\left(\frac{\delta_3}{t_3}\right)^{\operatorname{Re}\theta_3}$$

where $A_2 = \overset{*}{\delta}_2 \overset{*}{t}_2$ with $\overset{*}{\delta}_2 = \max(\widetilde{\delta}_2, \delta_2, \widetilde{\delta}_2^{-1}, \delta_2^{-1})$, $\overset{*}{t}_2 = \max(\widetilde{t}_2, t_2, \widetilde{t}_2^{-1}, t_2^{-1})$. Hence by Condition A_2 we get

$$|\widetilde{H}_{i_2}(z,\theta'')| \le \widetilde{C}(z,\theta^{(3)}) A_2^{C|\theta_3|}\left(\frac{\delta_3}{t_3}\right)^{\operatorname{Re}\theta_3} \le \widetilde{\widetilde{C}}(z,\theta^{(3)}) A_2^{C|\operatorname{Im}\theta_3|}\left(\frac{A_2^C\delta_3}{t_3}\right)^{\operatorname{Re}\theta_3}$$

Since the denominator in Q_{i_2} vanishes only on the set $\{\widetilde{\Delta}_{3,2}(P)(z_1,\theta^{(3)}) = 0\}$ which is discrete in the variable θ_3 we conclude that for any $\kappa > 0$,

$$\text{(6.6)} \qquad |Q_{i_2}(z,\theta'')| \le C_\kappa(z,\theta^{(3)}) e^{\rho|\operatorname{Im}\theta_3|} \overset{*}{r}^{-\operatorname{Re}\theta_3} \qquad \text{for } \operatorname{Re}\theta_3 > \overset{*}{\alpha}_3,\ \theta \notin \mathbb{L}^3_\kappa$$

where

$$\mathbb{L}^3 = \{\widetilde{\Delta}_{3,2}(P)(z_1,\theta^{(3)}) = 0\} + \overline{\mathbb{R}}_+$$

and

$$\rho = C\ln A_2, \quad \overset{*}{r} = (A_2^C\delta_3/2)^{-1}.$$

We want to improve (6.6) by applying Corollary 5.1. To this end we must only check that

$$|J_2(z,\overset{*}{\alpha}_3 + i\beta_3, \theta^{(3)})| \le C_j(z,\theta^{(3)})\langle\beta_3\rangle^{-j}$$

for $j \in \mathbb{N}$. This, however, is obvious from (6.3) and (6.4) since for such θ_3, J_2 is given by (6.5). Now it follows from Corollary 5.1 that for any $j \in \mathbb{N}$ and any $\varepsilon, \kappa > 0$,

$$|J_2(z,\theta_3,\theta^{(3)})| \le \frac{C_{j,\varepsilon,\kappa}(z,\theta^{(3)})}{\langle\operatorname{Im}\theta_3\rangle^j}(\overset{*}{r}e^{-\varepsilon})^{-\operatorname{Re}\theta_3} \qquad \text{for } \operatorname{Re}\theta_3 \ge \overset{*}{\alpha}_3,\ \theta \notin \mathbb{L}^3_\kappa.$$

Noting that $\overset{*}{r}^{-1}e^\varepsilon = (A_2^C e^\varepsilon/t_3)\delta_3$, we see that condition (ii) of Theorem 3.1 is satisfied if $\delta_3 < t_3/(A_2^C e^\varepsilon)$ for some $\varepsilon > 0$.

By iterating the above procedure $n-2$ times we arrive, by Corollary 3.2, at

$$\text{(6.7)} \quad \widetilde{C}'_{z_1}(z') = \frac{1}{(2\pi i)^{n-1}} \sum_{(i_2,\dots,i_n)\in I^+(z_1)} \sum_{\sigma''\in\{+,-\}^{n-2}} \operatorname{sgn}\sigma'' \times \int_{\mathbb{R}_+^{n-2}} b_{\sigma''}(Q_{i_2,\dots,i_n}(z_1,z',\gamma''))\,d\gamma''$$

where

$$(6.8)\quad Q_{i_2,\dots,i_n}(z_1,z',\gamma'') = \frac{K'(z'-\theta')F(z_1,\theta')}{a_{m_1}(z_1,\theta'')\prod_{j\neq i_2}(c^2_{i_2}(z_1,\theta'')-c^2_j(z_1,\theta''))} \circ E_{i_2,\dots,i_n}(z_1,\gamma'')$$

with the mapping $E_{i_2,\dots,i_n}$ given by

$$(z_1,\gamma_3,\dots,\gamma_n)\mapsto (z_1, c^2_{i_2}(z_1,c^3_{i_3}(z_1,\dots,c^{n-1}_{i_{n-1}}(z_1,c^n_{i_n}(z_1)+\gamma_n) +\gamma_{n-1}),\dots)+\gamma_3)),\dots,c^n(z_1)+\gamma_n)$$

and

$$I^+(z_1)=\{(i_2,\dots,i_n):\ \text{for } j=2,\dots,n, \operatorname{Re} c^j_{i_2}(z_1,\overset{*}{\alpha}^{(j)}+i\beta^{(j)})>\overset{*}{\alpha}_j \ \text{for } \beta_1>\operatorname{Im} z_1 \text{ close to } \operatorname{Im} z_1\}.$$

Note that by Condition A_1, $I^+(z_1)$ is independent of β'.

Our aim now is to apply Corollary 5.1 and Theorem 5.2 in the variable z_1. It follows from Theorem 3.1 that the functions[9] $\widetilde{C}'_{z_1}(z')$ are holomorphic on $\widetilde{\mathbb{C}\setminus V_P}$ where $V_P=\{z_1\in\mathbb{C}:\widetilde{\Delta}_{n,\dots,2}(P)(z_1)=0\}$. We need to estimate the growth of $C'_{z_1}(z_1)$ as $z_1\to\infty$ with $\operatorname{Re} z_1>\overset{*}{\alpha}_1$. By (6.3), (6.4) and Condition A_2 we get for fixed $z'\in\mathbb{C}^{n-1}$ and all $\gamma''\in\mathbb{R}^{n-2}_+$,

$$\begin{aligned}
&|K'(z'-\theta')F(z_1,\theta')\circ E_{i_2,\dots,i_n}(z_1,\gamma'')|\\
&\quad\le C(z')\max(\delta'^{\operatorname{Re}\theta'},\widetilde{\delta}'^{\operatorname{Re}\theta'})t_1^{-\operatorname{Re} z_1}\max(t'^{\operatorname{Re}\theta'},\widetilde{t}'^{\operatorname{Re}\theta'})\circ E_{i_2,\dots,i_n}(z_1,\gamma'')\\
&\quad\le \widetilde{C}(z')t_1^{-\operatorname{Re} z_1}\delta''^{\gamma''}\prod_{j=2}^n A_j^{C|z_1|+C\sum_{l=j+1}^n\gamma_l}\\
&\quad\le \widetilde{C}(z')\Big(\frac{t_1}{\prod_{j=2}^n A_j^C}\Big)^{-\operatorname{Re} z_1}\Big(\prod_{j=2}^n A_j\Big)^{C|\operatorname{Im} z_1|}\prod_{j=3}^n\Big(\delta_j\prod_{l=2}^{j-1}A_l^C\Big)^{\gamma_j}
\end{aligned}$$

where for $j=2,\dots,n$, $A_j=\overset{*}{\delta}_j\overset{*}{t}_j$ with $\overset{*}{\delta}_j=\max(\widetilde{\delta}_j,\delta_j,\widetilde{\delta}_j^{-1},\delta_j^{-1})$, $\overset{*}{t}_j=\max(\widetilde{t}_j,t_j,\widetilde{t}_j^{-1},t_j^{-1})$. Now choosing $\delta'\in\mathbb{R}^{n-2}_+$ so that $\delta_j\prod_{l=1}^{j-1}A_l^C<1$ for $j=3,\dots,n$ we get for $\operatorname{Re} z_1>\overset{*}{a}_1$ with $|\operatorname{Im} z_1-B^j|>\varepsilon_0$ $(j=1,\dots,k)$ and $z_1\notin(V_P+\overline{\mathbb{R}}_+)_{\varepsilon_0}$ ($\varepsilon_0>0$ fixed),

$$(6.9)\quad |\widetilde{C}'_{z_1}(z')|\le C(z')\prod_{j=3}^n\int_{\mathbb{R}_+}\Big(\delta_j\prod_{l=2}^{j-1}A_l^C\Big)^{\gamma_j}d\gamma_j \times\Big(\frac{t_1}{\prod_{j=2}^n A_j^C}\Big)^{-\operatorname{Re} z_1}e^{C(\ln\prod_{j=2}^n A_j)|\operatorname{Im} z_1|}$$

[9] $\widetilde{C}'_{z_1}(z')$ depends on the component of the set $\overset{*}{\alpha}_1+i(\mathbb{R}\setminus\{B^1,\dots,B^k\})$ from which the initial point $\overset{\circ}{z}_1$ is chosen.

provided the boundary values $b_{\sigma''}(Q_{i_2,\dots,i_n}(z_1, z', \gamma''))$ are bounded on $\mathbb{R}^{n-2}_+$. Otherwise we consider integrals over the polycurve $\Gamma^{\sigma''} = \Gamma^{\sigma_3} \times \dots \times \Gamma^{\sigma_n}$ where Γ^{σ_j} is a curve in a small tubular neighbourhood of $\mathbb{R}_+$ with $\sigma_j \operatorname{Im} z_j > 0$ starting at the point $-\varepsilon_0$.

We next consider the behaviour of $C'_{z_1}(z')$ near the half lines $\overset{\circ}{z}_1 + \mathbb{R}_+$ starting at the points $\overset{\circ}{z}_1 \in V_P$. It follows by combining the inequalities in Condition A_3 that in a small tubular neighbourhood of $\overset{\circ}{z}_1 + \mathbb{R}_+$ we have

$$|(c^2_{i_2}(z_1, \theta'') - c^2_j(z_1, \theta'')) \circ E_{i_2,\dots,i_n}(z_1, w'')| \geq C(w'') \frac{1}{|\operatorname{Im} z_1 - \operatorname{Im} \overset{\circ}{z}_1|^{\nu}}$$

with $C(w'')$ bounded for $w'' \in \mathbb{R}^{n-2}_+$ (or $w'' \in \Gamma^{\sigma''}$ as above avoiding the singular points). Analogous estimates hold for $\operatorname{Re} z_1 \geq \overset{*}{a}_1$ and $\operatorname{Im} z_1$ close to B_j.

Now we consider the behaviour of $C^+(z_1, z')$ as $\operatorname{Re} z_1 \to +\infty$ with z' fixed. Since $(1/P)_{\overset{*}{\alpha}}$ is a tempered distribution of order not exceeding m we obtain by Theorem 5.1[10]

$$\begin{aligned}
|\widetilde{C}^+(z_1, z')| &\leq C \sup_{\gamma \in \mathbb{R}^n} \langle \gamma \rangle^m \Big| \sum_{|\alpha| \leq m} \frac{\partial^{\alpha}}{\partial \gamma^{\alpha}} K(z - \overset{*}{\alpha} - i\gamma) \Big| \\
&\leq \widetilde{C}\widetilde{t}^{-\operatorname{Re} z_1} \sup_{\gamma' \in \mathbb{R}^{n-1}} \langle \gamma' \rangle^m \Big| \sum_{|\alpha'| \leq m} \frac{\partial^{\alpha'}}{\partial \gamma'^{\alpha'}} \mathcal{M}\sigma(z' - \overset{*}{\alpha}' - i\gamma') \Big| \\
&\leq C' \widetilde{t}^{-\operatorname{Re} z_1} \qquad \text{for } \operatorname{Re} z_1 > \overset{*}{\alpha}_1.
\end{aligned}$$

Now by Theorem 5.4 we get the following expression for the holomorphic extension $\Psi(z_1, z')$ of $\widetilde{C}^-(z_1, z')$:

$$\Psi(z_1, z') = \begin{cases} \widetilde{C}^-(z_1, z') & \text{for } \operatorname{Re} z_1 < \overset{*}{a}_1, z' \in \mathbb{C}^{n-1}, \\ \widetilde{C}^+(z_1, z') + \widetilde{C}'_{z_1}(z') & \text{for } \operatorname{Re} z_1 > \overset{*}{a}_1, z' \in \mathbb{C}^{n-1}. \end{cases} \tag{6.10}$$

It follows from the estimates proved above that conditions (i) and (ii) of Theorem 5.3 are satisfied for $\Psi(\cdot, z')$. Condition (iii) is also satisfied except perhaps for the points $\overset{*}{\alpha}_1 + iB^j$. However, in a neighbourhood of such a point $\widetilde{C}^{\pm}$ can be reduced to the standard Cauchy transformation and the result follows from Theorem 5.6 (details can be found in [15]). Consequently, by Theorem 5.3, $u(\cdot, [\sigma])$ is a generalized analytic function in x_1 and by Theorem 5.2 the Laplace distribution $\widetilde{T}^{e_1}[\sigma]$ can be computed as the difference of the boundary values (in the sense of (5.1)) of the function $\mathcal{M}'^{-1}_{\overset{*}{\alpha}'} \Psi(z_1, \cdot)$ where $\mathcal{M}'^{-1}_{\overset{*}{\alpha}'}$ is the inverse Mellin transformation in x'.

[10] Since F is entire the estimates (5.1) also hold for the derivatives of F.

7. The Fourier-Taylor representation and asymptotic expansions

In this section, by assuming Condition A_4 we will geometrize the result of §6 and thus prove the Main Theorem formulated in §1.

We have proved in the Main Lemma that any solution $u \in \mathfrak{M}_{\underset{\alpha}{*}}$ of $P(x\partial/\partial x)u = f$ can, roughly speaking, be represented as a sum over all vertices e of

$$\int_{\mathbb{R}_+^{n-2}} \widetilde{T}^e(\gamma'')[x_1^{z_1}]x''^{\gamma''}\,d\gamma''$$

where $\widetilde{T}^e(\gamma'')$ is a Laplace distribution in z_1 (with parameters γ''). The formula (6.1) can be regarded as the Taylor expansion of u in x_1 (with x' as parameters). We call such a representation of u the *Taylor representation* in x_1 (see [15]). The case where $\operatorname{supp}\widetilde{T}^e(\gamma'') \subset e_1 + \overline{\mathbb{R}}_+$ is optimal from the point of view of Taylor (= asymptotic) representation. The Taylor character of $\widetilde{T}^e(\gamma'')[x_1^{\alpha_1}]$ will not change if (taking advantage of the analyticity and growth properties of $\widetilde{T}^e(\gamma)$ away from e) we modify the half line $e_1 + \mathbb{R}_+$, outside a neighbourhood of e_1, to a curve $\Gamma_{v_1}^{e_1}$ which asymptotically goes in the direction of a vector v_1 with $\operatorname{Re} v_1 > 0$, and replace $\widetilde{T}^e(\gamma'')$ by the Laplace distribution supported by $\Gamma_{v_1}^{e_1}$ obtained by analytic continuation of $\widetilde{T}^e(\gamma'')$.

Observe that we do not admit purely imaginary vectors v_1 ($\operatorname{Re} v_1 = 0$) since then we would have a representation in terms of the oscillatory functions $x_1^{i\beta_1}$. This is a Fourier type representation which looses its Taylor character. Accordingly, the intermediate cases corresponding to v_1 with $\operatorname{Re} v_1 > 0$, $\operatorname{Im} v_1 \neq 0$ are called the *Fourier-Taylor representations.*

Now, we return to formula (6.1). One of its drawbacks is that $\widetilde{T}^e$ is not a Laplace distribution in all variables. Indeed, (6.1) only holds for x' away from zero. We hope that the situation will improve if we modify the sets L_e by modifying $\overline{\mathbb{R}}_+^{n-1}$ as we have done with $\overline{\mathbb{R}}_+$. Thus assume Condition A_4 and take a vertex $e \in \Sigma_{\alpha_1}(P)$.

In computing the distribution $\widetilde{T}^e$ we took the difference of the boundary values in z_1 of the function $\Psi(z_1, \dots, z_n)$ defined by (6.10). This in turn depended on the function $\widetilde{C}'_{z_1}(z')$ defined by means of the boundary values in γ'', $b_{\sigma''}(Q_{i_2,\dots,i_n}(z, z', \gamma''))$ on $\mathbb{R}_+^{n-2}$. By (6.8) the latter corresponds to a choice of branches $c_{i_2}, \dots, c_{i_n}$ of the iterated discriminant roots. Let $v_1, v_3, \dots, v_n$ be the vectors corresponding to this choice of the branches, which exist by Condition A_4. We may assume by slightly perturbing those vectors that $\operatorname{Re} v_j > 0$ for $j = 1, 3, \dots, n$. By Remark 3.1 we can modify each $\overline{\mathbb{R}}_+$ by the vector v_j (as we did at the beginning of this section) to arrive at

$$\Gamma_{v''}^{\sigma''} = \Gamma_{v_3}^{\sigma_3} \times \dots \times \Gamma_{v_n}^{\sigma_n}.$$

The boundary value of $Q_{i_2,\dots,i_n}$ on $\Gamma^e_{v''}$ is obtained by means of analytic continuation from that on $\overline{\mathbb{R}}^{n-2}_+$. Performing this procedure for all $\sigma'' \in \{+,-\}^{n-2}$ we obtain

$$\widetilde{C}'_{z_1}(z') = \sum_{(i_2,\dots,i_n)\in I^+(z_1)} \sum_{\sigma''\in\{+,-\}^{n-2}} \operatorname{sgn}\sigma'' \int_{\Gamma^{\sigma''}_{v''}} b_{\sigma''}(Q_{i_2,\dots,i_n}(z_1,z',\theta''))\,d\theta''.$$

Proceeding as in the proof of the Main Lemma we observe that the above integrals satisfy estimates analogous to (6.9). Further, it is seen that the estimates will remain essentially the same with respect to the variable $\widetilde{z}_1 = v_1\cdot z_1$. Thus we can compute the boundary values with respect to the curve $\Gamma^{\sigma_1}_{v_1}$ and the result will be a Laplace distribution at $+\infty$.

We now describe the final effect of what we have done. For every vertex e and every $\sigma = (\sigma_1,\sigma_3,\dots,\sigma_n) \in \{+,-\}^{n-1}$ we have found a set $\Gamma^\sigma_v = \Gamma^{\sigma_1}_{v_1} \times \Gamma^{\sigma_3}_{v_3} \times \dots \times \Gamma^{\sigma_n}_{v_n}$ which is a deformation of $\mathbb{R}^{n-1}_+$ such that for the corresponding function $E_{i_2,\dots,i_n}$ the set

$$L^e_\sigma = E_{i_2,\dots,i_n}(\Gamma^\sigma_v)$$

is such that $L^\varepsilon_\sigma \cap \{(z_1,\theta') : \|(z_1,\theta')\| \ge r\} \subset v\overline{\mathbb{R}}^{n-1}_+$ for r large enough. Next proceeding as in the proof of the Main Lemma we obtain for every vertex e and γ a distribution $\widetilde{T}^e_\sigma$ on Γ^σ_v which is a Laplace distribution in $z_1 \in \Gamma^{\sigma_1}_{v_1}$. We also observe that outside the boundary of Γ^σ_v, the distribution $\widetilde{T}^e_\sigma$ can be chosen to coincide with the boundary value of the multivalued function

$$\text{(7.1)} \quad R_{i_2,\dots,i_n}(z_1,\gamma'') = \frac{F(z_1,\theta'')}{a_{m_1}(z_1,\theta'')\prod_{k\ne i_2}(c^2_{i_2}(z_1,\theta'') - c^2_k(z_1,\theta''))} \circ E_{i_2,\dots,i_n}(z_1,\gamma'').$$

To estimate the function $x^{E_{i_2,\dots,i_n}(z_1,\gamma'')} R_{i_2,\dots,i_n}$ on the set Γ^σ_v we recall that by Theorem 5.1,

$$|F(z_1,\theta')| \le C\langle(z,\theta')\rangle^s \widetilde{t}_1^{\,-\operatorname{Re} z_1}\widetilde{t}'^{\,-\operatorname{Re}\theta'} \qquad \text{for } \operatorname{Re} z_1 \ge 0, \operatorname{Re}\theta' \ge 0.$$

Hence for $(z_1,\gamma'') \in \Gamma^\sigma_v$ large we have by Condition A_4 (we may assume $\mathring{E} = 0$)

$$\begin{aligned}\text{(7.2)} \quad & |x^{E_{i_2,\dots,i_n}(z_1,\gamma'')} F \circ E_{i_2,\dots,i_n}(z_1,\gamma'')| \\ & \le C\langle E_{i_2,\dots,i_n}(z_1,\gamma'')\rangle^s \widetilde{t}^{\,-\operatorname{Re} E_{i_2,\dots,i_n}(z_1,\gamma'')} \\ & \le C\langle(z_1,\gamma'')\rangle^s \prod_{j=1}^n \left(\frac{x_j}{\widetilde{t}_j}\right)^{\operatorname{Re} E^j_{i_2,\dots,i_n}(z_1,\gamma'')} \le C\langle(z_1,\gamma'')\rangle^s e^{-C_\varepsilon\|(z,\gamma'')\|^{\tilde{\kappa}}}\end{aligned}$$

if $x_j < \widetilde{t}_j e^{-\varepsilon_i}$, $j = 1,\dots,n$, for some $\varepsilon = (\varepsilon_1,\dots,\varepsilon_n) \in \mathbb{R}^n_+$.

Now by Condition A_3 the denominator in (7.1) is estimated from below by $(\operatorname{dist}((z_1,\gamma''),\Gamma^\sigma_v))^\nu$ for some $\nu \in \mathbb{R}$, which together with (7.2) leads to the conclusion that $\widetilde{T}^e_\sigma$ is a Laplace distribution with convergence multiradius not less than $\widetilde{t}$.

Finally, to obtain the geometric formula (1.4) of §1 we consider $\widetilde{T}^e_\sigma$ as a Laplace n-current and define an n-current T^e_σ on L^e_σ as the image of $\widetilde{T}^e_\sigma$ under the mapping E. Then outside the boundary of L^e_σ, T^e_σ coincides with a branch of the Leray residue form res $\left(\frac{F(\theta)}{P(\theta)}d\theta\right)$ which in local coordinates (z_1, θ'') is given by

$$\frac{F(z_1, c^2_{i_2}(z_1,\theta''))}{a_{m_1}(z_1,\theta'')\prod_{k\neq i_2}(c^2_{i_2}(z_1,\theta'') - c^2_k(z_1,\theta''))}.$$

8. Examples

The example presented below gives the main motivation for the paper. The theoretical background presented above serves to provide formal justification for the developments in this section.

Let $\Delta_n = \partial^2/\partial s_1^2 + \ldots + \partial^2/\partial s_n^2$ be the Laplace operator in $\mathbb{R}^n$, $n > 2$. We are interested in fundamental solutions of Δ_n, i.e. in v satisfying the equation

$$\Delta_n v = \delta_{(0)}.$$

v will belong to a certain space of generalized functions and we are interested in the behaviour of v as $(s_1, \ldots, s_n) \to \infty$. It will be convenient to change the variables $s_1 = -\ln x_1, \ldots, s_n = -\ln x_n$ and consider instead the equation

$$\sum_{j=1}^{n}\left(x_j\frac{\partial}{\partial x_j}\right)^2 u_{\overset{*}{\alpha}} = \delta_{(1)}$$

where $u_{\overset{*}{\alpha}} \in \mathfrak{M}'_{\overset{*}{\alpha}}$ for fixed $\overset{*}{\alpha} \in \mathbb{R}^n$. For $\overset{*}{\alpha} = 0$, u_0 will be the standard fundamental solution of Δ_n in logarithmic coordinates,

$$u_0(x) = \frac{-1}{(n-2)|S^{n-1}|}((\ln x_1)^2 + \ldots + (\ln x_n)^2)^{(2-n)/2} \qquad \text{for } x \in \mathbb{R}^n_+,$$

while for $\overset{*}{\alpha} \neq 0$ no explicit formulas for $u_{\overset{*}{\alpha}}(x)$ are available, due to the lack of spherical symmetry.

Let $P(z) = \sum_{j=1}^n z_j^2$ and $n > 2$. Fix $\overset{*}{a} = (\overset{*}{a}_1, 0) \in \mathbb{R}^n$ and consider the equation

$$P(\overset{*}{a}_1 + ib_1, i\gamma') = 0 \qquad \text{with } \gamma' \in \mathbb{R}^{n-1}.$$

This is equivalent to the system

$$\overset{*}{a}{}_1^2 = b_1^2 + \gamma_2^2 + \ldots + \gamma_n^2, \qquad \overset{*}{a}_1 b_1 = 0.$$

Hence we observe that if $b_1 \neq 0$ we have $P(\overset{*}{a}_1 + ib_1, i\gamma') \neq 0$ for all $\gamma' \in \mathbb{R}^{n-1}$.

According to the general theory we are interested in the analyticity (and ramification) properties of the integral

$$J(z) = \frac{1}{(2\pi i)^{n-1}}\int_{i\mathbb{R}^{n-1}} \frac{K'(z' - \theta')}{P(z_1, \theta')}\,d\theta'$$

where $K'(z') = \mathcal{M}'\sigma(z')$ is the Mellin transform in $z' = (z_2, \dots, z_n)$ of a function $\kappa' \in C_0^\infty(\mathbb{R}_+^{n-1})$. Let $\sqrt{\zeta}$ be the branch of the square root function holomorphic on $\mathbb{C} \setminus \overline{\mathbb{R}}_+$, i.e. defined by

$$\mathbb{C} \setminus \overline{\mathbb{R}}_+ \ni \zeta \mapsto |\zeta|^{1/2} e^{i \operatorname{Arg} \zeta}$$

where $0 \le \operatorname{Arg} \zeta < 2\pi$. Thus if we write

$$c_2(z_1, \theta'') = \sqrt{-z_1^2 - \theta_3^2 - \dots - \theta_n^2},$$

where $\theta'' = (\theta_3, \dots, \theta_n)$, we get

$$P(z_1, \theta_2, \theta'') = (\theta_2 - c_2(z_1, \theta''))(\theta_2 + c_2(z_1, \theta'')).$$

Fix $z_1 = \overset{*}{a}_1 + ib_1$ with $b_1 \neq 0$ and $\theta'' \in \mathbb{R}^{n-2}$. We first compute the integral

$$J_2(z, \theta'') = \frac{1}{2\pi i} \int_{i\mathbb{R}} \frac{K'(z_2 - \theta_2, z'' - \theta'')}{(\theta_2 - c_2(z_1, \theta''))(\theta_2 + c_2(z, \theta''))} \, d\theta_2.$$

Due to the growth estimates for K' the integral reduces to the residue terms in θ_2. To compute the residues we must first learn when $\operatorname{Re} c_2^\pm > 0$. To this end note that $\operatorname{Re} \sqrt{\zeta} > 0$ if and only if $0 < \operatorname{Arg} \zeta < \pi$, i.e. if and only if $\operatorname{Im} \zeta > 0$. On the other hand, for $\zeta = -z_1^2 - \theta_3^2 - \dots - \theta_n^2$ with $\theta'' \in i\mathbb{R}^{n-2}$, we have $\operatorname{Im} \zeta = -\overset{*}{a}_1 b_1$. Thus we get

$$J_2(z, \theta'') = \begin{cases} \frac{K'(z_2 + c_2(z_1, \theta''), z'' - \theta'')}{2c_2(z_1, \theta'')} & \text{for } \overset{*}{a}_1 b_1 > 0, \\ -\frac{K'(z_2 - c_2(z_1, \theta''), z'' - \theta'')}{2c_2(z_1, \theta'')} & \text{for } \overset{*}{a}_1 b_1 < 0. \end{cases}$$

Next we introduce the function

$$c_3(z_1, \theta^{(3)}) = \sqrt{-z_1^2 - \theta_4^2 - \dots - \theta_n^2}$$

with the branch of the square root chosen as before. We are interested in computing the integrals

$$(8.1) \qquad J_3^\pm(z, \theta^{(3)}) = \frac{1}{2\pi i} \int_{i\mathbb{R}} \frac{K'(z_2 \pm c_2(z_1, \theta''), z'' - \theta'')}{2\sqrt{-(\theta_3 - c_3(z_1, \theta^{(3)}))(\theta_3 + c_3(z_1, \theta^{(3)}))}} \, d\theta_3.$$

This time we do not have only simple poles at $\pm c_3(z_1, \theta^{(3)})$ but ramification points. Due to the estimates for K' the integral (8.1) can be reduced (cf. the Painlevé theorem) to the integrals over $\pm c_3(z_1, \theta^{(3)}) + \overline{\mathbb{R}}_+$ of the difference of the boundary values (the jumps) of the integrand across those lines. Of course as in the case of J_2 we must see when $\operatorname{Re} c_3 > 0$. But the situation is the same as with c_2. Next to compute the jumps we observe that for $\zeta = \gamma \in \mathbb{R}_+$ we have $\sqrt{\zeta} = \sqrt{\gamma}$, the usual square root on $\mathbb{R}_+$. On the other hand, for $\zeta = \gamma e^{2\pi i}$ we have $\sqrt{\zeta} = -\sqrt{\gamma}$. Since

$$(8.2) \qquad c_2(z_1, \mp c_3(z_1, \theta^{(3)}) + \gamma_3, \theta^{(3)}) = \sqrt{\gamma_3} \sqrt{\pm 2c_3(z_1, \theta^{(3)}) - \gamma_3}$$

we thus get for $z_1 = \overset{*}{a}_1 + ib_1$ with $\overset{*}{a}_1 > 0$,

$$2J_3(z,\theta^{(3)}) =$$
$$\int_{\mathbb{R}_+} \frac{K'(z_2 + c_2(z_1, -c_3(z_1,\theta^{(3)}) + \gamma_3, \theta^{(3)}), z_3 + c_3(z_1,\theta^{(3)}) - \gamma_3, z^{(3)} - \theta^{(3)})}{c_2(z_1, -c_3(z_1,\theta^{(3)}) + \gamma_3, \theta^{(3)})} d\gamma_3$$
$$+\int_{\mathbb{R}_+} \frac{K'(z_2 - c_2(z_1, -c_3(z_1,\theta^{(3)}) + \gamma_3, \theta^{(3)}), z_3 + c_3(z_1,\theta^{(3)}) - \gamma_3, z^{(3)} - \theta^{(3)})}{c_2(z_1, -c_3(z_1,\theta^{(3)}) + \gamma_3, \theta^{(3)})} d\gamma_3$$

if $\overset{*}{a}_1 b_1 > 0$, and

$$2J_3(z,\theta^{(3)}) =$$
$$-\int_{\mathbb{R}_+} \frac{K'(z_2 - c_2(z_1, c_3(z_1,\theta^{(3)}) + \gamma_3, \theta^{(3)}), z_3 - c_3(z_1,\theta^{(3)}) - \gamma_3, z^{(3)} - \theta^{(3)})}{c_2(z_1, c_3(z_1,\theta^{(3)}) + \gamma_3, \theta^{(3)})} d\gamma_3$$
$$-\int_{\mathbb{R}_+} \frac{K'(z_2 + c_2(z_1, c_3(z_1,\theta^{(3)}) + \gamma_3, \theta^{(3)}), z_3 - c_3(z_1,\theta^{(3)}) - \gamma_3, z^{(3)} - \theta^{(3)})}{c_2(z_1, c_3(z_1,\theta^{(3)}) + \gamma_3, \theta^{(3)})} d\gamma_3$$

if $\overset{*}{a}_1 b_1 < 0$.

The expressions for $\overset{*}{a}_1 < 0$ are completely analogous. Now by using (8.2) we find by induction (since $c_n(z_1) = iz_1$) the expression for $J(z) = J_n(z)$. To this end define

$$\rho^\gamma(c) = \sqrt{\gamma}\sqrt{2c+\gamma},$$

and let $\sigma = (\sigma_3, \dots, \sigma_n) \in \{+,-\}^{n-2}$.

We get

$$J(z) = \frac{1}{2} \sum_{(\sigma_3,\dots,\sigma_n)} \int_{\mathbb{R}_+^{n-2}} \frac{K'(\Xi_+)}{\sigma_3\rho^{\gamma_3} \circ \dots \circ \sigma_n\rho^{\gamma_n}(iz_1)} d\gamma_3 \dots d\gamma_n,$$
$$\Xi_+ = (z_2 + \sigma_3\rho^{\gamma_3} \circ \dots \circ \sigma_n\rho^{\gamma_n}(iz_1),$$
$$z_3 + \sigma_4\rho^{\gamma_4} \circ \dots \circ \sigma_n\rho^{\gamma_n}(iz_1) - \gamma_3, \dots, z_n + iz_1 - \gamma_n),$$

for $\overset{*}{a}_1 b_1 > 0$, and

$$J(z) = -\frac{1}{2} \sum_{(\sigma_3,\dots,\sigma_n)} \int_{\mathbb{R}_+^{n-2}} \frac{K'(\Xi_-)}{\sigma_3\rho^{\gamma_3} \circ \dots \circ \sigma_n\rho^{\gamma_n}(-iz_1)} d\gamma_3 \dots d\gamma_n,$$
$$\Xi_- = (z_2 - \sigma_3\rho^{\gamma_3} \circ \dots \circ \sigma_n\rho^{\gamma_n}(-iz_1),$$
$$z_3 - \sigma_4\rho^{\gamma_4} \circ \dots \circ \sigma_n\rho^{\gamma_n}(-iz_1) - \gamma_3, \dots, z_n - iz_1 - \gamma_n),$$

for $\overset{*}{a}_1 b_1 < 0$. In particular, for $n = 3$, we get

$$\text{(8.3)} \qquad J_3(z_1,z_2,z_3) = \frac{1}{2}\int_{\mathbb{R}_+} \frac{K'(z_2 + \sqrt{\gamma_3}\sqrt{2iz_1 - \gamma_3}, z_3 + iz_1 - \gamma_3)}{\sqrt{\gamma_3}\sqrt{2iz_1 - \gamma_3}} d\gamma_3$$
$$+ \frac{1}{2}\int_{\mathbb{R}_+} \frac{K'(z_2 - \sqrt{\gamma_3}\sqrt{2iz_1 - \gamma_3}, z_3 + iz_1 - \gamma_3)}{\sqrt{\gamma_3}\sqrt{2iz_1 - \gamma_3}} d\gamma_3$$

for $\overset{*}{a}_1 b_1 > 0$, and

$$(8.4)\qquad J_3(z_1,z_2,z_3) = -\frac{1}{2}\int_{\mathbb{R}_+} \frac{K'(z_2 - \sqrt{\gamma_3}\sqrt{-2iz_1-\gamma_3}, z_3 - iz_1 - \gamma_3)}{\sqrt{\gamma_3}\sqrt{-2iz_1-\gamma_3}}\,d\gamma_3 - \frac{1}{2}\int_{\mathbb{R}_+} \frac{K'(z_2 + \sqrt{\gamma_3}\sqrt{-2iz_1-\gamma_3}, z_3 - iz_1 - \gamma_3)}{\sqrt{\gamma_3}\sqrt{-2iz_1-\gamma_3}}\,d\gamma_3$$

for $\overset{*}{a}_1 b_1 < 0$.

For $n = 4$ we have

$$(8.5)\quad J_4(z_1,z_2,z_3,z_4) = \sum_{\sigma_3,\sigma_4} \frac{1}{2}\int_{\mathbb{R}^2_+} \frac{K'(\Upsilon_+)}{\sqrt{\gamma_3}\sqrt{\sigma_4 2\sqrt{\gamma_4}\sqrt{2iz_1-\gamma_4}-\gamma_3}}\,d\gamma_3\,d\gamma_4,$$

$$\Upsilon_+ = \Big(z_2 + \sigma_3\sqrt{\gamma_3}\sqrt{\sigma_4 2\sqrt{\gamma_4}\sqrt{2iz_1-\gamma_4}-\gamma_3},\; z_3 + \sigma_4\sqrt{\gamma_4}\sqrt{2iz_1-\gamma_4},\, z_4 + iz_1 - \gamma_4\Big),$$

for $\overset{*}{a}_1 b_1 > 0$, and

$$(8.6)\quad J_4(z_1,z_2,z_3,z_4) = -\frac{1}{2}\sum_{\sigma_3,\sigma_4} \int_{\mathbb{R}^2_+} \frac{K'(\Upsilon_-)}{\sqrt{\gamma_3}\sqrt{\sigma_4 2\sqrt{\gamma_4}\sqrt{-2iz_1-\gamma_4}-\gamma_3}}\,d\gamma_3\,d\gamma_4,$$

$$\Upsilon_- = \Big(z_2 - \sigma_3\sqrt{\gamma_3}\sqrt{\sigma_4 2\sqrt{\gamma_4}\sqrt{-2iz_1-\gamma_4}-\gamma_3},\; z_3 - \sigma_4\sqrt{\gamma_4}\sqrt{-2iz_1-\gamma_4},\, z_4 - iz_1 - \gamma_4\Big),$$

for $\overset{*}{a}_1 b_1 < 0$.

Observe that the function given by (8.3) is holomorphic in (a neighbourhood of) the set $\operatorname{Re} z_1 \geq \overset{*}{a}_1$, $\operatorname{sgn}\overset{*}{a}_1 \operatorname{Im} z_1 > 0$. Putting $z_1 = \alpha + i\beta$ we find that $2iz_1 + \gamma = -2\beta + \gamma + i\alpha$ does not vanish on that set. Indeed, if $\overset{*}{a}_1 > 0$ then $\alpha \geq \overset{*}{a}_1 > 0$, and if $\overset{*}{a}_1 < 0$ then we must have $\beta < 0$ hence $-2\beta + \gamma > 0$. By symmetry the function given by (8.4) is holomorphic in the set $\operatorname{Re} z_1 \geq \overset{*}{a}_1$, $\operatorname{sgn}\overset{*}{a}_1 \operatorname{Im} z_1 < 0$. The same statements (respectively) are true for J_4. Actually, after performing the indicated integrations the two functions extend from their domains of holomorphy to the whole complex plane with a ramification point at $x_1 = 0$, but we do not need this result.

Recalling now the general theory, we are interested in the jump (in z_1) of the function

$$\Psi(z_1,z') = \begin{cases} \widetilde{C}^-(z_1,z') & \text{for } \operatorname{Re} z_1 < \overset{*}{a}_1, \\ \widetilde{C}^+(z_1,z') + J(z_1,z') & \text{for } \operatorname{Re} z_1 > \overset{*}{a}_1, \end{cases}$$

where $\widetilde{\mathcal{C}}^{\pm}$ is the modified Cauchy transformation. By applying Lemma 17.1 of [15] we find that if $\overset{*}{a}_1 \neq 0$ the jump, in our case, is equal to

$$\widetilde{T} = \frac{\operatorname{sgn} \overset{*}{a}_1}{2} Y(\alpha_1 - \overset{*}{a}_1) \sum_{(\sigma_2,\dots,\sigma_n)} \int_{\mathbb{R}_+^{n-2}} \frac{K'(\Omega)}{\sigma_3\rho^{\gamma_3} \circ \dots \circ \sigma_n\rho^{\gamma_n}(\sigma_2 i\alpha_1)} \, d\gamma_3 \dots d\gamma_n,$$

$$\Omega = (z_2 + \sigma_3\rho^{\gamma_3} \circ \dots \circ \sigma_n\rho^{\gamma_n}(\sigma_2 i\alpha_1),$$
$$z_3 + \sigma_4\rho^{\gamma_4} \circ \dots \circ \sigma_n\rho^{\gamma_n}(\sigma_2 i\alpha_1) - \gamma_3, \dots, z_n + \sigma_2 i\alpha_1 - \gamma_n),$$

with Y being the Heaviside function. This in turn leads to the following asymptotic expansion of $u_{\overset{*}{a}}$:

$$\frac{\operatorname{sgn} \overset{*}{a}_1}{2} \sum_{(\sigma_2,\dots,\sigma_n)} \int_{\overset{*}{a}_1}^{\infty} \int_{\mathbb{R}_+^{n-2}} \frac{x_1^{\alpha_1} x_2^{-\sigma_3\rho^{\gamma_3}\circ\dots\circ\sigma_n\rho^{\gamma_n}(\sigma_2 i\alpha_1)}}{\sigma_3\rho^{\gamma_3} \circ \dots \circ \sigma_n\rho^{\gamma_n}(\sigma_2 i\alpha_1)} \times x_3^{-\sigma_4\rho^{\gamma_4}\circ\dots\circ\sigma_n\rho^{\gamma_n}(\sigma_2 i\alpha_1)+\gamma_3} \dots x_n^{-\sigma_n i\alpha_1+\gamma_n} \, d\alpha_1 \, d\gamma_3 \dots d\gamma_n. \tag{8.7}$$

For $n = 3$ it assumes the form

$$\begin{aligned} \frac{\operatorname{sgn} \overset{*}{a}_1}{2} \Bigg(& \int_{\overset{*}{a}_1}^{\infty} \int_0^{\infty} \frac{x_1^{\alpha_1} x_2^{-\sqrt{\gamma_3(2i\alpha_1-\gamma_3)}} x_3^{-i\alpha_1+\gamma_3}}{\sqrt{\gamma_3(2i\alpha_1-\gamma_3)}} \, d\alpha_1 \, d\gamma_3 \\ & + \int_{\overset{*}{a}_1}^{\infty} \int_0^{\infty} \frac{x_1^{\alpha_1} x_2^{\sqrt{\gamma_3(2i\alpha_1-\gamma_3)}} x_3^{-i\alpha_1+\gamma_3}}{\sqrt{\gamma_3(2i\alpha_1-\gamma_3)}} \, d\alpha_1 \, d\gamma_3 \\ & + \int_{\overset{*}{a}_1}^{\infty} \int_0^{\infty} \frac{x_1^{\alpha_1} x_2^{\sqrt{\gamma_3(-2i\alpha_1-\gamma_3)}} x_3^{i\alpha_1+\gamma_3}}{\sqrt{\gamma_3(-2i\alpha_1-\gamma_3)}} \, d\alpha_1 \, d\gamma_3 \\ & + \int_{\overset{*}{a}_1}^{\infty} \int_0^{\infty} \frac{x_1^{\alpha_1} x_2^{-\sqrt{\gamma_3(-2i\alpha_1-\gamma_3)}} x_3^{i\alpha_1+\gamma_3}}{\sqrt{\gamma_3(-2i\alpha_1-\gamma_3)}} \, d\alpha_1 \, d\gamma_3 \Bigg). \end{aligned} \tag{8.8}$$

For $n = 4$ we get

$$\frac{\operatorname{sgn} \overset{*}{a}_1}{2} \sum_{\sigma_2,\sigma_3,\sigma_4} \int_{\overset{*}{a}_1}^{\infty} \int_0^{\infty} \int_0^{\infty} \frac{x_1^{\alpha_1} x_2^{-\sigma_2\sqrt{\gamma_3(\sigma_4 2\sqrt{\gamma_4(\sigma_2 2i\alpha_1-\gamma_4)}-\gamma_4)}}}{\sqrt{\gamma_3(\sigma_4 2\sqrt{\gamma_4(\sigma_2 2i\alpha_1-\gamma_4)}-\gamma_3)}} \times x_3^{-\sigma_3\sqrt{\gamma_4(\sigma_2 i\alpha_1-\gamma_4)}} x_4^{-\sigma_2 i\alpha_1-\gamma_4} \, d\alpha_1 \, d\gamma_3 \, d\gamma_4. \tag{8.9}$$

Observe that the formula given by (8.7) also makes sense[11] for $\overset{*}{a}_1 = 0$ (we put $\operatorname{sgn} 0 = 1$). It can be proved that the function (8.7) for $\overset{*}{a}_1 = 0$ coincides (up to a constant factor) with the standard fundamental solution for the Laplace operator in logarithmic variables, i.e. with $u_0(x)$.

It is seen that (8.7) provides an asymptotic expansion only in x_1, since $\alpha_1 \geq \overset{*}{a}_1$. All other exponents have real part zero. However, by the analyticity of the functions in the exponents it is possible to rotate the variables $\alpha_1, \gamma_3, \dots, \gamma_n$

[11] After a suitable regularization of the integrals at 0.

(in the respective complex planes) so that the real parts of all functions in the exponents will grow to $+\infty$ as the corresponding variables become large positive. We describe this phenomenon in detail for $n = 3$ and $\overset{*}{a}_1 = 0$. For the first integral in (8.8) we perform the rotation

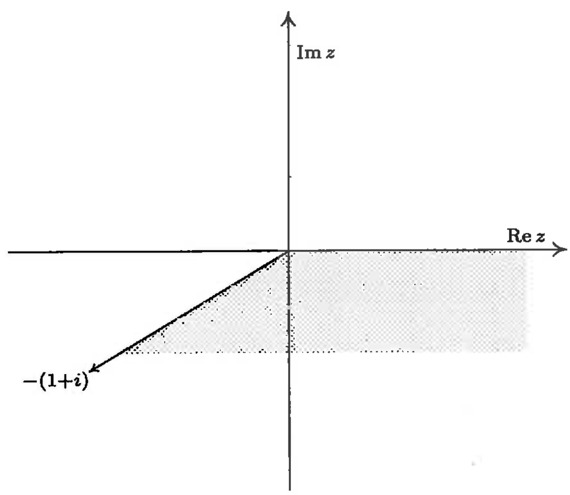
$$\alpha_1 \mapsto \frac{1+i}{2}\alpha_1, \quad \gamma_3 \mapsto i\gamma_3.$$

We obtain an integral of the form

$$\text{(8.10)} \qquad \operatorname{reg} \int_0^\infty \int_0^\infty \frac{x_1^{(1+i)\alpha_1/2} x_2^{-\sqrt{\gamma_3^2-(1+i)\gamma_3\alpha_1}} x_3^{(1-i)\alpha_1/2+i\gamma_3}}{\sqrt{\gamma_3^2-(1+i)\gamma_3\alpha_1}} \, d\alpha_1 \, d\gamma_3.$$

Observe that for $\alpha \in \mathbb{R}_+$, $\gamma \in \mathbb{R}_+$ (see figure below)

$$\pi + \pi/4 < \operatorname{Arg}(\gamma^2 - (1+i)\gamma\alpha) < 2\pi.$$



Hence

$$\operatorname{Re}\sqrt{\gamma^2-(1+i)\gamma\alpha} = ((\gamma^2-\gamma\alpha)^2+(\gamma\alpha)^2)^{1/4}\cos\theta$$

where $\pi/2 + \pi/8 < \theta < \pi$, and $\cos\theta$ is negative for such θ. Consequently, the real parts of the exponents of x_1, x_2, x_3 have the following growth property:

For any $r > 0$, if $\alpha_1 \geq r$ and $\gamma_3 \geq r$ then

$$\operatorname{Re}(1+i)\alpha_1/2 \geq r/2, \qquad \operatorname{Re} -\sqrt{\gamma_3^2-(1+i)\gamma_3\alpha_1} \geq r(-\cos(\pi/2+\pi/8)),$$
$$\operatorname{Re}((1-i)\alpha_1/2 + i\gamma_3) \geq r/2,$$

which shows that the representation (8.10) is a Taylor-Fourier expansion.

Similarly, after the rotation

$$\alpha_1 \mapsto \frac{1+i}{2}\alpha_1, \quad \gamma_3 \mapsto -i\gamma_3,$$

the second integral in (8.8) becomes

$$\text{(8.11)} \qquad \operatorname{reg} \int_0^\infty \int_0^\infty \frac{x_1^{(1+i)\alpha_1/2} x_2^{\sqrt{\gamma_3^2+(1+i)\gamma_3\alpha_1}} x_3^{(1-i)\alpha_1/2-i\gamma_3}}{\sqrt{\gamma_3^2+(1+i)\gamma_3\alpha_1}}\, d\alpha_1\, d\gamma_3.$$

Now we note that for $\alpha, \gamma \in \mathbb{R}_+$,

$$0 \le \operatorname{Arg}(\gamma^2 + (1+i)\alpha\gamma) \le \pi/4,$$

hence

$$\operatorname{Re} \sqrt{\gamma^2 + (1+i)\alpha\gamma} = ((\gamma^2 + \alpha\gamma)^2 + (\alpha\gamma)^2)^{1/4} \cos\theta$$

where $0 < \theta < \pi/8$. Then we have $\cos\theta > \cos\pi/8 > 0$, which as above leads to the conclusion that (8.11) is a Taylor-Fourier expansion.

Next, we perform the rotation

$$\alpha_1 \mapsto \frac{1-i}{2}\alpha_1, \quad \gamma_3 \mapsto -i\gamma_3,$$

in the third integral in (8.8) bringing it thus to the form

$$\text{(8.12)} \qquad \operatorname{reg} \int_0^\infty \int_0^\infty \frac{x_1^{(1-i)\alpha_1/2} x_2^{\sqrt{\gamma_3^2-(1-i)\gamma_3\alpha_1}} x_3^{(1+i)\alpha_1/2-i\gamma_3}}{\sqrt{\gamma_3^2-(1-i)\gamma_3\alpha_1}}\, d\alpha_1\, d\gamma_3.$$

We have for $\alpha, \gamma \in \mathbb{R}_+$,

$$0 \le \operatorname{Arg}(\gamma^2 - (1-i)\alpha\gamma) < \pi/2 + \pi/4,$$

hence

$$\operatorname{Re} \sqrt{\gamma^2 - (1-i)\gamma\alpha} = ((\gamma^2 - \gamma\alpha)^2 + (\gamma\alpha)^2)^{1/4} \cos\theta$$

where $0 < \theta < \pi/4 + \pi/8$. Now $\cos\theta > \cos(\pi/4 + \pi/8) > 0$, which proves the Taylor-Fourier character of the expansion (8.12).

Finally, we consider the fourth integral in (8.8). By symmetry we put

$$\alpha_1 \mapsto \frac{1-i}{2}\alpha_1, \quad \gamma_3 \mapsto i\gamma_3.$$

We get

$$\text{(8.13)} \qquad \operatorname{reg} \int_0^\infty \int_0^\infty \frac{x_1^{(1-i)\alpha_1/2} x_2^{-\sqrt{\gamma_3^2+(1-i)\gamma_3\alpha_1}} x_3^{(1+i)\alpha_1/2+i\gamma_3}}{\sqrt{\gamma_3^2+(1-i)\gamma_3\alpha_1}}\, d\alpha_1\, d\gamma_3.$$

This time we have for $\alpha, \gamma \in \mathbb{R}_+$,

$$2\pi - \pi/4 < \operatorname{Arg}(\gamma^2 + (1-i)\gamma\alpha) < 2\pi, \qquad 0 < \operatorname{Arg} z < \pi/8,$$

hence

$$\operatorname{Re} \sqrt{\gamma^2 + (1-i)\gamma\alpha} = ((\gamma^2 + \gamma\alpha)^2 + (\gamma\alpha)^2)^{1/4} \cos\theta$$

where $\pi < \theta < \pi - \pi/8$. Clearly $-\cos\theta > -\cos(\pi - \pi/8) > 0$, which shows that the fourth integral also has a Taylor-Fourier character.

Summing up we conclude that the function $u_0(x)$ for $n = 3$ is a GAF of convergence multiradius not less than $(1, 1, 1)$. By iterating the above reasoning the same result can be proved in any dimension n.

Clearly the result is valid for $u_{\overset{*}{\alpha}}$ for $\overset{*}{\alpha} = (\overset{*}{\alpha}_1, 0)$ with $\overset{*}{\alpha}_1 \in \mathbb{R}$. Returning to the variables s by the formula $x_1 = e^{-s_1}, \ldots, x_n = e^{-s_n}$ we obtain Laplace integral representations for the corresponding fundamental solutions $v_{\overset{*}{\alpha}}$ to Δ_n, which are valid for $s \in \mathbb{R}^n_+$. The choice of the positive orthant $\mathbb{R}^n_+$ is completely insignificant and after a suitable rotation of the variables (α_1, γ'') we can make the representation convergent in any fixed set $A(\mathbb{R}^n_+)$, where A is a rotation matrix. At this point we can also get rid of the restriction on $\overset{*}{\alpha}$. Indeed, fix arbitrarily $\overset{*}{\alpha} \in \mathbb{R}^n$ and let A be an orientation preserving real rotation matrix such that $A(\overset{*}{\alpha}) = (\|\overset{*}{\alpha}\|, 0, \ldots, 0)$. Then in view of the duality $(s, A^{-1}\theta) = ((A^{-1})^{\mathrm{tr}}s, 0)$ we get $v_{\overset{*}{\alpha}} = v_{A(\overset{*}{\alpha})}((A^{-1})^{\mathrm{tr}}y)$, which leads to the case already discussed.

9. Comments and open problems

1. The partial vertex set $\Sigma_{z_1}(P)$ was defined in an algebraic way. It should, however, have a geometric meaning related to the stratification of the algebraic set $\operatorname{char} P = \{P(z_1, \theta') = 0\}$.
2. The same applies to the relative vertex set $\Sigma_{\overset{*}{\alpha}_1}(P)$. This time also the set $\operatorname{char} P \cap (\overset{*}{\alpha} + i\mathbb{R}^n)$ should be relevant.
3. As observed in §§7 and 8 there is much freedom in choosing the $\overline{\mathbb{R}}^{n-1}_+$-type sets L_e. It seems natural to expect that L_ϵ should be elements of certain homology groups with bounds introduced by the stratifications of $\operatorname{char} P$ and $\operatorname{char} P \cap (\overset{*}{\alpha} + i\mathbb{R}^n)$.
4. It would be desirable to free oneself of the technical condition A_1. Indeed, a geometric proof of the Main Theorem should be possible.
5. It is interesting to investigate the hyperbolic counterparts of the phenomena observed in §7 for elliptic polynomials.
6. Investigate the geometric character of the ramified form R given locally by (7.1) which generalizes the Leray residue form.

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BOGDAN ZIEMIAN
Polish Academy of Sciences
Institute of Mathematics
Śniadeckich 8
00-950 Warszawa, POLAND