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COMPLEX GEOMETRIC ASYMPTOTICS FOR NONLINEAR SYSTEMS ON COMPLEX VARIETIES

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Dedicated to Jean Leray

1. Introduction

The method of geometric asymptotics was introduced to investigate semiclassical asymptotic solutions of the wave and Schrödinger equations in the presence of caustics (that is, focal surfaces of the corresponding geodesic flow). For example, this was done in [23] to explain the whispering gallery phenomenon of acoustics. This method developed into one of the main areas of research in geometric analysis by Leray, Hörmander, Guillemin and Sternberg, Kostant, Weinstein, Arnold, Duistermaat, Souriau and many others. (For details about the method of geometric asymptotics see [22], [12], [13], [33], [20], [26], [21], [25]).

Methods of complex analysis have been applied to the theory of Feynman path integrals and its relation to the semiclassical theory. In particular, McLaughlin [31] introduced the idea of using path integrals with complex time to obtain WKB barrier penetration.

In [3], [4] and [9], angle representations and complex geometric asymptotics for nonlinear problems are investigated using multi-valued functions of several complex variables on the moduli of Jacobi varieties. This is a new approach to the study of geometric asymptotics that naturally fits into the scheme of

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algebraic-geometric methods for nonlinear problems including the sine-Gordon and nonlinear Schrödinger equations. It is shown that the construction of such complex solutions gives new insight into the investigation of many phenomena basic to geometric asymptotics such as the index of a curve lying in a Lagrangian submanifold of a cotangent bundle. This index is related to the Maslov class, which is an obstruction to the transversality of two Lagrangian submanifolds. (This class has become a part of the theory of secondary characteristic classes and Chern-Simons classes [17]). This approach results in a particular form of the quantum conditions on the moduli of n-dimensional Jacobi varieties, which leads to the introduction of semiclassical geometric phases; see Berry [15]. At the same time, there is a new additional phase in the averaged shift of the quantum conditions after transporting a system along certain closed curves in the space of parameters, which can be linked to a symplectic representation of the braid group; for details, see [4] and [9].

In this paper we describe the general method of complex geometric asymptotics and illustrate it by constructing semiclassical modes for three types of systems. The first type concerns families of geodesics on n-dimensional quadrics and in domains bounded by quadrics in the context of problems of diffraction. We also construct semiclassical modes for umbilic billiards and for the n-dimensional complex spherical pendulum. This last example also illustrates the phenomena of semiclassical monodromy.

2. Geometric asymptotics

An important part of geometric asymptotics is the establishment of a link between the Schrödinger equation (using the Laplace-Beltrami operator in its kinetic part plus a potential part, in the usual way) and certain integrable non-linear Hamiltonian systems. One does this by considering a class of solutions of the Schrödinger equation of the form

$$U = \sum_k A_k(\mu_1, \dots, \mu_n) \exp(iwS_k(\mu_1, \dots, \mu_n)), \tag{2.1}$$

where the μ variables evolve in time according to the phase flow of an associated Hamiltonian system

$$\frac{dW}{dt} = \{W, H\}, \qquad W = (\mu_1, \dots, \mu_n, P_1, \dots, P_n). \tag{2.2}$$

Here $\{\,,\}$ are the standard canonical Poisson brackets and H is a Hamiltonian function of the form kinetic plus potential energy corresponding to the quantum

Hamiltonian; this Hamiltonian determines a flow on the phase space that we denote by

$$g_t : M^{2n} \to M^{2n}, \tag{2.3}$$

which is a 1-parameter group of diffeomorphisms of the phase space manifold M^{2n}. We will describe this manifold below and also take it to be a *complex* $2n$-dimensional manifold.

The function A_k is the so-called amplitude, which contains all the information about caustics (that is, the set of focal or conjugate points of the extremal, or geodesic, field). The function S_k is called the phase function and one can show that it is the generating function of the Lagrangian submanifold of the phase space obtained by transporting an initial Lagrangian submanifold by the Hamiltonian flow. Here w is a parameter, and in WKB theory, one normally takes $w = 1/\hbar$ where $\hbar$ is Planck's constant. Keep in mind that S_k can be multiple valued and that A_k, while single valued, generally blows up at a caustic. Index k indicates that A_k and S_k are amplitude and phase on the kth-sheet of the covering of the Lagrangian submanifold. (For details see below).

To resolve the difficulties posed by singularities and to deal with the multivaluedness of the phase function S, [23] introduced the method of geometric asymptotics in the case of 2-dimensional invariant varieties. This method together with the boundary-layer method [14] was then developed to treat problems of diffraction. In particular, imaginary rays and the corresponding wave fields which are defined in shadow domains were described in [24] as part of a geometric theory of diffraction. (For details about general method of geometric asymptotics see [20]).

Solutions of the form (2.1) with complex phase functions S and associated completely integrable systems with complex Hamiltonians H were studied in [3]. Complex geometric asymptotics in shadow domains were constructed in [4] and [5] in the context of the geometric theory of diffraction.

In particular, these references suggested a method for constructing local semiclassical solutions (modes) in the form of functions of several complex variables on the moduli of Jacobian varieties of compact multisheeted Riemann surfaces. Quantum conditions were defined as conditions of finiteness on the number of sheets of the Riemann surface. This method enables one to use, in the neighborhood of a caustic, a circuit in the complex plane. By gluing together different pieces of the solution in this fashion, one can obtain global geometric asymptotics. This procedure, together with the transport theorem for integrable problems on Riemannian manifolds, facilitates the construction of geometric asymptotics for a whole class of quasiperiodic solutions of integrable systems on hyperelliptic Jacobi varieties (see [3] and [4]). This class includes some of the most important problems such as the Jacobi problem of geodesics on quadrics and billiards in domains bounded by quadrics, as well as the KdV and Dym-type equations, the C. Neumann problem for the motion of particles on an n-dimensional sphere in the field of a quadratic potential and the sine-Gordon and nonlinear Schrödinger equations. In particular, whispering gallery modes and bouncing ball modes were constructed in [3] for the Jacobi problem of geodesics in the n-dimensional case. Similar modes were introduced in the 2- and 3-dimensional case by [23] to explain the whispering gallery phenomenon of acoustics and to describe waveguides.

Recall that quasiperiodic solutions of integrable nonlinear equations can be described in terms of finite dimensional Hamiltonian systems on $\mathbb{C}^{2n}$. In these problems, there is a complete set of first integrals that are obtained, for example, by the method of generating equations, as explained in [7] and [8]. The method of generating equations has associated with it a finite dimensional complex phase space $\mathbb{C}^{2n}$ and two commuting Hamiltonian flows. One of these gives the spatial evolution and the other gives the temporal evolution of special classes of solutions of the original PDE. The level sets of the common first integrals are Riemann surfaces $\mathcal{R}$. These surfaces have branch points that are parameterized by the choice of values of the first integrals.

We think of $\mathbb{C}^{2n}$ as being the cotangent bundle of $\mathbb{C}^n$, with configuration variables $\mu_1, \ldots, \mu_n$ and with canonically conjugate momenta $P_1, \ldots, P_n$. The two relevant Hamiltonians on $\mathbb{C}^{2n}$ both have the form

$$H = \frac{1}{2} g^{jj} P_j^2 + V(\mu_1, \ldots \mu_n), \tag{2.4}$$

where g^{jj} is a Riemannian metric on $\mathbb{C}^n$. The two Hamiltonians are distinguished by different choices of the diagonal metric.

These two Hamiltonians have the same set of first integrals, which are of the form

$$P_j^2 = K(\mu_j), \qquad j = 1, \ldots, n,$$

where K is a rational function of μ_j. Thus, we get two commuting flows on the symmetric product of n copies of the Riemann surface $\mathcal{R}$ defined by

$$P^2 = K(\mu).$$

These Riemann surfaces can be regarded as complex Lagrangian submanifolds of $\mathbb{C}^{2n}$. We call this the *μ-representation* of the problem. Recall that a Hamiltonian system is linearized when written in action-angle variables on the complex Jacobian.

For every spatial (stationary) Hamiltonian (2.4) there is a corresponding stationary Schrödinger equation which has the form

$$\nabla^j \nabla_j U + w^2(E - V)U = 0. \tag{2.5}$$

Here w is a parameter as before, and ∇^j and ∇_j are covariant and contravariant derivatives defined by the metric tensor g^{jj}. Equation (2.5) can be represented in the equivalent form

$$\sum_{j=1}^{n} \frac{g^{jj}}{\sqrt{\prod_{l=1}^{n} |g_{ll}|}} \frac{\partial}{\partial \mu_j} \left(g^{jj} \sqrt{\prod_{l=1}^{n} |g_{ll}|} \frac{\partial U}{\partial \mu_j} \right) + w^2(E - V)U = 0. \tag{2.6}$$

We consider geometric asymptotics to be solutions of equation (2.5) of the form (2.1) defined on the covering of the Jacobi variety in the phase space of the integrable problem. Substituting (2.1) into (2.5), (2.6) and equating the coefficients of w and w^2, respectively, one obtains the system

$$\nabla^j (A_k^2 \nabla_j S_k) = 0 \qquad \text{(transport equation)}, \tag{2.7}$$

$$\nabla^j S_k \nabla_j S_k - V = -E \qquad \text{(eikonal equation)}. \tag{2.8}$$

We can interpret the eikonal equation as the Hamilton-Jacobi equation of the corresponding problem. Solutions can be constructed using symmetry properties of the Riemann metric, which in turn determines the quantum equation as was shown in [3]. This method of construction is related to the general method of separation of variables in Schrödinger operators. As a result, we obtain an action function S, which is, at the same time, a phase function for the geometric asymptotics and that can be used to solve the transport equation for the amplitude function A in the form $A = U_0/\sqrt{D \det J}$. Here D is the volume element of the metric and J is the Jacobian of the change of coordinates from the μ-representation on the Riemann surface to the angle representation on the Jacobi variety, that is, on the level set of the first integrals in the phase space of the corresponding classical problem.

Then modes of the form (2.1) are constructed which link the Schrödinger operators on Riemannian manifolds with integrable systems corresponding to the class of metrics mentioned above. The methods of geometric asymptotics can be used in many problems including the whispering gallery phenomenon of acoustics and problems in diffraction, as is shown in the next section.

3. Diffracted modes

In this section we describe the collapsing construction introduced in [3] and its application to the problem of diffraction by an n-dimensional ellipsoid. This construction is of interest in a number of situations. For example, it was used in [7] to study peakon and billiard solutions in a shallow water equation.

The main idea of the collapsing construction is as follows. One first considers the geodesic flow on a quadric in $\mathbb{R}^{n+1}$. Associated with this flow is some underlying complex geometry (described in [3] and [8]), first integrals of the motion, and a complex Hamiltonian. We fix the value of the first integrals and let l_{n+1}, the shortest semiaxis (in the case of an ellipsoid and the semiaxis with the smallest absolute value in the case of a hyperboloid), tend to zero. This yields corresponding first integrals and Hamiltonians for the geodesic flow in a domain in $\mathbb{R}^n$ bounded by a quadric. This quadric develops from the limiting process. Also, the projections of the trajectories of the geodesic flow into $\mathbb{R}^n$ converge (as sets) to the trajectories of the billiard flow (in the elliptic case) in the domain. In the hyperbolic case, the trajectories may be regarded as complex billiards, as we will explain later. We note that the first integrals and Hamiltonian for billiards inside n-dimensional ellipsoids were obtained in this way; see [3] and [5].

When one fixes the first integrals for these geodesic flows, a special family of geodesics is picked out. Its envelope is, by definition, a caustic. As we will see, the amplitude of the associated semiclassical mode will blow up at each point of the caustic. We will use complexification of the problem to resolve these singularities and to extend the semiclassical mode into the shadow domain.

In Figure 1 we show families of geodesics (again with a fixed choice of first integral) obtained after collapsing a 2-dimensional ellipsoid (in (a) and (b)) and a 2-dimensional hyperboloid (in (c) and (d)). In (d) the solid straight lines are geodesics, but the dashed curved line is simply a schematic curve to indicate the behavior of a semiclassical mode called the diffraction mode, described below. The caustics are shown as dashed ellipses. For the elliptic case, (a) and (b) are distinguished by different choices of families of geodesics. In (a) the geodesics are quasiperiodic while in (b) they are umbilic, which is the particular family (choice of first integrals) whose (degenerate) caustic is the straight line segment between the foci, or just the two foci themselves, depending on how one interprets the notion of caustic. (See [10] and [8] for further details). The geodesics in (c) are called sliding geodesics. Each one of the families of geodesics gives rise to an interesting complex mode. For example, the mode associated with quasiperiodic

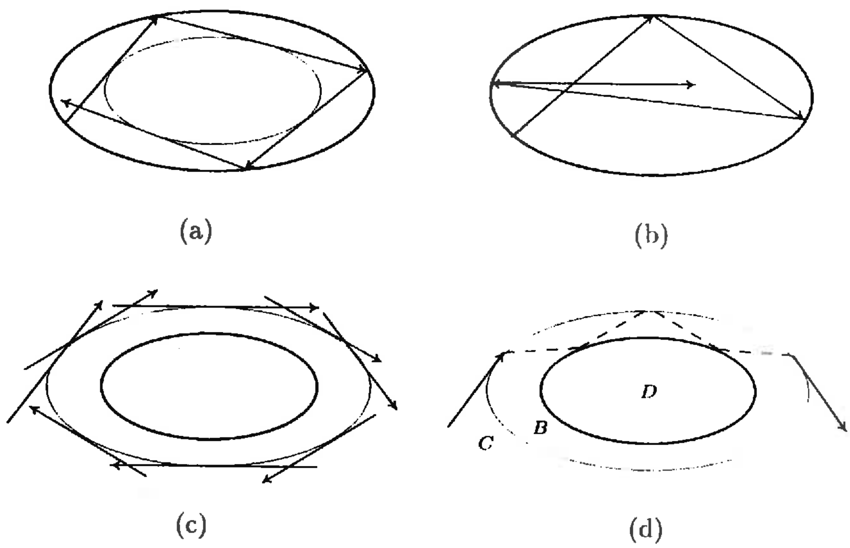

FIGURE 1. (a) shows a member of one of the families of elliptic billiards, (b) shows a member of the family of umbilic billiards, (c) shows one of the families of geodesics for sliding modes and (d) shows schematically a semiclassical mode for diffraction.

elliptic billiards ((a) of the figure) generates whispering gallery modes and sliding geodesics (c) produce luminous surfaces.

In what follows we apply the above construction to the geodesic flow on hyperboloids. The main difference with the elliptic case is that after collapsing the semiaxis with the smallest absolute value to zero, one obtains geodesics (straightlines) in the domain outside an n-dimensional ellipsoid, together with complex geodesics in the so-called shadow domain (the region B in (d)). In the shadow domain the momenta P_j are purely imaginary.

The first integrals and Hamiltonian for geodesics in the domain outside the $(n-1)$-dimensional ellipsoid have the form

$$P_j = \pm\sqrt{L_0 \prod_{k=1}^{2n-1} (\mu_j - m_k)}, \qquad j = 1, \dots, n. \tag{3.1}$$

and

$$H = \sum_{j=1}^{n} \frac{P_j^2 - L_0 \prod_{k=1, k\neq k_0}^{2n} (\mu_j - m_k)}{\prod_{r\neq j}(\mu_j - \mu_r)}. \tag{3.2}$$

The quantities $\mu_j = \mu_j(x)$ are functions of the variable x and the diagonal metric tensor has the expression

$$g^{jj} = \frac{1}{\prod_{r\neq j}(\mu_j - \mu_r)} \tag{3.3}$$

and the potential energy is given by

$$V = -\sum_{j=1}^{n} \frac{L_0 \prod_{k=1,k\neq k_0}^{2n}(\mu_j - b_k)}{\prod_{r\neq j}(\mu_j - \mu_r)}. \tag{3.4}$$

Now let $(\mu_j, j = 1, \dots, n)$ vary along cycles on the Riemann surface

$$W^2 = L_0 \prod_{k=1}^{2n-1} (\mu_j - m_k)$$

over the cuts with end points at m_r and let μ_n vary over an infinite cut from m_{2n-1} to $-\infty$. We call domains on the real axis other than cuts of the Riemann surface *shadow domains*. For example, after collapsing the hyperboloid by means of the limiting process $m_3 \to 0$, and making the choice of parameters and first integrals given by

$$m_4 < m_3 < 0 < m_2 < m_1,$$

one obtains the interval $]m_4, 0[$ as one of the shadow domains; this corresponds to domain B in Figure 1(d). Recall that the Hamiltonian system is defined on the complex Jacobian. In the real case, a projection onto a real subtorus is considered. In this situation, an extension back into the complex domain can be easily done. The variables P_j become purely imaginary in the shadow domains and therefore give rise to an exponentially decreasing factor in the expression for the semiclassical mode.

Applying the method of geometric asymptotics, as described above, one obtains a diffracted mode that has the following form:

$$U = U_0 \sum_{k=(k_1,\dots,k_n)} \left[\prod_{l=1,l\neq j_0}^{n+1} \prod_{j=1}^{n} (\mu_j - m_l) \right]^{-1/4} \times \exp\left(i\frac{\pi}{4} - i\frac{\pi}{2} D_0 - wD_1 + iwD_2 \right). \tag{3.5}$$

The mode (3.5) can be constructed independently in each domain. These domains for the two dimensional case are labeled C and B in Figure 1(d). Here D_0 is a vector of Maslov indices, D_1 is given by

$$D_1 = (-1)^{r_n} T_n(m_n, \mu_n) + k_n T_n(m_n, 0); \quad T_n(a,b) = \int_a^b P_n d\mu_n,$$

and D_2 is the real part of the phase function S.

In the shadow domain B, the amplitude A of the mode U has an exponentially decreasing factor as t, the parameter along the geodesic, increases, for the reasons indicated above. One glues together solutions from the domains C and B in the neighborhood of a caustic (the dashed ellipse shown in Figure 1(d)) by going along a half circuit in the complex plane. This procedure produces a phase shift of $\pi/4$. After going into the shadow domain B, the mode can be reflected several times on the boundary of the ellipsoid E before returning into the exterior domain C and leaving the ellipsoid; in the course of this, it is possible for the mode to go several times around the ellipsoid. In the shadow domain B, the amplitude of the mode is exponentially decreasing and, correspondingly, the mode is loosing energy; presumably the loss of energy due to this exponential decay in the shadow domain is accounted for by radiation in the exterior domain. We keep track of the number of reflections on the boundary of the quadric and number of tangent points with caustics by introducing indices k and Maslov indices r.

4. Umbilic modes

In [10] complex angle representations and Hamiltonian systems were obtained for a family of singular umbilic geodesics on quadrics. These systems have Hamiltonians of exponential type. Recently this singular geodesic flow was shown in [7] and [8] to generate a new class of soliton-like solutions of nonlinear Dym-type equations.

The collapsing construction $l_{n+1} \to 0$, where l_{n+1} is the shortest semiaxis of the n-dimensional ellipsoid, applied to the family of umbilic geodesics leads to a special type of umbilic billiards in domains bounded by $(n-1)$-dimensional ellipsoids. The Hamiltonians for these billiards have the form

$$H = \sum_{j=1}^{n} \frac{e^{M(\mu_j)P_j} - L_0 \prod_{k=1,k\neq j_0}^{n}(\mu_j - b_k)}{\prod_{r\neq j}(\mu_j - \mu_r)}, \tag{4.1}$$

where

$$M(\mu) = \sqrt{\mu - a}. \tag{4.2}$$

A complete set of first integrals for this system is as follows:

$$P_j = \sum_{k=1,k\neq j_0}^{n} \left(\frac{\log(\mu_j - b_k)}{M(\mu_j)} + \frac{\log L_0}{M(\mu_j)} \right). \tag{4.3}$$

Complex modes based on these billiards are different from both the whispering gallery modes and the bouncing ball modes obtained in [23], [3] and [5] and are similar to the soliton modes described in [9]. In the 2-dimensional case they can be represented as follows:

$$U = U_0(L_0(\mu_1 - b)(\mu_2 - b))^{-1/2}((\mu_1 - l_3)(\mu_2 - l_3))^{-1/4}$$
$$\times \exp iw\left[\sum_{j=1}^{2}\int_{\mu_j^0}^{\mu_j}\left(\frac{\log(\mu_j - b)}{M(\mu_j)} + \frac{\log L_0}{M(\mu_j)}\right)d\mu_j\right].$$

Here μ_1 and μ_2 are varying along cycles over the cuts $[0, b]$ and $[b, l_3]$ on the corresponding Riemann surfaces defined by the form of first integrals described above. An interesting thing about these modes is that they are self-focusing, namely they asymptotically approach the mode associated with the geodesic through the two foci of the ellipse.

5. The N-dimensional spherical pendulum

The Hamiltonian of the n-dimensional spherical pendulum in Cartesian coordinates Q_j and their conjugate momenta P_j has the form:

$$H = \frac{1}{2}\left(\sum_{j=1}^{n+1} P_j^2 - \left(\sum_{j=1}^{n+1} P_j Q_j\right)^2\right) + Q_{n+1}. \tag{5.1}$$

Here the acceleration due to gravity is taken to be unity. We also constrain the length of Q to be one. The same Hamiltonian in the n-dimensional spherical coordinates can be expressed in the following "nested" form

$$H = \frac{1}{2R^2}\sum_{j=1}^{n} P_{\theta_j}^2\left(\prod_{k=j+1}^{n}\frac{1}{(\sin\theta_j)^2}\right) + R\cos\theta_n; \tag{5.2}$$

i.e.,

$$H = \frac{1}{2R^2}\left(\frac{1}{(\sin\theta_n)^2}\left(P_{\theta_{n-1}}^2 + \frac{1}{(\sin\theta_{n-1})^2}\left(P_{\theta_{n-2}}^2 + \frac{1}{(\sin\theta_{n-2})^2}\right.\right.\right.$$
$$\left.\left.\left.\times\left(\cdots + \frac{1}{(\sin\theta_3)^2}\left(P_{\theta_2}^2 + \frac{1}{(\sin\theta_2)^2}P_{\theta_1}^2\right)\right)\cdots\right)\right) + \frac{1}{2R^2}P_{\theta_n}^2 + R\cos\theta_n. \tag{5.3}$$

The change of coordinates

$$\begin{cases} z_j = (\cos\theta_j)^2, \\ P_{z_j}^2(1 - z_j)z_j = P_{\theta_j}^2, \quad j = 1, \ldots, n-1, \\ z_n = \cos\theta_n, \\ P_{z_n}^2(1 - z_n^2) = P_{\theta_n}^2, \end{cases} \tag{5.4}$$

results in the Hamiltonian

$$H = \frac{1}{2}\sum_{j=1}^{n-1} P_{z_j}^2(1 - z_j)z_j\left(\prod_{k=j+1}^{n-1}\frac{1}{1 - z_k}\right)\frac{1}{1 - z_n^2} + P_{z_n}^2(1 - z_n^2) + z_n, \tag{5.5}$$

where (z_j, P_{z_j}) and (θ_j, P_{θ_j}) are pairs of conjugate variables.

The nested structure of the Hamiltonian (5.3) shows that one has the following first integrals for the n-dimensional pendulum:

$$\left\{\begin{array}{l} P_{\theta_1}^2 = \beta_1^2, \\ P_{\theta_2}^2 + \dfrac{\beta_1^2}{(\sin\theta_2)^2} = \beta_2^2, \\ \dots \\ P_{\theta_{n-1}}^2 + \dfrac{\beta_{n-2}^2}{(\sin\theta_{n-1})^2} = \beta_{n-1}^2, \\ \dfrac{1}{2}\left(P_{\theta_n}^2 + \dfrac{\beta_{n-1}^2}{(\sin\theta_n)^2}\right) + \cos\theta_n = \beta_n^2. \end{array}\right. \tag{5.6}$$

Here β_j are constants along solutions of the corresponding Hamiltonian system. Let

$$K_j(z) = \beta_j^2(1-z) - \beta_{j-1}^2, \qquad j = 2, \dots, n-1,$$

and

$$K_n(z) = 2(\beta_n^2 - z)(1 - z^2) - \beta_{n-1}^2.$$

In what follows we extend our system into the complex domain by considering β_j^2 to be complex numbers and let the variables z_j be defined on the associated Riemann surfaces:

$$\left\{\begin{array}{l} \Re_1 : W_1^2 = \dfrac{\beta_1^2}{z_1(1-z_1)}, \\ \Re_2 : W_2^2 = \dfrac{K_2(z_2)}{z_2(1-z_2)^2}, \\ \dots \\ \Re_{n-1} : W_{n-1}^2 = \dfrac{K_{n-1}(z_{n-1})}{z_{n-1}(1-z_{n-1})^2}, \\ \Re_n : W_n^2 = \dfrac{K_n(z_n)}{(1-z_n^2)^2}. \end{array}\right. \tag{5.7}$$

We call the Hamiltonian system with Hamiltonian (5.5) and first integrals (5.6) on the Riemann surfaces (5.7) a *complex n-dimensional spherical pendulum.*

To make things concrete, we shall apply the general construction of geometric asymptotics to the case of the 2-dimensional spherical pendulum. (See [11] for further information.) In this case the action function S can be represented in terms of angle variables (α_1, α_2) as follows:

$$S = -\beta_1\alpha_1^0 - \beta_2^2\alpha_2 - 2\int_{z_2^0}^{z_2} \frac{z_2 dz_2}{\sqrt{M(z_2)}}. \tag{5.8}$$

The last two terms correspond to the holomorphic and meromorphic parts of the action function. The holomorphic part is proportional to the angle variable of the classical problem. The amplitude A can be found after calculating D and J. We find that

$$D = \sqrt{g_{11}g_{22}} = \frac{1}{\sqrt{(1-z_1)z_1}}$$

and

$$\det J^{-1} = \left|\frac{\partial \alpha_i}{\partial z_j}\right| = -\frac{2\beta_2}{\sqrt{M(z_2)z_1(1-z_1)}}.$$

This results in the following form of the function U:

$$U = \sum_{k=(k_1,k_2)} A_0\sqrt{2\beta_2}(M(z_2))^{-1/4}\exp\left[iw\sum_{j=1}^{2} S_{kj}(z_j)\right], \tag{5.11}$$

where

$$S_{k_1}(z_1) = \int_{z_1^0}^{z_1}\sqrt{\frac{\beta_1^2}{(1-z_1)z_1}}\,dz_1 + k_1T_1 \tag{5.12}$$

and

$$S_{k_2}(z_2) = \int_{z_2^0}^{z_2}\sqrt{\frac{M(z_2)}{(1-z_2^2)^2}}\,dz_2 + k_2T_2 + \frac{r_2\pi}{2}, \tag{5.13}$$

where r_2 is the Maslov index and

$$\left\{\begin{aligned} T_1 &= \oint_{l_1}\sqrt{\frac{\beta_1^2}{(1-z_1)z_1}}\,dz_1,\\ T_2 &= 2\beta_2^2\oint_{l_2}\frac{dz_2}{\sqrt{M(z_2)}} - 2\oint_{l_2}\frac{z_2dz_2}{\sqrt{M(z_2)}} - \beta_1^2\oint_{l_2}\frac{dz_2}{(1-z_2^2)\sqrt{M(z_2)}}.\end{aligned}\right. \tag{5.14}$$

The amplitude A has singularities at the branch points $z_2 = m_1, m_2, m_3$ of the Riemann surface $W^2 = M(z)$. Each time a trajectory approaches one of these singularities, we continue in the complex time and go around a small circle in complex plane, enclosing the singularity. This results in a phase shift $(\pm i\frac{\pi}{2})$ of the phase function S, which is common in geometric asymptotics. The indices k_1 and k_2 keep track of the number of oriented circuits for z_1 and z_2 around l_1 and l_2. The complex mode (see (5.11)) is defined on the covering space of the complex Jacobi variety. In the real case, it is defined on the covering space of a real subtorus. Keeping this in mind, quantum conditions of Bohr-Sommerfeld-Keller type can be imposed as conditions on the number of sheets of the covering space of the corresponding Riemann surface for each coordinate z_j:

$$\left\{\begin{aligned} wk_1T_1 &= 2\pi N_1,\\ \frac{\pi}{2}r_2 + wk_2T_2 &= 2\pi N_2.\end{aligned}\right. \tag{5.15}$$

Here N_1, N_2 are integer quantum numbers. The quantum conditions (5.15) include a monodromy part after transport along a closed loop in the space of parameters (β_1 and β_2). This semiclassical monodromy consists of a classical part as well as a contribution from complex monodromy and the Maslov phase.

Classical monodromy may be explained briefly as follows. We consider two different cases, namely the case $-1 < \beta_2^2 < 1$ and $\beta_2^2 > 1$. In the first case, one considers a cycle l_2 over the cut $[-1, \beta_2^2]$ and in the second case, one considers a cycle l_2 over the cut $[-1, 1]$. There is a closed curve in the space of parameters that leads one from one case to the other. Evidently, there is a difference in the values of the third integral in the expression for T_2 between the two cases that is given by the residue of the integrand at $z_2 = 1$.

Complex monodromy is present if the roots m_1 and m_2 of the basic polynomial $M(z)$ approach each other. This singularity can be resolved by interchanging these two roots in the complex plane so as to avoid a real singularity. This leads to the change of orientation of the cycle l_2, and in the general case can be described by the generator of the symplectic representation of the braid group. It results in an additional shift in the quantum conditions.

The third type of shift in the quantum conditions comes from the integral representation for the Maslov class.

The complex mode (5.11), which corresponds to a particular choice of parameters in (5.15), is similar to an acoustic mode that occurs in the whispering gallery phenomenon described in Keller and Rubinow [23] and Alber [3], [5].

The n-dimensional system can be treated in a similar way. The complex mode U has the form

$$U = \sum_{k=(k_1,\dots,k_n)} \frac{A_0\sqrt{2\beta_2\dots\beta_n}}{((-1)^n K_2(z_2)\dots K_n(z_n)(1-z_3)(1-z_4)^2\dots(1-z_n)^{n-2})^{1/4}}$$
$$\times \exp\left[iw\sum_{j=1}^{n} S_{kj}(z_j)\right],$$

which yields the following quantum conditions:

$$\begin{cases} wk_1T_1 = 2\pi N_1, \\ \dfrac{\pi}{2}r_2 + wk_2T_2 = 2\pi N_2, \\ \dots \\ \dfrac{\pi}{2}r_n + wk_nT_n = 2\pi N_n. \end{cases} \tag{5.16}$$

Since the form of T_j, $j = 2, \dots, n-1$, is different from both T_1 and T_n, one gets additional new types of monodromy in the n-dimensional case.

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