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A GRANAS TYPE APPROACH TO SOME CONTINUATION THEOREMS AND PERIODIC BOUNDARY VALUE PROBLEMS WITH IMPULSES

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Dedicated to Ky Fan

1. Introduction

In this paper we study periodic solutions of a second order differential equation

$$x'' = f(t, x, x') \qquad \text{for a.e. } t \in [0, 1],$$

subject to some impulses at certain points. Our work was inspired by a paper by Capietto–Mawhin–Zanolin [1], where the case of no impulses was treated. The major difference between paper [1] and ours is that instead of topological degree, we use the elementary method based on essential maps. In this context, we also give some new contributions to Granas' theory of continuation principles.

The famous Leray–Schauder continuation principle, a very efficient tool in proving the existence of solutions for operator equations, can be stated, in one of its variants, as follows:

Let X be a real Banach space, K a subset of X and Ω an open subset of K. Whenever we shall be concerned with a subset of K or of $K \times [0, 1]$, all topological notions (open set, compact set, closure, boundary) will be understood with respect the topology induced on K and $K \times [0, 1]$, respectively.

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Proposition A (Leray–Schauder). *Assume K is a retract of X and $H : \overline{\Omega} \times [0,1] \to K$ is compact and such that*

(a) $H(x,\lambda) \neq x$ *for all* $x \in \partial\Omega$ *and* $\lambda \in [0,1]$*;*
(b) $i(H(\cdot,0),\Omega,K) \neq 0$.

Then, for each $\lambda \in [0,1]$, there exists at least one fixed point of $H(\cdot,\lambda)$ in Ω. Moreover, $i(H(\cdot,\lambda),\Omega,K)$ does not depend on λ.

We have denoted by $i(F,\Omega,K)$ the fixed point index over Ω with respect to K for the compact map $F : \overline{\Omega} \to K$ with $F(x) \neq x$ on $\partial\Omega$, that is, $\deg(I - FR, R^{-1}(\Omega), 0)$, where deg means the Leray–Schauder degree, I is the identity map of X and $R : X \to K$ is any retraction of X into K (see [3, 20.1]).

There are several known elementary approaches and extensions of this principle which do not use the subtle notion of degree (see, for example, [13], [6], [14], [8], [12]). One of them is due to Granas and is based on the notion of essential map. It can be described as follows:

Suppose K is convex. A compact map $F : \overline{\Omega} \to K$ is called *admissible* if it is fixed point free on $\partial\Omega$. An admissible map F is said to be *essential* on Ω provided that any admissible extension of $F|_{\partial\Omega}$ to all of $\overline{\Omega}$ has at least one fixed point in Ω. Now we can state Granas' variant of the Leray–Schauder principle.

Proposition 1 (Granas). *Assume K is convex and $H : \overline{\Omega} \times [0,1] \to K$ is compact and such that conditions* (a) *and*

(b′) $H(\cdot,0)$ *is essential on* Ω

hold. Then, for each $\lambda \in [0,1]$, there exists at least one fixed point of $H(\cdot,\lambda)$ in Ω. Moreover, the maps $H(\cdot,\lambda)$, $\lambda \in [0,1]$, are all essential.

The proof of Proposition 1 is elementary and is based only on Urysohn' characterization of the normal topological spaces.

There is an equivalent statement of Proposition 1 in terms of homotopic maps, which is known as the *topological transversality theorem*. Two admissible maps F and G are called *homotopic* on Ω if there is a compact homotopy $H : \overline{\Omega} \times [0,1] \to K$ for which $F = H(\cdot,1)$, $G = H(\cdot,0)$ and $H(\cdot,\lambda)$ is admissible for each $\lambda \in [0,1]$. Now, Proposition 1 can be formulated as follows:

Proposition 1′ (Granas). *Assume K is convex and F and G are homotopic on Ω. Then one of these maps is essential on Ω if and only if the other is.*

Recall that any constant map x_0, where x_0 is an arbitrary point in Ω, is essential on Ω (see [7, 1.2]).

Both variants of the Leray–Schauder principle are concerned with a family of maps $H(\cdot,\lambda)$, $\lambda \in [0,1]$, which are defined on the same domain $\overline{\Omega}$. The way this continuation principle applies in proving the existence of solutions for operator equations can be described as follows. Suppose we have to solve the equation $F(x) = x$ in a convex subset K of a Banach space X, where $F : K \to K$ is a completely continuous map. We first embed the map F in a one-parameter family of the form $H : K \times [0,1] \to K$, where H is completely continuous, $H(\cdot,1) = F$, while $H(\cdot,0) = G$ is a "simpler" operator. Then we try to find a bounded open subset Ω of K such that conditions (a) and (b) or (b′) are satisfied. Finally, we conclude that there is in Ω at least one solution to $F(x) = x$. This method has been intensively applied in the study of boundary value problems (see [7], [5], [10]). However, as was shown in [1], there are various examples of boundary value problems, especially those where no a priori bounds of solutions can be obtained, where the above method fails. In this case, an extension of the continuation principle to maps $H(\cdot,\lambda)$ having different domains often applies successfully. Such an extension is known in the context of degree theory [9]:

Proposition B (Leray–Schauder). *Assume K is a retract of X, $U \subset K \times [0,1]$ is open and $H : \overline{U} \to K$ is compact. Define $U_\lambda = \{x \in K : (x,\lambda) \in U\}$. If the following two conditions hold:*

(c) *$H(x,\lambda) \neq x$ for every $(x,\lambda) \in \partial U$;*
(d) *$i(H(\cdot,0), U_0, K) \neq 0$,*

then, for each $\lambda \in [0,1]$, there exists at least one fixed point of $H(\cdot,\lambda)$ in U_λ. Moreover, $i(H(\cdot,\lambda), U_\lambda, K)$ does not depend on λ.

To our knowledge, no version of this result in terms of essential maps has been given yet. We shall fill in this gap in the first part of the present paper. So, we shall be able to state and prove a variant "without degree" of a continuation theorem due to Capietto–Mawhin–Zanolin [1], and then, in the second part of the paper, to apply it to periodic boundary value problems with impulses.

2. Elementary approach to some continuation theorems

Throughout this section, X is a real Banach space, K is a convex subset of X and $U \subset K \times [0,1]$ is open in $K \times [0,1]$. For any $V \subset X \times [0,1]$ we denote by $V_\lambda = \{x \in X : (x,\lambda) \in V\}$ the section of V at λ.

We start with a consequence of Proposition 1, which can be considered as an elementary variant of Proposition B.

Proposition 2. *Assume $H : \overline{U} \to K$ is compact and such that conditions* (c) *and*

(d′) *$\mathcal{F} : \overline{U} \to K \times [0,1]$, $\mathcal{F}(x,\lambda) = (H(x,\lambda), 0)$, is essential on U*

hold. Then, for each $\lambda \in [0,1]$, there exists at least one fixed point of $H(\cdot,\lambda)$ in U_λ. Moreover, the map

$$\mathcal{H}(\cdot,\mu):\overline{U}\to K\times[0,1],\qquad \mathcal{H}(x,\lambda,\mu)=(H(x,\lambda),\mu),$$

is essential on U *for every* $\mu\in[0,1]$.

Proof. We apply Proposition 1 to the Banach space $X\times\mathbb{R}$, to its convex subset $K\times[0,1]$, to $\Omega=U$ and to the compact map

$$\mathcal{H}:\overline{U}\times[0,1]\to K\times[0,1],$$
$$\mathcal{H}(x,\lambda,\mu)=(H(x,\lambda),\mu)\qquad \text{for } (x,\lambda)\in\overline{U} \text{ and } \mu\in[0,1].$$

By assumption (c), we easily check that

$$\mathcal{H}(x,\lambda,\mu)\neq(x,\lambda)\qquad \text{for all } (x,\lambda)\in\partial U \text{ and } \mu\in[0,1].$$

Thus, $\mathcal{H}$ satisfies both conditions (a) and (b$'$) and we can apply Proposition 1. It follows that for each $\mu\in[0,1]$, there exists a fixed point $(x,\lambda)\in U$ of $\mathcal{H}(\cdot,\mu)$. Hence, $H(x,\lambda)=x$ and $\mu=\lambda$, and so, $x\in U_\mu$ and $H(x,\mu)=x$. The result is therefore proved.

Remark 1. In case U is of the form $U=\Omega\times[0,1]$, where Ω is an open subset of K, condition (d$'$) implies (b$'$). Indeed, if $F:\overline{\Omega}\to K$ is any admissible map such that F and $H(\cdot,0)$ coincide on $\partial\Omega$, then the map $\mathcal{F}:\overline{\Omega}\times[0,1]\to K\times[0,1]$, $\mathcal{F}(x,\lambda)=(F(x),0)$, is admissible on $\Omega\times[0,1]$ and

$$\mathcal{F}(x,\lambda)=(H(x,0),0)\qquad \text{for } (x,\lambda)\in\partial U.$$

On the other hand, the maps $(H(x,0),0)$ and $(H(x,\lambda),0)$ are homotopic via the homotopy

$$\overline{U}\times[0,1]\ni(x,\lambda,\mu)\to(H(x,\mu\lambda),0).$$

Thus, by (d$'$), the map $(H(x,0),0)$ is essential on U, whence it follows that $\mathcal{F}$ is also essential on U. Consequently, $\mathcal{F}$ has at least one fixed point in U, that is, F has at least one fixed point in Ω. Therefore, $H(\cdot,0)$ is essential on Ω, as claimed.

The next result is concerned with a sufficient condition for (d$'$) to hold, namely that $H(\cdot,0)$ be homotopic on U_0 to a constant map x_0, for some $x_0\in U_0$.

Corollary 1. *Assume* $H:\overline{U}\to K$ *is compact and satisfies* (c) *and*

(e) $(1-\mu)x_0+\mu H(x,0)\neq x$ *for all* $(x,0)\in\partial U$, $0<\mu<1$,

for some $x_0\in U_0$. *Then, for each* $\lambda\in[0,1]$, *there exists at least one fixed point of* $H(\cdot,\lambda)$ *in* U_λ.

Proof. We prove that (e) implies (d$'$) and we apply Proposition 1. For this, we consider the homotopy

$$\overline{U}\times[0,1]\ni(x,\lambda,\mu)\to((1-\mu)x_0+\mu H(x,\lambda),0)\in K\times[0,1],$$

which, by (e) and (c), is admissible. This homotopy connects the map $(H(x,\lambda),0)$ with the constant map $(x_0,0)$, essential on U, because $(x_0,0)\in U$. Thus, according to Proposition 1, the map $(H(x,\lambda),0)$ is also essential on U. Thus, (d′) is checked. Now the conclusion follows from Proposition 2.

Next we use Corollary 1 to prove a variant without degree of a continuation theorem due to Capietto–Mawhin–Zanolin [1].

Let $H : K\times[0,1]\to K$ be a completely continuous map. Define

$$S=\{(x,\lambda)\in K\times[0,1] : H(x,\lambda)=x\}.$$

For any fixed $x_0\in K$, we set

$$S(x_0)=\{(x,0)\in K\times[0,1] : (1-\mu)x_0+\mu H(x,0)=x \text{ for some } \mu\in[0,1]\}.$$

Also consider a continuous functional $\phi : K\times[0,1]\to\mathbb{R}$.

THEOREM 1. *Assume there are constants c_- and c_+, $c_-<c_+$, such that, if we set $V=\phi^{-1}(]c_-,c_+[)$, the following conditions are satisfied:*

(i1) *$S\cap V$ is bounded;*

(i2) $\phi(S)\cap\{c_-,c_+\}=\emptyset$;

(i3) *there is $x_0\in K$ such that $S(x_0)$ is bounded and included in V.*

Then, for each $\lambda\in[0,1]$, there exists at least one fixed point of $H(\cdot,\lambda)$ in V_λ.

PROOF. Consider the set

$$S^*=\phi^{-1}([c_-,c_+])\cap S\subset K\times[0,1].$$

Since ϕ is continuous and (i1) and (i2) hold, we see that S^* is compact in $K\times[0,1]$, V is open in $K\times[0,1]$ and $S^*\subset V$. It follows that there is a bounded open subset U_1 of $K\times[0,1]$ such that

$$S^*\subset U_1\subset \overline{U}_1\subset V.$$

On the other hand, by (i3), $S(x_0)$ is another compact set included in V. Thus, there is again a bounded open subset U_2 of $K\times[0,1]$ such that

$$S(x_0)\subset U_2\subset\overline{U}_2\subset V.$$

Now we apply Corollary 1 with the choice $U=U_1\cup U_2$. To do this, first observe that, since H is completely continuous on $K\times[0,1]$ and U is bounded, it follows that H is compact on $\overline{U}$. Next, using $\partial U\subset V\setminus U_1$ and $\partial U\subset V\setminus U_2$, we easily check conditions (c) and (e). Thus, Corollary 1 can be applied and the result follows.

We now state a consequence of Theorem 1 which is more suitable in applications. The result is also a slightly modified version, without degree, of a result by Capietto–Mawhin–Zanolin [1].

Recall that the functional ϕ is said to be *proper* on S provided that $\phi^{-1}(]a,b[) \cap S$ is bounded (and so, relatively compact) for each bounded interval $]a,b[$.

COROLLARY 2. *Assume*

(i1′) *ϕ is proper on S;*

(i2′) *ϕ is bounded below on S and there is a sequence (c_j) of real numbers such that $c_j \to \infty$ and $c_j \notin \phi(S)$ for all $j \in \mathbb{N}$;*

(i3′) *there is $x_0 \in K$ such that $S(x_0)$ is bounded.*

Then, for each $\lambda \in [0,1]$, there exists at least one fixed point of $H(\cdot,\lambda)$ in K.

PROOF. In order to apply Theorem 1 we need to find two constants c_- and c_+, $c_- < c_+$, such that assumptions (i1)–(i3) be satisfied. For this, observe that, from (i3′) and complete continuity of H, it follows that $S(x_0)$ is, in fact, compact. Hence, since ϕ is continuous, there are constants a and b, $a < b$, such that $a < \phi(x,\lambda) < b$ for every $(x,\lambda) \in S(x_0)$. Now, taking into account (i2′), we can choose c_- and j sufficiently large that

$$c_- < \inf\{\phi(x,\lambda) : (x,\lambda) \in S\}, \quad c_- \le a \quad \text{and} \quad c_+ = c_j \ge b.$$

Finally, it is easy to check (i1)–(i3). Hence, Theorem 1 can be applied and the result follows.

REMARK 2. For U bounded, all the results of this section remain true if instead of completely continuous maps, we use more general condensing maps. Notice that similar results can be stated for set-valued maps and also for maps in Hausdorff locally convex spaces.

3. Application to periodic solutions for second order differential equations with impulses

In this section we shall give an application of Corollary 2 to the existence of solutions for the following periodic boundary value problem with impulses:

$$\text{(P)} \qquad \begin{aligned} &x'' = f(t,x,x') \qquad \text{for a.e. } t \in [0,1],\\ &x(0) = x(1), \qquad x'(0) = x'(1),\\ &x(t_k^+) = \alpha_k(x(t_k)),\\ &x'(t_k^+) = \beta_k(x(t_k), x'(t_k)), \qquad k = 1,\dots,m, \end{aligned}$$

where the points t_k are fixed and such that

$$\text{(T)} \qquad 0 < t_1 < \ldots < t_m < 1.$$

We first require that the following condition holds:

(h1) $\alpha_k : \mathbb{R} \to \mathbb{R}$ and $\beta_k : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions, while $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is an L^1-Carathéodory function.

(Recall that f is said to be an L^1-*Carathéodory function* provided that $f(\cdot, u, v)$ is Lebesgue measurable for each $(u, v) \in \mathbb{R}^2$, $f(t, \cdot, \cdot)$ is continuous for a.e. $t \in [0, 1]$, and for each $r > 0$, there exists $\gamma_r \in L^1[0, 1]$ such that $|f(t, u, v)| \leq \gamma_r(t)$ for a.e. $t \in [0, 1]$ and $u^2 + v^2 \leq r^2$.)

The solutions to (P) are assumed to be C^1 on each interval $[0, t_1]$, $]t_k, t_{k+1}]$ $(k = 1, \dots, m-1)$ and $]t_m, 1]$, with possible discontinuities of the first kind for x and x' at points (T).

The topological transversality method of Granas was recently used by Erbe and Krawcewicz [4] in the study of some systems of impulsive differential inclusions. One of the main hypotheses in [4] is a modified form of the sign (coercivity) condition: $uf(t, u, 0) > 0$ for large $|u|$. Our approach to (P) does not use this condition and is essentially based on the results of Capietto–Mawhin–Zanolin [1], [2]. The difference to [1] will be that, in the case of equations with impulses, the values of the "winding number" functional ϕ on large solutions are not necessarily integers. Nevertheless, we can arrange that ϕ takes values in some disjoint intervals and, by this, that condition (i2$'$) in Corollary 2 be satisfied.

Our goal is to make transparent the use of the abstract theory just described in the previous section. So, we do not consider here the most general assumptions for the solvability of (P). We only deal with maps with linear growth and we find the analogue for problems with impulses of the classical nonresonance condition.

First we give the operator form of (P). We work in the function space

$$C_T^1 = \{x : [0,1] \to \mathbb{R} : x \text{ and } x' \text{ are everywhere continuous except possibly at points (T) of discontinuity of first kind, at which } x \text{ and } x' \text{ are left continuous}\},$$

endowed with the usual C^1-norm, $\|x\|^2 = \sup\{x^2(t) + x'^2(t) : t \in [0, 1]\}$. Notice that C_T^1 can be identified with the Banach space $\prod_{k=0}^m C^1[t_k, t_{k+1}]$ $(t_0 = 0, t_{m+1} = 1)$. Thus, C_T^1 is also a Banach space. Moreover, write $L^1 = L^1[0, 1]$ and

$$W_p^{2,1} = \{x \in C_T^1 : x' \text{ is absolutely continuous on each }]t_k, t_{k+1}[, \\ k = 0, 1, \dots, m \text{ and } x(0) = x(1),\ x'(0) = x'(1)\}.$$

It is clear that, if $x \in W_p^{2,1}$, then x belongs to the Sobolev space $W^{2,1}[t_k, t_{k+1}]$ for $k = 0, 1, \dots, m$.

Now, for each $c > 0$, we consider the map

$$L : W_p^{2,1} \to L^1 \times \mathbb{R}^m \times \mathbb{R}^m,$$
$$L(x) = (x'' + c^2 x,\ \{x(t_k^+)\}_{k=1}^m,\ \{x'(t_k^+)\}_{k=1}^m).$$

This map is invertible and its inverse,

$$L^{-1} : L^1 \times \mathbb{R}^m \times \mathbb{R}^m \to C_T^1,$$

is linear bounded. Indeed, to get its inverse, we have to solve m initial value problems:

$$\begin{aligned} &x'' + c^2 x = y \qquad \text{for a.e. } t \in [t_k, t_{k+1}],\\ &x(t_k) = u_k, \quad x'(t_k) = v_k \quad (k = 1, \dots, m-1), \end{aligned}$$

and

$$\begin{aligned} &x'' + c^2 x = \widetilde{y} \qquad \text{for a.e. } t \in [t_m, 1 + t_1],\\ &x(t_m) = u_m, \qquad x'(t_m) = v_m, \end{aligned}$$

where $\widetilde{y}(t) = y(t)$ for $t \in [t_m, 1]$, $\widetilde{y}(t) = y(t-1)$ for $t \in [1, 1+t_1]$ and $y \in L^1$, $u = \{u_k\}_{k=1}^m \in \mathbb{R}^m$, $v = \{v_k\}_{k=1}^m \in \mathbb{R}^m$.

Thus, the unique solution $x \in C_T^1$ to $L(x) = (y, u, v)$ is the function

$$(*) \quad x(t) = \begin{cases} u_k \cos c(t - t_k) + v_k c^{-1} \sin c(t - t_k) + c^{-1} \displaystyle\int_{t_k}^{t} \sin c(t-s) y(s)\, ds \\ \hfill \text{for } t \in \,]t_k, t_{k+1}], \ k = 1, \dots, m-1, \\ u_m \cos c(t - t_m) + v_m c^{-1} \sin c(t - t_m) + c^{-1} \displaystyle\int_{t_m}^{t} \sin c(t-s) y(s)\, ds \\ \hfill \text{for } t \in \,]t_m, 1], \\ u_m \cos c(1 + t - t_m) + v_m c^{-1} \sin c(1 + t - t_m) \\ \qquad + c^{-1} \displaystyle\int_{t_m}^{1+t} \sin c(1 + t - s) y(s)\, ds \quad \text{for } t \in [0, t_1]. \end{cases}$$

Also we define a family of nonlinear maps N_λ, $\lambda \in [0,1]$,

$$\begin{gathered} N_\lambda : C_T^1 \to L^1 \times \mathbb{R}^m \times \mathbb{R}^m,\\ N_\lambda(x) = \lambda(f(\cdot, x, x') + c^2 x, \{\alpha_k(x(t_k))\}_{k=1}^m, \{\beta_k(x(t_k), x'(t_k))\}_{k=1}^m). \end{gathered}$$

From the assumption that f is an L^1-Carathéodory function, we deduce that N_λ is well-defined, continuous and bounded. Moreover, again by this assumption and by $(*)$, it follows, via the Ascoli–Arzelà theorem, that the map

$$H : C_T^1 \times [0,1] \to C_T^1, \qquad H(\cdot, \lambda) = L^{-1} N_\lambda,$$

is completely continuous. We observe that the operator equation $x = H(x, \lambda)$ in C_T^1 is equivalent to the following problem:

$$(\mathrm{P}_\lambda) \qquad \begin{aligned} &x'' + c^2 x = \lambda(f(t, x, x') + c^2 x) \qquad \text{for a.e. } t \in [0,1],\\ &x(0) = x(1), \qquad x'(0) = x'(1),\\ &x(t_k^+) = \lambda \alpha_k(x(t_k)),\\ &x'(t_k^+) = \lambda \beta_k(x(t_k), x'(t_k)), \qquad k = 1, \dots, m. \end{aligned}$$

Hence, the equivalent operator form of (P) is

$$x = H(x, 1) \qquad (x \in C_T^1).$$

Now, since $H(\cdot, 0) \equiv 0$, we remark that condition (i3′) trivially holds with $x_0 = 0$, the null element of C_T^1.

Next, as in [1], we consider the functional

$$\phi : C_T^1 \times [0,1] \to \mathbb{R}_+,$$
$$\phi(x,\lambda) = c\pi^{-1}\left|\int_0^1 [x'^2 - \lambda x f(t,x,x') + (1-\lambda)c^2x^2]\theta(cx,x')\,dt\right|,$$

where $\theta(a,b) = \min\{1,\ 1/(a^2+b^2)\}$. It is immediate to check that ϕ is continuous.

Suppose that $(x,\lambda) \in S$ satisfies

$$c^2x^2(t) + x'^2(t) \ge 1 \qquad \text{for all } t \in [0,1].$$

Then x has a finite number of simple zeroes in $[0,1]\backslash\{t_k : 1 \le k \le m\}$. Assume, for the moment, that $x(t_k^+) \ne 0$ and $x(t_{k+1}) \ne 0$ for any $k \in \{0,\dots,m\}$ and denote by n_k the number of zeroes in $]t_k, t_{k+1}[$. Then we have

$$\begin{aligned}
J_k &= c\int_{t_k}^{t_{k+1}} \{[x'^2 - \lambda x f(t,x,x') + (1-\lambda)c^2x^2]/(c^2x^2 + x'^2)\}\,dt \\
&= \int_{t_k}^{t_{k+1}} (-\arctan c^{-1}x'/x)'\,dt \\
&= n_k\pi - \arctan c^{-1}x'(t_{k+1})/x(t_{k+1}) \\
&\qquad + \arctan c^{-1}\beta_k(x(t_k), x'(t_k))/\alpha_k(x(t_k))
\end{aligned}$$

for $1 \le k \le m$, while for $k = 0$,

$$J_0 = n_0\pi - \arctan c^{-1}x'(t_1)/x(t_1) + \arctan c^{-1}x'(0)/x(0).$$

In order to make precise the behaviour of x at the possible zeroes from (T), we require that the following condition holds:

(h2) $\alpha_k(a) = 0$ if and only if $a = 0$, and there is $q_1 > 0$ such that $b\beta_k(0,b) > 0$ whenever $|b| \ge q_1$ $(1 \le k \le m)$.

Thus, if for some k, one has $x(t_k) = 0$ and $|x'(t_k)| \ge q_1$, then

$$\alpha_k(x(t_k)) = 0 \quad \text{and} \quad x'(t_k)\beta_k(0, x'(t_k)) > 0,$$

i.e., x is continuous at t_k and x' has the same sign to its left as to its right. Under the above conditions, we have

$$\phi(x,\lambda) = \left| n + \pi^{-1}\sum_{x(t_k)\ne 0}\left(\arctan\frac{\beta_k(x(t_k), x'(t_k))}{c\alpha_k(x(t_k))} - \arctan\frac{x'(t_k)}{cx(t_k)}\right)\right|,$$

where n denotes the number of zeroes of x in $]0,1]$.

Let us assume that the following condition is satisfied:

(h3) there exist $0 \le \delta < c\pi/(2m)$ and $q_2 > 0$ such that $|\beta_k(a,b)/\alpha_k(a) - b/a| \le \delta$ whenever $a \neq 0$, $a^2 + b^2 \ge q_2^2$, $1 \le k \le m$.

Then, since $|\arctan s - \arctan s'| \le |s - s'|$ for all $s, s' \in \mathbb{R}$, we get

$$\phi(x,\lambda) \in [n - m\delta/(c\pi), n + m\delta/(c\pi)] \subset]n - 1/2, n + 1/2[$$

whenever $x^2(t) + x'^2(t) \ge q^2$ for all $t \in [0,1]$, where $q = \max\{1, c^{-1}, q_1, q_2\}$. It is now clear that (i2′) holds with $c_j = j + 1/2$, $j \ge j_0$ and j_0 sufficiently large, provided that the following condition, already introduced in [1]–[2], is satisfied:

(h4) for each $r_1 > 0$ there is $r_2 \ge r_1$ such that if $(x,\lambda) \in S$ and $\inf\{x^2(t) + x'^2(t) : t \in [0,1]\} \le r_1^2$, then $\|x\| \le r_2$.

The last hypothesis is

(h5) for each $n \in \mathbb{N}$ there is $R_n \ge 0$ such that if $(x,\lambda) \in S$ and $\phi(x,\lambda) \in [n - m\delta/(c\pi), n + m\delta/(c\pi)]$ then $\inf\{x^2(t) + x'^2(t) : t \in [0,1]\} \le R_n^2$.

Then (i1′) follows and so we have proved the following result:

THEOREM 2. *Suppose that conditions* (h1)–(h5) *hold for some* $c > 0$. *Then there exists at least one solution* $x \in C_T^1$ *to* (P).

EXAMPLE. Let us consider equations with linear growth:

$$f(t,u,v) = -c^2 u + g(t,u,v), \tag{LG}$$

where $g(t,u,v)/(u^2+v^2)^{1/2} \to 0$ as $u^2 + v^2 \to \infty$, uniformly a.e. in $t \in [0,1]$, and $c > 0$.

First we check that (h4) is satisfied. Indeed, let $(x,\lambda) \in S$ be such that

$$\inf\{x^2(t) + x'^2(t) : t \in [0,1]\} \le r_1^2.$$

Suppose that this infimum equals $\inf\{x^2(t) + x'^2(t) : t \in]t_k, t_{k+1}]\}$ for some k, $0 \le k \le m$. Then, as in the proof of Proposition 3 in [1], one can find $r_{2,k} \ge r_1$ depending only on r_1 and k such that

$$\sup\{x^2(t) + x'^2(t) : t \in]t_k, t_{k+1}]\} \le r_{2,k}^2.$$

In particular, $|x(t_{k+1})| \le r_{2,k}$ and $|x'(t_{k+1})| \le r_{2,k}$. By the continuity of α_k and β_k, we can find $r_{1,k+1} \ge 0$ depending only on $r_{2,k}$ and k such that

$$\alpha_{k+1}^2(x(t_{k+1})) + \beta_{k+1}^2(x(t_{k+1}), x'(t_{k+1})) = x^2(t_{k+1}^+) + x'^2(t_{k+1}^+) \le r_{1,k+1}^2.$$

Hence,

$$\inf\{x^2(t) + x'^2(t) : t \in]t_{k+1}, t_{k+2}]\} \le r_{1,k+1}^2.$$

Next, we apply the same reasoning for the interval $]t_{k+1}, t_{k+2}]$ to get $r_{2,k+1} \geq r_{1,k+1}$, $r_{2,k+1} \geq r_{2,k}$ and $r_{1,k+2} \geq 0$ such that

$$\sup\{x^2(t) + x'^2(t) : t \in \,]t_{k+1}, t_{k+2}]\} \leq r^2_{2,k+1}$$

and

$$x^2(t^+_{k+2}) + x'^2(t^+_{k+2}) \leq r^2_{1,k+2}.$$

Thus, the successive application of these arguments yields two systems of numbers

$$r_1 \leq r_{2,k} \leq r_{2,k+1} \leq \ldots \leq r_{2,k+m}$$

and

$$r_{1,k+1}, r_{1,k+2}, \ldots , r_{1,k+m}$$

such that

$$\begin{aligned} x^2(t^+_j) + x'^2(t^+_j) &\leq r^2_{1,j}, && j = k+1, \ldots , m, \\ x^2(t^+_j) + x'^2(t^+_j) &\leq r^2_{1,m+j+1}, && j = 0, \ldots , k-1, \\ \sup\{x^2(t) + x'^2(t) : t \in \,]t_j, t_{j+1}]\} &\leq r^2_{2,j}, && j = k, \ldots , m, \\ \sup\{x^2(t) + x'^2(t) : t \in \,]t_j, t_{j+1}]\} &\leq r^2_{2,m+j+1}, && j = 0, \ldots , k-1. \end{aligned}$$

It is clear that $r_2 = \max\{r_{2,k+m} : k = 0, \ldots , m\}$ satisfies (h4).

To check (h5), we use the arguments from the proof of Theorem 3 in [1] (with $h_1(t, x_1, x_2) = -c^{-1}f(t, x_1, cx_2), h_2(t, x_1, x_2) = cx_2,\ f_1(t, x_1, x_2; \lambda) = cx_2$, $f_2(t, x_1, x_2; \lambda) = c^{-1}(\lambda f(t, x_1, cx_1) - (1-\lambda)c^2x_1)$). Thus, we can prove that a sufficient condition for (h5) is

$$c/\pi \notin [n - m\delta/(c\pi), n + m\delta/(c\pi)] \qquad \text{for all } n \in \mathbb{N},$$

or equivalently,

$$\text{(NR)} \qquad \pi\mathbb{N} \cap [c - m\delta/c, c + m\delta/c] = \emptyset.$$

Recall that $0 \leq \delta < c\pi/(2m)$, so $0 \leq m\delta/c < \pi/2$.

Therefore, for maps of the form (LG), conditions (h4)–(h5) are satisfied provided that (NR) holds.

Notice that if, in addition, the inequality

$$a\alpha_k(a) > 0 \qquad \text{for all } a \in \mathbb{R}\backslash\{0\},\ k = 1, \ldots , m,$$

holds, then the number of (simple) zeroes in $]0, 1]$ of any large solution x of (P_λ) is even and, in consequence, it suffices to demand, instead of (NR), that

$$\text{(NR}'\text{)} \qquad 2\pi\mathbb{N} \cap [c - m\delta/c, c + m\delta/c] = \emptyset.$$

The conditions (NR), (NR$'$) generalize to the case of impulses the classical hypothesis of nonresonance

$$c \neq 2n\pi \qquad \text{for every } n \in \mathbb{N},$$

which corresponds to the case $m = 0$ (no impulses).

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