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A SURVEY ON RENORMING AND SET CONVERGENCE

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Dedicated to Ky Fan

Introduction

This survey is intended to complement the authoritative book of Beer [6] as well as his survey paper on Wijsman and other forms of set convergence [9]. These works provide ample historical references as well as applications for various forms of set convergence. With this in mind, let us briefly outline our objectives in this paper.

First, we will present some of the basic properties of dual Kadec–Klee norms that in the context of our study of set convergence are extremely useful. We will often include rather complete arguments because we have yet to find a convenient reference containing the basic facts. The remainder of the first section will discuss some further properties of Kadec–Klee norms. Since *Kadec–Klee* norms, as defined below, play (explicitly or implicitly) an absolutely key role in geometric Banach space theory and its applications, this discussion is significant in its own right. Their utility in the context of set convergence will be abundantly illustrated in our discussion.

After presenting some basic properties of Kadec–Klee norms, we will discuss their interplay with set convergence. The second section gives a brief account of how they entered into the study of set convergence.

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In the third section, we present some full arguments for results that tie together our work from [15] and [16] showing that under the existence of certain Kadec–Klee renormings some weaker forms of epi-convergence actually imply much stronger and more useful forms of epi-convergence. Many of the results in the third section have not been explicitly stated elsewhere, although the techniques are a natural combination of those from [15, 16]. In particular, the implications that a Kadec–Klee norm on the predual space has on the convergence of dual functions is more fully explored than has previously been done.

The fourth section mentions briefly Beer's result concerning Wijsman convergence with respect to each equivalent norm and some of its implications.

We will be primarily interested in the following forms of set convergence. A net of closed convex sets $\{C_\lambda\}$ is said to *converge Wijsman* (*resp. weak compact gap, slice*) to a closed convex set C if $d(W, C_\lambda) \to d(W, C)$ for each singleton (resp. weak compact convex, bounded convex) set $W \subset X$. We will denote this by $\{C_\lambda\} \to_{\mathrm{W}} C$ (resp. $\{C_\lambda\} \to_{\mathrm{WG}} C$, $\{C_\lambda\} \to_{\mathrm{S}} C$). If the sets C, C_λ are weak* closed in X^* and $d(W, C_\lambda) \to d(W, C)$ for each weak* compact convex set W and the distance is measured with respect to the dual norm, then the convergence is said to be *w^*-slice*, and we write $\{C_\lambda\} \to_{\mathrm{S}^*} C$. With Wijsman and weak compact gap convergence it is crucial to stipulate which norm on X is being used for measuring the "gaps" between the sets, while for slice convergence it is not [7]. For lsc convex functions f_λ, f on X, we will say $\{f_\lambda\}$ converges Wijsman (resp. weak compact gap, slice) to f if $\{\operatorname{epi} f_\lambda\}$ converges Wijsman (resp. weak compact gap, slice) to $\operatorname{epi} f$ in $X \times \mathbb{R}$ endowed with the norm $\|(x,t)\| = \max\{\|x\|, |t|\}$. In [15] there are results detailing other allowable norms on $X \times \mathbb{R}$ as well as limitations.

Wijsman convergence was introduced by Wijsman in the seminal paper [26] where it was shown to be preserved under Fenchel conjugation in finite-dimensional spaces. In the fundamental paper [21], Mosco introduced a form of convergence which is extremely useful in reflexive spaces; for example, as Mosco showed in a subsequent article, it is preserved under Fenchel conjugation in reflexive spaces (see [1, 6]). A sequence of lsc convex functions $\{f_n\}$ is said to *converge Mosco* to the lsc convex function f if for each $x \in X$, $f(x) \le \liminf f_n(x_n)$ whenever $x_n \to_w x$ and if there exists $x_n \to x$ such that $f(x) = \lim f_n(x_n)$. This definition can also be formulated in terms of convergence of convex sets, and it is not hard to check it is equivalent to slice convergence in reflexive spaces; see [6] for further information.

We will consider real Banach spaces X and their continuous duals will be denoted by X^*. The collection of proper lsc convex functions on X will be denoted by $\Gamma(X)$. The closed unit ball of X will be denoted by B_X where the norm under consideration will be clear from the context. By the *Mackey topology* on X^*, we mean the topology of uniform convergence on weakly compact sets of X.

1. Some properties of Kadec–Klee norms

In this section we gather a few facts concerning Kadec–Klee norms. Let τ_1 and τ_2 denote two vector topologies on X with τ_2 at least as strong as τ_1 (we will deal with the weak*, weak, Mackey and norm topologies). We will say a norm is (*sequentially*) τ_1-τ_2-Kadec–Klee if the τ_1 and τ_2 topologies coincide (sequentially) on the sphere. If τ_2 is the norm topology, we simply say the norm is τ_1-*Kadec–Klee*. The utility of a Kadec–Klee norm is that it allows us to exploit compactness in a weak topology in the context of a stronger original topology. The reader is referred to [18] for most prerequisite unreferenced Banach space information.

Theorem 1.1. *For a Banach space X, the following are equivalent.*

(a) *The dual norm on X^* is w^*-τ-Kadec–Klee (w^*-Kadec–Klee).*

(b) *For each $Y \subset X$, the dual norm on Y^* is w^*-τ-Kadec–Klee (w^*-Kadec–Klee).*

(c) *For each separable $Y \subset X$, the dual norm on Y^* is sequentially w^*-τ-Kadec–Klee (sequentially w^*-Kadec–Klee).*

Proof. The implication (a)⇒(b) is simple and well-known (see for example [14, Proposition 1.4]), and (b)⇒(c) is straightforward. The implication (c)⇒(a) is contained in [14, Theorems 2.1 and 2.3] where it is derived as a consequence of several results concerning set convergence. Because of this, we will present a direct proof here.

(c)⇒(a). We will prove this in the w^*-τ-Kadec–Klee case, the other case is analogous. Suppose the dual norm on X^* is not w^*-τ-Kadec–Klee. Then we can choose a net $\{x_\lambda^*\}$ and a weakly compact set $K \subset X$ such that $\{x_\lambda^*\} \to_{w^*} x^*$, $\|x_\lambda^*\| = \|x^*\| = 1$ and

$$\sup_K |x_\lambda^* - x^*| > \varepsilon \qquad \text{for all } \lambda \text{ and some } \varepsilon > 0.$$

Let $\{u_n\} \subset X$ be such that $\|u_n\| = 1$ and $x^*(u_n) > 1 - 1/n$. Let $\lambda_1 \le \lambda_2 \le \ldots$ be chosen so that $x_\lambda^*(u_n) \ge 1 - 1/n$ whenever $\lambda \ge \lambda_n$. Now choose $\{x_i^*\} \subset \{x_\lambda^*\}$ and $\{x_i\} \subset K$ as follows. Let $x_1^* = x_{\lambda_1}^*$ and $x_1 \in K$ be chosen so that $|(x_1^* - x^*)(x_1)| > \varepsilon$. Suppose $\{x_1^*, \ldots, x_{k-1}^*\}$ and $\{x_1, \ldots, x_{k-1}\}$ have been chosen so that $|(x_j^* - x^*)(x_j)| > \varepsilon$ for $1 \le j \le k-1$ and $|(x_i^* - x_j^*)(x_j)| \ge \varepsilon$ for $1 \le j < i \le k-1$. Because $x_\lambda^* \to_{w^*} x^*$, we can choose $x_k^* = x_\lambda^*$ where $\lambda \ge \lambda_k$ and

$$|(x_k^* - x_j^*)(x_j)| \ge \varepsilon \qquad \text{for } j = 1, \ldots, k-1.$$

Now choose $x_k \in K$ such that $|(x_k^* - x^*)(x_k)| > \varepsilon$.

Let $Y = \overline{\operatorname{span}}(\{x_i\}_{i=1}^{\infty} \cup \{u_i\}_{i=1}^{\infty})$. Then Y is separable and $K_1 = K \cap Y$ is a weakly compact subset of Y. Now let $y_i^* = x_i^*|_Y$. Because Y is separable, B_{Y^*} is w^*-sequentially compact and so $y_{i_j} \to_{w^*} y^*$ for some subsequence and some $y^* \in B_{Y^*}$. Observe that

$$y^*(u_n) = \lim_j x_{i_j}^*(u_n) \geq 1 - \frac{1}{n}.$$

From this, we know $\|y\| = 1$. Moreover, for $n > m$, $|(y_n^* - y_m^*)(x_m)| \geq \varepsilon$. Because $x_m \in K_1$, this means that $\{y_{i_j}\}$ does not converge Mackey to y^*. Thus the dual norm on Y^* is not sequentially w^*-τ-Kadec–Klee. □

The following result follows easily from the above theorem; it is also easy to prove directly from Alaoglu's theorem.

Corollary 1.2. *Suppose B_{X^*} is w^*-sequentially compact. If the dual norm on X^* is sequentially w^*-τ-Kadec–Klee (sequentially w^*-Kadec–Klee), then it is w^*-τ-Kadec–Klee (w^*-Kadec–Klee).*

The next result shows connections between the equivalence of topologies and the existence of Kadec–Klee norms. Such results have been known for some time (cf. Troyanski's paper [24] and its references).

Theorem 1.3. *The topologies τ_1 and τ_2 agree (sequentially) on B_X if and only if every equivalent norm on X is (sequentially) τ_1-τ_2-Kadec–Klee.*

Proof. If τ_1 and τ_2 agree (sequentially), then it is clear that every norm on X is (sequentially) τ_1-τ_2-Kadec–Klee.

Suppose τ_1 is strictly weaker than τ_2 (sequentially). Then there is a (sequence) $\{x_\lambda\} \subset B_X$ and $x_0 \in B_X$ such that $\{x_\lambda\} \to_{\tau_1} x_0$ but $\{x_\lambda\} \not\to_{\tau_2} x_0$. By translation, we may assume $x_0 \neq 0$. Choose $\phi \in (X, \tau_1)^*$ such that $\phi(x_0) > 0$. Perturbing $\{x_\lambda\}$ by an appropriate norm convergent net, we may assume $\phi(x_\lambda) = \phi(x_0)$ for all λ and $x_\lambda, x \in 2B_X$. Now let $B = \{x : \|x\| \leq 2,\ |\phi(x)| \leq \phi(x_0)\}$. Then B is the unit ball of an equivalent norm $|||\cdot|||$ and $|||x_\lambda||| = |||x_0|||$, while $\{x_\lambda\} \not\to_{\tau_2} x_0$. □

Observe that the unit ball B of the new norm constructed in the above proof is τ_1-closed provided the unit ball of the original norm is τ_1-closed. Because dual norms always have w^*-closed unit balls, we are able to deduce the following corollary which probably contains the more interesting cases of Theorem 1.3. First recall that a space is said to be *Schur* if its weak and norm topologies agree sequentially. The typical examples of Schur spaces are the spaces $\ell_1(\Gamma)$, where Γ is any nonempty set.

Corollary 1.4. *Let X be a Banach space. Then:*

(a) *X is finite-dimensional if and only if weak* and norm convergence agree sequentially in X^* if and only if each (dual) norm on X^* is sequentially w^*-Kadec–Klee.*

(b) *$X \not\supset \ell_1$ if and only if Mackey and norm convergence agree sequentially in X^* if and only if each (dual) norm on X^* is sequentially τ-Kadec–Klee.*

(c) *Weak* and Mackey convergence agree sequentially in X^* if and only if each (dual) norm on X^* is sequentially w^*-τ-Kadec–Klee.*

(d) *X is a Schur space if and only if each equivalent norm on X is sequentially w-Kadec–Klee.*

Proof. Part (a) is due to Josefson and Nissenzweig who independently showed the deep result that weak* and norm convergence differ sequentially in the dual ball of any infinite-dimensional Banach space (see [18, Chapter XII]). Part (b) is a result based on Rosenthal's powerful ℓ_1 theorem that is derived in the unpublished manuscript [22]; see [12, Theorem 5] for the details of the proof. □

With these results at hand, we can provide relationships between dual Kadec–Klee norms and Asplund spaces (every separable subspace possesses a norm separable dual). Let us recall that an Asplund space cannot contain an isomorphic copy of ℓ_1. On the other hand, James constructed a space that does not contain ℓ_1 and yet it is not Asplund. We refer the reader to van Dulst's book [20] for a comprehensive discussion on this topic.

Theorem 1.5. *For a Banach space X, the following are equivalent.*

(a) *The dual norm on X^* is w^*-Kadec–Klee.*

(b) *X is Asplund and the dual norm on X^* is sequentially w^*-Kadec–Klee.*

(c) *B_{X^*} is w^*-sequentially compact and the dual norm on X^* is sequentially w^*-Kadec–Klee.*

(d) *$X \not\supset \ell_1$ and the dual norm on X^* is w^*-τ-Kadec–Klee.*

Proof. (a)⇒(b). Let Y be a separable subspace of X and suppose $\|\cdot\|$ is a dual w^*-Kadec–Klee norm on X^*. Let $\{\phi_n\}_{n=1}^{\infty}$ be w^*-dense in B_{Y^*}. For a fixed $\phi \in S_{Y^*}$, we choose $\phi_j \to_{w^*} \phi$. Because the dual norm is sequentially w^*-Kadec–Klee on Y^*, $\phi_j \to \phi$. Therefore Y^* is separable and so X is Asplund (see [23, Theorem 2.34]). The implication (b) implies (c) follows because Asplund spaces have w^*-sequentially compact dual balls [18, p. 230]. According to Corollary 1.2, (c) implies (a) and so clearly (c) implies (b) and hence (d). If (d) holds, then for every separable subspace Y of X, the dual norm on Y^* is sequentially w^*-τ-Kadec–Klee. Because $Y \not\supset \ell_1$, according to Corollary 1.4(b) the dual norm on Y^* is, in fact, sequentially w^*-Kadec–Klee. Invoking Theorem 1.1 shows that (a) holds. □

The next remark outlines some further results involving dual w^*-Kadec–Klee norms. We refer the reader to [17, Chapter VII] for the proofs of these assertions.

Remark 1.6. Let X be a Banach space.

(a) If X is weakly compactly generated and Asplund, then X admits a norm whose dual norm is locally uniformly rotund and hence w^*-Kadec–Klee.
(b) If X^* is weakly compactly generated, then X admits a norm whose dual norm is locally uniformly rotund.
(c) There are spaces admitting Fréchet differentiable norms, such as $C[0,\omega_1]$, that cannot be renormed so that the dual norm is sequentially w^*-Kadec–Klee.
(d) A norm on X whose dual norm is w^*-Kadec–Klee need not be Gateaux differentiable. The usual norm on c_0 is a typical example of this, as is any nonsmooth norm on a finite-dimensional space.

Let us mention that weakly compactly generated spaces include all reflexive and all separable spaces. In fact, results much stronger than (a) and (b) can be found in [17]. However, as far as we know, the following question is open.

Question 1.7. If the norm on X has a dual norm that is sequentially w^*-Kadec–Klee, is the dual norm w^*-Kadec–Klee? Equivalently, if X admits a norm whose dual is sequentially w^*-Kadec–Klee, is X an Asplund space?

In contrast to this question, using Theorems 1.1 and 1.5 one can prove the following example. The details of the proof can be found in [14, Example 4.3].

Example 1.8. On ℓ_∞ there is a norm whose dual norm is sequentially w^*-τ-Kadec–Klee, but not w^*-τ-Kadec–Klee.

In the case of τ-Kadec–Klee norms, we obtain the following example which also contrasts with Theorem 1.1.

Remark 1.9. Consider any nonreflexive Banach space such that X^{**} is separable, for example the James space. Then every double dual norm on X^{**} is sequentially τ-Kadec–Klee, while no such norm is τ-Kadec–Klee. Hence τ-Kadec–Klee norms are not sequentially separably determined.

Proof. Because X^{**} is separable, X^* cannot contain ℓ_1 and so [12, Theorem 5] says Mackey and norm convergence agree sequentially on X^{**}. However, an easy modification of the Lindenstrauss–Tzafriri proof of the Milman–Pettis theorem (see [18, p. 131]) shows that a double dual τ-Kadec–Klee norm would force reflexivity. □

The existence of w-Kadec–Klee norms on X has implications regarding w^*-slice convergence in X^*, as we will discuss at the end of Section 3. At present, we wish to highlight a few of the more prominent properties of such norms. Rather large classes of spaces admit norms that are w-Kadec–Klee (see [17, Chapter VII]) as was established over twenty years ago in the following significant result of Troyanski.

Theorem 1.10 (Troyanski). *If X is weakly compactly generated, then X admits a locally uniformly rotund norm, and in particular a w-Kadec–Klee norm.*

As is thoroughly discussed in [17], Troyanski's result has been extended to broader classes of spaces than the weakly compactly generated spaces, but we will not pursue that level of generality here. However, the following result is definitely worth mentioning (see [17, Chapter VII]).

Theorem 1.11. *If X^* is weakly compactly generated, then X admits a norm that is locally uniformly rotund and whose dual is locally uniformly rotund.*

From Theorem 1.3 it is clear that one can have sequentially w-Kadec–Klee norms on ℓ_1 that are not w-Kadec–Klee. In contrast to this, one has the following result (see [24, Lemma 1.1]).

Proposition 1.12. *Suppose X is separable and does not contain ℓ_1. If the norm on X is sequentially w-Kadec–Klee, then it is also w-Kadec–Klee.*

The following deep result of Troyanski shows what needs to be added to a w-Kadec–Klee norm to ensure the existence of a locally uniformly rotund norm on a Banach space (see [24, 17]).

Theorem 1.13 (Troyanski). *A Banach space X admits an equivalent locally uniformly rotund norm if and only if it admits a w-Kadec–Klee norm and a strictly convex norm.*

The space ℓ_∞ is the classical example of a space that has a strictly convex norm, but does not admit any sequentially w-Kadec–Klee norm (see [17, Chapter II]). Relatedly, Haydon, in a string of striking examples, constructed Asplund spaces that admit strictly convex norms but do not admit sequentially w-Kadec–Klee norms and also Asplund spaces that admit w-Kadec–Klee norms but no strictly convex norms (see [17, Chapter VII]). For some further interesting results concerning Kadec–Klee norms on certain classes of spaces, we refer the reader to the paper of Dowling *et al.* [19] and its references.

2. Renorming and set convergence

Notions of set convergence arise naturally in many different mathematical settings. In turn, properties of the norm often play a crucial role in allowing comparison of various forms of set convergence. This is well illustrated in [25] where Tsukada obtains necessary and sufficient conditions for the best approximations to a sequence of closed convex sets to converge. In the language of set convergence, the main result is as follows.

Theorem 2.1 (Tsukada). *Suppose X is Banach space renormed so that both the norm on X and its dual norm are Fréchet differentiable. Then Wijsman and Mosco convergence coincide for sequences of closed convex sets in X.*

Subsequently, the work of Beer [4] showed that it was necessary for the dual norm to be sequentially w^*-Kadec–Klee for Tsukada's result to hold (and so in the reflexive setting for the dual norm to be w-Kadec–Klee). Moreover, in reflexive spaces, the connection between w-Kadec–Klee norms and Fréchet differentiability is the following long-known result (see [17]).

Theorem 2.2. *Let X be a Banach space. Then the dual norm on X^* is Fréchet differentiable if and only if X is reflexive and the norm on X is both w-Kadec–Klee and strictly convex.*

We note that reflexive spaces admit Fréchet differentiable norms whose dual norms are not locally uniformly rotund (see [27]).

From here, Borwein and Fitzpatrick produced a complete answer as to when Wijsman and Mosco convergence coincide sequentially with a chain of equivalences that at first glance do not appear to be related [13].

Theorem 2.3 (Borwein–Fitzpatrick). *For a Banach space X, the following are equivalent.*

(a) *X is reflexive and its dual norm is* (*sequentially*) *w^*-Kadec–Klee.*
(b) *Wijsman and Mosco convergence coincide for sequences of closed convex sets in X.*
(c) *For every closed nonempty subset $F \subset X$, there is a dense, equivalently a generic, set of points in $X^* \setminus F$ that admit nearest points in C.*

At about the time the preceding result appeared, it was discovered that Mosco convergence is not preserved under duality in nonreflexive spaces; see [10]. Shortly thereafter, slice convergence became more prominent because of papers such as [2, 5]; see also [6]. Indeed, it was known (and is straightforward to check) that slice convergence coincides with Mosco convergence in reflexive spaces. In [5], Beer showed that slice convergence is preserved under polarity/Fenchel duality provided one considers dual slice convergence on the dual space. Because of this duality result, the question as to whether the Borwein–Fitzpatrick theorem [13] could be extended to nonreflexive spaces with slice convergence replacing Mosco convergence became rather interesting. In [11], a detailed analysis of convergence of level sets and dual topologies was given continuing the original work of Beer in [4]. In particular, it was observed Wijsman and slice convergence agree for level sets of linear functionals if and only if the dual norm is w^*-Kadec–Klee, and it was conjectured that the result would hold for general closed convex sets. The full result was obtained by the authors in [14]; notice its similarities with Theorem 1.1.

THEOREM 2.4. *For a Banach space X, the following are equivalent.*

(a) *The dual norm on X^* is w^*-Kadec–Klee.*
(b) *For each separable subspace Y of X, Wijsman and slice convergence coincide sequentially with respect to the inherited norm.*
(c) *Wijsman and slice convergence coincide in every subspace of X.*

The proof of this result relied heavily on separation techniques and a characterization of slice convergence given by Attouch and Beer [2] and some subtle variations thereof. We will discuss this in the next section. We should also mention that analogues of this result concerning Wijsman versus Mosco and weak compact gap convergence are also valid provided we use w^*-τ-Kadec–Klee norms; see [14, Theorem 2.3].

3. Characterizations of epi-convergence

We begin with Attouch's and Beer's particularly nice characterization of slice convergence used in their path to determining the relationship between slice convergence of a sequence of convex functions and the epigraphical convergence of their subdifferentials. First, we single out the following two conditions from their work which we have chosen to call (AB_1) and (AB_2).

(AB_1) If $x_0 \in \mathrm{dom}(\partial f)$, there exist $x_n \to x_0$ such that $\limsup_{n\to\infty} f_n(x_n) \leq f(x_0)$.

(AB_2) If $y_0 \in \mathrm{range}(\partial f)$, there exist $y_n \in X^*$ with $\|y_n - y_0\| \to 0$ and
$$\limsup_{n\to\infty} f_n^*(y_n) \leq f^*(y_0).$$

The importance of these conditions is demonstrated in the following result [2, Theorem 3.1] and some of its applications; see [2, 6].

THEOREM 3.1 (Attouch–Beer). *Let X be a Banach space and suppose $f_n, f \in \Gamma(X)$. Then $\{f_n\} \to_{\mathrm{S}} f$ if and only if* (AB_1) *and* (AB_2) *are satisfied.*

A natural question is whether an analogous characterization holds for Wijsman or other forms of gap convergence. For this it is natural to consider the following variations of (AB_2).

(AB_2^T) If $y_0 \in \mathrm{range}(\partial f)$, there exist $y_n \in X^*$ with $\|y_n\| \to \|y_0\|$ such that $\{y_n\}$ converges to y_0 in the topology T on X^* and
$$\limsup_{n\to\infty} f_n^*(y_n) \leq f^*(y_0).$$

We will use (AB_2^*) or (AB_2^τ) to denote this condition when T is respectively the weak* or Mackey topology on X^*.

Let us note that it is relatively easy to show (AB_1) and (AB_2^*) (resp. (AB_2^τ)) imply Wijsman (resp. weak compact gap) convergence. The converse is far more subtle, and is based on the following theorem which was obtained by the present authors in set form in [14, Remark 2.2] and in its current form in [15, Corollary 3.2] (along with further information concerning the properties of the corresponding product norm on $X \times \mathbb{R}$).

Theorem 3.2. *Let X be a separable Banach space and suppose $f_n, f \in \Gamma(X)$. Then $\{f_n\} \to_{\mathrm{W}} f$ if and only if* (AB_1) *and* (AB_2^*) *are satisfied.*

Observe that if the dual norm is sequentially w^*-Kadec–Klee, then (AB_2^*) becomes (AB_2). This, along with the above theorem and a separable/sequential reduction argument, was the original approach for proving Wijsman and slice convergence coincide when the dual norm is w^*-Kadec–Klee. This naturally leads to the following questions. First, is Theorem 3.2 valid in arbitrary nonseparable spaces? Second, is it sufficient to have a *sequentially* w^*-Kadec–Klee dual norm for Wijsman convergence to imply slice convergence for sequences? To our knowledge the second question remains open, while it follows that the first has a negative answer by the functional analogue of [16, Theorem 3.1]. Moreover, the functional analogue of [16, Proposition 3.2] also shows that (AB_1) and (AB_2^τ) can be properly stronger than weak compact gap convergence even in separable spaces.

We thus turn to the task of presenting analogues of the Attouch–Beer conditions which do characterize Wijsman, weak compact gap or slice convergence. These characterizations are based on combining arguments from [15, 16] so we are including rather complete details, which also give a flavour of the techniques involved in the quoted arguments.

The following functional versions of conditions given in [16, Theorem 3.3] are what we believe are the appropriate variations of the Attouch–Beer conditions; they will enable us to characterize certain forms of "gap-convergence" for nets of functions.

(3.1) If $x_0 \in \mathrm{dom}(\partial f)$, $d((x_0, f(x_0)), \operatorname{epi} f_\lambda) \to 0$.

(3.2^T) If $y_0 \in \mathrm{range}(\partial f)$, then there exists $\{y_{\lambda,\mu}\}$ converging to y_0 with respect to a given topology T on X^* such that $\|y_{\lambda,\mu}\| \to \|y_0\|$ and

$$\limsup_\mu(\limsup_\lambda f_\lambda^*(y_{\lambda,\mu})) \le f^*(y_0).$$

We will write (3.2), (3.2^τ) or (3.2^*) when T is respectively the norm, Mackey or weak* topology on X^*. We will also say that a norm on $X \times \mathbb{R}$ is *fully compatible* with the original norm on X if its dual norm on $X^* \times \mathbb{R}$ satisfies

$$\|(y_\lambda, t)\| \to \|(y_0, t)\| \qquad \text{whenever } y_\lambda \to_{w^*} y_0 \text{ and } \|y_\lambda\| \to \|y_0\|.$$

Notice that the ℓ_p-product norms on $X \times \mathbb{R}$ for $1 < p \le \infty$ are fully compatible, while the ℓ_1-product norm is not.

THEOREM 3.3. *Suppose f_λ, f are functions in $\Gamma(X)$.*

(a) $\{f_\lambda\} \to_{\mathrm{W}} f$ *(with respect to each fully compatible norm on $X \times \mathbb{R}$) if and only if* (3.1) *and* (3.2*) *are satisfied.*

(b) $\{f_\lambda\} \to_{\mathrm{WG}} f$ *(with respect to each fully compatible norm on $X \times \mathbb{R}$) if and only if* (3.1) *and* (3.2$^\tau$) *are satisfied.*

(c) $\{f_\lambda\} \to_{\mathrm{S}} f$ *if and only if* (3.1) *and* (3.2) *are satisfied.*

PROOF. Let us prove (b), for example, because all the cases have similar proofs. Throughout we will let $C := \operatorname{epi} f$ and $C_\lambda = \operatorname{epi} f_\lambda$.

$\Rightarrow$: Suppose $\{f_\lambda\}$ converges weak compact gap to f, that is, $\{C_\lambda\}$ converges weak compact gap to C in $X \times \mathbb{R}$ endowed with the norm $\|(x,t)\| = \max\{\|x\|, |t|\}$. Fix $(x_0, f(x_0)) \in C$. Then Wijsman convergence forces $d((x_0, f(x_0)), C_\lambda) \to 0$, so (3.1) holds.

Let us now show that (3.2$^\tau$) holds. We let $y_0 \in \partial f(x_0)$ and so $(y_0, -1)$ attains its supremum on C at $(x_0, f(x_0))$. By [16, Theorem 3.3] there is a net $\{(y_{\lambda,\beta}, t_{\lambda,\beta})\}$ converging to $(y_0, -1)$ in the Mackey topology on $X^* \times \mathbb{R}$ such that

$$\limsup_\beta (\limsup_\lambda \{\sup_{C_\lambda} (y_{\lambda,\beta}, t_{\lambda,\beta})\})) \le \sup_C (y_0, -1).$$

Now $t_{\lambda,\beta} \to -1$ and so letting $\widetilde{y}_{\lambda,\beta} := |t_{\lambda,\beta}|^{-1} y_{\lambda,\beta}$ we have

$$\limsup_\beta (\limsup_\lambda \{\sup_{C_\lambda} (\widetilde{y}_{\lambda,\beta}, -1)\}) \le \sup_C (y_0, -1) \tag{3.3}$$

and $\|\widetilde{y}_{\lambda,\beta}\| \to \|y_0\|$ and $\widetilde{y}_{\lambda,\beta} \to_\tau y_0$. But now (3.3) implies

$$\limsup_\beta (\limsup_\lambda \ f_\lambda^*(y_{\lambda,\beta})) \le f^*(y_0).$$

$\Leftarrow$: Suppose (3.1) and (3.2$^\tau$) are satisfied. We will show $\{C_\lambda\}$ converges weak compact gap to C with respect to every fully compatible norm on $X \times \mathbb{R}$. First, regardless of the norm on $X \times \mathbb{R}$, it follows from the Brønsted–Rockafellar theorem (see [23, Theorem 3.17]) and (3.1) that $\lim_\lambda d((x,r), C_\lambda) = 0$ for each $(x,r) \in C$. Thus, for any set $W \subset X \times \mathbb{R}$, $\limsup_\lambda d(W, C_\lambda) \le d(W, C)$. Now suppose $X \times \mathbb{R}$ is endowed with a fully compatible norm and W is a weakly compact set. We may suppose $d(W,C) > r > 0$ and it suffices to show $\liminf_\lambda d(W, C_\lambda) \ge r$. Thus for $F = W + B_r$, we have $d(F, C) > 0$. Consequently, [5, Lemma 4.10] shows there is a $y_0 \in \partial f(x_0)$ such that F lies below the graph of $f(x_0) + y_0(\cdot - x_0)$. In other words,

$$\sup_C \phi_0 + r\|\phi_0\| \le \inf_W \phi_0 \qquad \text{where} \quad \phi_0 := (y_0, -1). \tag{3.4}$$

By (3.2$^\tau$), we fix $y_{\lambda,\beta} \to_\tau y_0$ such that $\|y_{\lambda,\beta}\| \to \|y_0\|$ and

$$\limsup_\beta (\limsup_\lambda f^*(y_{\lambda,\beta})) \le f^*(y_0).$$

Because the norm on $X \times \mathbb{R}$ is fully compatible with $\|\cdot\|$, we have $\|\phi_{\lambda,\beta}\| \to \|\phi_0\|$ where $\phi_{\lambda,\beta} := (y_{\lambda,\beta}, -1)$. For $\varepsilon > 0$, we fix (λ_0, β_0) with

$$\sup_W |\phi_{\lambda,\beta_0} - \phi_0| < \varepsilon\|\phi_0\| \quad \text{and} \quad f^*(y_{\lambda,\beta_0}) \le f^*(y_0) + \varepsilon\|\phi_0\| \quad \text{for } \lambda \ge \lambda_0. \tag{3.5}$$

Therefore, for $\lambda \ge \lambda_0$, we have

$$\begin{aligned}
\|\phi_{\lambda,\beta_0}\| d(W, C_\lambda) &\ge \inf_W \phi_{\lambda,\beta_0} - \sup_{C_\lambda} \phi_{\lambda,\beta_0} = \inf_W \phi_{\lambda,\beta_0} - f^*_{\lambda,\beta_0}(y_{\lambda,\beta_0}) \\
&\ge \inf_W \phi_{\lambda,\beta_0} - f^*(y_0) - \varepsilon\|\phi_0\| && \text{[by (3.5)]} \\
&\ge \inf_W \phi_0 - f^*(y_0) - 2\varepsilon\|\phi_0\| && \text{[by (3.5)]} \\
&= \inf_W \phi_0 - \sup_C \phi_0 - 2\varepsilon\|\phi_0\| \\
&\ge (r - 2\varepsilon)\|\phi_0\|. && \text{[by (3.4)]}
\end{aligned}$$

Because $\|\phi_{\lambda,\beta}\| \to \|\phi_0\|$, using an appropriately chosen β_0 in the above inequality establishes that $\liminf_\lambda d(W, C_\lambda) \ge r$. □

Examples showing that some restrictions, such as full compatibility, are needed on the norm to obtain Theorem 3.3 are presented in [15].

The next result links Wijsman convergence with the convergence of Lipschitz regularizations (we use the term *Lipschitz regularization* to denote the infimal convolution of f with a multiple of the norm: $f(x, \mu) := \inf\{f(y) + \mu\|x - y\| : y \in X\}$). In his influential monograph [1], Attouch was the first to explore connections between epi-convergence and convergence of certain infimal convolutions of the functions. Subsequently, Azé [3] produced a characterization of Mosco convergence in terms of certain Lipschitz regularizations in "nicely" renormed reflexive spaces. This work in turn had a strong influence on [15]. In fact, Beer, who in [8] obtained characterizations of slice convergence that involved the convergence of Lipschitz regularizations of both the functions and their conjugates, conjectured to us that Azé's theorem could be extended to nonreflexive spaces renormed so that the dual norm is w^*-Kadec–Klee. The work in [15] substantiates Beer's conjecture and is one of the main by-products of the following theorem; see Corollary 3.6(a).

THEOREM 3.4. *Let X be a Banach space. Then the following are equivalent:*

(a) $f_\lambda(\,\cdot\,,\mu)$ *converges pointwise to* $f(\,\cdot\,,\mu)$ *for all* $\mu > \mu^* := d(0, \operatorname{dom} f^*)$*;*

(b) (3.1) *and* (3.2*) *are satisfied;*

(c) $\{f_\lambda\}$ *converges Wijsman to* f (*with respect to each fully compatible norm on* $X \times \mathbb{R}$).

PROOF. (a)⇒(b). One obtains (3.1) directly from [15, Prop. 1.5]. To show that (3.2*) holds, we first let $y_0 \in \partial f(x_0)$ for some x_0. For all $r \geq \|y_0\|$ we have $y_0 \in \partial f(x_0, r)$ and $f(x_0, r) = f(x_0)$ (see [15, Lemma 1.1]). Let $r = \|y_0\|$ and let $r_n \downarrow r$. Index the norm compact convex sets by inclusion: $\{K_\alpha\}$. Fix α_0 such that $x_0 \in K_\alpha$ for $\alpha > \alpha_0$. Now $f_\lambda(\,\cdot\,, r_n)$ is r_n-Lipschitz and so $f_\lambda(\,\cdot\,, r_n)$ converges uniformly to $f(\,\cdot\,, r_n)$ on compact sets. Now we form the net $\beta = (\alpha, n)$ ordered so that $\beta' \geq \beta$ if $\alpha' \geq \alpha$ and $n' \geq n$. Because $f_\lambda(\,\cdot\,, r_n)$ converges uniformly on compact sets to $f(\,\cdot\,, r_n)$ for each n, there exist $\lambda(\alpha, n)$ such that

$$f(x, r_n) + \frac{1}{n} \geq f_\lambda(x, r_n) \geq f(x_0) + y_0(x - x_0) - \frac{1}{2n} \quad \text{for } \lambda \geq \lambda(\alpha, n),\ x \in K_\alpha. \tag{3.6}$$

Let α_0 be chosen so that $x_0 \in K_{\alpha_0}$. Consequently, (3.6) implies

$$f(x_0) + \frac{1}{n} \geq f_\lambda(x_0, r_n) \geq f(x_0) - \frac{1}{2n} \qquad \text{for } \lambda \geq \lambda(\alpha, n), \alpha \geq \alpha_0. \tag{3.7}$$

Now for $\beta = (\alpha, n)$, $\alpha \geq \alpha_0$ and $\lambda \geq \lambda(\alpha, n)$, one uses (3.6) with the sandwich theorem to find functionals $y_{\lambda,\beta}$ and numbers $a_{\lambda,\beta}$ such that

$$f(x_0) + y_0(x - x_0) - \frac{1}{n} \leq y_{\lambda,\beta}(x) + a_{\lambda,\beta} \qquad \text{for } x \in K_\alpha, \tag{3.8}$$

$$y_{\lambda,\beta}(x) + a_{\lambda,\beta} \leq f_\lambda(x, r_n) \qquad \text{for all } x. \tag{3.9}$$

Because $f_\lambda(x, r_n)$ is r_n-Lipschitz, we know $\|y_{\lambda,\beta}\| \leq r_n$. If $\lambda \not\geq \lambda(\alpha, n)$, we let $y_{\lambda,\beta} = y_0$.

We now show that the net $\{y_{\lambda,\beta}\}$ converges weak* to y_0. This will be accomplished by showing $\liminf_{\lambda,\beta} y_{\lambda,\beta}(h) \geq y_0(h)$ for each $h \in X$. For this, we fix $h \in X$ and choose $\alpha_1 \geq \alpha_0$ such that $h \in K_\alpha$ for all $\alpha \geq \alpha_1$. We may restrict our attention to the tail of the net where $\alpha \geq \alpha_1$. If $\lambda \not\geq \lambda(\alpha, n)$, then $y_{\lambda,\beta} = y_0$. Thus, we need only further concern ourselves when $\lambda \geq \lambda(\alpha, n)$. Now (3.7), (3.8) and (3.9) imply

$$\begin{gathered} f(x_0) + y_0(h - x_0) - \frac{1}{n} \leq y_{\lambda,\beta}(h) + a_{\lambda,\beta}, \\ y_{\lambda,\beta}(x_0) + a_{\lambda,\beta} \leq f(x_0) + \frac{1}{n}. \end{gathered} \tag{3.10}$$

The second inequality in (3.10) implies $y_{\lambda,\beta}(x_0) + a_{\lambda,\beta} - 1/n \leq f(x_0)$. Using this on the left side of the first inequality yields $y_{\lambda,\beta}(x_0) + a_{\lambda,\beta} + y_0(h - x_0) - 2/n \leq$

$y_{\lambda,\beta}(h) + a_{\lambda,\beta}$, which in turn implies

$$y_0(h - x_0) - \frac{2}{n} \leq y_{\lambda,\beta}(h - x_0).$$

Therefore, $\liminf_{\lambda,\beta} y_{\lambda,\beta}(x) \geq y_0(x)$ for all $x \in X$. Hence $\{y_{\lambda,\beta}\}$ converges weak* to y_0 as desired. Consequently, $\liminf_{\alpha,\beta} \|y_{\alpha,\beta}\| \geq \|y_0\|$ and so $\lim_{\alpha,\beta} \|y_{\alpha,\beta}\| = \|y_0\|$.

Finally, we show $\limsup_\beta(\limsup_\lambda f_\lambda^*(y_{\lambda,\beta})) \leq f^*(y_0)$. From (3.7) and (3.8) we have

$$y_{\lambda,\beta}(x_0) + a_{\lambda,\beta} \geq f(x_0) - \frac{1}{n} \geq f_\lambda(x_0, r_n) - \frac{2}{n} \quad \text{for } \lambda \geq \lambda(\alpha, n),\ \alpha \geq \alpha_0. \tag{3.11}$$

Now (3.9) and (3.11) imply $y_{\lambda,\beta} \in \partial_{2/n} f_\lambda(x_0, r_n)$. Since $f_\lambda(\,\cdot\,, r_n) \leq f_\lambda(\,\cdot\,)$, this yields

$$f_\lambda^*(y_{\lambda,\beta}) \leq (f_\lambda(\,\cdot\,, r_n))^* \leq y_{\lambda,\beta}(x_0) - f_\lambda(x_0, r_n) + \frac{2}{n}. \tag{3.12}$$

Suppose (λ, β) is such that $\alpha \geq \alpha_0$ and $|(y_{\lambda,\beta} - y_0)(x_0)| < 1/n$ by weak* convergence. Then for $\lambda \geq \lambda(\alpha, n)$, we have

$$\begin{aligned} f_\lambda^*(y_{\lambda,\beta}) &\leq y_{\lambda,\beta}(x_0) - f(x_0) + \frac{3}{n} \qquad \text{[by (3.12) \& (3.7)]} \\ &\leq y_0(x_0) - f(x_0) + \frac{4}{n} \\ &= f^*(y_0) + \frac{4}{n}. \end{aligned}$$

This completes the proof that (3.2*) holds.

(b)⇒(c). This follows from Theorem 3.3(a).

(c)⇒(a). Because $\{f_\lambda\}$ converges Wijsman to f, this follows from the equivalence of (a) and (c) in [15, Theorem 4.3]. □

We now gather some consequences first of Theorem 3.3 and then of Theorem 3.4.

Corollary 3.5. *Suppose X is a Banach space and $f_\lambda, f \in \Gamma(X)$.*

(a) *If $\{f_\lambda\} \to_{\mathrm{W}} f$ and the dual norm on X^* is w^*-Kadec–Klee, then $\{f_\lambda\} \to_{\mathrm{S}} f$.*
(b) *If $\{f_\lambda\} \to_{\mathrm{W}} f$ and the dual norm on X^* is w^*-τ-Kadec–Klee, then $\{f_\lambda\} \to_{\mathrm{W}G} f$.*
(c) *If $\{f_\lambda\} \to_{\mathrm{W}G} f$ and the dual norm on X^* is τ-Kadec–Klee, then $\{f_\lambda\} \to_{\mathrm{S}} f$.*

Proof. This follows directly from Theorem 3.3 and the properties of Kadec–Klee norms. □

COROLLARY 3.6. *Let X be a Banach space, $f, f_\lambda \in \Gamma(X)$ and suppose $f_\lambda(\,\cdot\,,\mu)$ converges pointwise to $f(\,\cdot\,,\mu)$ for all $\mu > \mu^* := d(0, \operatorname{dom} f^*)$.*

(a) *If the dual norm on X^* is w^*-Kadec–Klee, then $\{f_\lambda\}$ converges slice to f.*

(b) *If the dual norm on X^* is w^*-τ-Kadec–Klee, then $\{f_\lambda\}$ converges weak compact gap to f with respect to each fully compatible norm on $X \times \mathbb{R}$.*

PROOF. By Theorem 3.4 we know $\{f_\lambda\}$ converges Wijsman to f. Hence the result follows from Corollary 3.5. □

Contrast the failure of Wijsman convergence of sequences of convex functions to imply (AB_1) and (AB_2^*) with the characterization of Theorem 3.3. This shows why it would be difficult, if indeed possible, to prove a variant of Corollary 3.5(a) for sequences of convex functions if the norm were only assumed sequentially w^*-Kadec–Klee. We leave this as the following open question.

QUESTION 3.7. (a) If the dual norm on X^* is sequentially w^*-Kadec–Klee, does Wijsman convergence of sequences of closed convex sets in X imply slice convergence?

(b) If the dual norm on X^* is sequentially w^*-τ-Kadec–Klee, does Wijsman convergence imply weak compact gap convergence for sequences of closed convex sets in X, or even less arduously Mosco convergence?

Notice, of course, that if the answer to Question 1.7 turns out to be positive, then Theorem 2.4 has already solved Question 3.7(a). However, in light of Example 1.8, ℓ_∞ would be natural place to begin looking at Question 3.7(b). We should point out that the sequential variant of Corollary 3.5(c) fails in a separable space such that all norms have dual norms that are sequentially τ-Kadec–Klee. Indeed, this was shown in [16, Proposition 3.2], where a sequence of w^*-compact convex sets converging weak compact gap (in fact w^*-slice) but not slice to a w^*-compact convex set was constructed in the dual of the James space; cf. Remark 1.9.

Finally, let us outline how some of the previous techniques can be applied to w^*-slice convergence.

THEOREM 3.8. $\{f_\lambda^*\} \to_{S^*} f^*$ *if and only if*

(i) *if $y_0 \in \operatorname{dom}(\partial f^*)$, then $d((y_0, f^*(y_0)), \operatorname{epi} f_\lambda^*) \to 0$.*

(ii) *if $x_0 \in \operatorname{range}(\partial f^*) \cap X$, then there exist $x_{\lambda,\mu} \to x_0$ such that*

$$\limsup_\mu (\limsup_\lambda f_\lambda(x_{\lambda,\mu})) \le f(x_0).$$

PROOF. $\Rightarrow$: For this, one can first follow the argument of [16, Theorem 3.3] to provide a modified w^*-slice version where the functionals are w^*-continuous because all sets used in the proof are w^*-closed so the separating functionals can be chosen from X. One then argues as in Theorem 3.3 to convert to functional form.

$\Leftarrow$: One argues as in Theorem 3.3 and uses a w^*-continuous separating subgradient. □

Corollary 3.9. *Suppose the norm on X is w-Kadec–Klee and that $\{f_\lambda^*\} \to_{\mathrm{W}} f^*$ with respect to the dual norm on X^*. Then $\{f_\lambda^*\} \to_{\mathrm{S}^*} f^*$.*

Proof. It is clear that Theorem 3.8(i) holds, so it remains to verify Theorem 3.8(ii). For this, one can check that the proof of [16, Theorem 3.3] along with the proof of Theorem 3.3 shows that for $x_0 \in \mathrm{range}(\partial f^*) \cap X$, there exists $\{x_{\lambda,\mu}\} \to_{w^*} x_0$ such that $\|x_{\lambda,\mu}\| \to \|x_0\|$ and

$$\limsup_{\mu}(\limsup_{\lambda} f_\lambda(x_{\lambda,\mu})) \leq f^*(x_0).$$

Because the norm on X is w-Kadec–Klee, we have $\|x_{\lambda,\mu} - x_0\| \to 0$. This shows Theorem 3.8(ii) holds, and completes the proof. □

Corollary 3.10. *Suppose $f_\lambda, f \in \Gamma(X)$ and the norm on X is w-Kadec–Klee. If $f_\lambda^*(\,\cdot\,,\mu)$ converges pointwise to $f^*(\,\cdot\,,\mu)$ for all $\mu > d(0, \mathrm{dom}\, f)$, then $\{f_\lambda\} \to_{\mathrm{S}} f$.*

Proof. It follows from Theorems 3.4 and 3.8 that $\{f_\lambda^*\} \to_{\mathrm{S}^*} f^*$. Thus the result follows from Beer's bicontinuity result [5]. □

It is apparently unknown whether sequential variants of Corollaries 3.9 and 3.10 are valid when one only posits sequentially w-Kadec–Klee norms.

4. Convergence with respect to all norms

From Theorems 1.10 and 2.3, one can deduce that if a sequence of closed convex sets converges Wijsman with respect to each equivalent norm in a reflexive space, then it convergences slice. In fact, much more is true as is shown by the following interesting result of Beer [7, Theorem 3.1].

Theorem 4.1 (Beer). *For a normed linear space X, a net of closed convex sets $\{C_\lambda\}$ converges Wijsman to C with respect to each equivalent norm on X if and only if $\{C_\lambda\}$ converges slice to C.*

From this result and its proof, Beer was also able to show that the slice topology on the closed convex sets of a Banach space X is Polish when X^* is separable. Theorem 4.1 is also closely related to the stability of various forms of gap convergence in superspaces. It is not that difficult to check that slice convergence is preserved in superspaces (see [14, Prop. 4.1]). In contrast, however, [14, Example 4.2] constructs a sequence of convex sets converging Wijsman in a space

X, isomorphic to c_0, which does not converge Wijsman in $X \times \mathbb{R}$ with respect to some norm extending the norm on X. Using Theorem 4.1, [16, Theorem 2.2] completely determines when such examples can exist:

THEOREM 4.2. *For a Banach space $(X, \|\cdot\|)$ and C_λ, C closed convex sets in X, the following are equivalent.*

(a) $\{C_\lambda\} \to_{\mathrm{W}} C$ *with respect to each norm on* $X \times \mathbb{R}$ *extending* $\|\cdot\|$.
(b) $\{C_\lambda\} \to_{\mathrm{W}} C$ *with respect to every norm on* X.
(c) $\{C_\lambda\} \to_{\mathrm{S}} C$ *in every superspace of* X.

There is also a dual version of this for w^*-slice convergence (see [16, Theorem 2.5]), which is, naturally, based on the dual version of Theorem 4.1. Theorem 4.2 serves as a further reminder of the superior permanence properties possessed by slice convergence compared to weaker forms of gap convergence; and whence as to why Wijsman convergence with respect to norms whose dual norms are w^*-Kadec–Klee is so well behaved.

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