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SOLUTIONS WITH SHOCKS IN SEVERAL VARIABLES

Marek Burnat

Dedicated to Louis Nirenberg on the occasion of his 70th birthday

0. Introduction

We consider autonomous systems of the form

$$\sum_{j=1}^{l}\sum_{i=1}^{n} a_{ijs}(u)\partial_{x_i}u_j = 0, \qquad s = 1,\dots,r \geq l. \tag{1}$$

It is known ([1], [8]) that if $r = l$ and (1) is hyperbolic, then the solutions $u : \mathbb{R}^n \supset D \to \mathbb{R}^l$, $u \in C^1(D)$, are characterized by the following condition for their Jacobi matrix:

$$Du(x) = \sum_{i=1}^{q} \alpha_i(x)\lambda^i(u(x)) \otimes \gamma^i(u(x)), \qquad q < \infty,\ x \in D, \tag{2}$$

where

$$\sum_{j=1}^{l}\sum_{i=1}^{n} a_{ijs}(u)\lambda_i\gamma_j = 0, \qquad s = 1,\dots,r,$$

for $\lambda^i(u) = (\lambda_1,\dots,\lambda_n)$, $\gamma^i(u) = (\gamma_1,\dots,\gamma_n)$, and $\alpha_i : D \to \mathbb{R}$ are appropriate functions.

In the present paper we deal with systems (1), $r \geq l$, which are hyperbolic in the sense that the solutions are described by the property (2). An example

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of an overdetermined system (1) having this property is given by the system describing the magnetohydrodynamic flow of the ideal conducting inviscid fluid:

$$
\begin{aligned}
&\partial_t \varrho + \operatorname{div}(\varrho v) = 0,\\
&\varrho\Big(\partial_t v + \sum_{i=1}^{3} v_i \partial_{x_i} v\Big) + \operatorname{grad} p + \frac{1}{4\pi} H \times \operatorname{rot} H = 0,\\
&\partial_t H = \operatorname{rot}(v \times H),\\
&\operatorname{div} H = 0,\\
&\partial_t\Big(\frac{p}{\varrho\kappa}\Big) + \sum_{i=1}^{3} v_i \partial_{x_i}\Big(\frac{p}{\varrho\kappa}\Big) = 0.
\end{aligned}
\tag{3}
$$

It is a system of $r = 9$ equations and $l = 8$ unknown functions $u = (q, p, v, H)$ where $v = (v_1, v_2, v_3)$ is the fluid velocity, $H = (H_1, H_2, H_3)$ the magnetic field, $E = -v \times H$ the electric field, $\kappa = \text{const}$, and $(t, x) = (t, x_1, x_2, x_3)$ the $n = 4$ independent variables.

As a consequence of the property (2) one can derive a qualitative theory in which just from the assumption that the required solution $u : \mathbb{R}^n \supset D \to \mathbb{R}^l$ exists one gets a construction of the image $u(D) \subset \mathbb{R}^l$. From the image $u(D)$ one obtains important qualitative information (see [2]).

Moreover, the property (2) provides essential information about the parametrization of $u(D)$ by the independent variables $x_1, \ldots, x_n$ leading to the solution $u : \mathbb{R}^l \supset D \to u(D) \subset \mathbb{R}^l$.

In the present paper we confine ourselves to the simplest conical parametrization (see [3], [5]). For the constructed image $u(D)$ we find a family of linear subspaces

$$\Sigma(u) \subset \mathbb{R}^n, \quad \dim \Sigma(u) = \text{const} \quad \text{for } u \in u(D),$$

such that the mapping

$$u(x) := u, \qquad x \in \Sigma^y(u) \cap D,\ u \in u(D),$$

where for some $y \in \mathbb{R}^n$, $\Sigma^y(u) \subset \mathbb{R}^n$ denotes the plane through y tangent to $\Sigma(u)$, represents the required solution.

The main goal of the present paper is to show how, basing on the property (2), one can construct k-dimensional manifolds $M_k \subset \mathbb{R}^l, k < l, k \leq n$, with some singularities, so that after conical parametrization of M_k by the independent variables $x_1, \ldots, x_n$ we obtain a solution with a prescribed system of interacting shocks satisfying the corresponding Hugoniot conditions. For simplicity we shall outline this possibility in detail only for the system

$$(4)\qquad \begin{aligned} &\partial_t c + v_1\partial_{x_1} c + v_2\partial_{x_2} c + kc(\partial_{x_1} v_1 + \partial_{x_2} v_2) = 0,\\ &\partial_t v_1 + v_1\partial_{x_1} v_1 + v_2\partial_{x_2} v_1 + \frac{c}{k}\partial_{x_1} c = 0,\\ &\partial_t v_2 + v_1\partial_{x_1} v_2 + v_2\partial_{x_2} v_2 + \frac{c}{k}\partial_{x_2} c = 0, \end{aligned}$$

where $k = \text{const}$, describing the two-dimensional time-dependent isentropic and polytropic gas flow. Moreover, the jump conditions on the shock fronts will satisfy the Hugoniot conditions following from the gradient form of the system (4) only, and the corresponding conservation laws (see (36), (39)). This means that we do not take into account the energy conservation law and so our shocks are not quite physical ones[1]. Nevertheless, this illustrates quite well and clearly enough the possibilities of the suggested method. Moreover, in this case we give a fairly simple and exact mathematical construction of solutions describing the interaction of two regular waves producing a prescribed shock, or a prescribed system of interacting shocks.

The same possibilities are available, keeping all physical requirements, in the general case of non-isentropic gas flows, or for the system (3).

Finally, let us mention that almost all considerations and results of [6] are in fact consequences of the property (2) in the case $n = l = r = 2$. Hence we are going to give a natural extension of the methods used in [6] to the case of several variables.

In what follows, if no other requirements are formulated, we shall assume that the coefficients a_{ijs}, solutions and all functions considered are of class C^1.

1. Regular solutions

In our considerations we adopt a geometrical point of view for autonomous systems of the form

$$(5)\qquad \sum_{i=1}^{n} A_i(u)\partial_{x_i} u = f(u)$$

which represents the natural generalization of the well known qualitative theory for autonomous systems of ordinary differential equations

$$(6)\qquad \frac{du}{dx} = f(u).$$

The basic features of the qualitative theory of (6) may be formulated in the following two facts:

1′. The system (6) gives explicit information about the derivative Du of the mapping $u : \mathbb{R} \supset D \to \mathbb{R}^l$ representing the solution.

[1] Only in the case of regular solutions of (4) is the energy conservation law automatically satisfied.

2′. We have at our disposal a simple geometrical interpretation of that information.

In the general case of the system (5) we start with the explicit information about the derivative $Du = (\partial_{x_i} u_j)$ (Jacobi matrix) of the solution $u : \mathbb{R}^n \supset D \to \mathbb{R}^l$, given by (5). For this purpose, let $\mathbb{R}^{nl}$ denote nl-dimensional Euclidean space of $n \times l$ matrices $N = (N_{ij})$, $i = 1, \ldots, n$, $j = 1, \ldots, l$. For $u \in \mathbb{R}^l$ consider the plane $F(u) \subset \mathbb{R}^{nl}$ given by the system (5):

$$F(u) = \Big\{ N : \sum_{j=1}^{l} \sum_{i=1}^{n} a_{isj}(u) N_{ij} = f_s(u),\ s = 1, \ldots, r \Big\},$$

where $A_i(u) = (a_{ijs})$, $i = 1, \ldots, n$, $j = 1, \ldots, l$, $s = 1, \ldots, r \geq l$, and $f(u) = (f_1, \ldots, f_r)$. Obviously the mapping $u : \mathbb{R}^n \supset D \to \mathbb{R}^l$ is a solution of (5) if and only if

$$Du(x) \in F(u(x)), \qquad x \in D, \tag{7}$$

and this is our explicit information about the derivatives Du of the mappings $u : \mathbb{R}^n \supset D \to \mathbb{R}^l$ which are solutions of (5).

Using the differential inclusion (7) one can construct various classes of solutions as well obtain qualitative information about solutions of (7). The information can be of the following form: for a given boundary value problem we construct a k-dimensional manifold $M_k \subset \mathbb{R}^l, k < l$, such that for the solution $u : \mathbb{R}^n \supset D \to \mathbb{R}^l$ of the problem the image $u(D)$ satisfies the condition

$$u(D) \subset M_k.$$

In some cases the image $u(D)$ can be exactly constructed independently of the construction of the solution (see [2]).

For a successful application of (7) we shall use some additional algebraic properties of $F(u)$, together with the geometrical meaning of (7).

To this end we shall introduce two kinds of characteristic cones for the system (1), $\Lambda(u) \subset \mathbb{R}^n$ and $\Gamma(u) \subset \mathbb{R}^l$. First,

$$\Lambda(u) = \Big\{ \lambda = (\lambda_1, \ldots, \lambda_n) : \operatorname{rank} \Big[\sum_{i=1}^{n} a_{ijs}(u) \lambda_i \Big] < l \Big\}.$$

If the system is not overdetermined, i.e. $r = l$, then

$$\Lambda(u) = \Big\{ \lambda = (\lambda_1, \ldots, \lambda_n) : \det \Big[\sum_{i=1}^{n} a_{ijs}(u) \lambda_i \Big] = 0 \Big\}$$

and $\Lambda(u)$ is the usual cone of characterisitic vectors of the system (1). Next,

$$\Gamma(u) = \Big\{ \gamma = (\gamma_1, \ldots, \gamma_l) : \operatorname{rank} \Big[\sum_{j=1}^{l} a_{ijs}(u) \gamma_j \Big] < n \Big\}.$$

For two characteristic vectors $\lambda \in \Lambda(u) \subset \mathbb{R}^n, \lambda \neq 0$, and $\gamma \in \Gamma(u) \subset \mathbb{R}^l, \gamma \neq 0$, we shall say that the relation $\lambda \rightleftharpoons \gamma$ holds at $u \in \mathbb{R}^l$ if and only if

$$\sum_{i=1}^{n} \sum_{j=1}^{l} a_{ijs}(u)\lambda_i \gamma_j = 0, \qquad s = 1, \ldots, r.$$

Obviously if $\lambda \rightleftharpoons \gamma$ then $\lambda \in \Lambda(u), \gamma \in \Gamma(u)$. Moreover,

$$\Lambda(u) = \{\lambda : \exists_\gamma \ \lambda \rightleftharpoons \gamma \text{ at } u\}, \qquad \Gamma(u) = \{\gamma : \exists_\lambda \ \gamma \rightleftharpoons \lambda \text{ at } u\}.$$

If $\lambda \rightleftharpoons \gamma$ at u, then the matrix

$$\lambda \otimes \gamma := (\lambda_i \gamma_j) \in \mathbb{R}^{nl}$$

obviously satisfies $\lambda \otimes \gamma \in F(u)$. Moreover, the plane $F^{\otimes}(u) \subset \mathbb{R}^{nl}$ defined by

$$F^{\otimes}(u) := \Big\{ N : \sum_{i=1}^{q} \lambda_i \otimes \gamma_i, \ \lambda_i \rightleftharpoons \gamma_i, \ q < \infty \Big\} \tag{8}$$

satisfies $F^{\otimes}(u) \subset F(u)$. For a homogeneous hyperbolic system (1) with $r = l$,

$$F^{\otimes}(u) = F(u) \tag{9}$$

(see [1], [8]). This condition will be used as the most general definition of a general overdetermined hyperbolic system (1).

We shall consider now the following two questions.

1. For which k-dimensional manifolds $M_k \subset \mathbb{R}^l$, do there exist solutions $u : \mathbb{R}^n \supset D \to M_k$ such that for some neighbourhood $Q \subset \mathbb{R}^l$,

$$u(D) \cap Q = M_k \cap Q\ ?$$

2. How to construct these solutions by means of conical parametrization of M_k by the independent variables $x_1, \ldots, x_n$?

From (2) it follows that each vector of the tangent space $T_u(M_k)$, $u \in M_k \cap Q$, is a linear combination of some vectors $\gamma \in \Gamma(u)$. Hence from the point of view of our questions, it is reasonable to consider manifolds having the following property:

$$T_u(M_k) = \operatorname{lin}[\overset{1}{\gamma}, \ldots, \overset{k}{\gamma}], \qquad \overset{i}{\gamma} \in \Gamma(u), \ i = 1, \ldots, k.$$

Suppose now that for some manifold $M_k \in \mathbb{R}^l$, we have $2k$ functions $\gamma^i : M_k \to \mathbb{R}^l$, $\lambda^i : M_k \to \mathbb{R}^n$, $i = 1, \ldots, k$, such that

$$\Gamma(u) \ni \gamma^i(u) \rightleftharpoons \lambda^i(u) \in \Lambda(u) \qquad \text{for } u \in M_k,$$

and γ^i as well as λ^i are linearly independent. We can now consider the parametrization of the manifold $M_k \subset \mathbb{R}^l$, $M_k : u = \Phi(\mu_1, \ldots, \mu_k)$, such that

$$\partial_{\mu_i} \Phi = \gamma^i(\Phi(\mu)), \qquad i = 1, \ldots, k, \qquad \Phi : \mathbb{R}^k \supset M \to \mathbb{R}^l.$$

Let us additionally introduce for $u \in M_k$ the spaces

$$\Sigma(u) = \mathrm{lin}[\sigma^1(u), \dots, \sigma^{n-k}(u)] := (\mathrm{lin}[\lambda^1(u), \dots, \lambda^k(u)])^{\perp} \tag{10}$$

and for $y \in \mathbb{R}^n$ denote by $\Sigma^y(u) \subset \mathbb{R}^n$ the plane through y tangent to $\Sigma(u)$. We ask for which manifolds M_k the mapping

$$u_{\mathrm{con}}(x) := u, \qquad x \in \Sigma^y(u),\ u \in M_k, \tag{11}$$

is, in some region $D \subset \mathbb{R}^n$, a conical solution $u_{\mathrm{con}} : \mathbb{R}^n \supset D \to M_k$ of the system (1).

To address this question let us consider the mapping $\mathfrak{M} : \mathbb{R}^n \to \mathbb{R}^n, s \mapsto x$, given by

$$x = x(s) = y + \sum_{\nu=1}^{n-k} s_{k+\nu}\sigma^{\nu}(\Phi(s_1, \dots, s_k)).$$

We shall say that the family of planes $\Sigma(u), u \subset M_k$, is *conical* iff for some $d > \varepsilon > 0$ the mapping $\mathfrak{M}$ is one-to-one in the set

$$S = M \times [(\varepsilon, d) \times \dots \times (\varepsilon, d)] \subset \mathbb{R}^k \times \mathbb{R}^{n-k}.$$

Note that the family $\Sigma(u), u \in M_k$, is conical if and only if it represents a foliation of the region $\mathfrak{M}(S) \subset \mathbb{R}^n$, or if and only if the mapping u_{con} is well defined in all regions $D \subset \mathfrak{M}(S)$.

For $a = (a_1, \dots, a_q) \in \mathbb{R}^q$ and $f : \mathbb{R}^q \to \mathbb{R}^p, f = f(x_1, \dots, x_q)$, write $\partial_a f = \sum_{i=1}^q a_i \partial_{x_i} f$.

Theorem. *If for all* $u \in M_k$,

$$\partial_{\gamma^i(u)}\lambda^j(u) \in \mathrm{lin}[\lambda^1(u), \dots, \lambda^k(u)], \qquad i \neq j,\ i, j = 1, \dots, k, \tag{12}$$

and the family $\Sigma^y(u), u \in M_k$, *is conical, then the mapping* (11)

$$u_{\mathrm{con}} : \mathfrak{M}(S) \supset D \to M_k$$

is a solution of (1).

For the proof we introduce subspaces $E(u) \subset F^{\otimes}(u)$ for $u \in M_k$,

$$E(u) := \Big\{ N : N = \sum_{i=1}^{k} \alpha_i \gamma^i(u) \otimes \lambda^i(u),\ \alpha_i \in \mathbb{R} \Big\}.$$

It is sufficient to prove that $Du_{\mathrm{con}}(x) \in E(u_{\mathrm{con}}(x))$ for $x \in \mathfrak{M}(S)$. In order to determine the matrix $Du_{\mathrm{con}}(x)$ we shall use the identity

$$\Phi(s) = u_{\mathrm{con}}(x(s)), \qquad s \in S,$$

which is a system of l identities. Differentiating the jth identity with respect to s we obtain

$$\begin{aligned}
&\partial_{s_i}\Phi_j = \gamma_j^i = \partial_{s_i} u_{\mathrm{conj}}(x(s)) = \Big\langle \nabla u_{\mathrm{conj}}, \sum_\nu s_{k+\nu}\partial_{s_i}\sigma^\nu(\Phi(s))\Big\rangle, \\
&\hspace{20em} i = 1, \dots, k, \\
&\partial_{s_i}\Phi_j = 0 = \partial_{s_i} u_{\mathrm{conj}}(x(s)) = \langle \nabla u_{\mathrm{conj}}, \sigma^{i-k}(\Phi(s))\rangle, \qquad i = k+1, \dots, n,
\end{aligned} \tag{13}$$

where $\langle\cdot\,,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^n$. Keeping $j = \text{const}$ and defining

$$\begin{aligned}
&a = (a_1, \dots, a_n) := (\gamma_j^1, \dots, \gamma_j^k, 0, \dots, 0), \\
&w^i := \begin{cases} \sum_{\nu=1}^{n-k} s_{k+\nu}\partial_{s_i}\sigma^\nu(\Phi(s)), & i = 1, \dots, k, \\ \sigma^{i-k}(\Phi(s)), & i = k+1, \dots, n, \end{cases} \\
&z = \nabla u_{\mathrm{conj}},
\end{aligned}$$

we can write (13) in the form

$$a_i = \langle z, w^i\rangle, \qquad i = 1, \dots, n. \tag{14}$$

Since the determinant of the system (14) is the nonvanishing Jacobian of the mapping $\mathfrak{M}$ the system (14) has exactly one solution of the form $z = \sum_{\nu=1}^n a_\nu \tau^\nu$, where $\langle \tau^i, w^j\rangle = \delta_{ij}$. Hence in our case $z = \nabla u_{\mathrm{conj}} = \sum_{\nu=1}^k \gamma_j^\nu \tau^\nu$ or

$$\partial_{x_i} u_{\mathrm{conj}} = \sum_{\nu=1}^k \gamma_j^\nu \tau_i^\nu = \Big(\sum_{\nu=1}^k \tau^\nu \otimes \gamma^\nu\Big)_{ij}, \tag{15}$$

where

$$\begin{aligned}
&\Big\langle \tau^i, \sum_{\nu=1}^{n-k} s_{k+\nu}\partial_{\gamma^j}\sigma^\nu\Big\rangle = \delta_{ij}, \qquad i = 1, \dots, k,\ j = 1, \dots, k, \\
&\qquad\qquad \langle \tau^i, \sigma^{j-k}\rangle = 0, \qquad i = 1, \dots, k,\ j = k+1, \dots, n.
\end{aligned} \tag{16}$$

We now use the following simple lemma.

Lemma. *The condition* (12) *is equivalent to*

$$\partial_{\gamma^i(u)}\sigma^j(u) \in \operatorname{lin}[d^i(u), \sigma^1(u), \dots, \sigma^{n-k}(u)], \qquad j = 1, \dots, n-k, \tag{17}$$

for all $i = 1, \dots, k$ and $u \in M_k$, where

$$\langle d^i(u), \lambda^j(u)\rangle = \delta_{ij}, \quad i, j = 1, \dots, k, \quad \operatorname{lin}[d^1, \dots, d^k] = \operatorname{lin}[\lambda^1, \dots, \lambda^k].$$

From (17) it follows that for some $\alpha_i, \beta_{1i}, \dots, \beta_{n-k,i} \in \mathbb{R}$, $i = 1, \dots, k$,

$$\sum_{\nu=1}^{n-k} s_{k+\nu}\partial_{\gamma^i}\sigma^\nu = \alpha_i d^i + \sum_{\nu=1}^{n-k} \beta_{\nu i}\sigma^\nu.$$

Since (16) can be written as

$$\begin{aligned} \langle \tau^i, d^j \rangle &= \frac{1}{\alpha_i}\delta_{ij}, \qquad i = 1, \dots, k,\ j = 1, \dots, k, \\ \langle \tau^i, \sigma^{j-k} \rangle &= 0, \qquad i = 1, \dots, k,\ j = k+1, \dots, n, \end{aligned}$$

this shows that τ^i is parallel to λ^i for $i = 1, \dots, k$. By (15), $Du_{\mathrm{con}} \in E(u_{\mathrm{con}})$, which completes the proof of the theorem.

PROOF OF THE LEMMA. Differentiating the equality $\langle \sigma^i(u), \lambda^j(u) \rangle = 0$, where $u \in M_k$, $i = 1, \dots, n-k$, $j = 1, \dots, k$, we obtain

$$\langle \partial_{\gamma^m} \sigma^i, \lambda^j \rangle + \langle \sigma^i, \partial_{\gamma^m} \lambda^j \rangle = 0, \qquad i = 1, \dots, n-k,\ j, m = 1, \dots, k,$$

and hence we can write (12) in the following equivalent way:

$$\langle \partial_{\gamma^m} \sigma^i, \lambda^j \rangle = 0, \qquad m \neq j,\ j, m = 1, \dots, k,\ i = 1, \dots, n-k,$$

or

$$\partial_{\gamma^m} \sigma^i \in (\mathrm{lin}[\lambda^1, \dots, \lambda^{m-1}, \lambda^{m+1}, \dots, \lambda^k])^\perp, \qquad i = 1, \dots, n-k,\ m = 1, \dots, k.$$

But obviously

$$(\mathrm{lin}[\lambda^1, \dots, \lambda^{m-1}, \lambda^{m+1}, \dots, \lambda^k])^\perp = \mathrm{lin}[d^m, \sigma^1, \dots, \sigma^{n-k}],$$

which completes the proof of the Lemma.

The condition (12) gives the possibility of constructing conical solutions for autonomous hyperbolic systems (1).

Suppose we have a function $\lambda : \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}^n$, $\lambda = \lambda(u, \gamma)$, satisfying for (1) the following condition:

$$\Gamma(u) \ni \gamma \rightleftharpoons \lambda(u, \gamma) \in \Lambda(u). \tag{18}$$

Then the construction of manifolds $M_k \subset \mathbb{R}^l$, $M_k : u = \Phi(\mu_1, \dots, \mu_k)$, such that the conical parametrization (11) gives a solution of (1), is, according to the Theorem, equivalent to the construction of a function $\Phi : \mathbb{R}^k \supset M \to \mathbb{R}^l$ satisfying

$$\begin{aligned} &\partial_{\mu_i} \Phi \in \Gamma(\Phi), \qquad i = 1, \dots, k, \\ &\partial_{\mu_i} \lambda(\Phi, \partial_{\mu_j} \Phi) \in \mathrm{lin}[\lambda(\Phi, \partial_{\mu_1} \Phi), \dots, \lambda(\Phi, \partial_{\mu_k} \Phi)], \quad i \neq j,\ i, j = 1, \dots, k. \end{aligned} \tag{19}$$

In this formulation the functions $\gamma^i(u), \lambda^i(u), u \in M_k, i = 1, \dots, k$, appearing in the Theorem are

$$\gamma^i(u) = \partial_{\mu_i} \Phi(\Phi^{-1}(u)), \qquad \lambda^i(u) = \lambda(u, \gamma^i(u)).$$

This is a PDE system for $\Phi_1(\mu), \dots, \Phi_l(\mu)$ which may be used as the parametric representation of the required manifolds M_k. The parametrization (11) of these manifolds now takes the form

$$u_{\mathrm{con}}(x) = \Phi(\mu), \qquad x \in \Sigma^y(\Phi(\mu)),\ \mu \in M \subset \mathbb{R}^k, \tag{20}$$

where

$$\Sigma(\Phi(\mu)) = (\mathrm{lin}[\lambda(\Phi, \partial_{\mu_1}\Phi), \dots, \lambda(\Phi, \partial_{\mu_k}\Phi)])^{\perp}.$$

Note that for $k = 1$, the condition (12) is automatically satisfied and therefore for any curve $M_1 \subset \mathbb{R}^l$ with

$$M_1 : \quad u = \Phi(\mu), \quad \frac{d\Phi}{d\mu} \in \Gamma(\Phi),$$

the parametrization (20) of M_1, where

$$\Sigma(\Phi(\mu)) = (\mathrm{lin}[\lambda(\Phi, d\Phi/d\mu)])^{\perp},$$

leads to a solution. These are the simple wave solutions.

2. Example

We now give in more detail an example of an application of (19) to the construction of two-dimensional manifolds $M_2 \subset \mathbb{R}^l$, $M_2 : u = \varphi(\mu_1, \mu_2)$, satisfying the assumptions of the Theorem.

Consider a pair of subcones $\mathring{\Lambda}(u) \subset \Lambda(u)$, $\mathring{\Gamma}(u) \subset \Gamma(u)$ with

$$\mathring{\Gamma}(u) : \quad \Psi_\omega(u, \gamma) = 0, \quad \omega = 1, \dots, p, \tag{21}$$

such that there exists a function $\mathring{\lambda} : \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}^n$, satisfying

$$\mathring{\Gamma}(u) \ni \gamma \rightleftharpoons \mathring{\lambda}(u, \gamma) \in \mathring{\Lambda}(u)$$

for $\gamma \in \mathring{\Gamma}(u)$.

Moreover, assume that the vectors $\mathring{\lambda}(u, \gamma^1)$ and $\mathring{\lambda}(u, \gamma^2)$ are linearly independent if γ^1, γ^2 are linearly independent. In this case the parametric representation $u = \varphi(\mu_1, \mu_2)$ of M_2 has to satisfy the following differential conditions:

$$\begin{aligned} &\Psi_\omega(\varphi, \partial_{\mu_i}\varphi) = 0, \qquad \omega = 1, \dots, p,\ i = 1, 2, \\ &\partial_{\mu_i}[\mathring{\lambda}(\varphi, \partial_{\mu_j}\varphi)] \in \mathrm{lin}[\mathring{\lambda}(\varphi, \partial_{\mu_1}\varphi), \mathring{\lambda}(\varphi, \partial_{\mu_2}\varphi)], \qquad i \neq j,\ i, j = 1, 2. \end{aligned} \tag{22}$$

If

$$[\mathrm{lin}(\mathring{\lambda}(u, \gamma^1), \mathring{\lambda}(u, \gamma^2))]^{\perp} = \mathrm{lin}[g^1(u, \gamma^1, \gamma^2), \dots, g^{n-2}(u, \gamma^1, \gamma^2)]$$

then the conditions (22) can be written in the form of the following PDE system of $2p+2(n-2)$ equations with l unknown functions $\varphi_1(\mu),\dots,\varphi_l(\mu)$:

$$
\begin{aligned}
&\Psi_\omega(\varphi,\partial_{\mu_i}\varphi)=0, \qquad \omega=1,\dots,p,\ i=1,2,\\
(23)\qquad &\langle\partial_{\mu_i}[\overset{\circ}{\lambda}(\varphi,\partial_{\mu_j}\varphi)],g^\sigma(\varphi,\partial_{\mu_1}\varphi,\partial_{\mu_2}\varphi)\rangle=0,\\
&\hspace{6cm}\sigma=1,\dots,n-2,\ i,j=1,2,\ i\neq j.
\end{aligned}
$$

Solutions of (23) give manifolds $M_2: u=\varphi(\mu_1,\mu_2)$ for which the construction of solutions $u:\mathbb{R}^m\supset D\to M_2$ by means of conical parametrization (20) is possible.

If $2p+2(n-2)\le l$, then performing the differentiations with respect to $\mu_i, i=1,2$, in (23), and additionally differentiating the first $2p$ equations of (23) in the following way:

$$\partial_{\mu_i}\Psi_\omega(\varphi,\partial_{\mu_j}\varphi)=0, \qquad i,j=1,2,\ i\neq j,\ \omega=1,\dots,p,$$

one can eliminate the derivatives $\partial_{\mu_1}\partial_{\mu_2}\varphi$ and assuming $\varphi\in C^2$ reduce the system (23) to a hyperbolic system

$$\partial_{\mu_1}\partial_{\mu_2}\varphi=F(\varphi,\partial_{\mu_1}\varphi,\partial_{\mu_2}\varphi). \tag{24}$$

Basing on the well known local and global existence theorems for this system (see [7]) one obtains the existence as well as the numerical construction of a broad class of $M_2\subset\mathbb{R}^l$ satisfying the assumptions of our Theorem.

As an example illustrating this possibility we take the system (3) of magnetohydrodynamics (see [9], [10]).

The characteristic vectors of the system (3) will be denoted by

$$
\begin{aligned}
&\Gamma(u)\ni\gamma=(\gamma_\varrho,\gamma_p,\overline{\gamma},\overline{h})\in\mathbb{R}^l, && l=8,\\
&\Lambda(u)\ni\lambda=(\lambda_0,\overline{\lambda})\in\mathbb{R}^n, && n=4,
\end{aligned}
$$

where $\overline{\gamma}=(\gamma_1,\gamma_2,\gamma_3)\in\mathbb{R}^3$, $\overline{h}=(h_1,h_2,h_3)\in\mathbb{R}^3$, $\overline{\lambda}=(\lambda_1,\lambda_2,\lambda_3)\in\mathbb{R}^3$. For the system (3) there exists the following three basic pairs (21) of characteristic subcones $\overset{\circ}{\Lambda}(u),\overset{\circ}{\Gamma}(u)$:

- The *entropic pair* $\Lambda^{\mathrm{e}}(u),\Gamma^{\mathrm{e}}(u)$, where

$$
\begin{aligned}
&\Lambda^{\mathrm{e}}(u): \quad \lambda_0+\langle v,\overline{\lambda}\rangle=0,\\
&\Gamma^{\mathrm{e}}(u): \quad \begin{cases}\Psi_1^{\mathrm{e}}(u,\gamma)=2\pi\gamma_\varrho+\langle H,\overline{h}\rangle=0,\\ \Psi_2^{\mathrm{e}}(u,\gamma)=\langle H,\overline{\gamma}\times\overline{h}\rangle=0,\end{cases}
\end{aligned}
$$

and

$$\lambda^{\mathrm{e}}(u,\gamma)=(-\langle v,\overline{\gamma}\times\overline{h}\rangle,\overline{\gamma}\times\overline{h}).$$

Since the corresponding system (23) has $2p+2(n-2)=8$ equations with $l=8$ unknown functions, it can be reduced to a hyperbolic system (24).

• The *Alfen pairs* $\Lambda^{\mathrm{A},\varepsilon}(u), \Gamma^{\mathrm{A},\varepsilon}(u), \varepsilon = \pm 1$, where

$$\Lambda^{\mathrm{A},\varepsilon}(u): \quad \lambda_0 + \langle v, \overline{\lambda}\rangle - \varepsilon \frac{\langle H, \overline{\lambda}\rangle}{\sqrt{4\pi\varrho}} = 0,$$

$$\Gamma^{\mathrm{A},\varepsilon}(u): \quad \begin{cases} \Psi_1^{\mathrm{A},\varepsilon}(u,\gamma) = \gamma_\varrho^2 + \gamma_p^2 + |\varepsilon\sqrt{4\pi\varrho}\,\overline{\gamma} + \overline{h}|^2 = 0, \\ \Psi_2^{\mathrm{A},\varepsilon}(u,\gamma) = \langle H, \overline{h}\rangle = 0, \end{cases}$$

and

$$\lambda^{\mathrm{A},\varepsilon}(u,\gamma) = \left(\frac{\varepsilon\langle H, l(\overline{h})\rangle}{\sqrt{4\pi\varrho}} - \langle v, l(\overline{h})\rangle, l(\overline{h}) \right),$$

where $l(\overline{h})$ denotes an arbitrary function $l : \mathbb{R}^3 \to \mathbb{R}^3$ satisfying the condition $l(\overline{h}) \perp \overline{h}$.

As in the entropic pair, the corresponding system (23) reduces to a hyperbolic system (24).

• The *magneto-acoustic pair* $\Lambda^{\mathrm{ma}}(u)$, $\Gamma^{\mathrm{ma}}(u)$, with

$$\Lambda^{\mathrm{ma}}(u): \quad (\lambda_0 + \langle v, \overline{\lambda}\rangle)^2 - \frac{1}{4}\left(\left| c\overline{\lambda} + \frac{H|\overline{\lambda}|}{\sqrt{4\pi\varrho}} \right| \pm \left| c\overline{\lambda} - \frac{H|\overline{\lambda}|}{\sqrt{4\pi\varrho}} \right| \right)^2 = 0,$$

where $c^2 = \kappa p/\varrho$ is the sound speed,

$$\Gamma^{\mathrm{ma}}(u): \quad \begin{cases} \Psi_1^{\mathrm{ma}}(u,\gamma) \\ \qquad = \dfrac{|\overline{h}|^2}{4\pi\varrho}\left(\dfrac{\gamma_\varrho}{\varrho}|H|^2 - \langle H, \overline{h}\rangle \right) - \dfrac{c^2\gamma_\varrho}{\varrho}\left(|\overline{h}|^2 - \dfrac{\gamma_\varrho}{\gamma}\langle H, \overline{h}\rangle \right) = 0, \\ \Psi_2^{\mathrm{ma}}(u,\gamma) = (\gamma_p - c^2\gamma_\varrho)^2 + \left| \overline{\gamma} - \dfrac{|\overline{h}|(\overline{h} - (\gamma_\varrho/\varrho)H)}{\sqrt{4\pi\varrho|\overline{h}|^2 - (\gamma_\varrho/\varrho)\langle H, \overline{h}\rangle}} \right|^2 = 0, \end{cases}$$

and

$$\lambda^{\mathrm{ma}}(u,\gamma) = \left(\frac{|H \times \overline{h}|^2|\overline{h}|}{\sqrt{4\pi\varrho(|\overline{h}|^2 - (\gamma_\varrho/\varrho)\langle H, h\rangle)}} - \langle v, \overline{h} \times (H \times \overline{h})\rangle, \overline{h} \times (H \times \overline{h}) \right).$$

Hence as in both previous cases the corresponding system (23) reduces to a hyperbolic system (24).

3. Regular isentropic flow

Now, using our Theorem, we describe in more detail the construction of regular solutions of the system (4).

We start with the construction of solutions describing the phenomena of regular interaction (without shocks) of two regular waves.

For the system (4), we use the following notations:

$$x = (t, \overline{x}) = (x_0, x_1, x_2) \in \mathbb{R}^n, \qquad \lambda = (\lambda_0, \overline{\lambda}) = (\lambda_0, \lambda_1, \lambda_2) \in \mathbb{R}^n,$$
$$u = (u_0, \overline{u}) = (u_0, u_1, u_2) = (c, v_1, v_2) \in \mathbb{R}^l, \qquad \gamma = (\gamma_0, \overline{\gamma}) = (\gamma_0, \gamma_1, \gamma_2) \in \mathbb{R}^l,$$

where $n = l = 3$, and the following pair of subcones $\Lambda^0(u) \subset \Lambda(u)$, $\Gamma^0(u) \subset \Gamma(u)$:

$$\begin{aligned} &\Lambda^0(u): \quad (\lambda_0 + \langle v, \overline{\lambda}\rangle)^2 - c^2|\overline{\lambda}|^2 = 0, \\ &\Gamma^0(u): \quad \Psi(\gamma) = \gamma_0^2 - k^2|\overline{\gamma}|^2 = 0. \end{aligned} \tag{25}$$

Then the function $\lambda^0 : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$, $\lambda^0 = \lambda^0(u,\gamma)$, satisfying the condition

$$\Gamma^0(u) \ni \gamma \rightleftharpoons \lambda^0(u,\gamma) \in \Lambda^0(u)$$

can be defined by

$$\lambda^0(u,\gamma) = \left(\frac{c}{k}\gamma_0 + \langle v, \overline{\gamma}\rangle, -\overline{\gamma}\right).$$

Hence in our case the system (23) takes the form

$$\begin{aligned} &(\partial_{\mu_i} c)^2 - k^2(\partial_{\mu_i} v_1)^2 - k^2(\partial_{\mu_i} v_2)^2 = 0, \qquad i = 1,2, \\ &\langle \partial_{\mu_i}[\lambda^0(\varphi, \partial_{\mu_j}\varphi)], \sigma\rangle = 0, \qquad i \neq j,\ i,j = 1,2, \end{aligned} \tag{26}$$

where $\varphi = (\varphi_0, \overline{\varphi}) = (c, v_1, v_2)$,

$$\lambda^0(\varphi, \partial_{\mu_i}\varphi) = \left(\frac{c}{k}\partial_{\mu_i} c + \langle v, \partial_{\mu_i} v\rangle, -\partial_{\mu_i} v_1, -\partial_{\mu_i} v_2\right)$$

and

$$\sigma = (\sigma_0, \overline{\sigma}) \parallel \lambda^0(\varphi, \partial_{\mu_1}\varphi) \times \lambda^0(\varphi, \partial_{\mu_2}\varphi).$$

In our case

$$\partial_{\mu_j}[\lambda^0(\varphi, \partial_{\mu_i}\varphi)] = \partial_{\mu_i}[\lambda^0(\varphi, \partial_{\mu_j}\varphi)].$$

Hence upon assuming $\varphi \in C^2$ and performing the appropriate differentiations with respect to μ_1, μ_2, the system (26) takes the form

$$\begin{aligned} &\partial_{\mu_1} c\partial_{\mu_1}\partial_{\mu_2} c - k^2\partial_{\mu_1} v_1\partial_{\mu_1}\partial_{\mu_2} v_1 - k^2\partial_{\mu_1} v_2\partial_{\mu_1}\partial_{\mu_2} v_2 = 0, \\ &\partial_{\mu_2} c\partial_{\mu_1}\partial_{\mu_2} c - k^2\partial_{\mu_2} v_1\partial_{\mu_1}\partial_{\mu_2} v_1 - k^2\partial_{\mu_2} v_2\partial_{\mu_1}\partial_{\mu_2} v_2 = 0, \\ &\sigma_0\frac{c}{k}\partial_{\mu_1}\partial_{\mu_2} c + (\sigma_0 v_1 - \sigma_1)\partial_{\mu_1}\partial_{\mu_2} v_1 + (\sigma_0 v_2 - \sigma_2)\partial_{\mu_1}\partial_{\mu_2} v_2 \\ &\qquad\qquad = -\sigma_0\left(\frac{1}{k}\partial_{\mu_1} c\partial_{\mu_2} c + \langle \partial_{\mu_1} v, \partial_{\mu_2} v\rangle\right). \end{aligned} \tag{27}$$

Taking into account that

$$\sigma_0\left(\frac{c}{k}\partial_{\mu_i} c + \langle v, \partial_{\mu_i} v\rangle\right) = \langle \overline{\sigma}, \partial_{\mu_i} v\rangle$$

and setting $\sigma_0 = k$, one can evaluate $\det A$ after writing the system (27) in the form $A\partial_{\mu_1}\partial_{\mu_2}\varphi = d$. As a result we find that $\det A \neq 0$ if and only if

$$c \neq \frac{1}{2}\left|\frac{\partial_{\mu_1} v}{|\partial_{\mu_1} v|} + \frac{\partial_{\mu_2} v}{|\partial_{\mu_2} v|}\right| \cos\alpha(\partial_{\mu_1} v, \partial_{\mu_2} v), \tag{28}$$

where $\alpha(\partial_{\mu_1} v, \partial_{\mu_2} v)$ denotes half the angle between $\partial_{\mu_1} v$ and $\partial_{\mu_2} v$. Obviously this condition is satisfied in particular if $c > 1$.

In this way assuming (28) we reduce the system (26) to a hyperbolic system of the second order:

$$\partial_{\mu_1}\partial_{\mu_2}\varphi = F(\varphi, \partial_{\mu_1}\varphi, \partial_{\mu_2}\varphi). \tag{29}$$

Solutions of (29) satisfy the condition

$$\Psi(\partial_{\mu_i}\varphi) = c_i = \text{const}, \qquad i = 1, 2,$$

where $\Psi(\gamma) = \gamma_0^2 - k^2|\overline{\gamma}|^2$ (cf. (25)), and only those solutions satisfy (26) for which both constants are zero.

To prove the existence of the manifolds M_2 satisfying the assumptions of the Theorem for the system (4) we use the local existence theorems for the Cauchy problem

$$\varphi(\tau, 1-\tau) = U(\tau), \qquad \partial_\nu \varphi(\tau, 1-\tau) = \eta(\tau) \tag{30}$$

for the system (29) (see [7]) posed on the noncharacteristic line

$$l: \quad \mu_1 = \tau, \quad \mu_2 = 1 - \tau, \quad \tau \in (0, 1),$$

where $\nu = (1, 1)$. On the line l we have

$$\partial_{\mu_1}\varphi = \frac{1}{2}\left(\frac{dU}{d\tau} + \eta\right), \qquad \partial_{\mu_2}\varphi = \frac{1}{2}\left(-\frac{dU}{d\tau} + \eta\right).$$

Hence the initial conditions (30) should be chosen in such a way that

$$\Psi\left(\pm \frac{dU}{d\tau} + \eta\right) = 0 \tag{31}$$

and the condition (28) is satisfied for

$$c = U_0, \quad \partial_{\mu_1} v = \frac{d\overline{U}}{d\tau} + \overline{\eta}, \quad \partial_{\mu_2} v = -\frac{d\overline{U}}{d\tau} + \overline{\eta},$$

where $U = (U_0, \overline{U})$, $\eta = (\eta_0, \overline{\eta})$.

Suppose now that in some neighbourhood $Q \subset \mathbb{R}^2$ of the initial line l we have a solution $\varphi(\mu_1, \mu_2)$ of (29) satisfying the initial conditions (30) such that $M_2 : u = \varphi(\mu_1, \mu_2)$ is a two-dimensional surface $M_2 \subset \mathbb{R}^3$, and the initial data satisfy (31), (28). Then the next step is to establish for which part $M_2^{\text{con}} \subset M_2$,

$$M_2^{\text{con}}: \quad u = \varphi(\mu_1, \mu_2), \quad \mu \in M \subset Q,$$

the family

$$\Sigma(u) = (\text{lin}[\lambda^1(u), \lambda^2(u)])^\perp \subset \mathbb{R}^3, \qquad u \in M_2^{\text{con}},$$

is conical, where $\lambda^i(u) = \lambda^0(u, \gamma^i(u))$ and $\gamma^i(u) = \partial_{\mu_i}\varphi(\varphi^{-1}(u))$, $i = 1, 2$.

To this end we have to determine a region M, $l \subset M \subset Q$, such that the mapping

$$\mathfrak{M} : M \times (\varepsilon, d) \to \mathbb{R}^3, \qquad 0 < \varepsilon < d, \tag{32}$$

given by $x = x(\mu_1, \mu_2, \mu_3) = y + \mu_3\sigma(\mu_1, \mu_2)$ is one-to-one, where the function $\sigma(\mu_1, \mu_2) = \sigma = (k, \overline{\sigma})$ is given by

$$(33)\qquad \langle \sigma, \lambda^0(\varphi, \partial_{\mu_i}\varphi)\rangle = \varphi_0\partial_{\mu_i}\varphi_0 + k\langle\overline{\varphi}, \partial_{\mu_i}\overline{\varphi}\rangle - \langle\overline{\sigma}, \partial_{\mu_i}\overline{\varphi}\rangle = 0, \qquad i = 1, 2.$$

Since $\sigma_0 = k = \mathrm{const} > 0$, the mapping (32) is one-to-one if and only if so is the mapping

$$(34)\qquad \overline{\sigma} : \mathbb{R}^2 \supset M \to \mathbb{R}^2.$$

Differentiating (33) with respect to μ_1, μ_2 one obtains

$$(35)\qquad \langle \partial_{\mu_i}\overline{\sigma}, \partial_{\mu_j}\varphi\rangle = a^i_j(\mu), \qquad i, j = 1, 2,$$

where $a^i = (a^i_1, a^i_2)$, $i = 1, 2$, are simple polynomials of $\sigma, \varphi, \partial_{\mu_1}\varphi, \partial_{\mu_2}\varphi, \partial^2_{\mu_1}\varphi, \partial^2_{\mu_2}\varphi, \partial_{\mu_1}\partial_{\mu_2}\varphi$. The Jacobians of the mappings (34) and (32) do not vanish at $\mu \in M$ if and only if the vectors $a^1(\mu), a^2(\mu)$ are linearly indpendent.

Setting the Cauchy problem (30) we have to ensure that for the solution $\varphi(\mu_1, \mu_2)$ of this problem, the region M and the corresponding manifold M_2^{con} are not empty. To this end one has to compute the derivatives $\partial_{\mu_1}\varphi, \partial_{\mu_2}\varphi, \partial^2_{\mu_1}\varphi, \partial^2_{\mu_2}\varphi, \partial_{\mu_1}\partial_{\mu_2}\varphi$ on the initial line $l : \mu_1 = \tau, \mu_2 = 1 - \tau, \tau \in (0, 1)$, as functions of τ and keep the corresponding vectors $a^1(\mu), a^2(\mu)$ linearly independent. In that case we shall call the Cauchy problem (30) *conical*.

From the above considerations it easily follows that there exists a broad class of initial data (30) satisfying (28) and (31) which are conical and such that the vectors

$$\partial_{\mu_1}\varphi = \frac{1}{2}\bigg(\frac{dU}{d\tau} + \eta(\tau)\bigg), \qquad \partial_{\mu_2}\varphi = \frac{1}{2}\bigg(-\frac{dU}{d\tau} + \eta(\tau)\bigg).$$

are linearly independent. In what follows we call such initial problems *admissible*.

Summing up we can formulate the following

FACT. *If the Cauchy problem* (30) *is admissible then for some region* $M \subset \mathbb{R}^2, l \subset M$*, the solution* $\varphi(\mu_1, \mu_2)$*,* $\mu \in M$*, of* (29), (30) *defines a two-dimensional manifold* $M_2 \subset \mathbb{R}^3$*,* $M_2 : u = \varphi(\mu_1, \mu_2), \mu \in M$*, which satifies the assumptions of the Theorem with*

$$\gamma^i(u) = \partial_{\mu_i}\varphi(\varphi^{-1}(u)), \quad \lambda^i(u) = \lambda^0(u, \gamma^i(u)), \quad u \in M_2,\ i = 1, 2.$$

Denoting by $\mathcal{C}^{(1)}$ ($\mu_2 = \mathrm{const}$), $\mathcal{C}^{(2)}$ ($\mu_1 = \mathrm{const}$) the two kinds of characterisitic lines of the system (29) we see that the curves $M_1^{(i)} = \varphi(\mathcal{C}^{(i)}) \subset M_2$ are tangent to $\gamma^i(u)$, $i = 1, 2$. Therefore the curves $M_1^{(i)}$ are one-dimensional manifolds satisfying the conditions of the Theorem, with

$$\Sigma^{(i)}(u) = (\mathrm{lin}[\lambda^i(u)])^\perp, \qquad i = 1, 2.$$

Now, take an arbitrary closed rectangle $M' \subset M$ with sides parallel to the μ_1 and μ_2 axes. We denote by $1, 2, 3, 4$ the vertices of M' and by $\{i, j\}$ the sides of the rectangle M' joining the vertices i, j.

Moreover, we consider the closed manifold $M_2' \subset M_2$, $M_2' = \varphi(M')$ cut out of M_2 by the curves

$$M_1(i,j) = \varphi(\{i,j\}), \qquad \{i,j\} = \{1,2\}, \{2,3\}, \{3,4\}, \{4,1\}.$$

In the closed region

$$D' = \bigcup_{u \in M_2'} \Sigma^y(u),$$

where $\Sigma(u) = (\mathrm{lin}[\lambda^1(u), \lambda^2(u)])^\perp$, we now have a conical solution of the system (4) defined by $u_{\mathrm{con}}(t,x) := u$, $(t,x) \in \Sigma^y(u)$, $u \in M_2'$. Figure 1 ilustrates this construction with $y = 0$, $u^i = \varphi(i)$, $i = 1, 2, 3, 4$.

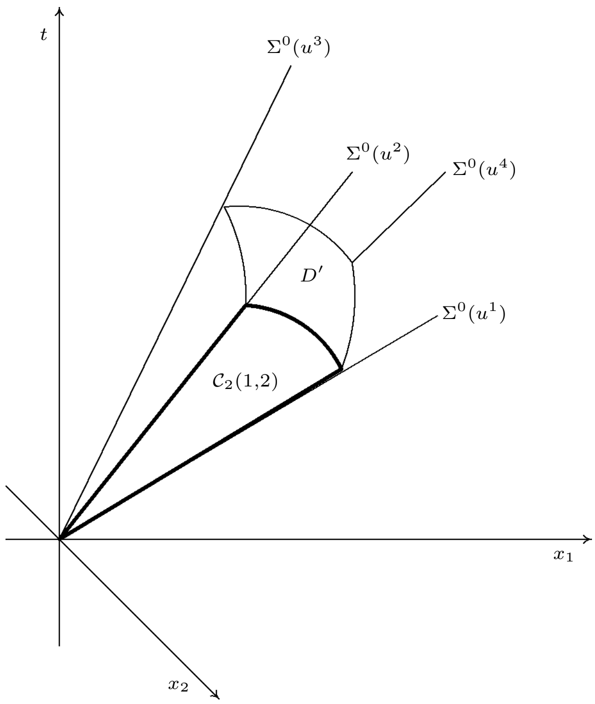


Figure 1

This solution satisfies the differential inclusion

$$Du \in \mathrm{lin}[\gamma^1(u) \otimes \lambda^1(u), \gamma^2(u) \otimes \lambda^2(u)].$$

A simple consequence of this inclusion is that for the curves $M_1^{(i)} \subset M_2'$, $i = 1, 2$, the conical surfaces

$$\mathcal{C}_2^{(i)} = u_{\mathrm{con}}^{-1}(M_1^{(i)}) \subset D', \qquad i = 1, 2,$$

are characteristic surfaces for the solution u_{con}, and at $x \in \mathcal{C}_2^{(i)}$ the surface $\mathcal{C}_2^{(i)}$ is perpendicular to $\lambda^j(u_{\mathrm{con}}(x))$, $i \neq j$, $i, j = 1, 2$.

The boundary of the conical region D' consists of the conical characteristic surfaces (see Figure 1)

$$\mathcal{C}_2(i, j) = u_{\mathrm{con}}^{-1}(M_1(i, j)), \qquad (i, j) = (1, 2), (2, 3), (3, 4), (4, 1).$$

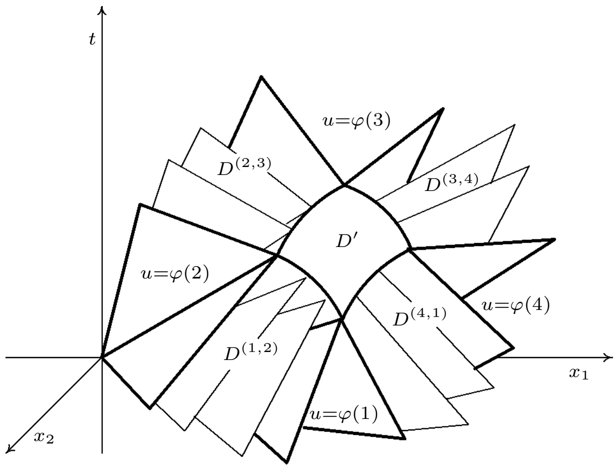


FIGURE 2

We now extend the solution u_{con} defined in D' through the characteristic surfaces $\mathcal{C}_2(i, j)$ by the simple wave solutions (see Figure 2)

$$u^{(i,j)} : \mathbb{R}^3 \supset D^{(i,j)} \to M_1(i, j), \qquad (i, j) = (1, 2), (2, 3), (3, 4), (4, 1),$$

For example, $u^{(1,2)}(x) = u$, $x \in \Sigma^0(u)$, $u \in M_1(1, 2)$, where $\Sigma(u) = (\mathrm{lin}[\lambda^1(u)])^{\perp}$, so that the curve $M_1(1, 2)$ is of the type $M_1^{(1)}$. The planes $\Sigma^0(u'), \Sigma^0(u'')$, $u' \neq u'' \in M_1$, may have common lines $l(u', u'') \not\subset D^{(i,j)}$. If M' and $M_2' = \varphi(M')$ are chosen small enough, then the lines $l(u', u'')$ are sufficiently far from $D' \setminus \{x : |x| < R\}$, $R > 0$.

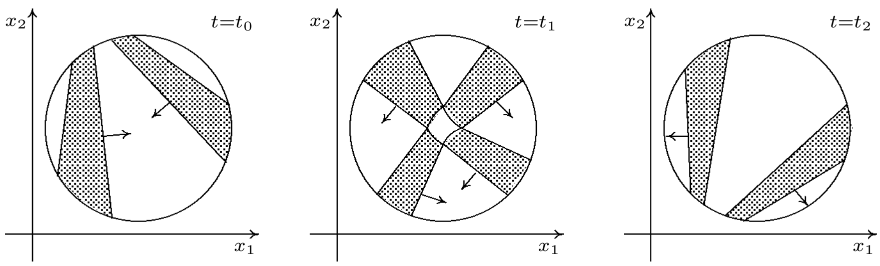
In a neighbourhood of D' the solution is still undefined in the four corners bounded by the two corresponding planes Σ^0 (see Figure 2). Defining the solution to be constant in these corners, $u(x) = \varphi(i)$, $i = 1, 2, 3, 4$, we obtain, in some neighbourhood D of $D' \setminus \{x : |x| < R\}$, a regular solution admitting maybe some weak discontinuities.

Looking now at the solution at

$$0 < t_0 < t_1 < t_2,$$

under the conditions that M' and $M_2' = \varphi(M')$ are small enough and $\partial_{\mu_i} v(\mu_1, \mu_2)$, $i = 1, 2$, $\mu \in M'$, are linearly independent, we get, in an appropriate disc, the film of Figure 3. It represents the regular interaction of two regular waves. Figure 3(a), $t = t_0$, shows the two waves before interaction, moving towards each other. Figure 3(b), $t = t_2$, shows the full interaction. Finally, Figure 3(c) shows the situation after the interaction: the two simple waves formed in the interaction are moving away of each other.

Moreover, in our case of a continuous solution the analysis of the region of uniqueness shows that the flow $t = t_0$ uniquely determines the flow inside the circles for $t > t_0$.



(a) (b) (c)

FIGURE 3

4. Interaction of two regular waves with a shock

We now come to the natural idea of constructing discontinuous manifolds M_2, satisfying the conditions of the Theorem at regular points, so that the above described conical parametrization gives a solution with shocks. We construct solutions of (4) describing the interaction of two regular waves with a prescribed linear shock. We shall not address the problem of uniqueness for solutions with shocks of our class[2].

[2]Nevertheless, we mention that using our method one can construct examples of non-unique solutions with shocks.

Let the two-dimensional surface $F \subset \mathbb{R}^3$ be the shock front of some solution of the system (4), and

$$F(t) = \{x : (t, x) \in F\} \subset \mathbb{R}^2$$

the corresponding time-dependent moving shock front in $\mathbb{R}^2$. Our solutions are conical and therefore we only consider the conical fronts

$$F : (t, x) = s\vartheta(\tau), \qquad s > 0,\ \tau \in (0, 1),$$

where $\vartheta = (1, \overline{\vartheta}(\tau))$ and $\overline{\vartheta}(\tau) = (\vartheta_1, \vartheta_2)$ (arbitrary). We have $F(t) : x = t\overline{\vartheta}(\tau)$, $\tau \in (0, 1)$. Hence the normal and tangent vectors to $F(t)$ can be taken in the form

$$n(\tau) = |\dot{\overline{\vartheta}}|^{-1}(-\dot{\vartheta}_2, \dot{\vartheta}_1), \qquad d(\tau) = |\dot{\overline{\vartheta}}|^{-1}(\dot{\vartheta}_1, \dot{\vartheta}_2).$$

The velocity of the shock front $F(t)$ in the direction $n(\tau)$ is

$$D(\tau) = |\dot{\overline{\vartheta}}|^{-1}(\vartheta_2\dot{\vartheta}_1 - \vartheta_1\dot{\vartheta}_2).$$

Let us denote the values of the solutions on the two sides of the shock front by

$$u^i(t, x) = (c^i, v^i), \qquad i = 1, 2,$$

and put, for $x \in F(t)$,

$$v^i(t, x) = v^i(\tau) = v^i_n n + v^i_d d, \qquad i = 1, 2,$$

where

$$v^i_n(\tau) = \langle v^i(\tau), n(\tau)\rangle, \qquad v^i_d(\tau) = \langle v^i(\tau), d(\tau)\rangle.$$

The divergence form of the system (4) is

$$\begin{aligned} &\partial_t c^{1/k} + \operatorname{div}(c^{1/k} v) = 0, \\ &\partial_t(c^{1/k} v_i) + \operatorname{div}\left(c^{1/k} v_i v + \frac{1}{2k+1} c^{2+1/k} e^i\right) = 0, \qquad i = 1, 2. \end{aligned} \tag{36}$$

where $e^1 = (1, 0)$, $e^2 = (0, 1)$.

The Hugoniot jump conditions for this system take the form

$$\begin{aligned} &\varrho^1(v^1_n - D) - \varrho^2(v^2_n - D) = 0, \\ &\varrho^1 v^1_i(v^1_n - D) + p^1 n_i - \varrho^2 v^2_i(v^2_n - D) - p^2 n_i = 0, \qquad i = 1, 2. \end{aligned} \tag{37}$$

where $n = (n_1, n_2)$. The pressures p^i, $i = 1, 2$, and the densities ϱ^i, $i = 1, 2$, should be expressed by the sound speeds c^i, $i = 1, 2$, in the following way:

$$\varrho^i = \left[\frac{1}{(2k+1)A}\right]^{1/(2k)} (c^i)^{1/k}, \quad p^i = A(\varrho^i)^{2k+1}, \quad A = \text{const}. \tag{38}$$

If $(v_n^1 - D)^2 + (v_n^2 - D)^2 \neq 0$, that is, our shock does not represent a contact discontinuity, then the conditions (37) can be written in the form

$$
(39) \qquad \begin{aligned}
&(v_n^2 - v_n^1)^2 = (p^2 - p^1)\left(\frac{1}{\varrho^1} - \frac{1}{\varrho^2}\right),\\
&(v_n^1 - D)^2 = \frac{\varrho^2}{\varrho^1} \cdot \frac{p^2 - p^1}{\varrho^2 - \varrho^1}, \qquad v_d^2 = v_d^1,
\end{aligned}
$$

where the pressures p^i, $i = 1, 2$, and the densities ϱ^i, $i = 1, 2$, must be expressed by the sound speeds c^i, $i = 1, 2$, according to (38).

Now we prescribe an arbitrary conical shock front $F : (t, x) = s\vartheta(\tau)$. Our first goal is to construct, in some neighbourhood $N(F)$ of F, solutions of the system (4) admitting on F a shock discontinuity which satisfies the Hugoniot jump conditions (39).

To this end consider two admissible Cauchy problems (30) for the system (29),

$$
(40) \qquad \varphi^i(\tau, 1 - \tau) = U^i(\tau), \quad \partial_\nu \varphi^i(\tau, 1 - \tau) = \eta^i(\tau), \quad 0 \leq \tau \leq 1,\ i = 1, 2.
$$

In some neighbourhood $Q \subset \mathbb{R}^2$ of the initial line l there exist two solutions $\varphi^i(\mu_1, \mu_2)$, $\mu \in Q$, $i = 1, 2$, of (29) satisfying the initial conditions (40). Let

$$
Q = Q^1 \cup l \cup Q^2, \qquad Q^1 \cap Q^2 = \emptyset.
$$

We now consider two manifolds $M_2^i \subset \mathbb{R}^3$, $i = 1, 2$,

$$
M_2^i : \quad u = \varphi^i(\mu_1, \mu_2), \quad \mu \in Q^i,
$$

satisfying the conditions of the Theorem.

Performing the conical parametrization of M_2^1 and M_2^2 we obtain in some conical regions

$$
D^{i'} = \bigcup_{\mu \in Q^i} \Sigma^{i,0}(\varphi^i(\mu)), \qquad i = 1, 2,
$$

two conical solutions

$$
u^i(t, x) = u, \qquad i = 1, 2,\ (t, x) \in \Sigma^{i,0}(u),\ u \in M_2^i.
$$

Our problem is to find initial conditions (40) such that:

1′. The regions $D^{i'}$ stick together exactly along the prescribed shock front F.
2′. The solutions $u^i(t, x)$, $i = 1, 2$, satisfy on F the Hugoniot jump conditions (39).

The condition 1′ is obviously satisfied if the initial data (40), apart from

$$
(41) \qquad \Psi(\pm\dot{U}(\tau) + \eta(\tau)) = (\pm\dot{U}_0 + \eta_0)^2 - k^2 |\pm\dot{\overline{U}} + \overline{\eta}|^2 = 0,
$$

satisfy

$$\vartheta(\tau) \parallel \lambda^0(U, \dot{U} + \eta) \times \lambda^0(U, -\dot{U} + \eta).$$

Simple computations show that for $\vartheta = (1, \overline{\vartheta})$ the condition $\langle \vartheta, \lambda^0(U, \pm\dot{U} + \eta)\rangle = 0$ is equivalent to

$$\frac{1}{k}U_0\dot{U}_0 + \langle \overline{U}, \dot{\overline{U}}\rangle - \langle \overline{\vartheta}, \dot{\overline{U}}\rangle = 0, \tag{42}$$

$$\frac{1}{k}U_0\eta_0 + \langle \overline{U}, \overline{\eta}\rangle - \langle \overline{\vartheta}, \overline{\eta}\rangle = 0. \tag{43}$$

If we put $\eta = \alpha\zeta$, then (41) is equivalent to

$$(\dot{U}_0)^2 - k^2|\dot{\overline{U}}|^2 + \alpha^2(\zeta_0^2 - k^2|\overline{\zeta}|^2) = 0, \tag{44}$$

$$\dot{U}_0\zeta_0 - k^2\langle \dot{\overline{U}}, \overline{\zeta}\rangle = 0. \tag{45}$$

Hence for the construction of the functions $U^i(\tau)$, $\eta^i(\tau) = \alpha^i(\tau)\zeta^i(\tau)$, $i = 1, 2$, such that the data (40) satisfy the condition $1'$, we have to act in the following way. First we construct functions U^1, U^2 satisfying (42), and then we construct functions $\eta^1 = \alpha^1\zeta^1, \eta^2 = \alpha^2\zeta^2$ satisfying (43), (45) and finally (44).

Now we discuss the requirement $2'$. First observe that Hugoniot conditions (39) only concern the functions

$$U^i(\tau) = (c^i(\tau), v^i(\tau)), \qquad i = 1, 2. \tag{46}$$

That is, in order to meet the requirement $2'$, we have to find two different solutions (46) of (42) such that the functions $c^i(\tau)$, $v^i_n(\tau) = \langle v^i, n\rangle$, $v^i_d(\tau) = \langle v^i, d\rangle$, $i = 1, 2$, satisfy the jump conditions (39).

To this end put $v^1_d = v^2_d = v_d(\tau)$, where $v_d(\tau)$ is an arbitrary real-valued function. In this way the last condition of (39) is satisfied. The first two conditions of (39) can be written as

$$v^i_n = E^i(\tau, c^1, c^2), \qquad i = 1, 2, \tag{47}$$

where $E^i : \mathbb{R}^3 \to \mathbb{R}$, $E^1 \neq E^2$, are appropriate functions. On putting now in (46),

$$v^i(\tau) = v^i(\tau, c^1, c^2) := n(\tau)E^i(\tau, c^1, c^2) + d(\tau)v_d(\tau), \qquad i = 1, 2,$$

the condition (42) takes the form

$$\frac{1}{k}c^i\frac{dc^i}{d\tau} + \left\langle v^i(\tau, c^1, c^2), \frac{d}{d\tau}v^i(\tau, c^1, c^2)\right\rangle - \left\langle \overline{\vartheta}(\tau), \frac{d}{d\tau}v^i(\tau, c^1, c^2)\right\rangle = 0. \tag{48}$$

This is a system of two ordinary differential equations of first order with two unknown functions c^1, c^2.

Finally, the construction of the initial data (4) satisfying the requirements $1'$, $2'$ should be performed in the following way.

First, we take an arbitrary function $v_d(\tau)$, which determines the functions $v^i(\tau, c^1, c^2)$, $i = 1, 2$, and the system (48).

Second, we take an arbitrary solution $c^1(\tau), c^2(\tau)$ of (48) and put

$$U^i(\tau) = (c^i(\tau), v^i(\tau, c^1(\tau), c^2(\tau))), \qquad i = 1, 2.$$

Third, we determine $\eta^i(\tau) = \alpha^i(\tau)\zeta^i(\tau)$, $i = 1, 2$, satisfying (43), (45) and (44) with $U(\tau) = U^i(\tau)$, $i = 1, 2$.

Suppose now we have two admissible Cauchy problems (40) satisfying $1'$ and $2'$. Then for some domain $Q \subset \mathbb{R}^2$, $Q = Q^1 \cup l \cup Q^2$, we have two solutions of the system (29), $\varphi^i : \mathbb{R}^2 \supset Q^i \to \mathbb{R}^3$, $i = 1, 2$, satisfying the respective initial conditions (40). Now take the closed square $M \subset Q$ with sides parallel to the μ_1 and μ_2 axes with vertices $1, 2, 3, 4$.

We put $M = M^1 \cup l \cup M^2$ where $M^1 = M \cap Q^1$, $M^2 = M \cap Q^2$, $l' = M \cap l$, $2 \in Q^2$, $4 \in Q^1$ and $3, 1 \in l'$. Moreover, consider the manifolds $M_2^{(1)} = \varphi^1(\overline{M}^1), M_2^{(2)} = \varphi^2(\overline{M}^2)$. Performing now our conical parametrization of the closed manifolds $M_2^{(1)}$ and $M_2^{(2)}$ we obtain, in two closed conical regions $D^{1'}, D^{2'}$, regular (except at the point $(t, x) = (0, 0)$) solutions $u^i = D^{i'} \to M_2^{(i)}$, $i = 1, 2$. Moreover, by $1'$, $D' = D^{1'} \cup F \cup D^{2'}$ is a closed, connected conical region (see Figure 4). By $2'$, the solution

$$u(t, x) := \begin{cases} u^1(t, x), & (t, x) \in D^{1'}, \\ u^2(t, x), & (t, x) \in D^{2'}, \end{cases}$$

admits on F a shock discontinuity satisfying the Hugoniot conditions (39).

Now we extend this solution to a neighbourhood of D' by simple wave solutions (see Figure 4) exactly as in the case of regular interaction of two waves. After this operation we have to define the solution in the four corners bounded by the two planes Σ^0 indicated with thick lines in Figure 4. Two of those corners touch the shock front $F \subset D$ (see Figure 4). In those two corners we continue the conical shock F by two planes $F^{(1)}$ and $F^{(3)}$ tangent to F (see Figure 4). On one side of $F^{(i)}$ we put, in those corners,

$$u(t, x) := \varphi^1(i) = \text{const}, \qquad i = 1, 3,$$

and on the other side

$$u(t, x) := \varphi^2(i) = \text{const}, \qquad i = 1, 3.$$

In this way the solution $u(t, x)$ has shock front $F^{(1)} \cup F \cup F^{(3)}$ with discontinuity satisfying the Hugoniot jump conditions.

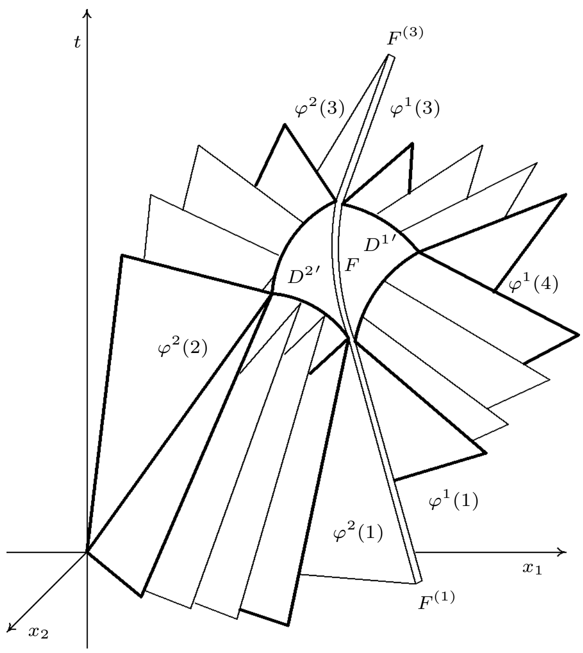


FIGURE 4

In the other two corners we put respectively (see Figure 4)

$$u(t,x) := \varphi^2(2) = \text{const}, \qquad u(t,x) := \varphi^1(4) = \text{const}.$$

In this way, in some neighbourhood D of $D' \setminus \{x : |x| < \varepsilon\}$, $\varepsilon > 0$, we get a solution of the system (4), which, off the shock front $F^{(1)} \cup F \cup F^{(3)}$, is regular with maybe some weak discontinuities. Obviously $u(D) = M_2^{(1)} \cup M_2^{(2)}$.

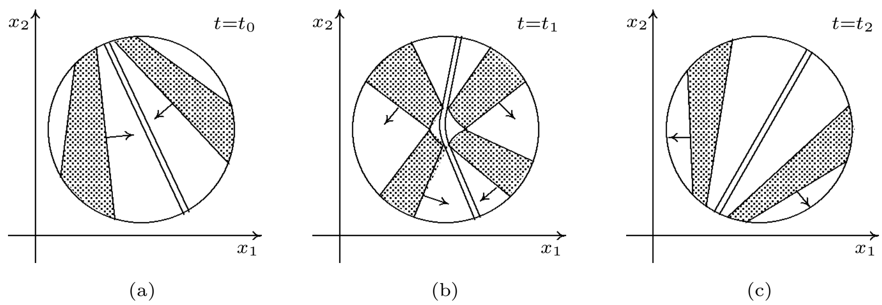


FIGURE 5

If the manifolds $M_2^{(1)}, M_2^{(2)}$ are taken small enough, and the conical surface F is chosen as in Figure 4 (F was arbitrary), then just as in the case of regular interaction, for $0 < t_0 < t_1 < t_2$ in an appropriate disc we obtain the film of Figure 5. The shadowed parts denote the interacting simple waves, the double lines and double curves denote the moving shock fronts.

Figure 5(a), $t = t_0$, shows two regular waves before the interaction moving toward each other and to the prescribed linear shock. Figure 5(b), $t = t_1$, shows the full interaction. Figure 5(c), $t = t_2$, shows the situation after the interaction. The two simple waves and the linear shock front formed in the interaction are moving away of each other.

5. Interaction of two regular waves producing the prescribed shock

We now consider the following problem: how the prescribed shock can be produced as a result of interaction of two regular waves?

As before we first ask for the image of the required solution of the system (4). To this end we construct, for the system (29), two Cauchy problems (40) such that

A. The data for $i = 1, 2$ are admissible and for the shock front $F : (t, x) = s\vartheta(\tau)$, $0 < s$, $0 < \tau < 1/2$, the requirements $1'$ and $2'$ are satisfied.

B. $U^1(\tau) \neq U^2(\tau)$, $0 < \tau < 1/2$,
$U^1(\tau) = U^2(\tau)$, $1/2 \leq \tau < 1$.

From the above considerations it easily follows that the construction of initial data (40) satisfying A and B is possible. Indeed, this follows from the observation that in the system (48) for the initial conditions $\tau_0 = 1/2,\ c_0^1 = c^1(\tau_0) = c_0^2 = c^2(\tau_0)$ we have $dc^1(\tau_0)/d\tau \neq dc^2(\tau_0)/d\tau$.

Now, consider the square $M(\varepsilon) \subset \mathbb{R}^2$ with sides parallel to the μ_1, μ_2 axes with vertices

$$1 = \left(\tfrac{1}{2}+\varepsilon, \tfrac{1}{2}-\varepsilon\right), \quad 2 = \left(\tfrac{1}{2}-\varepsilon, \tfrac{1}{2}-\varepsilon\right), \quad 3 = \left(\tfrac{1}{2}-\varepsilon, \tfrac{1}{2}+\varepsilon\right), \quad 4 = \left(\tfrac{1}{2}+\varepsilon, \tfrac{1}{2}+\varepsilon\right),$$

and put $M(\varepsilon) = M^1(\varepsilon) \cup l' \cup M^2(\varepsilon)$, $M^1(\varepsilon) \cap M^2(\varepsilon) = \emptyset$, $l' = l \cap M(\varepsilon)$. For $\varepsilon > 0$ small enough there exist solutions $\varphi^i(\mu_1, \mu_2)$, $\mu \in M^i(\varepsilon)$, $i = 1, 2$, satisfying the initial conditions (40) for $1/2 - \varepsilon \leq \tau \leq 1/2 + \varepsilon$. If the initial data satisfy A and B, then the image we seek for is the manifold

$$M_2 = \varphi^1(\overline{M^1(\varepsilon)}) \cup \varphi^2(\overline{M^2(\varepsilon)}) \tag{49}$$

shown in Figure 6.

Now we may parametrize M_2 as before. The result is a solution with shock front $F \cup F^{(3)}$, where $F^{(3)}$ is an appropriate plane tangent to F. For an appropriate disc one obtains, for $0 < t_0 < t_1 < t_2$ and sufficiently small $\varepsilon > 0$, the film of Figure 7. Figure 7(a), $t = t_0$, shows the situation before the interaction.

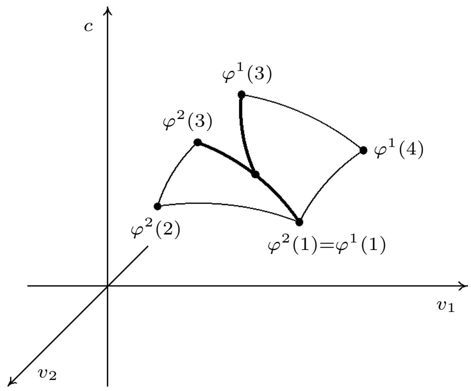
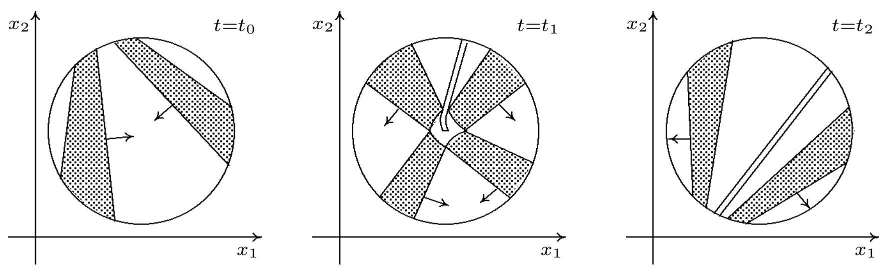


Figure 6



(a) (b) (c)

Figure 7

Figure 7(b), $t = t_1$, shows the full interaction. Finally, Figure 7(c), $t = t_2$, shows the situation after the interaction.

6. Systems of interacting shocks

Our qualitative method allows the construction of a broad class of interacting shock systems. We confine ourselves to one example in which the interaction of two regular waves produces two interacting shock waves. We hope that knowing that example one can quite easily construct many other much more sophisticated interacting shock systems.

Let us return to the manifold (49), and consider the submanifold $M_2' = \varphi^1(M^{1'}(\varepsilon)) \cup \varphi^2(M^{2'}(\varepsilon)) \subset M_2$, where $M^{i'}(\varepsilon)$ are disjoint triangles such that

$$M' := \left[\tfrac{1}{2} - \varepsilon, \tfrac{1}{2}\right] \times \left[\tfrac{1}{2}, \tfrac{1}{2} + \varepsilon\right] = M^{1'}(\varepsilon) \cup l' \cup M^{2'}(\varepsilon),$$

where $l' = M' \cap l$. In Figure 8 the vertices of the triangle $M^{1'}(\varepsilon)$ are denoted by $1, 3, 4$ and the vertices of $M^{2'}(\varepsilon)$ are denoted by $1, 2, 3$. Parametrizing conically

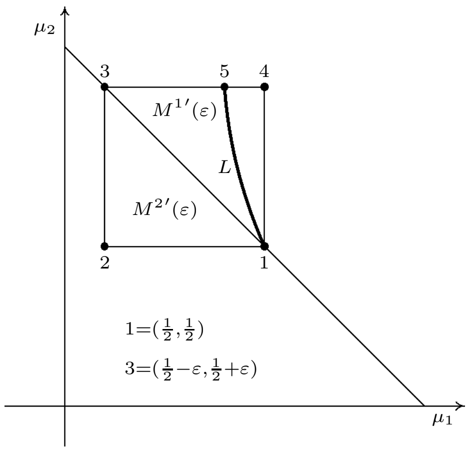


Figure 8

the manifolds

$$M_2^{1'} = \varphi^1(\overline{M^{1'}(\varepsilon)}), \qquad M_2^{2'} = \varphi^2(\overline{M^{2'}(\varepsilon)})$$

we obtain, in some conical regions $D^{1'}$ and $D^{2'}$, solutions

$$u^i : D^{i'} \to M_2^{i'}(\varepsilon), \qquad i = 1, 2, \tag{50}$$

so that the solution

$$u(t,x) := \begin{cases} u^1(t,x), & (t,x) \in D^{1'}, \\ u^2(t,x), & (t,x) \in D^{2'}, \end{cases}$$

admits a shock on the shock front

$$F: \quad (t,x) = s\vartheta(t), \quad 0 < s,\ \tfrac{1}{2} - \varepsilon \le \tau \le \tfrac{1}{2},$$

which is a part of the front F constructed in Section 5, Figures 6, 7.

Now we construct another shock front F^* interacting with F. To this end consider the curve $L \subset M^{1'}(\varepsilon)$,

$$L: \quad \mu = m(\tau), \quad 0 \le \tau,\ m(0) = \left(\tfrac{1}{2}, \tfrac{1}{2}\right),$$

shown in Figure 8, and the function $\vartheta^*(\tau) := \sigma(m(\tau))$, where

$$\sigma(\mu) \parallel \lambda^0(\varphi^1(\mu), \partial_{\mu_1}\varphi^1(\mu)) \times \lambda^0(\varphi^1(\mu), \partial_{\mu_2}\varphi^1(\mu)), \qquad \mu \in M^{1'}(\varepsilon).$$

Note that these vectors are parallel to the lines appearing in our conical parametrization of the manifold $M_2^{1'}$ leading to the solution $u^1(t,x)$ of (50).

We look for a curve L such that for the conical surface $F^* : (t,x) = s\vartheta^*(\tau)$, $0 \le s$, $0 \le \tau$, there exists a shock transition of the form

$$\begin{gathered} c^1(\tau) = \varphi_0^1(m(\tau)) = c^1(s\vartheta^*(\tau)), \\ v^1(\tau) = \overline{\varphi}^1(m(\tau)) = v^1(s\vartheta^*(\tau)),\ c^2(\tau),\ v^2(\tau), \end{gathered} \tag{51}$$

satisfying (39) and the system (48) with the initial conditions $c^1(0) = c^2(0)$.

Note that for c^1, v_1^1, v_2^1 we took the values of the solution (50), $i = 1$, on F^*.

Suppose that the required curve L does exist. Then we cut the region $D^{1'}$ by F^* into two disjoint parts,

$$D^{1'} = D^{1'}(1,3,5) \cup F^* \cup D^{1'}(1,4,5),$$

where $D^{1'}(1,3,5)$ corresponds to the curvilinear triangle $\{1,3,5\}$ and $D^{1'}(1,4,5)$ corresponds to the curvilinear triangle $\{1,4,5\} \subset M^{1'}(\varepsilon)$ (see Figure 8).

We now want to extend the solution

$$u^1 : D^{1'}(1,3,5) \to \varphi^1(\{1,3,5\})$$

through F^* by means of another solution

$$u^* : D^*(1,4,5) \to \varphi^*(\{1,4,5\})$$

so that $D^{1'}(1,3,5) \cup F^* \cup D^*(1,4,5)$ is a connected region, and the solution

$$u(t,x) = \begin{cases} u^1(t,x), & (t,x) \in D^{1'}(1,3,5), \\ u^*(t,x), & (t,x) \in D^*(1,4,5), \end{cases}$$

admits on F^* a shock transition.

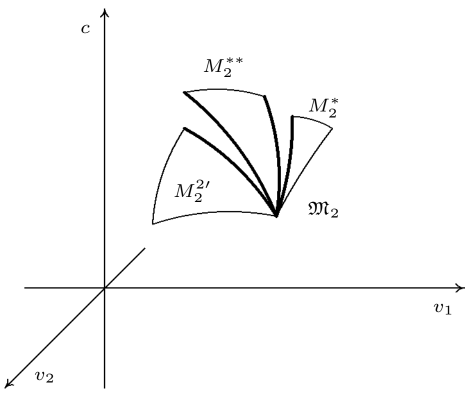


Figure 9

We construct $D^*(1,4,5)$ and u^* in the following way. We find a solution $\varphi^*(\mu_1,\mu_2)$, $\mu \in \{1,4,5\}$, of the system (29) satisfying on L the initial conditions

$$\varphi^*(m(\tau)) = U^2(\tau) = (c^2(\tau), v^2(\tau)), \qquad \partial_\nu \varphi^*(m(\tau)) = \eta^2(\tau),$$

where $c^2(\tau)$, $v^2(\tau)$ are taken from (51) and $\eta^2(\tau)$ is chosen appropriately. The solution u^* is then obtained by conical parametrization of the manifold $M_2^* = \varphi^*(\{1,4,5\})$.

Let us introduce additionally the manifold $M_2^{**} \subset M_2^{1'}$, $M_2^{**} = \varphi^1(\{1,3,5\})$, and the manifold $\mathfrak{M}_2 = M_2^* \cup M_2^{**} \cup M_2^{2'}$ shown in Figure 9.

Performing now the conical parametrization of $\mathfrak{M}_2$, and then extension by simple waves and appropriate constants, we obtain a solution which, under our assumptions about L, admits two interacting shocks.

If $\varepsilon > 0$ is small enough, then in an appropriate disc we obtain for $0 < t_0 < t_1 < t_2$ the film of Figure 10. Figure 10(a), $t = t_0$, shows the situation before the interaction, the interaction begins with the creation of two interacting shocks. Figure 10(b), $t = t_1$, shows the full interaction. Finally, Figure 10(c), $t = t_2$, shows the situation after the interaction.

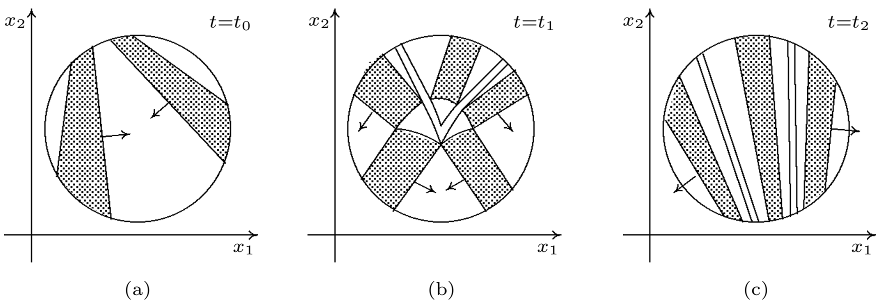


Figure 10

If the operation performed above for the solution $u^1(t,x)$ and the triangle $\{1,3,4\} \subset M^{1'}(\varepsilon)$ is applied to the last solution with two shocks and to the curvilinear triangle $\{1,4,5\} \subset M^{1'}(\varepsilon)$ we get a solution with three interacting shocks, and so on.

There remains the problem of whether a curve L satisfying our requirements does exist.

The first requirement is that the functions (51) must satisfy the Hugoniot conditions (39):

$$(v_n^2 - v_n^1)^2 = (p^2 - p^1)\left(\frac{1}{\varrho^1} - \frac{1}{\varrho^2}\right), \quad (v_n^1 - D^*)^2 = \frac{\varrho^2}{\varrho^1} \cdot \frac{p^2 - p^1}{\varrho^2 - \varrho^1}, \quad v_d^2 = v_d^1,$$

where $p^i = p(c^i)$, $\varrho^i = \varrho(c^i)$, $i = 1, 2$, are given by (38), $D^*(\tau)$ denotes the velocity of the time-dependent shock front $F^*(t)$, and $n(\tau), d(\tau)$ are the unit normal and tangent vectors to $F^*(t)$.

The second requirement is that the system (48) written for $\vartheta = \vartheta^*$ is satisfied for the functions (51). But the equations (48) are equivalent to the assumption that the lines $\Pi(\tau) \subset F^*$ are identical with the lines $\Sigma^{i,0}(U^i(\tau))$, $U^i = (c^i, v^i)$, $i = 1, 2$, along which our conical solutions are constant. So if we take $\vartheta^*(\tau) \parallel \Sigma^{i,0}(U^1(\tau)) = \Pi(\tau)$, then the first equation of (48), with $i = 1$, is automatically satisfied.

Finally, both requirements on L reduce in fact to

$$\begin{gathered}\left[v_n^1(\tau) - D^*\left(m_1(\tau), m_2(\tau), \frac{dm_1}{d\tau}, \frac{dm_2}{d\tau}\right)\right] = \frac{\varrho(c^2)}{\varrho(c^1)} \cdot \frac{p(c^2) - p(c^1)}{\varrho(c^2) - \varrho(c^1)}, \\ \frac{1}{k} c^2 \frac{dc^2}{d\tau} + \left\langle v^2(\tau, c^1, c^2), \frac{d}{d\tau} v^2(\tau, c^1, c^2) \right\rangle = \left\langle \overline{\vartheta}^*(\tau), \frac{d}{d\tau} v^2(\tau, c^1, c^2) \right\rangle,\end{gathered} \tag{52}$$

where $c^1(\tau)$ is given in (51) and $v_n^1(\tau) = \langle \overline{\varphi}^1(m(\tau)), n(\tau) \rangle$.

This means that we have to determine three real-valued functions $c^2(\tau)$, $m_1(\tau), m_2(\tau)$ such that (52) is satisfied, $m_1(0) = m_2(0) = 1/2$, and the curve $L : \mu = m(\tau),\ 0 \le \tau$, is as in Figure 8. Performing some elementary computations one can check that prescribing the function $m_2(\tau)$, $0 \le \tau$, in an appropriate but almost arbitrary way, (52) is equivalent to a system of two ordinary differential equations with two unknown functions $c^2(\tau), m_1(\tau)$. This system can be solved, so that the resulting curve L satisfies our requirements if $\varepsilon > 0$ is small enough.

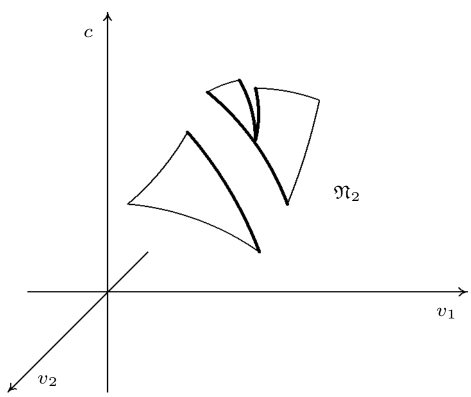
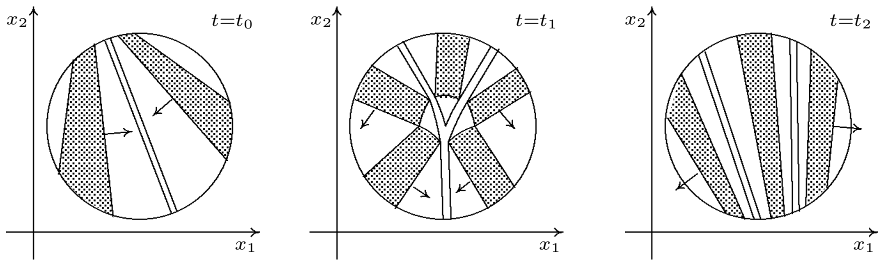


FIGURE 11

Acting quite analogously to the above construction of the manifold $\mathfrak{M}_2$ one can construct a manifold $\mathfrak{N}_2$ of the form shown in Figure 11. The parametrization of $\mathfrak{N}_2$ leads to the film $0 < t_0 < t_1 < t_2$ shown in Figure 12. The solution



(a) (b) (c)

FIGURE 12

describes the interaction of two regular waves with a prescribed shock producing a system of two interacting shocks. Many other possibilities are available.

7. Final remarks

The above examples show that for several variables it is possible to construct interesting solutions with shocks by means of qualitative methods obtained as a very special case by our Theorem. First we construct the image (hodograph) of the required solution, which allows us to set quite arbitrarily a number of qualitative properties (like properties of shocks) of the solution we are going to construct. Knowing the image $u(D) \subset \mathbb{R}^l$ of the required solution $u : \mathbb{R}^n \supset D \to \mathbb{R}^l$, we construct this solution by appropriate parametrization of $u(D)$.

To perform mathematically correct constructions we assumed that: the image $u(D)$ is a two-dimensional manifold, $u(D)$ is small enough and PDE system (19) reduces to a hyperbolic system

$$\partial_{\mu_1}\partial_{\mu_2}\Phi = f(\Phi, \partial_{\mu_1}\Phi, \partial_{\mu_2}\Phi).$$

The smallness assumption is far from necessary. The numerical implementation in each case may determine the full possibilities of the method.

For two-dimensional images $u(D) = M_2$ the system (19) allowing the construction of M_2 can often be reduced to a hyperbolic system. We gave an example of the M.H.D. system (3). But that is not always the case. For example, for the system

$$\begin{aligned}
&\partial_t\varrho + \operatorname{div}(\varrho v) = 0,\\
&\varrho\left(\partial_t v + \sum_{i=1}^{3} v_i\partial_{x_i} v\right) + \nabla p = 0,\\
&\partial_t\left(\frac{p}{\varrho\kappa}\right) + \sum_{i=1}^{3} v_i\partial_{x_i}\left(\frac{p}{\varrho\kappa}\right) = 0,
\end{aligned}$$

describing the nonisentropic gas flow, the system (19) for two-dimensional manifolds M_2 is overdetermined and must be treated in another way.

In the case of k-dimensional manifolds $M_k \subset \mathbb{R}^l$, $2 < k < n$, the system (19) is almost always overdetermined, but nevertheless gives some possibilities of construction of k-dimensional images of solutions. If $k = 3$, $n = 4$, and $M_3 \subset \mathbb{R}^l$ is a manifold satisfying (19), then the solution obtained via an appropriate conical parametrization of M_3 describes the interaction of three regular waves. The interaction may produce shocks or be regular.

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Marek Burnat
Institute of Applied Mathematics and Mechanics
Warsaw University
Banacha 2, 02-097 Warszawa, POLAND