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MORSE COMPLEX, EVEN FUNCTIONALS AND ASYMPTOTICALLY LINEAR DIFFERENTIAL EQUATIONS WITH RESONANCE AT INFINITY

Zalman Balanov — Efim Schwartzman

Introduction

I. Motivation. Let H be a Hilbert space and $f : H \to \mathbb{R}$ a C^1-functional. To study critical points of f in the framework of the classical approaches (Morse Theory [39], Ljusternik–Schnirelman theory [40], etc.) one needs to assume, in particular, that f satisfies the Palais–Smale condition (in short, PS-condition): any sequence $\{x_n\} \subset H$ with $\{f(x_n)\}$ bounded and $\nabla f(x_n) \to 0$ contains a convergent subsequence. In turn, the PS-condition is closely related to deformation properties of the flows associated with gradient vector fields. At the same time, as is well-known, there are many important variational problems, where the corresponding functionals fail to satisfy the PS-condition in any suitable sense. In addition, these functionals may not satisfy certain other conditions that are necessary for application of the classical methods.

The problem of weakening the PS-condition has attracted a considerable attention for a long time (see, for instance, [16], [20], [21], [49] and references therein). An essential step in this direction was done by C. Conley [21] who

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observed that important invariants of a functional can be obtained by taking into account the behavior of the functional in a *neighbourhood* of a *compact isolated invariant set* only. In the presence of group symmetries this approach was developed in details by T. Bartsch [12]. Developing Conley's approach A. Floer [23]–[26] constructed the Morse complex for important classes of functionals with ill-defined gradient fields. Note that in [23] Floer used a combination of the technique based on the Morse complex with Ljusternik–Schnirelman theory to study multiple periodic solutions of Hamiltonian systems on a compact simplectic manifold.

In this paper we suggest a Morse complex based method for studying variational problems that are degenerate in a certain sense and have group symmetries. We mean, in particular, the following degeneracies, which are often related to each other and appear simultaneously:

1. Lack of the PS-condition: (a) resonance at infinity [20]; (b) critical Sobolev exponents [16], [49]; (c) elliptic equations and Hamiltonian systems with singularities [2]; (d) unboundedness of the domain in elliptic equations [2], [42], etc.

2. Absence of (regular) flows: (a) functionals with ill-defined gradient fields [23]–[26]; (b) lack of smoothness [17], [22] (quasilinear elliptic equations); see also [20], [49] and references therein.

In other words, we are interested in situations for which the application of the classical methods is impossible or meets serious difficulties.

In the present paper the Morse complex technique is applied to study odd variational problems exhibiting resonance behavior at infinity. However, the authors believe that this technique extends to certain other degeneracies mentioned above as well as to group symmetries more complicated than the involution.

The basic ideas behind the approach presented in this paper can be traced back to [45], where the Morse complex based method was used to obtain a multiplicity result for finite-dimensional even functionals (cf. Theorem 2.1). Then, it was shown in [8] (see also [6] and [7]) that this approach is also effective for a wide class of infinite dimensional functionals and, moreover, provides a way to weaken the PS-condition as well as to involve approximation techniques in studying multiple critical points. The present paper contains some improvements and essentially new applications of our technique.

This paper is a revised version of [9]. Some of its results were announced in [10].

II. Approach. To be more specific, we will describe the main abstract result of the paper.

A finite-dimensional C^2-functional $f : \mathbb{R}^n \to \mathbb{R}$ is called *quadratic-like* if $\nabla f(x) = Ax + o(\|x\|)$ as x goes to infinity, where A is a non-degenerate self-adjoint linear operator. Let $\operatorname{ind}(\infty, f)$ (respectively, $\operatorname{ind}(x, f)$) denote the Morse index of A (respectively, the Morse index of f at a non-degenerate critical point x). If zero is a non-degenerate critical point of f, we set

$$\mathrm{indr}(\infty, f) = \mathrm{ind}\,(0, f) - \mathrm{ind}\,(\infty, f), \quad \mathrm{indr}(x, f) = \mathrm{ind}\,(0, f) - \mathrm{ind}\,(x, f).$$

Let, further, H be a separable Hilbert space, $E_1 \subset E_2 \subset \ldots \subset E_n \subset \ldots$ finite-dimensional subspaces of H such that $\bigcup_n E_n$ is dense in H, and $\{f_n : E_n \to \mathbb{R}\}$ a sequence of C^2-functionals such that each f_n has non-degenerate critical points only. Let K_n be the set of all critical points of f_n, and K_n^p the set of critical points x of f_n with $0 < \mathrm{indr}(x, f_n) \leq p$. Set

$$K_\infty = \bigcap_{j=1}^{\infty} \left(\mathrm{cl}\left(\bigcup_{n=j}^{\infty} K_n \right)\right), \quad K_\infty^p = \bigcap_{j=1}^{\infty} \left(\mathrm{cl}\left(\bigcup_{n=j}^{\infty} K_n^p \right)\right).$$

Assume that the sequence $\{f_n\}$ satisfies the following conditions.

(f_1) There exists n_0 such that for every $n \geq n_0$ the functionals f_n are even (in particular, 0 is a critical point of f_n), quadratic-like, have only non-degenerate critical points, and $\mathrm{indr}(\infty, f_n) \geq p > 0$, where p is independent of n.

(f_2) The set $\bigcup_n K_n^p$ is bounded in H (in other words, $\infty \notin K_\infty^p$) and $0 \notin K_\infty^p$.

(f_3) There exists $\lim f_{n_k}(x_k)$, for any sequence $\{x_k,\ x_k \in K_{n_k}\}$ convergent in H.

(f_4) Any *bounded* sequence $\{x_k\}$ with $x_k \in E_{n_k}$ and $\nabla f_{n_k}(x_k) \to 0$ contains a convergent subsequence, and, moreover, if $\{x_k\}$ is convergent then its limit point belongs to K_∞.

THEOREM 0. *Assume that conditions* (f_1)–(f_4) *hold. Then K_∞ contains at least p nontrivial distinct pairs of symmetric points.*

Moreover, if K_∞ consists of isolated points only, then, in addition to the above statement, some homological (stability) information for p distinguished distinct pairs from K_∞^p can be obtained (see Theorem 2.4 for the precise formulation).

Conditions (f_1)–(f_4) define a Galerkin-type approximation of a "limit" functional f_*, which has not been mentioned explicitly. From this point of view elements of K_∞ should be thought of as critical points of f_*. It is important to understand that nothing in conditions (f_1)–(f_4) prevents the limit functional from behaving very irregularly. In particular, f_* may not be well-defined or have not well-defined gradient field; or, in the case when the gradient is well-defined, the PS-condition may fail, etc. The conditions also allow critical points to travel both to infinity and to zero. All this makes us believe that Theorem 0 is qualitatively distinct from what can be obtained by methods of Ljusternik–Schnirelman Theory (cf. [11], [20], [40], [49]).

The core of the proof of Theorem 0 (Sections 4–7) is the Normal Form Graph Lemma (in short, NFG Lemma), which is a sort of classification result on a set of graded graphs closely related to Morse complexes (see Sections 4 and 7). Observe that the combinatorial arguments involved into the proof of the NFG Lemma go beyond the scope of (co)homological methods (cf. [11], [20]) and may be interesting independently. It is worth noting that we get multiplicity results by means of Morse complex arguments only (with Ljusternik–Schnirelman theory not being addressed whatsoever).

III. Variational problems with resonance at infinity. The technique developed in the present paper is illustrated by two asymptotically linear problems having resonance at infinity. It should be pointed out that in both cases the associated functionals do not satisfy the Palais–Smale condition in any appropriate sense (in particular, the set of critical points belonging to the same level surface may be unbounded). From this point of view the settings of the both examples are beyond the scope of (the standard) Ljusternik–Schnirelman theory.

In the first example we are dealing with periodic solutions of nonautonomous odd Hamiltonian systems with resonance at infinity. Note that asymptotically linear Hamiltonian systems (with/without resonance at infinity) were studied by many authors (see, for instance, [4], [5], [20], [31], [34], [36], [40], [49] and references therein).

Under our assumptions the system has an a priori unbounded (in the L_2-metric) set of solutions. We are interested in *limit* solutions of this system, that is to say, in the solutions that come from the non-resonant systems approximating the initial system in an appropriate sense. Under the additional assumption that each limit solution is isolated, we obtain a multiplicity result for limit solutions and the corresponding homological information (see Theorem 3.3).

In the second example we are dealing with elliptic semilinear boundary value problems (BVP) having an odd nonlinearity and resonance at infinity. This problem was studied intensively in many papers (see, for instance, [1], [3], [11], [13], [16], [20], [33], [35], [38], [41], [48], [50], and references therein). Using our technique we obtain a multiplicity result (Theorem 3.6). This result can be applied, for instance, to the following BVP:

$$(*)\qquad \begin{cases} -\Delta u - \lambda u = \psi(x,u), & x \in \Omega, \\ u|\partial\Omega = 0, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with C^2-smooth boundary and λ is resonant; the nonlinearity ψ is odd and of class C^1 in u with $\psi'_u(x,u) \to 0$ as $u \to \infty$ uniformly in x and $\psi'_u(x,0) = \nu > 0$. These conditions include, in particular, a class of nonlinearities in $(*)$ that, apparently, cannot be treated by the known methods (cf. [11], [20], [41], [50]).

IV. Overview. The paper is organized as follows. In Section 1 we fix the terminology (regular pairs, MS-type, Morse complex, etc.), and recall certain auxiliary results. In Section 2 we present formulations of the main abstract result (see Theorem 2.4) and its consequence dealing with asymptotically linear odd equations with resonance at infinity (see Theorem 2.7).

In the third section we present applications to asymptotically linear resonant Hamiltonian systems (see Theorem 3.3), and to asymptotically linear resonant elliptic BVP (see Theorem 3.6).

In Section 4 we introduce a combinatorial operation defined over the set of all (Morse) complexes (see Definitions 4.1, cf. [37], [15]). Furthermore, with any regular pair (Φ, L) (see Definition 1.2) having Φ of the MS-type we associate a graded graph of a special form (see Definition 4.4), which inherits all properties of the corresponding Morse complex (over $\mathbb{Z}_2$).

Sections 5–7 are devoted to the proof of Theorem 2.4. In Section 5 we reduce the proof of Theorem 2.4 to the proof of the so-called Combinatorial Lemma (see Lemma 5.8) dealing with the above mentioned graphs. In Section 6 we reduce the Combinatorial Lemma to the above mentioned NFG Lemma. The NFG Lemma is proved in Section 7.

In Sections 8 and 9 we deduce Theorem 2.7 from Theorem 2.4. Finally, Sections 10 and 11 contain the proofs of Theorems 3.3 and 3.6, respectively.

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1. Preliminaries

1.1. Regular pairs. Let U be an open domain in $\mathbb{R}^n$, $L : U \to \mathbb{R}$ a C^2-smooth functional and K the set of critical points of L.

Definition 1.1. A C^1-vector field Φ on U is said to be a *pseudo-gradient field* (*pg-vector field*) for L (respectively, (Φ, L) is said to be a *pair*) if the following conditions are fulfilled:

(i) $\|\Phi(x)\| \le \|\nabla L(x)\|$, $x \in U$,
(ii) $(\nabla L(x), \Phi(x)) > \|\nabla L(x)\|^2/2$ for any $x \in U \setminus K$.

To introduce the Morse complex we need the following

Definition 1.2. A *pg*-vector field Φ for L (respectively, a pair (Φ, L)) is called *regular* on U if there exist open bounded sets $Q_0,\ Q_1$ such that

(a) $K \subset Q_0 \subset Q_1 \subset U$ and
(b) if the endpoints of an integral curve μ of Φ belong to Q_0 then $\mu \subset Q_1$.

Definition 1.3. A homotopy of pairs $(\Phi_\lambda, L_\lambda)$, $\lambda \in [0,1]$, is said to be *regular* on U if every pair $(\Phi_\lambda, L_\lambda)$ is regular on U and, moreover, the relevant sets Q_0 and Q_1 can be chosen to be independent of λ.

Observe that the regularity condition is a sort of "compactness conditions" allowing to some extent to deal with (regular) functionals as defined on a compact manifold.

Recall that the Morse index of a non-degenerate singular point x of a regular field Φ coincides with the dimension of the stable invariant manifold W^s_x.

Definition 1.4. Let Φ be a *pg*-vector field (for C^2-functional L) defined and regular on U. The field Φ is said to be of the *Morse–Smale type* (in short, of the MS-*type*) if:

(i) all the critical points of Φ are non-degenerate,
(ii) for any pair of critical points x, y with $\operatorname{ind} y \leq \operatorname{ind} x + 1$, the unstable manifold W^u_x meets transversally the stable manifold W^s_y.

Standard dimensional and compactness arguments yield (cf. [37], [24], [15]):

Proposition 1.5. *If Φ is of the* MS-*type then:*

(i) *Φ has finitely many (non-degenerate) critical points,*
(ii) *no connecting trajectory joins critical points of the same index,*
(iii) *there are only finitely many connecting trajectories joining the critical points of the neighbouring indices k and $k-1$ for any integer $k \geq 1$, and their number does not depend on sufficiently C^1-small perturbations of Φ.*

Theorem 1.6. *Let (Φ, L) be an arbitrary pair defined and regular on U. There exists a homotopy of pairs (Φ_t, L_t), $t \in [0,1]$, regular on U such that:*

(i) *$\Phi_0 = \Phi$, $L_0 = L$,*
(ii) *Φ_1 is of the* MS-*type.*

Theorem 1.6 as well as Proposition 1.5 are a sort of "general position results", and in this sense are similar to the corresponding statements from [15], [24], [37], [43], [44], [47].

The sketch of proof of Theorem 1.6 for a class of functionals defined on an open domain of a Hilbert space was presented in [6]. We refer to [8] for the complete proof of this result.

1.2. Homology groups of regular pairs. Recall the construction of the Morse complex (see, for instance, [6], [8], [24], [43], [44]). Let Φ be a *pg*-vector field of the MS-type, $x_1^k, \dots, x_{p_k}^k$ be the set of all critical points of the Morse index k and $\widetilde{\sigma}_{ij}^k$ the number of trajectories connecting x_j^k to x_i^{k-1}. The Morse complex $C_* = C_*(\Phi)$ (over $\mathbb{Z}_2$) is the finite complex

$$0 \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0,$$

where the critical points of index k form the distinguished basis of $C_k = \mathbb{Z}_2^{p_k}$ and ∂_k is generated in the bases by the matrix $\{\widetilde{\sigma}_{ij}^k \text{ (modulo 2)}\} = \{\sigma_{ij}^k\}$. By the standard argument, under the above conditions $\{(C_n, \partial_n)\}$ is a chain complex. Denote by $H_*(\Phi) = H_*(L) = H_*(\Phi, L)$ homology groups of this complex.

The following invariance result on homology groups is essential for our technique.

Theorem 1.7. *Let $(\Phi_\lambda, L_\lambda)$ be a regular homotopy of pairs defined on U, $\lambda \in [0,1]$. Suppose the fields Φ_0 and Φ_1 are of the* MS*-type. Then the homology groups $H_*(\Phi_0)$ and $H_*(\Phi_1)$ coincide.*

This statement is a direct consequence of the main result of [21] on the invariance of the Conley index and Theorem 1 from [24] on the connection between the Conley index and homology groups of the Morse complex. The union of all critical points and connecting trajectories is considered here as the isolated invariant set involved in the corresponding Conley index (cf. [24]).

REMARKS 1.8. (i) Although, to the best of our knowledge, Theorem 1.7 as stated, has not appeared in the literature, it is a part of mathematical folklore that can be traced to [37].

(ii) The method used in [15] can be easily adjusted to prove Theorem 1.7 as the regularity condition means that the behavior of a functional regular in the above sense is quite similar to the behavior of a smooth function defined on a compact manifold.

(iii) A generalization of Theorem 1.7 for a class of infinite-dimensional functionals is proved in [8] (see also [6], [7]).

(iv) Note also that the homotopy invariance of Morse complex homology groups was proved by A. Floer for certain important classes of infinite dimensional functionals in his famous papers [25], [26] (see also [43]). A complete presentation of the Floer approach in finite-dimensional case can be also found in [44].

Let (Φ_0, L_0) and (Φ_1, L_1) be two pairs defined and regular on U. We say that these pairs are equivalent if there exists a homotopy defined and regular on U joining (Φ_0, L_0) to (Φ_1, L_1). It is easy to see that this relation determines the equivalence.

Let (Φ_0, L_0) be a pair defined and regular on U. Denote by E the equivalence class of all pairs $\{(\Phi, L)\}$ for which there exists a regular homotopy joining (Φ, L) to (Φ_0, L_0). By Theorem 1.6, the class E contains pairs with the fields of the Morse-Smale type. By Theorem 1.7, the homology groups of the Morse complexes associated with these fields coincide. Thus one can associate with (Φ_0, L_0) the homology groups of the Morse complexes of any suitable pair from E.

Let x_0 be an isolated critical point of L (possibly degenerate). Denote by U_0 an *isolating* neighbourhood of x_0 (containing no other critical point).

Proposition 1.9. *The pair $(\nabla L, L)$ is regular on U_0.*

Proof. It is enough to set $Q_1 = U_0$ and to put Q_0 to be a sufficiently small neighbourhood of x_0. $\square$

The homology groups associated with the pair $(\nabla L, L)$ restricted to a neighbourhood U_0 isolating a critical point x_0 are said to be *homology groups* of x_0. It should be noted also that one can define the (co)homology groups of an isolated degenerate critical point via the Gromoll–Meyer pair ([19], [20], [28]) or by means of the Conley index (see [12], [21], [43]).

We complete this section with a corollary of Theorem 1.7 which will be essentially used in what follows.

Let us consider a functional

$$f(x) = \frac{1}{2}(Ax, x) + \psi(x),$$

where $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear non-degenerate self-adjoint operator and $\psi : \mathbb{R}^n \to \mathbb{R}$ is a C^2-functional such that $||\nabla\psi(x)|| < q||Ax||$ for large x with $q < 1$.

Corollary 1.10. *Let k be the dimension of the negative eigenspace* (*the Morse index*) *of the operator A. Then*

$$\dim H_i(f) = \begin{cases} 0 & \text{if } i \neq k, \\ 1 & \text{if } i = k. \end{cases} \tag{1.1}$$

Proof. Set $f_t(x) = (1-t)\cdot f(x) + t\cdot(Ax, x)/2$. It is easy to see that $(\nabla f_t, f_t)$ is a regular homotopy on $\mathbb{R}^n$ connecting $(\nabla f, f)$ to $(Ax, (Ax, x)/2)$. Then the application of Theorem 1.7 reduces the study of the functional f in question to the study of the non-degenerate quadratic form $(Ax, x)/2$, which is trivial. $\square$

2. Statement of results: abstract framework

2.1. Sequences of regular even functionals. In this subsection we use notions and notations defined in Subsection II of the Introduction.

Before giving a formulation of the main result let us consider a particular case, which is well-known in Ljusternik–Schnirelman theory (see, for instance, [19], [20], [40]).

Theorem 2.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be an even quadratic-like functional, zero a non-degenerate critical point and* $\operatorname{indr}(\infty, f) = p > 0$. *If f has finitely many critical points then there exist among them p pairs, say, $\pm x_1, \dots , \pm x_p$, such that each homology group $H_{\nu-i}(x_i)$, $i = 1, \dots , p$, is nontrivial; here ν is the Morse index of zero* ($\nu = \operatorname{ind}(0, f)$).

Because of the technical reasons, which will be clarified later, introduce the following notation.

Definition 2.2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a functional such that zero is a non-degenerate critical point of index ν. If x is an isolated critical point of f then we define the *shifted* homology groups by setting $H^s_k(x) = H_{\nu-k}(x)$. Similarly, if $U \subset \mathbb{R}^n$ is an open subset and a pair $(\Phi, f)|U$ is regular, then $H^s_k(f|U) = H_{\nu-k}(f|U)$.

Remark 2.3. Note that under the above notation Theorem 2.1 states the non-triviality of the shifted groups $H^s_i(x_i)$, $i = 1, \dots , p$.

With these preliminaries in hand Theorem 0 stated in the Introduction can be specified as follows. Assuming that we are in the setting of Subsection II from the Introduction, suppose that K_∞ is equipped with the induced metric.

Theorem 2.4. *Assume that conditions* (f_1)–(f_4) *from the Introduction hold. Then K_∞ contains at least p nontrivial distinct symmetric pairs. Moreover, if every point of K^p_∞ is isolated as a point of K_∞ then there exist p distinct pairs $\pm x_1, \dots , \pm x_p \in K^p_\infty$ satisfying the following condition: if W_i is a neighbourhood of x_i such that $W_i \cap K_\infty = \{x_i\}$ then for any n_0 there exists $n > n_0$ such that the functional $f_n|(W_i \cap E_n)$ together with its gradient field form a regular pair for which the shifted homology group $H^s_i(f_n|(W_i \cap E_n))$ is non-trivial.*

Remarks 2.5. (i) As was mentioned in the Introduction, Theorem 2.4 allows to study ill-defined functionals. This possibility is essentially exploited in what follows.

(ii) Note that the approximating functionals from Theorem 2.4 should not be finite-dimensional, in general. The key condition is that they admit the construction of the Morse complexes. From this point of view the class of infinite dimensional functionals studied in [6]–[8] can be useful.

2.2. Abstract asymptotically linear equations with resonance. In this section we present a corollary of Theorem 2.4 which deals with abstract asymptotically linear equations with resonance at infinity. We need some additional notation.

Let H be a separable Hilbert space and let V be a vector field on H. Take $x \in H$. Recall the definition of local Lipschitz constant $L(x) = L(x, V) \in [0, \infty]$

at x. Take a ball $B(x,r)$ of radius r centered at x, and denote by $L(x,V,r)$ the minimal Lipshitz constant for V on $B(x,r)$. Then $L(x,V)$ is a limit of $L(x,V,r)$ as $r \to 0$. If E is a linear subspace of H the number $L_E(x,V) = L(x, V|\{x+E\})$ is said to be a local Lipschitz constant of V at x along E.

Observe that if V is differentiable then $L(x,V) = \|V'(x)\|$.

Let $F : \operatorname{dom} F \subset H \to H$ be a linear self-adjoint operator densely defined on H and having a discrete spectrum only. Denote by $N(F)$ the dimension of the negative space in the spectral decomposition for F, and set $\overline{N}(F) = N(F) + \dim\ker F$.

Consider an asymptotically quadratic functional

$$h(x) = \frac{1}{2}(Ax,x) + \phi(x)$$

(in general, ill-defined on H). Here $A : \operatorname{dom} A \subset H \to H$ is assumed to be a linear, self-adjoint (in general, unbounded) operator densely defined on H with $\dim\ker A < \infty$ and $\phi \in C^1(H,\mathbb{R})$.

To study h in a neighbourhood of zero we will use another representation of h in the form

$$h(x) = \frac{1}{2}(Bx,x) + \phi_0(x),$$

where $B : \operatorname{dom} B \subset H \to H$ is a linear self-adjoint operator densely defined on H with $\dim\ker B < \infty$, $\operatorname{dom} B = \operatorname{dom} A$ and $\phi_0 \in C^1(H,\mathbb{R})$. In this representation the first term is assumed to be the principal part of h around zero.

Let $E_1 \subset E_2 \subset \ldots \subset E_n \subset \ldots$ be a sequence of finite-dimensional A-invariant subspaces of H such that $\operatorname{cl}(\bigcup E_n) = H$.

We will assume the following conditions to be fulfilled.

(h$_1$) The operator A has a discrete spectrum only, and any its eigenvalue is of finite multiplicity. Moreover, we suppose that A is closed, $B - A$ is bounded and every E_n is also B-invariant.

(h$_2$) The functional ϕ is even. The operator $\nabla\phi$ is bounded on any ball, satisfies the Lipschitz condition with the Lipschitz constant L outside a ball and $\nabla\phi(x)/\|x\| \to 0$ as $\|x\| \to \infty$.

Moreover, for any *finite-dimensional* subspace $E \subset H$ and any sequence $\{x_n\}$ with $\|x_n\| \to \infty$ and $\rho(x_n, \ker A)/\|x_n\| \to 0$, one has $L_E(x_n, \nabla\phi) \to 0$ (here ρ stands for the metric induced by the norm in H).

(h$_3$) The operator $\nabla\phi_0$ satisfies the Lipschitz condition in a neighbourhood of zero and $\nabla\phi_0(x)/\|x\| \to 0$ as $\|x\| \to 0$. Moreover, for any *finite-dimensional* subspace $E \subset H$ and any sequence $\{x_n\}$ with $\|x_n\| \to 0$ and $\rho(x_n, \ker B)/\|x_n\| \to 0$, one has $L_E(x_n, \nabla\phi_0) \to 0$.

Set $A_n = A|E_n$, $B_n = B|E_n$. It is easy to show that under condition (h_1) the limit

$$r = \lim_n [N(B_n) - \overline{N}(A_n)] \tag{2.1}$$

is well-defined. Under the assumptions (h_1)–(h_3) we are interested in solutions of the equation

$$Ax + \nabla\phi(x) = 0. \tag{2.2}$$

Observe that if H is infinite dimensional, then the operator A is *unbounded* (cf. condition (h_1)). Hence in this case one cannot speak about critical points of h in the classical sense. In what follows by *critical points* of h we mean, by definition, solutions of (2.2).

Definition 2.6. Let x be an isolated critical point of h and let W be an *isolating* neighbourhood of x (that is to say, x is the only critical point belonging to W). We say that h has *well-defined shifted homology groups at* x (associated to a sequence of subspaces $\{E_n\}$) if for every n large enough the functional $h|(W \cap E_n)$ together with its gradient field form a regular pair and the sequence of shifted homology groups $H_i^s(h|(W \cap E_n))$ stabilizes for n large enough. We set $H_i^s(x, h) = H_i^s(h|(W \cap E_n))$, where n is supposed to be large enough.

Theorem 2.7. *Assume that the above assumptions* (h_1)–(h_3) *hold and* $p = r - 1 > 0$ (*cf.* (2.1)). *Then the functional h has at least p distinct pairs of nontrivial critical points. Moreover, if every critical point of h is isolated, then there exist p distinct pairs of nontrivial critical points $\pm x_1, \dots, \pm x_p$ such that the functional h has well-defined shifted homology groups at each x_i with* $\dim H_i^s(x_i, h) \neq 0$, $i = 1, \dots, p$ (*cf. Definition* 2.6).

Corollary 2.8. *Assume that conditions* (h_1)–(h_3) *above hold. Assume also that $N(A) < \infty$. Then the conclusion of Theorem* 2.7 *is valid with* $p = N(B) - \overline{N}(A) - 1$ *provided that* $p > 0$.

Proof. It suffices to observe that for the operator sequences $\{A_n\}, \{B_n\}$ defined above one has

$$\lim_n [N(B_n) - \overline{N}(A_n)] = p + 1. \qquad \square$$

Remark 2.9. Suppose that under the assumptions of Theorem 2.7 h satisfies the PS-condition in the energy space associated to A at any level $c < h(0)$. Then, under this additional assumption, it can be proved in the framework of Ljusternik–Schnirelman theory that h has $r = p + 1$ distinct nontrivial pairs of critical points (cf. [50]). Observe that the same multiplicity result can be obtained by means of Theorem 2.4 (even under the assumption that h satisfies the PS-condition just with respect to the norm of H). Thus, in comparison with this result, the number of pairs of critical points provided by Theorem 2.7 (in the general setting) is lower by one. However, the authors have no examples when this "lost" pair of critical points really disappears.

Remark 2.10. The B-invariance of every E_n (see (h_1)) is assumed for the sake of simplicity and can be removed.

3. Statement of results: applications

3.1. Asymptotically linear Hamiltonian systems with resonance. As an application of Theorem 2.4 we obtain a multiplicity result for Hamiltonian systems with resonance at infinity (see Theorem 3.3).

The starting point of our discussion is the well-known paper by V. Benci [14]. Let $H \in C^1(\mathbb{R}^{2n} \times \mathbb{R}, \mathbb{R})$ and consider the Hamiltonian system of ordinary differential equations

$$\dot{z} = JH_z(z,t). \tag{3.1}$$

Let $A, B : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be symmetric linear operators. We will suppose that H satisfies the following conditions.

(H_1) $H(z,t)$ is τ-periodic in t.

(H_2) $H(z,t) = (Az,z)/2 + h(z,t)$, where A is non-degenerate and $h(z,t)$ has a "compact support" in the sense that there exists a ball $B_M \subset \mathbb{R}^{2n}$ such that $h(z,t) = 0$ for all $z \in \mathbb{R}^{2n} \setminus B_M$ and $t \in \mathbb{R}$.

(H_3) $H_z(z,t) = Bz + o(\|z\|)$ as $z \to 0$, where B is non-degenerate and, moreover, the linear system $\dot{z} = JBz$ has no τ-periodic solutions.

(H_4) $H(-z,t) = H(z,t)$.

Consider the spectrum of the operator $(\tau/2\pi)JA$. As it is well-known (see, for instance, [14]), if τ is resonant then

$$\sigma\left(\frac{\tau}{2\pi}JA\right) \bigcap i\mathbb{Z} \neq \emptyset. \tag{3.2}$$

In the setting defined by (H_1)–(H_4) we are interested in the following two problems:

(1) the behavior of solutions of system (3.1) for τ close to the resonant period,

(2) multiplicity results and stability (homological) information on solutions of the resonant system (3.1) that come from the non-resonant systems by passing to a limit.

To formulate our results, let us recall the definition of the integer-valued function θ (see [14]).

Consider the complexification of A, B denoted by the same symbols. We set

$$\begin{aligned} N(A) &= \{\text{number of negative eigenvalues of } A\}, \\ \overline{N}(A) &= \{\text{number of non-positive eigenvalues of } A\}, \\ \theta_j(B,A) &= N\Big(ijJ + \frac{\tau}{2\pi}B\Big) - \overline{N}\Big(ijJ + \frac{\tau}{2\pi}A\Big), \\ \theta(B,A) &= \sum_{j=-\infty}^{\infty} \theta_j(B,A). \end{aligned}$$

(The symbol "i" here stands for the imaginary unit).

Consider also a family of systems

$$\dot z = \lambda \cdot JH_z(z,t). \tag{3.3}$$

For $\lambda = 1$ we obtain (3.1). It is easy to see that the resonance condition (see (3.2)) for (3.3) depends on both λ and τ. Set

$$f(z(\cdot)) = \int_0^\tau (1/2(J\dot z(t), z(t)) + H(z(t),t))\,dt. \tag{3.4}$$

As it is well-known, τ-periodic solutions of (3.1) are critical points of f considered, for instance, on $W^{1,2}[0,\tau]$, and vice versa.

In contrast to the classical approach, which deals with the energy space $H^{1/2}$ (cf. [14], [20], [40]), we study systems (3.1) and (3.3) as operator equations in the space $L_2[0,\tau]$.

Bearing in mind Definition 2.6, let us introduce the following sequence of subspaces of L_2:

$$E_m = \operatorname{span}\{u\cos\nu kt,\ v\sin\nu kt;\ u,v\in\mathbb{R}^{2n},\ \nu = 2\pi/\tau,\ k = 0,\dots,m\}. \tag{3.5}$$

Lemma 3.1. *Assume that H satisfies conditions* (H_1)–(H_4)*, and consider non-resonant pairs (λ,τ) with fixed τ and $\lambda\in[1/2,3/2]$. Then there exists a constant $r = r(M,A) > 0$ independent of the above non-resonant pairs (τ,λ) such that all τ-periodic solutions of system* (3.3) *belong to the ball $B_r = \{z\in\mathbb{R}^{2n} \mid \|z\| < r\}$ in the phase space.*

Consider now system (3.1) assuming τ to be a resonant period. Involve (3.1) into a parametric family (3.3) assuming λ to be close to 1. Take a monotone sequence $\lambda_k \to 1$ ($\lambda_k \in [1/2,3/2]$) such that (λ_k,τ) is a non-resonant pair for each $k = 1,2,\dots$. From Lemma 3.1 it follows that for all $\lambda = \lambda_k$ the solutions of (3.3) belong to a fixed ball (independent of k) of the phase space. Then a simple analysis of system (3.3) easily yields that the union of the solutions is C-precompact.

Definition 3.2. Let $\lambda_k \to 1$ be a monotone sequence such that every (λ_k, τ) is non-resonant and let $\{z_k(t)\}$ be a C-convergent sequence of functions such that every $z_k(t)$ is a solution of (3.3) with a certain $\lambda = \lambda_k$. Then the limit function $z(t) = \lim z_k(t)$ (which is obviously a τ-periodic solution of (3.1)) is said to be a *limit* solution of (3.1).

It is easy to see that in the resonance case for λ close enough to 1 the numbers $\theta(\lambda B, \lambda A)$ are different for $\lambda < 1$ and $\lambda > 1$. At the same time, the numbers

$$\theta^+_{B,A} = \lim_{\lambda \to 1+} \theta(\lambda B, \lambda A) \quad \text{and} \quad \theta^-_{B,A} = \lim_{\lambda \to 1-} \theta(\lambda B, \lambda A)$$

are well-defined.

Theorem 3.3. *Assume that H satisfies conditions* (H_1)–(H_4). *Let τ be a resonant period for* (3.1) *and let $\lambda_k \to 1$ be a decreasing* (*respectively, increasing*) *monotone sequence such that every (λ_k, τ) is non-resonant. Suppose, furthermore, that every limit τ-periodic solution is an isolated solution of* (3.1). *Then there exist $p = p^+ = \theta^+_{B,A}$* (*respectively, $p = p^- = \theta^-_{B,A}$*) *distinct nontrivial pairs of limit solutions, say, $\pm z_1(t), \ldots, \pm z_p(t)$ such that the functional f defined by* (3.4) *has well-defined shifted homology groups at each z_i* (*associated to the sequence of subspaces defined by* (3.5)) *with* $\dim H^s_i(z_i, f) \neq 0$, *$i = 1, \ldots, p$* (*cf. Definition* 2.6).

Remarks 3.4. (i) It follows immediately from the assumptions of Theorem 3.3 (cf. (H_2)) that each τ-periodic solution of the linear system $\dot{z} = JAz$ with sufficiently large L_2-norm is also the τ-periodic solution of (3.1) and, moreover, belongs to the zero level of f defined by (3.4). It means, in particular, that the PS-condition is not fulfilled at the zero level. Therefore, there is the obstacle, in general, to application of the classical methods. At the same time, under the additional assumption $f(0) \leq 0$ Ljusternik–Schnirelman theory does provide the multiplicity result similar to that stated in Theorem 3.3 (cf. Remark 2.9, see also [50], [11]).

(ii) We will deduce Theorem 3.3 directly from Theorem 2.4 (see Section 10). Observe that Theorem 2.7 also can be applied in a setting close to the one of Theorem 3.3. However, the result provided by Theorem 2.7 is weaker (cf. Remark 2.9).

(iii) Observe that the non-resonance assumption with respect to the system $\dot{z} = JBz$ as well as the non-degeneracy of A and B can be removed from the assumptions of Theorem 3.3.

(iv) For the references related to Theorem 3.3 see the Introduction.

3.2. Asymptotically linear elliptic equations with resonance. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^2-smooth boundary, $W^{k,2}_0 = W^{k,2}_0(\Omega)$

the corresponding Sobolev space of functions vanishing on the boundary and $L_2 = L_2(\Omega)$.

Consider the following boundary value problem:

$$\begin{cases} -\Delta u = p(x,u), \\ u|\partial\Omega \equiv 0. \end{cases} \tag{3.6}$$

Assume that $p(x,t) \in C^0(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ and set $P(x,t) = \int_0^t p(x,s)\,ds$. As it is well-known (see, for instance, [40], [19]), if p is of subcritical growth in t, then the functional

$$f(u) = \int_\Omega \left(\frac{1}{2} |\nabla u(x)|^2 - P(x, u(x)) \right) dx \tag{3.7}$$

belongs to $C^1(W_0^{1,2}(\Omega), \mathbb{R})$ and its critical points are weak solutions of (3.6).

Assume, furthermore, that

$$p(x,t) = \widehat{\lambda} t + \psi(x,t),$$

where $\psi(x,t) = o(|t|)$ as $|t| \to \infty$ uniformly in $x \in \Omega$. Set

$$\Psi(x,t) = \int_0^t \psi(x,s)\,ds.$$

Let

$$\lambda_1 < \lambda_2 < \ldots < \lambda_k < \ldots$$

be eigenvalues of the operator $-\Delta : W_0^{2,2} \subset L_2 \to L_2$. We are interested in the resonant situation in (3.6), that is to say, $\widehat{\lambda} = \lambda_k$ for some k. Then functional (3.7) takes the form

$$f(u) = \int_\Omega \left(\frac{1}{2} (|\nabla u(x)|^2 - \lambda_k u^2(x)) - \Psi(x, u(x)) \right) dx. \tag{3.8}$$

As it is well-known (see, for instance, [11], [19]), functional (3.8) (as defined on $W_0^{1,2}$) does not satisfy the PS-condition in general, therefore, the application of classical methods to study functional (3.8) meets serious difficulties.

We will investigate (3.8) by means of Corollary 2.8 in the space L_2 , on which (3.8) is, evidently, ill-defined.

With these preliminaries in hand we rewrite (3.6) in the form

$$\begin{cases} -\Delta u - \lambda_k u = \psi(x,u), \\ u|\partial\Omega \equiv 0, \end{cases} \tag{3.9}$$

and assume that the nonlinearity ψ satisfies the following conditions.

- (ψ_1) The function $\psi(x,t)$ satisfies the Lipschitz condition in both arguments with constant l.
- (ψ_2) ψ has the zero derivative over t at infinity in the sense that the local Lipschitz constants in t go to zero as $|t| \to \infty$, uniformly in x.

(ψ_3) $\psi(x,t) = \nu t + \psi_0(x,t)$, where $\nu > 0$ and the local Lipshitz constants of $\psi_0(x,t)$ in t go to zero as $|t| \to 0$, uniformly in x.

(ψ_4) $\psi(x,-t) = -\psi(x,t)$.

Remark 3.5. Assume that $\psi \in C^1(\Omega \times \mathbb{R}, \mathbb{R})$. Then

(a) condition (ψ_2) means that $\psi'_t(x,t) \to 0$ as $t \to \infty$ uniformly in x,

(b) condition (ψ_3) means that $\psi'_t(x,0) = \nu$.

Denote by E_n the subspace of L_2 spanned by all eigenfunctions of $-\Delta$ corresponding to eigenvalues λ with $\lambda < n$.

Let r be the number of the eigenvalues of $-\Delta$ in the interval $(\lambda_k, \lambda_k + \nu)$ (taking into account their multiplicities). Using Corollary 2.8 we obtain the following

Theorem 3.6. *Under the assumptions* (ψ_1)–(ψ_4) *assume that* $p = r - 1 > 0$. *Then problem* (3.9) *has at least* p *nontrivial distinct pairs of classical solutions of class* $C^{2,\alpha}$ *with any* $\alpha < 1$. *Moreover, if* (3.9) *has a descrete* (*in the* L_2-*metric*) *set of solutions then there exist among them* p *distinct pairs of symmetric solutions, say,* $\pm u_1, \ldots, \pm u_p$ *such that the functional* f *defined by* (3.8) *has well-defined shifted homology groups at each* u_i (*associated to the sequence* $\{E_n\}$ *introduced above*) *with* $\dim H_i^s(u_i, f) \neq 0$, $i = 1, \ldots, p$ (*cf. Definition* 2.6 *and Theorem* 2.7).

Remarks 3.7. (i) Observe that the nontriviality of homology groups, which is stated in Theorems 3.3 and 3.6 means, in particular, the stability of the corresponding solutions with respect to perturbations (of the relevant data) that lead to sufficiently C^1-small perturbations of the functional.

(ii) Under the assumptions of Theorem 2.7 (see condition h_2)) the operator $\nabla\phi$ is required to satisfy the Lipschitz condition outside a certain ball and in a neighbourhood of zero only. This gives rise to the following question: whether the smoothness assumptions on ψ in t in Theorem 3.6 can be weakened? The authors do not know the answer to this question in the case of the Dirichlet problem (3.9). However, for the problem on periodic solutions of Hamiltonian systems as well as for the Neumann problem of the type $-\Delta u = \psi(x,u)$ with the zero boundary conditions and sublinear ψ, the answer is positive. Observe that Theorem 3.3 (where the Hamiltonian is supposed to be of class C^1) is, in fact, a result of this sort. Moreover, the study of periodic solutions to problem (3.1) in the spirit of Theorem 2.7 can provide the following result.

Suppose that H from (3.1) satisfies conditions (H_1), (H_3), (H_4), and the following one: $H(z,t) = (Az,z)/2 + h(z,t)$, where A is non-degenerate, and there exists $r > 0$ such that $h_z(z,t)$ satisfies the Lipschitz condition in z for $|z| > r$. Moreover, the local Lipschitz constants of h_z in z go to zero as $|z| \to \infty$ uniformly in t.

Then (3.1) has at least $p = \theta(B, A) - 1$ pairs of distinct limit τ-periodic solutions. Moreover, in the case of isolated limit solutions the relevant homological assertion is valid (cf. Theorem 3.6).

It is worth noting that the described problem is considered in the L_2-space because the study in the (energy) $H^{1/2}$-space meets serious difficulties.

Concluding this section let us consider some of known results related to Theorem 3.6. Observe, first, that all these results were obtained with making appeals to Ljusternik–Schnirelman (minimax) theory.

A satisfactory study of problem (3.6) in the non-resonant case has been carried out by various authors (see, for instance, [4], [11], [18], [19], [20], [31], [40], [49] and references therein).

Landesman and Lazer [33] were the first to consider resonant problems. They studied the existence of solutions of problem (3.9) in the non-symmetric case under the assumption that functional (3.8) is coercive on the kernel. Their result has been extended by a number of authors (see [1], [3], [20], [41], and references therein). In particular, P. Rabinowitz (see [41]) studied the problem under the Landesman–Lazer condition in the presence of (ψ_4).

The further development of the theory was mostly due to Chang–Liu [20], Bartsch–Liu [13], Mawhin [35], Solimini [48], Mizogushi [38] (non-symmetric case) and Bartolo–Benci–Fortunato [11], Thews [50] (symmetric case). In [11] and [50] it is assumed that ψ is independent of x and the real function $\int_{-\infty}^{t} \psi(s)\, ds$ is well defined. Also certain specific "integral" conditions with respect to ψ are required in [20], [35], [48]. The conditions of another nature can be found in [13] and [38]).

4. Additions over complexes and graphs

This section contains some preliminaries for the proof of Theorem 2.4.

4.1. Additions. We start with the definition of additions over complexes. Let (C, ∂) be a complex over $\mathbb{Z}_2$, where ∂_i is generated in the distinguished bases by the matrices $\{\sigma_{ij}^k\}$ (see Section 1). Fix a vector space C_k and choose two elements x_p and x_q from the distinguished basis in C_k.

DEFINITION 4.1. By an *addition* associated to the ordered pair (x_p, x_q) we mean the change of the coefficients σ_{ip}^k, σ_{qj}^{k+1} according to the following rules:

$$\begin{aligned} \overline{\sigma}_{ip}^k &= \sigma_{ip}^k + \sigma_{iq}^k \pmod 2, \\ \overline{\sigma}_{qj}^{k+1} &= \sigma_{pj}^{k+1} + \sigma_{qj}^{k+1} \pmod 2, \end{aligned}$$

while all the other coefficients remain unchanged (here the symbol "$\overline{\sigma}$" stands for the transformed coefficients).

Remark 4.2. Clearly, an addition corresponds to the change of the basis in the vector space C_k generated by the substitution $x_p \to (x_p + x_q)$. Observe also that in the case of Morse complexes defined over $\mathbb{Z}$ the above formulae take the following form: $\overline{\sigma}^k_{ip} = \sigma^k_{ip} \pm \sigma^k_{iq}$ and $\overline{\sigma}^{k+1}_{qj} = \sigma^{k+1}_{pj} \mp \sigma^{k+1}_{qj}$.

The following fact is trivial from the viewpoint of Remark 4.2.

Lemma 4.3. *Any addition preserves the chain property of a complex and leaves its homology groups unchanged.*

4.2. Graphs. We are now in a position to introduce a graph associated to a regular pair (see Definitions 1.1 and 1.2). This construction will be essentially used in the proof of Theorem 2.4.

Definition 4.4. Let (Φ, L) be a regular pair with Φ of the MS-type (see Definition 1.4). Symbols $\gamma(\Phi, L) = \gamma(\Phi) = \gamma(L) = \gamma$ stand for a *graded* graph defined by the following rules:

(a) The vertices of the graph $\gamma(\Phi)$ are in one-to-one correspondence with the critical points of L. Moreover, we keep the same symbol x^k (where k denotes the index of the vertex) for the vertex corresponding to a critical point x^k.
(b) Two vertices x^k and y^m of the graph $\gamma(\Phi)$ with $m \leq k$ are connected by a (single) edge if and only if
   (i) $m = k - 1$,
   (ii) $\widetilde{\sigma}(x^k, y^m) = 1 \pmod 2$, that is to say, there is an odd number of connected trajectories of Φ joining x^k to y^m.

We refer the reader to Figure 1 below for an example of such graph.

Definition 4.5. Let $\gamma(\Phi)$ be as in Definition 4.4. We denote by $\sigma_\gamma(x^k, y^m)$ the number of edges connecting the vertex x^k to the vertex y^m; in other words, $\sigma_\gamma(x^k, y^m) = 1$ if x^k and y^m are connected by an edge, and $\sigma_\gamma(x, y) = 0$, otherwise. We will drop the index γ in this notation if it does not lead to confusion.

Straightforward calculations using the chain property of the Morse complex yield the following useful

Proposition 4.6 (chain relation). *Let $\gamma(\Phi)$ be as in Definition 4.4, and let u^n, v^{n-2} be a pair of vertices of $\gamma(\Phi)$. Let $y_1, \ldots, y_m$ be all the vertices of $\gamma(\Phi)$ of index $n - 1$. Then*

$$\sum_{i=1}^{m} \sigma_\gamma(u, y_i)\sigma_\gamma(y_i, v) \equiv 0 \pmod 2.$$

REMARKS 4.7. (i) Observe that passing from the Morse complex (over $\mathbb{Z}_2$) to the corresponding graph is nothing else but another (and more convenient) way to keep the information about the complex (over Z_2). On the other hand, given a graph one can completely reproduce the corresponding Morse complex (over $\mathbb{Z}_2$). Because of this we will speak frequently about the chain property, homology groups and additions over graphs.

(ii) We will often abuse precise terminology in the following two cases:

- instead of "addition associated to an ordered pair of vertices x_p^k, x_q^k" we will use "addition x_p^k to x_q^k",
- when the index k of the vertex x^k is understood from the context or is not important we will drop the index and write x.

(iii) Additions are "antisymmetric" in the following sense: an addition x^k to y^k transforms edges connecting x to vertices of index $(k-1)$; at the same time it does not change the edges connecting x to the vertices of index $(k+1)$. On the contrary, this addition transforms the edges connecting y^k to vertices of index $(k+1)$ while it does not change the edges connecting y to vertices of index $(k-1)$.

DEFINITION 4.8. A vertex x of a graph is called *isolated* if no edge is incident with x.

A pair of vertices x^k and y^{k+1} of a graph is called *complementary* if there is an edge connecting x to y and this edge is the only one incident with x and the only one incident with y.

4.3. Graphs associated to even functionals. Consider graphs associated to a regular pair (Φ, L), where L is *even* and Φ is odd and of the MS-type. Since L is even, zero is a critical point. Denote by θ the vertex of $\gamma(L)$ corresponding to zero and call this vertex *central*.

Observe, if x is a non-zero critical point of L then so is $-x$. Since Φ is odd, connecting trajectories are also symmetric with respect to zero. This induces the natural involution on $\gamma(L)$. If x is a non-central vertex of $\gamma(L)$ denote by $-x$ its symmetric counterpart.

Let x_p^k, x_q^k be a pair of vertices of $\gamma(L)$. Keeping in mind the above involution on $\gamma(L)$ we will speak about the *pairs* of (symmetric) additions meaning by that the additions of x_p^k to x_q^k and $-x_p^k$ to $-x_q^k$.

Finally, according to Definition 2.2, one can speak about the shifted homology groups of $\gamma(L)$.

4.4. Projective graphs. Let γ be a graded graph with the involution introduced in the preceding subsection. Then the general "projectivization funktor" applied to γ gives rise to the following

Definition 4.9. By the *projective graph* $\gamma_{\rm pr}$ we mean the graph obtained from γ by the subsequent application of the following two operations:

(1) the identification of symmetric vertices and symmetric edges;
(2) reducing the number of edges connecting a pair of vertices in the resulting graph modulo 2.

Observe that the first operation transforms γ into the graph with vertices which can be connected by either no edge or one edge or two edges only. Observe also that due to the second operation the vertex of the projective graph that comes from the central one is always isolated (cf. Definition 4.8).

Furthermore, it is easy to see that every pair of symmetric additions on γ induces the (unique) addition on $\gamma_{\rm pr}$ and vice versa.

It follows immediately from Definition 4.9 and the chain relation (Proposition 4.6).

Proposition 4.10. *Suppose γ is a chain graph. Then so is $\gamma_{\rm pr}$.*

Definition 4.11 (cf. Definition 4.8).

(i) A vertex x of the graph γ is called *p-isolated* (projectively isolated) if its projective image $x_{\rm pr}$ is isolated in $\gamma_{\rm pr}$.
(ii) A pair of vertices x, y are called *p-complementary* if $\sigma(x,y) = 1$ and their projective images $x_{\rm pr}$, $y_{\rm pr}$ form the complementary pair in $\gamma_{\rm pr}$.

Observe that x is p-isolated if and only if so is $-x$. Similarly, a pair x, y is p-complementary if and only if so is the pair $-x$, $-y$.

Definitions 4.9 and 4.11 readily imply the following

Proposition 4.12.

(i) *A vertex x of the graph γ is p-isolated if and only if for any $y \neq \pm x$ one has $\sigma(x,y) = \sigma(x,-y)$.*
(ii) *A pair of vertices x, y of γ is p-complementary if and only if $\sigma(x,y) = 1$, $\sigma(x,-y) = 0$ and for any $z \neq \pm x, \pm y$ one has $\sigma(x,z) = \sigma(x,-z)$ and $\sigma(y,z) = \sigma(y,-z)$.*

5. Proof of Theorem 2.4: reduction to the Combinatorial Lemma

The proof of Theorem 2.4 consists of two parts: analytical and combinatorial. This section is devoted to the analytical part. The core of the section is Lemma 5.1, where we deal with connecting trajectories of *pg*-fields for f_n. Lemma 5.1 allows us to reduce the proof of the theorem to the study of special partitions on the set of vertices of graphs associated to a quadratic-like functional.

5.1. Connecting trajectories. Without loss of generality we will assume that every point of K_∞^p is isolated in K_∞ (cf. the Introduction) and $f_n(0) = 0$ for any n.

According to (f_3) the limit functional f_∞ is well-defined on K_∞ by $f_\infty(\lim u_k) = \lim f_{n_k}(u_k)$, $u_k \in K_{n_k}$. In particular, $f_\infty(0) = 0$. Denote the elements of K_∞^p by $x_1, \ldots, x_{2m}$ so that $x_i = -x_{m+i}$ and $f_\infty(x_1) \geq \ldots \geq f_\infty(x_m)$. In what follows we will prove, in particular, that $m \geq p$.

LEMMA 5.1. *Under the assumptions of Theorem* 2.4 *there exist a natural number N large enough and a disjoint system of neighbourhoods $U_1, \ldots, U_{2m}$ of the points $x_1, \ldots, x_{2m}$ respectively, such that $U_{m+i} = -U_i$, $i = 1, \ldots, m$, and for any $n > N$ and any pg-field Φ_n for f_n the following conditions are fulfilled.*

(i) *Suppose that for some different $x_i, x_j \in K_\infty^p$ we have $f_\infty(x_i) \geq f_\infty(x_j)$. Let $z \in U_i$ and $y \in U_j$ be critical points of f_n such that $f_n(z) \leq f_n(y)$. Then $\widetilde{\sigma}(z, y) = 0$, that is to say, no Φ_n-trajectory connects z to y (cf. Section* 1).

(ii) *Let z be a critical point of f_n outside $\bigcup U_i$. Let $f_n(z) \leq f_\infty(x_j)$ (respectively, $f_n(z) \geq f_\infty(x_j)$) for a certain $j = 1, \ldots, 2m$. Suppose that $y \in U_j$ is a critical point of f_n such that $f_n(z) \geq f_n(y)$ (respectively, $f_n(z) \leq f_n(y)$). Then $\widetilde{\sigma}(z, y) = 0$.*

(iii) *If x and y are critical points of f_n belonging to the same U_i, then any Φ_n-trajectory connecting x to y is contained in U_i.*

PROOF. Using conditions (f_2) and (f_4) choose a system of neighbourhoods $U_i^1 \subset U_i^0$ of x_i, $i = 1, \ldots, 2m$, with $U_{m+i}^j = -U_i^j$, $i = 1, \ldots, m$, $j = 0, 1$, such that:

(a) $\{U_i^0\}$ is a disjoint system,

(b) there exists $\delta > 0$ such that for any $x \in \bigcup_{i=1}^{2m} (U_i^0 \setminus U_i^1)$ and n large enough, we have:

$$\|\nabla f_n(x)\| \geq \delta. \tag{5.1}$$

To prove statement (i) we argue indirectly and assume that there exists a sequence of trajectories T_k of Φ_{n_k}, $n_k \to \infty$, joining critical points of f_{n_k}, say, z_k to y_k with $z_k \to x_i$ and $y_k \to x_j$. Then by (f_3),

$$\lim f_{n_k}(z_k) = f_\infty(x_i) \geq f_\infty(x_j) = \lim f_{n_k}(y_k). \tag{5.2}$$

From $y_k \notin U_i^0$ it follows that $T_k \cap (H \setminus U_i^0) \neq \emptyset$ and $T_k \cap U_i^1 \neq \emptyset$, therefore, the lengths of curves $T_k \cap (U_i^0 \setminus U_i^1)$ are bounded away from zero. Now, taking into account Definition 1.1 and (5.2) we obtain a contradiction with (5.1). Setting $U_i = U_i^0$ we get the first statement.

To show (ii) and (iii) one can use the similar argument. Lemma 5.1 is proved.

Remark 5.2. Observe that, in fact, statement (iii) means that for any $i = 1, \dots , 2m$ and any n large enough the pair $(\Phi_n, f_n)|U_i^0$ is regular. In this sense statement (iii) is similar to Proposition 1.9.

5.2. Admissible partitions. Let (Φ, L) be a regular pair with even L and odd Φ of the MS-type. Consider the graph γ associated to this pair (see Definition 4.4). The following notion plays a substantial role in the proof of Theorem 2.4.

Let T be a set. Recall that a covering $\{Q_i\}$ of T is called a partition of T if this collection of subsets is disjoint. As a matter of convenience, we admit that some of Q_i may be *empty*.

Definition 5.3. A partition of the vertices of γ into sets $Q_0, \dots , Q_{2s}$ is said to be *admissible* if:

(a) $Q_0 = \{\theta\}$ and the partition is symmetric meaning that $\pm Q_{s+i} = \mp Q_i$, $i = 1, \dots , s$,

(b) the partition $\{Q_0, \dots , Q_{2s}\}$ is (partially) ordered in such a way that the following conditions are fulfilled:

- (i) if vertices $x^k \in Q_i$ and $y^{k-1} \in Q_j$ are connected in γ by an edge then either $i = j$ or $Q_i > Q_j$,
- (ii) the order is symmetric in the following sense: if $Q_i > Q_j$ then $-Q_i > -Q_j$ and vice versa,
- (iii) the partition is "projectively linear" in the following sense: if $Q_j \neq \pm Q_i$, where $i > 0$, then either $Q_i > Q_j$ or $Q_j > Q_i$.

Observe that due to Definition 5.3(a), we may write $\{Q_0, Q_1, \dots , Q_{2s}\} = \{Q_0, \dots , \pm Q_s\}$. Let us state some immediate consequences of Definition 5.3.

Proposition 5.4. *If $\{Q_0, \pm Q_1, \dots , \pm Q_s\}$ is an admissible partition then*

(i) *The projective image of the* (*admissible*) *partition* (*cf. Definition* 4.9) *is a partition on the set of vertices of the projective graph.*

(ii) *The given* (*partial*) *order induces the well-defined linear order on the elements of the "projective partition" provided by* (i).

(iii) *Any two symmetric vertices belong to different* (*symmetric*) *elements of the partition.*

(iv) *If $Q_i > Q_j$ then no edge connects a vertex $u^m \in Q_i$ with a vertex $v^{m+1} \in Q_j$.*

(v) *The relations of type "$Q_i > -Q_i$" as well as "$-Q_i > Q_i$" are prohibited.*

Proof. (i) follows from Definition 5.3(a). (ii) follows from point (iii) of Definition 5.3(b). (iii) follows from Definition 5.3(a). (iv) follows from point (i) of Definition 5.3(b). (v) follows from point (ii) of Definition 5.3(b). $\square$

Consider an admissible partition $\{Q_0, \cdots, Q_{2s}\}$ of γ and fix a partial order provided by Definition 5.3(b). For any Q_i denote by γ_i the natural restriction of γ to the vertices of Q_i.

PROPOSITION 5.5. *γ_i is a chain graph.*

PROOF. Let A^k and B^{k-2} be two vertices of γ_i. Note that if for some point x^{k-1} of γ, $\sigma_\gamma(A,x) = \sigma_\gamma(x,B) = 1$, then $x \in Q_i$ (see Definition 5.3(b)). To complete the proof it suffices to use the fact that γ is a chain graph and to apply the chain relation (Proposition 4.6). $\square$

In the sequel we also need the following simple

PROPOSITION 5.6. *Let $x^k \in Q_i$, $y^{k-1} \in Q_j$ be a pair of vertices of γ for which*

$$\sigma_\gamma(\pm x, y) = 1. \tag{5.3}$$

Then $\pm Q_i > Q_j$ (in particular, x and y belong to different elements of the partition).

PROOF. If x and y do not belong to the same element of the partition then Definition 5.3 and (5.3) yield $\pm Q_i > Q_j$. Thus, it suffices to show that x and y do not belong to the same element.

Suppose that x and y do belong to the same element Q_i. Then, by Definition 5.3(b) and (5.3), we have: $-x, -y \in -Q_i$, $Q_i > -Q_i$ and $-Q_i > Q_i$. The contradiction yields the desired statement. $\square$

5.3. Special partitions associated to (Φ_n, f_n). Return to the assumptions (f_1)–(f_4) of Theorem 2.4 (see Introduction). Let us fix the number N and the system of neighbourhoods $U_1, \dots, U_{2m}$ provided by Lemma 5.1. Take $n > N$. Choose for f_n an odd *pg*-field Φ_n of the MS-type and consider the graph γ associated to (Φ_n, f_n) (by condition (f_1), the pair (Φ_n, f_n) is regular). Below we describe a partition on γ, which is essentially used in what follows.

Define the *special* partition $\{Q_0, \dots, Q_{2s}\} = \{Q_0, \pm Q_1, \dots \pm Q_s\}$ of the vertices of γ by the following conditions.

(i) For every $i = 1, \dots, m$ the set of all vertices corresponding to critical points of f_n from U_i (respectively, from $-U_i$) constitute a separate element Q_i (respectively, $-Q_i = Q_{s+i}$) of the partition.

(ii) Each vertex x of γ (including the central vertex θ) that corresponds to a critical point outside $\bigcup U_i$ constitutes itself a separate element $\{x\}$ of the partition; in the case $x \neq \theta$ we suppose that $\{x\} = Q_j$ and $\{-x\} = Q_{s+j}$ or, conversely ($j = m+1, \dots, s$).

Thus, we obtain the partition $\{Q_0, \pm Q_1, \dots, \pm Q_s\}$ with $Q_0 = \{\theta\}$ and $s \geq m$. To introduce the corresponding partial order on this partition we need an auxiliary function χ defined on the elements of the partition by the following rule: for every $i = 1, \dots, m$ set $\chi(\pm Q_i) = f_\infty(\pm x_i)$; for every one-vertex element $\{x\}$ of the partition defined in the point (ii) above, set $\chi(\{x\}) = f_n(x)$. In particular, $\chi(Q_0) = 0$.

Now we define the order on the partition as follows:

- (iii) For $i, j = 0, \dots, m$ we set $\pm Q_i > \pm Q_j$, if and only if either $\chi(Q_i) > \chi(Q_j)$ or $\chi(Q_i) = \chi(Q_j)$ and $i < j$.

It follows immediately from Lemma 5.1

Proposition 5.7. *The special partition introduced above is admissible provided that n is large enough.*

5.4. Reduction to the Combinatorial Lemma. The following statement is an important ingredient of the proof of Theorem 2.4.

Lemma 5.8 (Combinatorial Lemma). *Let f be a quadratic-like functional defined on $\mathbb{R}^\nu$ and Φ be an odd pg-field of the* MS*-type for f. Suppose* $\operatorname{indr}(\infty, f) = r$. *Let $Y = \{Q_0, \pm Q_1, \dots, \pm Q_s\}$ be an admissible partition* (*equipped with an order satisfying Definition* 5.3(b)) *of the vertices of the graph γ associated to* (Φ, f). *Then there exist* (*non-empty*) *elements $\pm Q_{l_1}, \dots, \pm Q_{l_r}$ in Y such that*

- (i) $Q_0 > Q_{l_1} > \dots > Q_{l_r}$,
- (ii) *if γ_i is a subgraph associated with $\{Q_{l_i}\}$, then the shifted homology group $H_i^s(\gamma_i)$ is non-trivial for every $i = 1, \dots, r$ (cf. Proposition* 5.5).

The proof of the Combinatorial Lemma will be given in Sections 6 and 7. Here we deduce Theorem 2.4 from Lemma 5.8.

Proof. Lemma 5.8 $\Rightarrow$ Theorem 2.4. Suppose we are in the setting of Subsection 5.3. Assume that throughout the proof n is large enough. Consider the special partition associated to (Φ_n, f_n). By Proposition 5.7, this partition is admissible.

By (f_1), we have $\operatorname{indr}(f_n, \infty) \geq p$. Observe that by the Combinatorial Lemma 5.8(ii) the homology group $H_i^s(\gamma_i)$ is nontrivial for every $i = 1, \dots, p$. Therefore, for each $i = 1, \dots, p$ there exists at least one critical point x of f_n corresponding to a vertex from Q_{l_i} such that $\operatorname{indr}(x, f_n) = i$. Furthermore, by conditions (f_2) and (f_4) and by construction, such a point x lies in a prescribed small neighbourhood of K^p_∞. Hence we may suppose that $x \in U_{m_i}$ for some m_i. Then, by construction (cf. the preceding subsection), Q_{l_i} coincides with the set of vertices corresponding to the critical points of f_n from U_{m_i}. Now the application of Lemma 5.8(i) provides the multiplicity result required while the application of Lemma 5.8(ii) gives us the corresponding homological information. Finally, since K^p_∞ is finite, by the above argument one can choose p pairs $\pm x_1, \dots, \pm x_p$ from K^p_∞ as is required by the theorem. $\square$

6. Proof of the Combinatorial Lemma: reduction to the Normal Form Graph Lemma (NFG Lemma)

6.1. Admissible additions and NFG Lemma. Throughout this subsection we denote by γ the graph associated to a regular pair (Φ, L) with even quadratic-like functional L and odd Φ of the MS-type. We also assume that γ is equipped with an admissible partition. Observe that the order on the partition induces the order on the set of vertices of γ. Namely, if $x \in Q_i$, $y \in Q_j$ and $Q_i > Q_j$, then we set $x > y$. In turn, the partition on the set of the vertices of γ induces the order on the set of vertices of the projective graph γ_{pr}.

Definition 6.1. Let $x^k, y^k \neq \theta$ be vertices of γ such that $x \neq \pm y$. A pair of symmetric additions of x to y and, respectively, $-x$ to $-y$ is said to be *admissible* if either x and y belong to the same element of the partition or $x > y$.

The projective image of a pair of admissible additions is also called an *admissible addition* (over the projective graph γ_{pr}).

Proposition 5.4(ii) (see also the beginning of the section) readily implies the following

Proposition 6.2. *Let x and y be vertices of the projective graph γ_{pr} of the same index. Then the addition of x to y is admissible if and only if $x \not< y$. Moreover, any admissible addition over the projective graph is the image of the only pair of symmetric admissible additions over the initial graph.*

Note that any addition transforms edges and does not change vertices. Combining this observation with Definitions 6.1 and 5.3 yields the following

Proposition 6.3. *Let γ_1 be a graph equipped with an admissible partition of its vertices. Let γ_2 be a graph obtained from γ_1 by means of a pair of symmetric admissible additions. Then the initial partition considered with respect to γ_2 is still admissible.*

The following lemma is crucial for the proof of the Combinatorial Lemma (and, respectively, Theorem 2.4).

Lemma 6.4 (Normal Form Graph Lemma). *Under the assumptions of the Combinatorial Lemma (Lemma 5.8) suppose that* $\operatorname{ind}\theta = n = k + r$. *Then there exists a finite sequence of pairs of symmetric admissible additions transforming the graph γ into a graph γ' that satisfies the following conditions:*

(i) *any vertex of γ' is either p-isolated or belongs to a p-complementary pair (see Definition 4.11),*

(ii) *γ' restricted to the set of all p-isolated vertices coincides with the graph pictured in Figure 1; in particular, for each $i = k, \ldots, k+r-1$ this graph* *contains the only pair of vertices* $\pm u^i$ *of index* i, *and* $\sigma_{\gamma'}(u^i, \pm u^{i+1}) = 1$, $i = k, \dots, k + r - 2$.

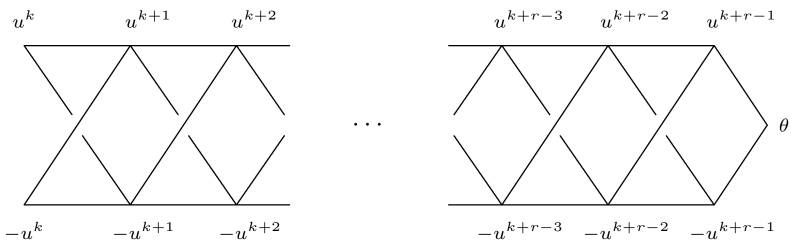

FIGURE 1

6.2. Reduction to the NFG Lemma. The proof of the NFG Lemma will be given in the next section. In this subsection we will deduce the Combinatorial Lemma from the NFG Lemma.

PROOF. NFG Lemma $\Rightarrow$ Combinatorial Lemma. Let us apply the NFG Lemma to the graph γ from the Combinatorial Lemma. Then the collection of admissible additions provided by the NFG Lemma makes the resulting graph γ' satisfy conditions (i) and (ii) from the NFG Lemma.

Observe first of all that by Proposition 5.6, for each $j = k, k+1, \dots, k+r-2$, p-isolated vertices u^j and u^{j+1} belong to different elements of the partition (cf. Figure 1 and Lemma 6.4(ii)). Denote by $Q_{l_{k+r-j}}$ the element of the partition containing u^j. Then, by Definition 5.3,

$$Q_0 > Q_{l_1} > \dots > Q_{l_r}.$$

Furthermore, by Proposition 4.12,

$$\sigma_{\gamma'}(u^j, z) = \sigma_{\gamma'}(u^j, -z)$$

for each p-isolated vertex u^j and any $z \neq \pm u^j$. Therefore, the application of Proposition 5.6 provides that $\sigma_{\gamma'}(u^j, z) = 0$ for every $z \in Q_{l_{k+r-j}}$. This fact implies that for the subgraph γ'_{k+r-j}, which denotes the restriction γ' to the set of the vertices from $Q_{l_{k+r-j}}$, the vertex u^j gives rise to a dimension in $H^s_{k+r-j}(\gamma'_{k+r-j})$. Hence $\dim H^s_i(\gamma'_i) \neq 0$ for every $i = 1, \dots, r$.

Since additions preserve homology groups, one has $\dim H^s_i(\gamma_i) \neq 0$, $i = 1, \dots, r$, for the corresponding subgraph of the initial graph γ. $\square$

7. Proof of the NFG Lemma

The proof of the NFG Lemma is going in two steps. In the first step (Lemma 7.1) we construct a sequence of admissible additions transforming the given graph into a graph that contains p-isolated vertices and vertices belonging to p-complementary pairs only (see Definition 4.11). At the second step (Lemma 7.2) we show that the graph containing only the above mentioned vertices satisfies the conclusion of the NGF Lemma.

Lemma 7.1. *Let γ satisfy the assumptions of the* NFG *Lemma. Then there exists a sequence of pairs of admissible additions transforming γ into a graph that contains p-isolated vertices and vertices belonging to p-complementary pairs only.*

Proof. We will construct a sequence of admissible additions transforming the corresponding projective graph γ_{pr} into a (projective) graph that is a disjoint sum of isolated vertices and (isolated) complementary pairs. Then, by Proposition 6.2, the corresponding "lifting" sequence provides the transformation required.

Denote by Q the set of vertices of the graph γ_{pr} that neither are isolated nor belong to (isolated) complementary pairs. Suppose m is the minimal index of vertices from Q. Let x_0^m be a *maximal* vertex in the set of all vertices of index m from Q. Let Y_{x_0} be the set of all vertices connected to x_0 by an edge (by construction, all of them are of index $m+1$), and let y_0 be a *minimal* vertex on Y_{x_0}. For every $y \in Y_{x_0} \setminus \{y_0\}$ we kill the edge connecting y to x_0 by means of the addition of y to y_0. Since y_0 is minimal, this addition is *admissible*. Similarly, by means of additions of x_0 to the vertices of index m that are connected to y_0 by an edge, we kill all the edges connecting y_0 to the vertices $x^m \neq x_0$. Since x_0 is maximal, all these additions are *admissible*. Now, by the chain relation (Proposition 4.6), the pair (x_0, y_0) is a new isolated complementary pair. The induction completes the proof. $\square$

Lemma 7.2. *Let γ be a graph satisfying the conditions of the* NFG *Lemma and such that each its vertex either is p-isolated or belongs to a p-complementary pair. Then the restriction of γ to all the p-isolated vertices is the subgraph* (*denoted by γ^**) *coinciding with the graph pictured in Figure* 1.

In turn, the proof of the preceding lemma is based on the following

Lemma 7.3. *Let γ satisfy all the assumptions of Lemma* 7.2. *Then the subgraph γ^* is a chain graph with the same homology groups as γ.*

Proof. We use Proposition 4.12 and a construction similar to that from the proof of Lemma 7.1. Namely, we construct a sequence of pairs of additions (not necessarily admissible) transforming γ into a graph that is a disjoint sum of γ^* and a number of (isolated) complementary pairs. Since an addition does not change homology groups, the conclusion of the lemma follows. Consider a couple of symmetric *non-isolated* p-complementary pairs (x^m, y^{m+1}) and $(-x^m, -y^{m+1})$. Observe that by the lemma assumptions and Proposition 4.12, for u, v different from $\pm x, \pm y$ we have

$$\sigma(v, x) = \sigma(v, -x), \quad \sigma(u, y) = \sigma(u, -y). \tag{7.1}$$

By means of *pairs of additions* of vertices $v^{m+1} \neq \pm y$ such that $\sigma(v, \pm x) = 1$ (see (7.1)) to y and to $-y$, we kill all edges connecting $\pm x$ to vertices $v^{m+1} \neq \pm y$. Consider now a couple of vertices u^m and v^{m+1} such that $u \neq \pm x$ and $v \neq \pm y$. According to the second relation from (7.1), the pairs of additions described above do not change $\sigma(u, v)$.

Similarly, we kill the edges connecting $\pm y$ to vertices $u^m \neq \pm x$ keeping all the other edges unchanged. Thus, taking into account the chain relation, we get two more isolated complementary pairs while the resulting graph still satisfies the assumptions of the lemma, and the subgraph γ^* remains the same. The induction completes the proof. □

Proof of Lemma 7.2. By Lemma 7.3, we can assume that the graph γ contains p-isolated vertices only (in other words, $\gamma = \gamma^*$).

Suppose that every symmetric pair is ordered: $\{\pm x\} = (x_1, x_2)$. Add in every such a pair the first vertex to the second one. (Observe that this transformation breaks the symmetry.) Denote the resulting graph by $\overline{\gamma}$. For every (symmetric) pair (x_1^m, x_2^m) and for any y^{m-1} we have (cf. Remark 4.7(iii))

$$\sigma_{\overline{\gamma}}(x_1, y) = 0. \tag{7.2}$$

To see this, we may suppose that we add vertices starting from the maximal possible index, and then continue to add following the direction of decreasing indices of the vertices. At the same time, x_2 and y can be connected by an edge.

Consider the complex corresponding to $\overline{\gamma}$. By the lemma assumptions and Corollary 1.10,

$$\dim H_j(\overline{\gamma}) = \begin{cases} 0 & \text{if } j \neq k, \\ 1 & \text{if } j = k. \end{cases} \tag{7.3}$$

It follows from (7.2) that for any symmetric pair (x_1, x_2) the vertex x_1 belongs to the kernel of the corresponding boundary homomorphism. Therefore, by (7.3) $\overline{\gamma}$ has no pairs of symmetric vertices $\pm x^m \neq \theta$ with indices $m \geq n = k + r$, where n is the index of the central vertex. In particular, the central vertex θ is the only vertex of index n.

If now $n > k$ (the case $n = k$ is trivial) we have $\dim(\operatorname{Ker} \partial_{n-1}) = \dim(\operatorname{Im} \partial_n) = 1$. Hence it is the only symmetric pair of vertices of index $n - 1$. Similarly, in the case $n > k + 1$ we have $\dim(\operatorname{Ker} \partial_{n-2}) = \dim(\operatorname{Im} \partial_{n-1}) = 1$ and so on. Thus for every m with $k \leq m < n$ we obtain the only symmetric pair of vertices of index m. Finally, by the symmetry argument, the initial graph γ (recall that $\gamma = \gamma^*$) coincides with the graph pictured in Figure 1. □

8. Proof of Theorem 2.7: smooth case

In this section we will deduce Theorem 2.7 from Theorem 2.4 under the additional assumption $\phi \in C^2(H,\mathbb{R})$. The crucial point of the proof is the verification of condition (f_2) (see Subsection 8(b)). To verify (f_2) we estimate indices of critical points of functionals f_n that travel to infinity or zero as n tends to infinity.

8.1. Verification of conditions (f_1)**,** (f_3) **and** (f_4)**.** Set $\phi_n = \phi|E_n$. Define an approximating sequence $\{f_n\}$ by setting

$$\forall x \in E_n \quad f_n(x) = \frac{1}{2}(A_n x, x) + \frac{||x||^2}{(2n)} + \phi_n(x), \tag{8.1}$$

and verify that sequence (8.1) satisfies conditions (f_1)–(f_4).

It is easy to see that for n large enough the second term in (8.1) provides the second derivative of f_n to be non-degenerate at zero and infinity. Combining this with Corollary 1.10 yields that f_n is a quadratic-like functional for n large enough. Another simple observation is that for n large enough formula (8.1) together with condition (h_4) yield the inequality $\operatorname{indr}(\infty, f_n) \geq p+1$. It remains to note that using a sufficiently C^2-small even perturbation outside a small neighbourhood of zero one can provide f_n to have non-degenerate critical points only. This completes the verification of condition (f_1).

To check conditions (f_2)–(f_4) we need the following standard

Lemma 8.1.

(i) *Suppose that A satisfies* (h_1) *and for a sequence $\{x_n\} \subset \operatorname{dom} A$ the sequence $\{Ax_n\}$ is bounded in H. Then $\{x_n\}$ is precompact in H.*

(ii) *If, in addition, $\{x_n\}$ has a limit x_0, then $x_0 \in \operatorname{dom} A$ and $(Ax_0, x_0) = \lim_n (Ax_n, x_n)$.*

Proof. Let $\lambda_1, \dots, \lambda_k, \dots$ be eigenvalues of A. Denote by $\widetilde{E}_k$ the eigenspace corresponding to λ_k, and let $\widetilde{P}_k : H \to \widetilde{E}_k$ be the orthogonal projection. Recall that by condition (h_1), we have $\lambda_k^2 \to \infty$ and $\widetilde{E}_k$ is finite-dimensional.

By assumptions, a sequence

$$\|Ax_n\|^2 = \sum_{k=1}^{\infty} \lambda_k^2 ||\widetilde{P}_k x_n||^2 \tag{8.2}$$

is bounded by a constant M. Since $\lambda_k^2 \to \infty$, it follows from the boundedness of (8.2) that the series

$$\sum_{k=1}^{\infty} ||\widetilde{P}_k x_n||^2, \tag{8.3}$$

and

$$\sum_{k=1}^{\infty} \lambda_k ||\widetilde{P}_k x_n||^2 \tag{8.4}$$

converge uniformly in n. In particular, the precompactness of the sequence $\{x_n\}$ follows from the uniform convergence of series (8.3). Statement (i) is proved.

Statement (ii) easily follows from the uniform convergence of series (8.4). $\square$

Recall that by critical points of h we mean solutions of the equation $Ax + \nabla\phi(x) = 0$.

Lemma 8.2. *Suppose that A satisfies* (h_1) *and the gradient field $\nabla\phi$ is bounded on any ball. Let $\{x_k\} \subset H$ be a bounded sequence such that $x_k \in E_{n_k}$ and $\nabla f_{n_k}(x_k) \to 0$. Then:*

(i) *$\{x_k\}$ is precompact in H,*
(ii) *every limit point of $\{x_k\}$ is a critical point of h,*
(iii) *if $\{x_n\}$ is convergent, then so is $f_{n_k}(x_k)$ and $\lim f_{n_k}(x_k) = h(\lim x_k)$.*

Proof. (i) It follows from (8.1) that for every k

$$Ax_k + x_k/n_k + \nabla\phi_{n_k}(x_k) \to 0. \tag{8.5}$$

Since $\{x_k\}$ is bounded, the set $\{\nabla\phi_{n_k}(x_k)\}$ is bounded as well. Now, from (8.5) it follows that the set $\{Ax_k\}$ is bounded. Then the precompactness of $\{x_k\}$ follows from Lemma 8.1(i).

(ii) Let x_0 be a limit point for $\{x_k\}$. For every $v \in \bigcup_n E_n$

$$(Ax_0 + \nabla\phi(x_0), v) = 0. \tag{8.6}$$

Since $\mathrm{cl}(\bigcup_n E_n) = H$, (8.6) implies (ii). Finally, (iii) follows from Lemma 8.1(ii) and the continuity of ϕ. Lemma 8.2 is proved. $\square$

Return to the verification of the conditions of Theorem 2.4. Since ϕ is continuous on H, condition (f_3) follows from Lemma 8.2(iii). Condition (f_4) is a direct consequence of Lemma 8.2(i).

Thus, we proved in the smooth case that all assumptions of Theorem 2.4 hold up to verification of (f_2).

8.2. Verification of condition (f_2)**.** To verify condition (f_2) we have to check that the set $\bigcup_n K_n^p$ of critical points x with $0 < \mathrm{indr}(x, f_n) \le p$ does not contain sequences going to infinity or zero as $n \to \infty$.

We need two propositions following below.

PROPOSITION 8.3.

(i) *Let* $\{x_k \in K_{n_k}\}$ *be a sequence of critical points such that* $\|x_k\| \to \infty$. *Then* $\rho(x_k, \ker A)/\|x_k\| \to 0$ *and for any finite-dimensional subspace* $E \subset H$ *one has* $\|\phi''_{n_k}(x_k)|E \cap E_{n_k}\| \to 0$.

(ii) *Let* $\{x_k \in K_{n_k}\}$ *be a sequence of critical points such that* $\|x_k\| \to 0$. *Then* $\rho(x_k, \ker B)/\|x_k\| \to 0$ *and for any finite-dimensional subspace* $E \subset H$ *one has* $\|\phi''_{0n_k}(x_k)|E \cap E_{n_k}\| \to 0$.

PROOF. Set $\overline{x}_k = x_k/\|x_k\|$. Since x_k is a critical point of f_{n_k}, we have

$$A\overline{x}_k + \overline{x}_k/n_k + \nabla\phi_{n_k}(x_k)/\|x_k\| = 0. \tag{8.7}$$

By condition (h_2), $\nabla\phi_{n_k}(x_k)/\|x_k\| \to 0$, hence (see (8.7)) $A\overline{x}_k \to 0$ as well. This implies $\rho(\overline{x}_k, \ker A) = \rho(x_k, \ker A)/\|x_k\| \to 0$. The application of ($h_2$) yields $\|\phi''_{n_k}(x_k)|E \cap E_{n_k}\| \to 0$. Statement (i) is proved.

The proof of (ii) utilizes the same argument with (h_3) instead of (h_2). $\square$

To formulate the next proposition we need certain preliminaries. For any subset $S \subset \mathbb{R}$ denote by $\widetilde{E}_S$ the sum of eigenspaces corresponding to A-eigenvalues belonging to S. Observe that for any $S \subset \mathbb{R}$ the subspace $E_n \cap \widetilde{E}_S$ is A- as well as A_n-invariant. Set

$$X_n = E_n \cap \widetilde{E}_{(-\infty,0]}, \quad Y_n = E_n \cap \widetilde{E}_{(0,\infty]}.$$

Clearly, $A_n|Y_n$ is strictly positive. Observe also that by definition, $\overline{N}(A_n) = \dim X_n$.

PROPOSITION 8.4. *Let* $\{x_k\}$ *be the same as in Proposition* 8.3(i) *or* (ii). *Then for any* k *large enough and any* $v \in Y_{n_k}$ *one has*

$$(f''_{n_k}(x_k)v, v) > 0.$$

PROOF. Let $\widehat{\lambda}_1$ be the smallest positive eigenvalue of A. Set

$$M = 2L(2L+1)^2/\widehat{\lambda}_1^2, \tag{8.8}$$

where L is the Lipschitz constant from condition (h_2). Then for any n and every $v \in Y_n$

$$(A_n v, v) = (Av, v) \geq \widehat{\lambda}_1 \cdot \|v\|^2. \tag{8.9}$$

For the sake of definiteness assume that $x_k \to \infty$ (one can treat the case $x_k \to 0$ using the same argument). The application of Proposition 8.3(i) with $E = \widetilde{E}_{(0,M]}$ yields $\|\phi''_{n_k}(x_k)|\widetilde{E}_{(0,M]} \cap E_{n_k}\| \to 0$, hence, for all k large enough, one has

$$\|\phi''_{n_k}(x_k)|\widetilde{E}_{(0,M]} \cap E_{n_k}\| < \widehat{\lambda}_1/2. \tag{8.10}$$

Fix k such that (8.10) is satisfied, and set

$$Y^1 = Y_{n_k} \cap \widetilde{E}_{(0,M]}, \quad Y^2 = Y_{n_k} \cap \widetilde{E}_{(M,\infty)}.$$

Take arbitrary $v \in Y_{n_k}$. Let $v = v_1 + v_2$ be the canonical decomposition with $v_1 \in Y^1$ and $v_2 \in Y^2$. Then, in particular,

$$(A_{n_k} v, v) = (A_{n_k} v_1, v_1) + (A_{n_k} v_2, v_2) \geq (A_{n_k} v_2, v_2).$$

Set,

$$\begin{aligned} G_1 &= \{v \in Y_{n_k} \mid \|v_2\| < \widehat{\lambda}_1 \|v\|/(2L+1)\}, \\ G_2 &= \{v \in Y_{n_k} \mid \|v_2\| > (L/M)^{1/2} \|v\|\}. \end{aligned}$$

By the choice of M (see (8.8)), one has: $G_1 \cup G_2 = Y_{n_k}$. Our goal is to show the inequality

$$(A_{n_k} v, v) > |(\phi''_{n_k}(x_k) v, v)|,$$

which easily implies the statement. To this end consider two cases.

Case 1. $v \in G_1$. By (8.10),

$$|(\phi''_{n_k}(x_k) v_1, v)| < (\widehat{\lambda}_1/2) \cdot \|v\|^2. \tag{8.11}$$

By definition of G_1 and condition (h_2),

$$|(\phi''_{n_k}(x_k) v_2, v) < L \cdot \widehat{\lambda}_1/(2L+1) \cdot \|v\|^2 < (\widehat{\lambda}_1/2) \cdot \|v\|^2. \tag{8.12}$$

Summing (8.11) and (8.12) and using (8.9) we get:

$$|(\phi''_{n_k}(x_k) v, v)| < \widehat{\lambda}_1 \|v\|^2 \leq (A_{n_k} v, v),$$

as required.

Case 2. $v \in G_2$. By condition (h_2) and definition of G_2, Y^2 and Y_{n_k},

$$(A_{n_k} v, v) \geq (A_{n_k} v_2, v_2) > M \|v_2\|^2 > L \|v\|^2 \geq |(\phi''_{n_k} v, v)|,$$

as required. $\square$

We are now in a position to verify condition (f_2). Denote by X^- the negative subspace in the spectral decomposition for $f''_{n_k}(x_k)$. By Proposition 8.4, $X^- \cap Y_{n_k} = \{0\}$. Hence $\operatorname{ind}(x_k, f_{n_k}) \leq \dim X_{n_k} = \overline{N}(A_{n_k})$. On the other hand, by virtue of (8.1), $\operatorname{ind}(0, f_{n_k}) \geq N(B_{n_k})$ provided k is large enough. Hence, for such k,

$$\operatorname{indr}(x_k, f_{n_k}) = \operatorname{ind}(0, f_{n_k}) - \operatorname{ind}(x_k, f_{n_k}) \geq N(B_{n_k}) - \overline{N}(A_{n_k}) = r > p.$$

By the same token, the set of critical points x of f_{n_k} with $\operatorname{indr}(x, f_{n_k}) \leq p$ is bounded, from which the first part of condition (f_2) follows.

To verify that $0 \notin K^p_\infty$ one should use the similar argument with Proposition 8.3(ii) instead of Proposition 8.3(i).

8.3. Homological statement. Thus all conditions (f_1)–(f_4) are fulfilled and we can apply Theorem 2.4. Since, by Lemma 8.2(ii), the set K_∞ coincides with the set of critical points of h, Theorem 2.4 provides the multiplicity result required. Now, in order to complete the proof of the theorem, it remains to show that the homological part of Theorem 2.4 implies the homological statement from Theorem 2.7. To this end it is enough to show that for any isolated critical point x of h one has $H_*^s(f_k|W) = H_*^s(h|W \cap E_k)$ provided k is sufficiently large and W is a sufficiently small neighbourhood of x. This follows from Theorem 1.7 and the following fact, which is closely related to Proposition 1.9 and Lemma 5.1(iii).

Proposition 8.5. *Suppose x_0 is an isolated critical point of h and W is an isolating neighbourhood of x_0 (that is, x_0 is the only critical point belonging to W). Then the linear homotopy connecting $(\nabla h, h)|(W \cap E_n)$ to $(\nabla f_n, f_n)|W$ is regular provided n is large enough.*

Proof. Denote by $f_n(\,\cdot\,, \lambda)$, $\lambda \in [0,1]$, the linear homotopy in question. Choose a ball $B(x_0, 2r)$ centered at x_0 such that $B(x_0, 2r) \subset W$. Due to Lemma 8.2, there exists n_0 such that for any $x \in B(x_0, 2r) \setminus B(x_0, r)$ and $n > n_0$ we have

$$\|\nabla f_n(x, \lambda)\| > \delta > 0.$$

To complete the proof one can apply Lemma 8.2(iii) and the standard arguments (cf. the proof of Lemma 5.1). $\square$

Finally, the application of Theorem 1.7 once again provides the stabilization of the homology groups.

9. Proof of Theorem 2.4: reduction to the smooth case

In this section we reduce the general case $\phi \in C^1(H, \mathbb{R})$ to the one considered in the previous section by means of a smoothing procedure. The procedure contains two steps: (i) smoothing in a neighbourhood of zero; (ii) smoothing outside a (smaller) neighbourhood of zero.

9.1. Preliminaries. In the non-smooth case we start as in the smooth case defining a sequence $\{f_n\}$ using formula (8.1). For any n set $\phi_{0n} = \phi_0|E_n$. It follows from (8.1) that

$$f_n(x) = \frac{1}{2}(B_n x, x) + \frac{\|x_n\|^2}{2n} + \phi_{0n}(x). \tag{9.1}$$

Set $\widehat{E}_n = E_n \cap \widetilde{E}_{(0,M]}$, where M is defined by (8.8) and $\widetilde{E}_{(0,M]}$ is defined before Proposition 8.4.

Observe, first of all, that using the methods of the previous subsection one can easily prove the following non-smooth analogue of Proposition 8.3.

Proposition 9.1.

(i) *Let* $\{x_k \in K_{n_k}\}$ *be a sequence such that* $\|x_k\| \to \infty$. *Then*

$$L_{\widehat{E}_{n_k}}(x_k, \nabla\phi_{n_k}) \to 0.$$

(ii) *Let* $\{x_k \in K_{n_k}\}$ *be a sequence such that* $\|x_k\| \to 0$. *Then*

$$L_{\widehat{E}_{n_k}}(x_k, \nabla\phi_{0n_k}) \to 0.$$

In the same way as in the preceding section we will use Proposition 9.1 to estimate indices of critical points going to infinity or zero.

The goal of the smoothing procedure described below is to produce from the sequence $\{f_n\}$ a new sequence, say, $\{\widetilde{f}_n : E_n \to \mathbb{R}\}$ satisfying the following conditions:

(a) for any n the functional $\widetilde{f}_n$ is smooth and $\nabla\widetilde{f_n}$ is asymptotically linear at infinity,
(b) critical points of $\{\widetilde{f}_n\}$ still approximate the critical points of h,
(c) the analogue of Proposition 9.1 holds.

9.2. Smoothing around zero. Fix n large enough. By construction, zero is an isolated critical point of f_n and K_n is bounded. Observe also that in a neighbourhood of zero

$$\phi_{0n}(x) = o(\|x\|^2) \tag{9.2}$$

and

$$\nabla\phi_{0n}(x) = o(\|x\|). \tag{9.3}$$

Let $\chi(t)$, $t \in \mathbb{R}$, be a non-negative smooth monotone function with the following properties: $\chi(t) = 0$ for $t < 1$ and $\chi(t) = 1$ for $t > 2$. Consider the family of functionals

$$\phi_{0n}(x,\delta) = \chi(||x||/\delta)\phi_{0n}(x),$$

where $\delta \in (0,1]$. It is easy to see that $\phi_{0n}(x,\delta) = \phi_{0n}(x)$, whenever $\|x\| > 2\delta$.

Proposition 9.2. *For any* $\varepsilon > 0$ *there exists* $\overline{\delta} > 0$ *such that for any* x *with* $\|x\| < 2\overline{\delta}$ *one has*

$$\|\nabla\phi_{0n}(x,\overline{\delta})\| < \varepsilon\|x\|. \tag{9.4}$$

Proof. We have

$$\nabla_x\phi_0(x,\delta) = (1/\delta)\chi'(||x||/\delta)\cdot\phi_0(x)\cdot x/||x|| + \chi(||x||/\delta)\cdot\nabla\phi_0(x). \tag{9.5}$$

Observe that both terms in (9.5) vanish inside the ball $\|x\| \le \delta$. Therefore, the application of (9.2) and (9.3) yields the desired result. The proposition is proved. $\square$

To realize the first step take $\varepsilon = 1/2n$, choose $\overline{\delta}$ provided by Proposition 9.2 and replace $\phi_{0n}(x)$ from (9.1) by $\phi_{0n}(x, \overline{\delta})$. Let us show that for n large enough the functional

$$\overline{f}_n(x) = \frac{1}{2}(B_n x, x) + \frac{\|x\|^2}{(2n)} + \phi_{0n}(x, \overline{\delta})$$

has the same collection of critical points as f_n.

Observe, first, that, by construction, $\overline{f}_n(x) = f_n(x)$ for x with $\|x\| > 2\overline{\delta}$. Hence we need to study $\overline{f}_n$ inside the ball $\|x\| \le 2\overline{\delta}$ only. Consider the quadratic functional $Q_n(x) = (B_n x, x)/2 + \|x\|^2/(2n)$. It is easy to see that for n large enough

$$\|[Q_n''(x)]^{-1}\| \le n. \tag{9.6}$$

Combining (9.4) with (9.6) and bearing in mind that $\varepsilon = 1/2n$ we see that $\overline{f}_n$ has no critical points different from zero inside the ball $\|x\| \le 2\overline{\delta}$. The first step is complete.

9.3. Smoothing outside a neighbourhood of zero. Let E be a finite-dimensional space and let $\psi_1, \psi_2 : E \to \mathbb{R}$ be continuous functions. Assume that ψ_2 has a compact support. Recall that the function

$$\psi_1 * \psi_2 = \int_E \psi_1(x-y)\psi_2(y)\, dy$$

is said to be a convolution of ψ_1 with ψ_2. As it is well-known, the above operation is commutative:

$$\psi_1 * \psi_2 = \int_E \psi_1(x-y)\psi_2(y)\, dy = \int_E \psi_1(z)\psi_2(x-z)\, dz = \psi_2 * \psi_1.$$

Let $\Omega(x)$ be a smooth non-negative cut-off function on E with the support in the unit ball and such that

$$\int_E \Omega(x)\, dx = 1.$$

For any $\mu \in (0,1]$ set

$$\Omega_\mu(x) = \mu^{-\dim E}\Omega(\mu^{-1}x).$$

Then the support of $\Omega_\mu(x)$ lies in the μ-neighbourhood of zero and

$$\int_E \Omega_\mu(x)\, dx = 1.$$

Consider a family of operators $C_\mu : C(E,\mathbb{R}) \to C(E,\mathbb{R})$, $\mu \in (0,1]$, defined by

$$C_\mu \psi = \psi * \Omega_\mu.$$

The following statements follow immediatly from the definition of convolution.

Proposition 9.3. *Let* $\psi \in C^1(E, \mathbb{R})$.

(i) $C_\mu \psi \in C^\infty(E, \mathbb{R})$,

(ii) *for any* $\varepsilon > 0$ *and radius* $R > 0$ *there exists* $\mu_0 > 0$ *such that*

$$|\psi(x) - C_\mu\psi(x)| + \|\nabla\psi(x) - \nabla C_\mu \psi(x)\| < \varepsilon,$$

whenever $\mu \le \mu_0$ *and* $\|x\| < R$,

(iii) *if* $\nabla\psi$ *satisfies the Lipschitz condition along a subspace* $\widehat{E} \subset E$ *in a ball* $B(x_0, r)$ *then* $\nabla C_\mu\psi$ *satisfies the Lipschitz condition along* $\widehat{E}$ *in the ball* $B(x_0, r-\mu)$ *with the same Lipschitz constant provided* $\mu < r$,

(iv) *if* $\nabla\psi(x) = o(\|x\|)$ *as* $\|x\| \to \infty$, *then* $\nabla_x C_\mu\psi(x) = o(\|x\|)$ *uniformly in* $\mu \in (0,1]$,

(v) *if* ψ *vanishes on a ball* $B(0,r)$ *then* $C_\mu\psi$ *vanishes on a ball* $B(0, r-\mu)$ *provided* $\mu < r$,

(vi) *if* $\psi(x) = (Qx,x)/2$, *where* Q *is a linear self-adjoint operator, then* $(C_\mu\psi)''(x) = Q$, *whenever* $x \in E$ *and* $\mu \in (0,1]$.

Take $\{\overline{f}_n\}$ obtained at the previous step. Consider the family of functionals $\{C_\mu \overline{f}_n\}$ $(\mu \in (0,1])$, which are smooth by Proposition 9.3(i). Set

$$\begin{aligned}
\phi_n(x,\mu) &= C_\mu \overline{f}_n(x) - \frac{1}{2}(A_n x, x) - \frac{\|x\|^2}{(2n)},\\
\phi_{0n}(x,\mu) &= C_\mu \overline{f}_n(x) - \frac{1}{2}(B_n x, x) - \frac{\|x\|^2}{(2n)}.
\end{aligned}$$

Then

$$\begin{aligned}
C_\mu \overline{f}_n(x) &= \frac{1}{2}(A_n x, x) + \frac{\|x\|^2}{(2n)} + \phi_n(x,\mu),\\
C_\mu \overline{f}_n(x) &= \frac{1}{2}(B_n x, x) + \frac{\|x\|^2}{(2n)} + \phi_{0n}(x,\mu).
\end{aligned}$$

Our goal is to choose a sequence $\{\mu_n\}$ in such a way that the critical points of $\{C_{\mu_n}\overline{f}_n\}$ approximate the critical points of h and the analogue of Proposition 9.1 holds for critical points of $\{C_{\mu_n}\overline{f}_n\}$ and for $\{\phi_n(\,\cdot\,,\mu_n)\}$ and $\{\phi_{0n}(\,\cdot\,,\mu_n)\}$, respectively.

Fix n large enough. By construction, in a small ball $B(0,r)$ one has $\overline{f}_n(x) = (B_n x, x)/2 + \|x\|^2/(2n)$. Using the linearity of the operator C_μ and Proposition 9.3(v), (vi) we see that $C_\mu\overline{f}_n$ has no critical points different from zero in $B(0, r-\mu)$ provided $\mu < r$. Similarly, Proposition 9.3(iv), (vi) yield that critical points of $C_\mu\overline{f}_n$ belong to a ball $B(0,R)$ with the radius R independent of $\mu \in (0,1]$. Now Proposition 9.3(ii) provides that for any $\delta > 0$ there exists $\overline{\mu} > 0$ such that for all $\mu \le \overline{\mu}$ if x is a nontrivial critical point of $C_\mu\overline{f}_n$ then there exists a critical point y of $\overline{f}_n$, as well as of f_n, with

$$\|x-y\| < \delta. \tag{9.7}$$

Using Proposition 9.3(iii), (vi) it is easy to see that if the above δ is small enough, then

$$\|\phi_n''(x,\mu)|\widehat{E_n}\| \leq 2L_{\widehat{E}_n}(y, \nabla\phi_n(y)) \tag{9.8}$$

and

$$\|\phi_{0n}''(x,\mu)|\widehat{E_n}\| \leq 2L_{\widehat{E}_n}(y, \nabla\phi_{0n}(y)), \tag{9.9}$$

where the subspace $\widehat{E_n}$ was defined at the beginning of the section. Choose μ_n such that the above δ is less than $1/n$ and (9.8), (9.9) hold with $\mu = \mu_n$.

Set $\widetilde{f}_n = C_{\mu_n}\overline{f}_n$, $\widetilde{\phi}_n = \phi_n(\cdot,\mu_n)$ and $\widetilde{\phi}_{0n} = \phi_{0n}(\cdot,\mu_n)$. Since, by construction, the critical points of $\{f_n\}$ approximate the critical points of h, by (9.7) the same is true for the sequence $\{\widetilde{f}_n\}$. In addition, formulae (9.7)–(9.9) provide that Proposition 9.1 is true for the new sequences $\{\widetilde{\phi}_n\}$ and $\{\widetilde{\phi}_{0n}\}$. It remains to observe that using a C^2-small even perturbation one can provide critical points for $\{\widetilde{f}_n\}$ to be non-degenerate.

10. Application to asymptotically linear Hamiltonian systems

In this section we prove Theorem 3.3 by means of Theorem 2.4. We study system (3.3) as an operator equation in the space $L_2[0,\tau]$ and follow the arguments from Subsection 8(a).

10.1. Preliminaries. Observe, first, that equation (3.3) can be rewritten as $\widetilde{A}_\lambda z + \lambda\nabla\phi(z) = 0$, where

$$\widetilde{A}_\lambda z = J\dot{z} + \lambda Az, \quad \nabla\phi(z)(t) = \nabla_z h(t, z(t)), \tag{10.1}$$

with $\phi(z) = \int_0^\tau h(t,z(t))\,dt$. For any $\lambda \in \mathbb{R}$ the operator $\widetilde{A}_\lambda$ is considered densely defined in $L_2 = L_2[0,\tau]$ with $\operatorname{dom}\widetilde{A}_\lambda = \{z \in W_2^1[0,\tau],\ z(0) = z(\tau)\}$. It is essentially used in what follows that for any $\lambda \in \mathbb{R}$ the operator $\widetilde{A}_\lambda$ *satisfies* condition (h_1) from Subsection 2.2 with the sequence of subspaces $\{E_n\}$ defined by (3.5). (Observe that for any $\lambda \in \mathbb{R}$ the subspace E_n is $\widetilde{A}_\lambda$-invariant for each n.) In particular, Lemma 8.1 can be applied to $\widetilde{A}_\lambda$ for any $\lambda \in \mathbb{R}$.

Another essential (and simple) fact is that due to (H_2) the gradient field $\nabla\phi$ is uniformly bounded:

$$\|\nabla\phi(z)\|_{L_2} \leq C, \quad z \in L_2. \tag{10.2}$$

All this makes the method developed in Subsection 8.1 applicable to the problem in question. This is the way to verify conditions (f_1), (f_3) and (f_4) from Theorem 2.4 (cf. the Introduction) for the approximating sequence $\{f_n\}$ constructed below.

Furthermore, we show that due to conditions (H_2) and (H_3), critical points of f_n cannot travel to infinity as well as to zero as $n \to \infty$. This will provide the verification of condition (f_2) of Theorem 2.4.

Let us start with some auxiliary results. Consider the linear system

$$\dot{z} = JAz. \tag{10.3}$$

Let B_M be the ball from condition (H_2) (see Subsection 3.1). To prove Lemma 3.1 we need

LEMMA 10.1. *There exists a ball $\widetilde{B} \subset \mathbb{R}^{2n}$, $\widetilde{B} \supset B_M$, such that for any $z_0 \notin \widetilde{B}$ the solution $z(t)$, $z(0) = z_0$, of system* (10.3) *satisfies at least one of the following conditions:*

(i) $z(-\infty, 0] \cap B_M = \emptyset$,
(ii) $z[0, \infty) \cap B_M = \emptyset$.

PROOF. The statement is trivial in the cases of purely elliptic as well as purely hyperbolic systems. The general case can be treated by considering the canonical decomposition into the two ones. □

Similarly, one can prove the following slight generalization of Lemma 10.1.

LEMMA 10.2. *For each fixed $\lambda \in [1/2, 3/2]$ consider the linear system*

$$\dot{z} = \lambda \cdot JAz. \tag{10.4}$$

There exists a ball $\widetilde{B} \supset B_M$ independent of $\lambda \in [1/2, 3/2]$ and such that for any $z \notin \widetilde{B}$ the solution $z(t)$, $z(0) = z$, of (10.4) *satisfies at least one of conditions* (i), (ii) *from Lemma* 10.1.

PROOF OF LEMMA 3.1. Suppose (λ, τ) is non-resonant and $\lambda \in [1/2, 3/2]$. Then equation (10.4) has no τ-periodic solutions. Therefore, by Lemma 10.2 and condition (H_2), all τ-periodic solutions of (3.3) belong to a ball $\widetilde{B}$ from Lemma 10.2, which is independent of $\lambda \in [1/2, 3/2]$. □

As an immidiate consequence of condition (H_3) we have

PROPOSITION 10.3. *There exist an L_2-neighbourhood U of zero and $\varepsilon > 0$ such that for any $\lambda \in [1 - \varepsilon, 1 + \varepsilon]$ system* (3.3) *has no nontrivial solutions inside U.*

The following fact is standard (cf. [14]).

PROPOSITION 10.4. *For m large enough the sequence* $\operatorname{indr}(\infty, f|E_m)$ *stabilizes* (*cf.* (3.4) *and* (3.5)), *and* $\lim_{m\to\infty} \operatorname{indr}(\infty, f|E_m) = \theta(B, A)$ (*cf. Subsection* 3.1).

10.2. Proof of Theorem 3.3. Observe, first, that under the assumptions of the theorem the number p from the statement is *finite*.

Consider the family of functionals

$$f(z,\lambda) = \int_0^\tau \left\{ \frac{1}{2}(J\dot{z}, z)_{\mathbb{R}^{2n}} + \lambda \cdot \frac{1}{2}(Az, z)_{\mathbb{R}^{2n}} + \lambda \cdot h(z,t) \right\} dt, \tag{10.5}$$

corresponding to the family of systems (3.3). In particular, $f(\,\cdot\,, 1) = f(\,\cdot\,)$ (cf. (3.4)) and $\nabla_z f(z,\lambda) = \widetilde{A}_\lambda z + \lambda \nabla\phi(z)$ (cf. (10.1)).

Let $\{\lambda_k\}$ be the sequence from the formulation of Theorem 3.3. Fix k and consider the sequence of finite-dimensional functionals $f(\,\cdot\,, \lambda_k)|E_m$. Combining Proposition 10.4 and estimate (10.2) with the the non-degeneracy of $\widetilde{A}_{\lambda_k}$, one can show (cf. the proof of Lemma 8.2) that for any k there exists an integer m_k satisfying the following conditions:

(a) $\operatorname{indr}(\infty, f(\,\cdot\,,\lambda_k)|E_{m_k}) \geq p$ (where p is from the statement),

(b) all the critical points of $f(\,\cdot\,,\lambda_k)|E_{m_k}$ belong to the open $(1/k)$-neighbourhood (in the L_2-norm) of the set of critical points of $f(\,\cdot\,,\lambda_k)$.

Set

$$f_k = f(\,\cdot\,,\lambda_k)|E_{m_k}.$$

Let us show that the sequence $\{f_k\}$ satisfies conditions (f_1)–(f_4). Lemma 3.1, Proposition 10.3 and the above condition (b) yield (f_2). Furthermore, it is easy to see that for each $\lambda \in \mathbb{R}$ Lemma 8.1 holds with $\widetilde{A}_\lambda$ instead of A. Therefore, following the scheme of the proof of Lemma 8.2, one can easily verify (f_3) and (f_4).

It remains to verify (f_1). By construction, every functional f_k is quadratic-like and $\operatorname{indr}(\infty, f_k) \geq p$. Applying the technique developed in Section 9 (cf. Proposition 9.3) we may smooth out every f_k. Further, we perturb the functionals in order to provide all critical points to be non-degenerate. Observe (cf. Proposition 9.3) that these arrangements can be carried out in such a way that the above conditions (a) and (b) will be still satisfied. Thus (f_1) is also fulfilled and we may apply Theorem 2.4, which provides the multiplicity result required.

To show the homological part of the statement we use the same argument as in Subsection 8.3.

11. Proof of Theorem 3.6

We prove Theorem 3.6 via Corollary 2.8. Set $H = L_2$, $Au = -\Delta u - \lambda_k u$, $Bu = -\Delta u - (\lambda_k + \nu)u$, $\operatorname{dom} A = \operatorname{dom} B = W_0^{2,2}(\Omega)$ and $\phi(u) = \int_\Omega \Psi(x, u(x))\,dx$. For any $n \geq 0$ define E_n as the sum of eigenspaces corresponding to A-eigenvalues less than n. Clearly, every E_n is also B-invariant.

To take advantage of Corollary 2.8 we must verify conditions (h_1)–(h_3) from Theorem 2.6 and verify that $N(A) < \infty$.

The standard Laplacian arguments (cf. Corollaries 8.2, 8.8 and 8.12 from [27]) yield (h_1) and $N(A) < \infty$.

Below we will verify condition (h_2). To verify condition (h_3) one can use the same arguments. Therefore, we omit the details of checking condition (h_3).

By condition (ψ_4), the functional ϕ is even. By (ψ_1), the operator $\nabla\phi$ is bounded on any bounded subset of L_2 and, moreover, satisfies the Lipschitz condition on the whole space H. Let us show that $\nabla\phi(u)_{L_2}/\|u\|_{L_2} \to 0$ as $\|u\|_{L_2} \to \infty$. Indeed, by conditions (ψ_1) and (ψ_2), one has $|\psi(x,t)| = o(|t|)$ as $|t| \to \infty$ uniformly in x. Therefore, for any $\varepsilon > 0$ we have $|\psi(x,t)| \le \varepsilon|t| + C_\varepsilon$ and hence

$$\begin{aligned}\|\nabla\phi(u)\|^2_{L_2} &= \int_\Omega \psi^2(x,u)\,dx \\ &\le 2\varepsilon^2 \int_\Omega u^2\,dx + 2C^2_\varepsilon \cdot \mathrm{mes}(\Omega) = 2\varepsilon^2\|u\|^2_{L_2} + 2C^2_\varepsilon \cdot \mathrm{mes}(\Omega).\end{aligned}$$

Since ε is arbitrarily small, the result follows.

It remains to check the last (and the most important) part of condition (h_2), which says that for any finite-dimensional subspace $E \subset L_2$ and any sequence $\{u_n\} \subset L_2$ with $\|u_n\|_{L_2} \to \infty$ and $\rho(u_n, \ker A)/\|u_n\| \to 0$ (in the L_2-metric) one has $L_E(u_n, \nabla\phi) \to 0$. To this end we need

Proposition 11.1. *Let $\{u_n\} \subset L_2$ be a sequence such that for any $M > 0$ one has* $\mathrm{mes}\{x \in \Omega \mid |u_n(x)| < M\} \to 0$. *Then, under the assumptions of Theorem* 3.6, *$L_E(u_n, \nabla\phi) \to 0$ for any finite-dimensional subspace $E \subset L_2$.*

Proof. For any $M > 0$ denote by δ_M the supremum of the local Lipschitz constants of ψ in t taken over t with $|t| > M$. By condition (ψ_2), one has $\delta_M \to 0$ as $M \to \infty$.

We need certain auxiliary inequalities. Fix a finite-dimensional subspace $E \subset H$. Further, take $u \in L_2$ and fix $M > 0$, $r \ge 0$. Set

$$\begin{aligned}X &= X(u,M,r) = \{x \in \Omega \mid |u(x) + v(x)| \ge M \ \forall v \in E,\ \|v\| \le r\}, \\ Y &= Y(u,M,r) = \overline{\Omega} \setminus X.\end{aligned}$$

Take some $v_1, v_2 \in E$ with $\|v_i\| \le r$ ($v_1 \ne v_2$) and set $v_0 = (v_1 - v_2)/\|v_1 - v_2\|$. From the definition of δ_M it follows:

$$\begin{aligned}(11.1)\qquad &\int_X |\psi(x, u(x) + v_1(x)) - \psi(x, u(x) + v_2(x))|^2\,dx \\ &\qquad\qquad \le \delta_M^2 \int_X |v_1(x) - v_2(x)|^2\,dx \le \delta_M^2 \cdot \|v_1 - v_2\|^2_{L_2}.\end{aligned}$$

By condition (ψ_1),

$$
\begin{aligned}
(11.2) \qquad & \int_Y |\psi(x,u(x)+v_1(x)) - \psi(x,u(x)+v_2(x))|^2\,dx \\
& \le l^2 \int_Y |v_1(x)-v_2(x)|^2\,dx = l^2 \|v_1-v_2\|_{L_2} \cdot \int_Y |v_0(x)|^2\,dx \\
& = l^2 \cdot \|v_0|Y\|_{L_2}^2 \cdot \|v_1-v_2\|_{L_2}^2,
\end{aligned}
$$

where l is the Lipschitz constant from condition (ψ_1). From (11.1) and (11.2) one gets

$$
\begin{aligned}
(11.3) \qquad & \|\nabla\phi(u+v_1) - \nabla\phi(u+v_2)\|_{L_2} \\
& = \|\psi(\,\cdot\,,u(\,\cdot\,)+v_1(\,\cdot\,)) - \psi(\,\cdot\,,u(\,\cdot\,)+v_2(\,\cdot\,))\|_{L_2} \\
& \le (\delta_M + l\cdot\|v_0|Y\|_{L_2}) \cdot \|v_1-v_2\|_{L_2}.
\end{aligned}
$$

Let us now prove Proposition 11.1. The standard continuous measure argument yields: for any $n\in\mathbb{N}$ there exists (small) $r_n>0$ such that

$$
(11.4) \qquad \operatorname{mes}\{Y(u_n,M,r_n)\} \le 2\cdot \operatorname{mes}\{Y(u_n,2M,0)\}.
$$

Fix a sequence $\{r_n>0\}$ satisfying (11.4) and set $Y_n = Y(u_n,M,r_n)$. By assumptions of Proposition 11.1, $\operatorname{mes}\{Y(u_n,2M,0)\}\to 0$, hence (see (11.4)) $\operatorname{mes}(Y_n)\to 0$. By construction, v_0 from (11.3) runs through the unit sphere $S(E)$ in the finite-dimensional space E. Since $S(E)$ is compact, $\|v_0|Y_n\|_{L_2}\to 0$ uniformly in v_0 (see, for instance, [30]). Recall also that $\delta_M\to 0$ as $M\to\infty$. Therefore, it follows from (11.3) that for any $\varepsilon>0$ there exists N such that

$$
\|\nabla\phi(u_n+v_1) - \nabla\phi(u_n+v_2)\|_{L_2} \le \varepsilon\|v_1-v_2\|_{L_2}
$$

whenever $n>N$ and $\|v_i\|_{L_2}<r_n$, $i=1,2$. $\square$

To take advantage of Proposition 11.1 we need

LEMMA 11.2. *Let $T=\{u\in\ker A \mid \|u\|_{L_2}=1\}$. For any $\varepsilon>0$ there exists $\alpha_\varepsilon>0$ independent of $u\in T$ and such that for any $u\in T$*

$$
\operatorname{mes}\{x\in\overline{\Omega} \mid |u(x)|\le\alpha_\varepsilon\} < \varepsilon.
$$

Before giving the proof of Lemma 11.2 let us complete the verification of condition (h_2).

Observe, first, that the standard continuous measure argument yields the following consequence of Lemma 11.2.

Corollary 11.3. *Under the notation of Lemma* 11.2 *for any* $\varepsilon > 0$ *there exists a* σ*-neighbourhood* U_σ *of* T *in* L_2 *and* $\alpha_\varepsilon > 0$ *such that for any* $u \in U_\sigma$

$$\operatorname{mes}\{x \in \overline{\Omega} \mid |u(x)| \le \alpha_\varepsilon\} < \varepsilon.$$

It is easy to see that Corollary 11.3 is equivalent to the following statement: for any $\varepsilon > 0$ and any $M > 0$ there exists $\sigma > 0$ such that $\operatorname{mes}\{x \in \overline{\Omega} \mid |u(x)| \le M\} < \varepsilon$ provided $\rho(u, \ker A)/\|u\| \le \sigma$ and $\|u\| > 1/\sigma$. Combining the last observation with Proposition 11.1 yields condition (h_2).

Proof of Lemma 11.2. Observe that every $u \in \ker A$ satisfies the equation $-\Delta u = \lambda u$ in Ω. Hence (see [29]) every $u \in \ker A$ is analytic in Ω. Take $\varepsilon > 0$ and choose a compact $P \subset \Omega$ such that $\operatorname{mes}\{\Omega \setminus P\} \le \varepsilon/2$. Consider the function

$$m_P(\alpha, u) = \operatorname{mes}\{x \in P \mid |u(x)| \le \alpha\},$$

where $u \in T$ and $\alpha \in \mathbb{R}$. Since every $u \in T$ is analytic, we can use Theorem 2 from [46] to provide the following result: $m_P(\alpha, u) \to 0$ as $\alpha \to 0$ uniformly in u. To complete the proof of Lemma 11.2 it remains to choose $\alpha = \alpha_\varepsilon$ for which $m_P(\alpha, u) < \varepsilon/2$ for every $u \in T$. $\square$

Thus, by Corollary 2.8, problem (3.9) has at least p distinct pairs of solutions with the corresponding homological information in the case of a discrete set of solutions. Now applying (ψ_1) and the standard regularity technique (see, for instance, [32]), one can easily show that each obtained solution is of class $C^{2,\alpha}$ with any $\alpha < 1$.

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Zalman Balanov
Department of Mathematics and Computer Sciences
Bar Ilan University
52900 Ramat-Gan, ISRAEL

Current address: Department of Mathematical Sciences
University of Alberta
Edmonton, Alberta, T6G 2G1, CANADA

E-mail address: balanov@macs.biu.ac.il

Efim Schwartzman
Department of Mathematics
Technion, 32000 Haifa, ISRAEL

E-mail address: fim@techunix.technion.ac.il