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UNIQUENESS OF PERIODIC SOLUTIONS FOR ASYMPTOTICALLY LINEAR DUFFING EQUATIONS WITH STRONG FORCING

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1. Introduction

In this work we investigate equations of the form

$$x'' + cx' + ax + g(x) = \lambda p(t), \tag{1}$$

where $g : \mathbb{R} \to \mathbb{R}$ is C^1, satisfies a Lipschitz condition, and

$$\lim_{|x|\to\infty} \frac{g(x)}{x} = 0, \tag{2}$$

so that we are dealing with an *asymptotically linear* problem. The forcing term p is T-periodic, and we are interested in T-periodic solutions of (1). We assume that the linear part of the equation is nonresonant, that is $a \neq 0$ and if $c = 0$ then $a \neq (2\pi m/T)^2$ for all integer m. Our result shows that for generic forcing term p, when the parameter λ, which measures the strength of the forcing, is sufficiently large, (1) has a *unique* T-periodic solution.

The existence of a solution (for all λ) under the assumptions made above is a well-known application of degree theory or Schauder's fixed point theorem (see [5] for existence results under much more general conditions), and it is the

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uniqueness of the solution that is of interest here. As far as we know, this result is new even for the case when g is bounded.

In contrast with other known uniqueness results (see e.g. [8] for the Duffing equation, [4] for abstract principles), there is no assumption here that the derivative of g does not "interact" with the spectrum of the linear part. The conclusion, on the other hand, is weaker than that obtained in the above-mentioned results, since the uniqueness is only for $|\lambda|$ large. We note that for $\lambda = 0$ the solutions of the algebraic equation $ax + g(x) = 0$ are equilibria of (1), and by continuation these generate T-periodic solutions for $|\lambda|$ sufficiently small, so that we may have an arbitrary number of T-periodic solutions, hence the assumption that $|\lambda|$ is sufficiently large is essential here.

We now formulate our result.

THEOREM 1. *Suppose:*

(i) *$a \neq 0$ and, if $c = 0$ then $a \neq (2\pi m/T)^2$ for all integers m.*

(ii) *$g : \mathbb{R} \to \mathbb{R}$ is C^1, satisfies (2), and g' is bounded:*

$$|g'(x)| \leq L \quad \textit{for all } x \in \mathbb{R}. \tag{3}$$

(iii) *$p : \mathbb{R} \to \mathbb{R}$ is continuous and T-periodic, and if we define u_0 as the unique T-periodic solution of*

$$u'' + cu' + au = p(t) \tag{4}$$

then the set of critical points of u_0 is of measure 0.

Then there exists $\lambda_0 \geq 0$ such that for $|\lambda| \geq \lambda_0$ (1) has a unique T-periodic solution. Assuming $c > 0$, we have, for $|\lambda|$ sufficiently large, that this solution is asymptotically stable if $a > 0$, and unstable if $a < 0$.

We now make some remarks about the hypotheses of Theorem 1.

1. The nonresonance assumption (i) cannot be dropped, as can be seen by looking at the case $g \equiv 0$.
2. Concerning (ii), a natural question, to which we do not know the answer, is whether the result remains true without (3). The growth assumption (2) on g certainly cannot be dropped: it is known (see [3]) that when g satisfies
$$\lim_{|x| \to \infty} \frac{g(x)}{x} = \infty$$
and $c = 0$, (1) has *infinitely* many T-periodic solutions for *any* λ.
3. Concerning assumption (iii) on p, we note that it holds for a dense G_δ subset in the space of continuous T-periodic functions p, and also that it automatically holds whenever p is real-analytic and non-constant. The fact that some restriction on p is necessary for the validity of the result

can be seen by looking at the case $g(x) = \sin(x)$ and $p \equiv 1$. Given any integer $n > 0$, it is easy to see that if we take a suffciently small, (1) will have at least n T-periodic (in this case constant) solutions for all λ. Of course in this case $u_0 \equiv 1/a$, so (iii) fails to hold.

The proof of Theorem 1, which is presented in Section 3, makes essential use of the weak topology in $L^2[0,T]$. The weak convergence of certain sequences of functions is proved by means of an elementary asymptotic result, which will be proved in the next section. We remark that other works have already used asymptotic results to obtain interesting information on nonlinear boundary value problems (see [2], [6], [10]), but the way in which the asymptotic result is used here, as well as the purpose to which it is applied, are entirely different from what is done in the works cited. Note also that our asymptotic lemmas (Section 2) pertain to sublinear nonlinearities in general, and not to periodic nonlinearities or those vanishing at infinity, as in the above-cited works.

In Section 4 we study the asymptotic form of the unique T-periodic solution as $|\lambda| \to \infty$ under some additional assumptions on the nonlinearity g.

2. Asymptotic lemmas

Lemma 1. *Suppose that $g : \mathbb{R} \to \mathbb{R}$ satisfies* (2) *and* (3). *Suppose $\gamma \in L^1[\alpha,\beta]$. Then*

$$\lim_{|\lambda|\to\infty} \int_\alpha^\beta g'(\lambda s)\gamma(s)\,ds = 0.$$

Proof. Assume first that γ is a characteristic function of a subinterval $[\alpha',\beta']$. Then, using (2),

$$\lim_{|\lambda|\to\infty} \int_\alpha^\beta g'(\lambda s)\gamma(s)\,ds = \lim_{|\lambda|\to\infty} \frac{1}{\lambda}(g(\lambda\beta') - g(\lambda\alpha')) = 0.$$

The result extends to the case when γ is a step function by linearity. If $\gamma \in L^1[\alpha,\beta]$ and $\varepsilon > 0$, we may choose a step-function $\widetilde{\gamma}$ with $\|\widetilde{\gamma} - \gamma\|_{L^1} \le \varepsilon$. We then have,

$$\left| \int_\alpha^\beta g'(\lambda s)\widetilde{\gamma}(s)\,ds - \int_\alpha^\beta g'(\lambda s)\gamma(s)\,ds \right| \le \varepsilon L,$$

so, from the validity of the result for $\widetilde{\gamma}$, we have

$$\limsup_{|\lambda|\to\infty} \left| \int_\alpha^\beta g'(\lambda s)\gamma(s)\,ds \right| \le \varepsilon L,$$

and since ε is arbitrary, we obtain the desired result. $\square$

LEMMA 2. *Suppose $g : \mathbb{R} \to \mathbb{R}$ satisfies* (2) *and* (3). *Suppose $\phi \in C^1[\alpha, \beta]$ and the set of critical points of ϕ has measure* 0, *and $\eta \in L^1[\alpha, \beta]$. Let $\{\lambda_n\} \subset \mathbb{R}$ and $\{\theta_n\} \subset C^1[\alpha, \beta]$ be sequences such that*

$$\lim_{n\to\infty} |\lambda_n| = \infty, \tag{5}$$

$$\lim_{n\to\infty} \frac{\|\theta_n\|_{C^1}}{\lambda_n} = 0. \tag{6}$$

Then

$$\lim_{n\to\infty} \int_\alpha^\beta g'[\lambda_n \phi(t) + \theta_n(t)]\eta(t)\, dt = 0. \tag{7}$$

PROOF. We first prove (7) under the assumption that $\phi'(t) \neq 0$ for all $t \in [\alpha, \beta]$, assuming without loss of generality that $\phi'(t) > 0$. We define $h_n(t) = \phi(t) + \theta_n(t)/\lambda_n$, and note that, by the positivity of ϕ' and by (6), h_n' is positive in $[\alpha, \beta]$ for n large, so we can perform the change of variable $s = h_n(t)$, obtaining

$$\int_\alpha^\beta g'[\lambda_n \phi(t) + \theta_n(t)]\eta(t)\, dt = \int_{h_n(\alpha)}^{h_n(\beta)} g'(\lambda_n s) \frac{\eta(h_n^{-1}(s))}{h_n'(h_n^{-1}(s))}\, ds. \tag{8}$$

Since $h_n \to \phi$ in $C^1[\alpha, \beta]$ as $n \to \infty$, and since g' is bounded, we have

$$\lim_{n\to\infty} \left| \int_{h_n(\alpha)}^{h_n(\beta)} g'(\lambda_n s) \frac{\eta(h_n^{-1}(s))}{h_n'(h_n^{-1}(s))}\, ds - \int_{\phi(\alpha)}^{\phi(\beta)} g'(\lambda_n s) \frac{\eta(\phi^{-1}(s))}{\phi'(\phi^{-1}(s))}\, ds \right| = 0. \tag{9}$$

By Lemma 1 we have

$$\lim_{n\to\infty} \int_{\phi(\alpha)}^{\phi(\beta)} g'(\lambda_n s) \frac{\eta(\phi^{-1}(s))}{\phi'(\phi^{-1}(s))}\, ds = 0. \tag{10}$$

Equations (8)–(10) imply (7).

Turning to the case of general ϕ, we denote

$$A = \{t \in (\alpha, \beta) \mid \phi'(t) \neq 0\}.$$

A is an open set, so it can be decomposed into a countable union of disjoint open intervals

$$A = \bigcup_{n=1}^\infty I_n.$$

We define

$$A_k = \bigcup_{n=1}^k I_n.$$

We fix $\varepsilon > 0$. Since, by our assumption on ϕ, A is of full measure, we can choose k_0 sufficiently large so that

$$\int_{A_{k_0}} |\eta(t)|\, dt \geq \|\eta\|_{L^1} - \varepsilon.$$

For each $1 \le n \le k_0$ we may choose a closed subinterval I'_n of I_n in such a way that, defining

$$A'_{k_0} = \bigcup_{n=1}^{k_0} I'_n,$$

we have

$$\int_{A'_{k_0}} |\eta(t)|\, dt \ge \|\eta\|_{L^1} - 2\varepsilon,$$

or, denoting the complement of A'_{k_0} in $[\alpha, \beta]$ by B'_{k_0},

$$\int_{B'_{k_0}} |\eta(t)|\, dt \le 2\varepsilon. \tag{11}$$

In each of the intervals I'_n $(1 \le n \le k_0)$ we may apply the previously proved case of our lemma to conclude that

$$\lim_{n\to\infty} \int_{A'_{k_0}} g'[\lambda_n \phi(t) + \theta_n(t)]\eta(t)\, dt = 0. \tag{12}$$

Since g' is bounded we also get, using (11),

$$\left| \int_{B'_{k_0}} g'[\lambda_n \phi(t) + \theta_n(t)]\eta(t)\, dt \right| \le 2\varepsilon L \tag{13}$$

for all n. Combining (12) and (13), we have

$$\limsup_{n\to\infty} \left| \int_\alpha^\beta g'[\lambda_n \phi(t) + \theta_n(t)]\eta(t)\, dt \right| \le 2\varepsilon L,$$

and since ε is arbitrary, we obtain the desired result. □

The following lemma is not used in the proof of Theorem 1, but rather in the asymptotic analysis of Section 4.

Lemma 3. *Suppose $g : \mathbb{R} \to \mathbb{R}$ satisfies* (3), *and the limits*

$$\mu_\pm = \lim_{x\to\pm\infty} \frac{g(x)}{x} \tag{14}$$

exist. Suppose $\phi \in C^1[\alpha, \beta]$ and the set of critical points of ϕ has measure 0, *and let $\eta \in L^1[\alpha, \beta]$. Define $\chi : [\alpha, \beta] \to \mathbb{R}$ by $\chi(t) = \mu_-$ when $\phi(t) < 0$ and $\chi(t) = \mu_+$ when $\phi(t) \ge 0$. Then we have*

$$\lim_{\lambda\to\infty} \int_\alpha^\beta g'(\lambda\phi(t))\eta(t)\, dt = \int_\alpha^\beta \chi(t)\eta(t)\, dt. \tag{15}$$

Proof. Let $g_0 : \mathbb{R} \to \mathbb{R}$ be a C^1-function satisfying

$$\lim_{x\to\pm\infty} g'_0(x) = \mu_\pm. \tag{16}$$

Define $g_1 = g - g_0$. From (16) and Lebesgue's dominated convergence theorem it follows that

$$\lim_{\lambda\to\infty}\int_\alpha^\beta g_0'(\lambda\phi(t))\eta(t)\,dt = \int_\alpha^\beta \chi(t)\eta(t)\,dt. \tag{17}$$

From (16) it follows that

$$\lim_{x\to\pm\infty}\frac{g_0(x)}{x} = \mu_\pm,$$

hence

$$\lim_{x\to\pm\infty}\frac{g_1(x)}{x} = 0,$$

so g_1 satisfies all the hypotheses of Lemma 2, hence

$$\lim_{\lambda\to\infty}\int_\alpha^\beta g_1'(\lambda\phi(t))\eta(t)\,dt = 0. \tag{18}$$

From (17), (18) we obtain (15). □

3. Proof of Theorem 1

Our first step is to write (1) in a different form. Defining u_0 as the unique (by assumption (i)) T-periodic solution of (4) and setting $x = \lambda u_0 + u$, we can rewrite (1) as

$$u'' + cu' + au + g(\lambda u_0 + u) = 0. \tag{19}$$

T-periodic solutions of (1) correspond to T-periodic solutions of (19), and these correspond to functions $u \in C^2[0,T]$ satisfying (19) and

$$u(0) - u(T) = u'(0) - u'(T) = 0. \tag{20}$$

We define a linear operator $K : L^2[0,T] \to C^1[0,T]$ as follows: for $y \in L^2[0,T]$, $K(y)$ denotes the unique (by assumption (i) of Theorem 1) function $u \in H^2[0,T]$ satisfying

$$u'' + cu' + au = y$$

almost everywhere in $[0,T]$ and (20). Since $H^2[0,T]$ is compactly embedded in $C^1[0,T]$, K is compact. We define a nonlinear operator (depending on the parameter λ) $G_\lambda : C^0[0,T] \to C^0[0,T]$ by

$$G_\lambda(u) = g(\lambda u_0 + u).$$

We can now rewrite (19) in the form

$$u + K \circ G_\lambda(u) = 0. \tag{21}$$

(21) is an equation in $C^0[0,T]$, but it is immediate that any solution of (21) is in fact in $C^2[0,T]$, and is T-periodic. We wish to consider (21) in the space $C^1[0,T]$, so we define $N_\lambda : C^1[0,T] \to C^1[0,T]$ by

$$N_\lambda = -K \circ G_\lambda|_{C^1[0,T]}$$

so that we may write (21) as

$$N_\lambda(u) = u. \tag{22}$$

Solutions of (22) are in one-to-one correspondence with periodic solutions of (1).

We shall denote by $B(r)$ the closed ball of radius r around the origin in $C^1[0,T]$, where the spaces $C^0[0,T]$, $C^1[0,T]$ are endowed with the norms

$$\|u\|_{C^0} = \max_{t\in[0,T]} |u(t)|, \quad \|u\|_{C^1} = \|u\|_{C^0} + \|u'\|_{C^0}.$$

The following lemma plays a key role in the proof of Theorem 1.

LEMMA 4. *Suppose the assumptions of Theorem* 1 *hold. Let* $R : \mathbb{R} \to [0,\infty)$ *be a function satisfying*

$$\lim_{|\lambda|\to\infty} \frac{R(\lambda)}{\lambda} = 0. \tag{23}$$

Then

$$\lim_{|\lambda|\to\infty} \sup_{\|u\|_{C^1} \le R(\lambda)} \|N'_\lambda(u)\|_{C^1,C^1} = 0 \tag{24}$$

where N'_λ *is the Gâteaux derivative of* N_λ (*the norm* $\|\cdot\|_{C^1,C^1}$ *denotes the norm in the space of linear operators from* $C^1[0,T]$ *to itself, induced by the* C^1*-norm*).

PROOF. We assume by way of contradiction that (24) does not hold. Then there exist sequences $\{u_n\}, \{v_n\} \subset C^1[0,T]$ and $\{\lambda_n\} \subset \mathbb{R}$ with $|\lambda_n| \to \infty$, and

$$\|u_n\|_{C^1} \le R(\lambda_n), \tag{25}$$

$$\|v_n\|_{C^1} = 1, \tag{26}$$

$$\|K \circ G'_{\lambda_n}(u_n)(v_n)\|_{C^1} = \|N'_{\lambda_n}(u_n)(v_n)\|_{C^1} \ge \varepsilon > 0 \tag{27}$$

for all n. Since, by (26), $\{v_n\}$ is bounded in C^1, we may invoke the Arzela–Ascoli theorem and assume, by taking a subsequence, that $\{v_n\}$ converges in $C^0[0,T]$, and denote its limit by $\overline{v}$. We have

$$\begin{aligned}
&\|G'_{\lambda_n}(u_n)(v_n) - G'_{\lambda_n}(u_n)(\overline{v})\|_{C^0} \\
&\qquad = \|g'[\lambda_n u_0(t) + u_n(t)](v_n(t) - \overline{v}(t))\|_{C^0} \\
&\qquad \le \|g'[\lambda_n u_0(t) + u_n(t)]\|_{C^0} \|v_n(t) - \overline{v}(t)\|_{C^0}.
\end{aligned}$$

The right-hand side goes to 0 as $n \to \infty$ since $v_n \to \overline{v}$ in $C^0[0,T]$ and g' is bounded. Therefore we have

$$G'_{\lambda_n}(u_n)(v_n) - G'_{\lambda_n}(u_n)(\overline{v}) \to 0$$

in $C^0[0,T]$, hence

$$K \circ G'_{\lambda_n}(u_n)(v_n) - K \circ G'_{\lambda_n}(u_n)(\overline{v}) \to 0 \tag{28}$$

in $C^1[0,T]$. On the other hand, we have, for all $w \in L^2[0,T]$

$$\int_0^T (G'_{\lambda_n}(u_n)(\overline{v}))(t)w(t)\,dt = \int_0^T g'[\lambda_n u_0(t) + u_n(t)]\overline{v}(t)w(t)\,dt, \tag{29}$$

and applying Lemma 2 with $\phi = u_0, \theta_n = u_n, \eta = \overline{v}w$, noting that, by (25) and (23), (6) holds, we obtain that the right-hand side of (29) tends to 0 as $n \to \infty$, so we have shown that $G'_{\lambda_n}(u_n)(\overline{v}) \to 0$ *weakly* in $L^2[0,T]$. Hence by the compactness of K, $K \circ G'_{\lambda_n}(u_n)(\overline{v}) \to 0$ *strongly* in $C^1[0,T]$. Together with (28), we get that $K \circ G'_{\lambda_n}(u_n)(v_n) \to 0$ in $C^1[0,T]$. But this contradicts (27), and this contradiction concludes the proof of (24). □

The following lemma gives a λ-dependent a priori bound on solutions of (22).

LEMMA 5. *Suppose the assumptions of Theorem* 1 *hold. Then there exists a function* $R : \mathbb{R} \to [0,\infty)$ *such that*

$$\lim_{|\lambda|\to\infty} \frac{R(\lambda)}{\lambda} = 0, \tag{30}$$

and such that for any $|\lambda|$ *sufficiently large, all solutions of* (22) *are in* $B(R(\lambda))$.

PROOF. We define

$$M(r) = \max_{|x|\le r} |g(x)|, \quad m(r) = \frac{M(r)}{r}.$$

Clearly, M is monotone nondecreasing, and from (2) it follows that

$$\lim_{r\to\infty} m(r) = 0. \tag{31}$$

Assume that there does not exist a function R as in the lemma. Then there exists a sequence $\{\lambda_n\}$ with

$$\lim_{n\to\infty} |\lambda_n| = \infty \tag{32}$$

and a sequence $\{u_n\} \subset C^1[0,T]$ such that

$$N_{\lambda_n}(u_n) = u_n \tag{33}$$

and

$$\|u_n\|_{C^1} \ge k|\lambda_n| \tag{34}$$

for all n, where $k > 0$. We shall derive a contradiction.

We have, for all n,

$$\begin{aligned}\|u_n\|_{C^1} &= \|N_{\lambda_n}(u_n)\|_{C^1} \le \|K\|_{C^0,C^1} M(\|\lambda_n u_0 + u_n\|_{C^0}) \\ &\le \|K\|_{C^0,C^1} M(\|\lambda_n u_0 + u_n\|_{C^1}) \le \|K\|_{C^0,C^1} M(\|\lambda_n u_0\|_{C^1} + \|u_n\|_{C^1}) \\ &= \|K\|_{C^0,C^1} m(\|\lambda_n u_0\|_{C^1} + \|u_n\|_{C^1})(\|\lambda_n u_0\|_{C^1} + \|u_n\|_{C^1})\end{aligned}$$

and, by (31), we have

$$\lim_{n\to\infty} m(\|\lambda_n u_0\|_{C^1} + \|u_n\|_{C^1}) = 0, \tag{35}$$

so, for n sufficiently large,

$$\|K\|_{C^0,C^1} m(\|\lambda_n u_0\|_{C^1} + \|u_n\|_{C^1}) < \frac{1}{2}.$$

Then we get

$$\|u_n\|_{C^1} \le \|K\|_{C^0,C^1} m(\|\lambda_n u_0\|_{C^1} + \|u_n\|_{C^1})\|\lambda_n u_0\|_{C^1} + \frac{1}{2}\|u_n\|_{C^1},$$

or

$$\|u_n\|_{C^1} \le 2\|K\|_{C^0,C^1} m(\|\lambda_n u_0\|_{C^1} + \|u_n\|_{C^1})\|u_0\|_{C^1}|\lambda_n|$$

for n sufficiently large. Using (35), we obtain

$$\lim_{n\to\infty} \frac{\|u_n\|_{C^1}}{|\lambda_n|} = 0,$$

contradicting (34), and finishing our proof. □

We are now ready for the

Proof of Theorem 1. We want to show that for $|\lambda|$ sufficiently large (22) has a unique solution in $C^1[0,T]$. The existence, as we noted, is a well-known result following from Schauder's fixed point theorem, so we only need to prove uniqueness.

By Lemma 5 we have that all solutions of (22) are in $B(R(\lambda))$, where $R : \mathbb{R} \to [0,\infty)$ satisfies (30). Hence it suffices to prove the uniqueness of the solution in $B(R(\lambda))$.

By Lemma 4, we can choose λ_0 so that $|\lambda| \ge \lambda_0$ implies that

$$\sup_{u\in B(R(\lambda))} \|N'_\lambda(u)\|_{C^1,C^1} < 1,$$

hence by the mean value theorem (e.g., [1, Theorem 1.8]) N_λ is a contraction in $B(R(\lambda))$ when $|\lambda| \ge \lambda_0$, so the fixed point is unique (we note that we have not shown that $B(R(\lambda))$ is invariant under N_λ, so that we have not proven existence, but this is already known).

To prove the statement on stability, we first recall the result of Ortega ([9, Section 2.5, Theorem 3]) which says that when $c > 0$, a periodic solution of (1) is asymptotically stable if and only if the corresponding fixed point of the Poincaré mapping with time $2T$ is isolated and of index 1.

We now note that since p is also $2T$ periodic, we may apply the uniqueness result which has just been proved to conclude that there exists λ_0' such that $|\lambda| \geq \lambda_0'$ implies that (1) has a unique $2T$-periodic solution. If $|\lambda| \geq \lambda_1 = \max\{\lambda_0, \lambda_0'\}$ then, since the unique T-periodic solution is also $2T$ periodic, the two coincide.

By a guiding function method it is shown (see [7, p. 55]) that the Poincaré operator for (1) of time kT ($k \geq 1$ an integer) on a sufficiently large disc around the origin is of index 1 if $a > 0$ and $c \neq 0$ and of index -1 if $a < 0$. Applying this in the case $k = 2$ we have, by the uniqueness of the $2T$-periodic solution, that the index of the corresponding fixed point of the time $2T$ Poincaré operator is equal to the global index in the disc, hence to 1 if $a > 0$ and to -1 if $a < 0$. The result follows by Ortega's theorem. □

4. Asymptotics of the solution

Assuming the conditions of Theorem 1 hold, we denote the unique solution ensured by the theorem by x_λ ($|\lambda| \geq \lambda_0$). Our aim now is to characterize the asymptotic form of x_λ as $\lambda \to \infty$ (we will assume λ positive; analogous formulas hold for $\lambda \to -\infty$). More precisely, we want to find a *simple* one-parameter family y_λ of T-periodic functions such that

$$\lim_{\lambda \to \infty} \|x_\lambda - y_\lambda\|_{C^1} = 0. \tag{36}$$

To obtain our result, we will need the further assumptions that g is bounded and has well-defined average values at $\pm\infty$, that is, the limits

$$\mu_\pm = \lim_{l \to \pm\infty} \frac{1}{l} \int_0^l g(x)\, dx \tag{37}$$

exist. We note that this holds in some important special cases: when g has limits at $\pm\infty$, in which case $\mu_\pm$ are equal to these limits, and in the case when g is almost periodic, in which case we have $\mu_+ = \mu_-$. In particular, when g is periodic with period ρ, we have

$$\mu_- = \mu_+ = \frac{1}{\rho} \int_0^\rho g(x)\, dx.$$

Of course (37) holds also in the case that g is a sum of the two types of nonlinearity mentioned.

Theorem 2. *Suppose the assumptions of Theorem* 1 *hold, and further that*

$$|g(x)| \leq R_0 \tag{38}$$

for all $x \in \mathbb{R}$*, and the limits* (37) *exist. Define* $\chi : [0,T] \to \mathbb{R}$ *by:* $\chi(t) = \mu_-$ *when* $u_0(t) < 0$ *and* $\chi(t) = \mu_+$ *when* $u_0(t) \geq 0$*, where* u_0 *is defined as in Theorem* 1*. Let* $u_1 \in H^2[0,T]$ *be the unique solution of the linear equation*

$$u_1'' + cu_1' + au_1 = -\chi(t)$$

satisfying

$$u_1(0) - u_1(T) = u_1'(0) - u_1'(T) = 0.$$

Let $y_\lambda = \lambda u_0 + u_1$*. Then* (36) *holds.*

Proof. We denote by z_λ the unique solution of (22) for $|\lambda| \geq \lambda_0$, so that

$$x_\lambda = \lambda u_0 + z_\lambda. \tag{39}$$

Our aim now will be to determine the asymptotic behaviour of z_λ. We first note that, using (38) gives

$$\|z_\lambda\|_{C^1} = \|N_\lambda(z_\lambda)\|_{C^1} \leq \|K\|_{C^0,C^1} R_0 \equiv R_1.$$

Using this and the mean value theorem, we have

$$\|z_\lambda - N_\lambda(0)\|_{C^1} = \|N_\lambda(z_\lambda) - N_\lambda(0)\|_{C^1} \leq R_1 \sup_{\|u\|_{C^1} \leq R_1} \|N_\lambda'(u)\|_{C^1,C^1}.$$

By Lemma 4 we have that the right-hand side of the above inequality converges to 0 as $|\lambda| \to \infty$, hence we have

$$\lim_{|\lambda| \to \infty} \|z_\lambda - N_\lambda(0)\|_{C^1} = 0. \tag{40}$$

This means that we can obtain the asymptotic form of z_λ by studying that of $N_\lambda(0)$. Recall that

$$N_\lambda(0) = -K \circ G_\lambda(0) = -K(g(\lambda u_0(t))).$$

Let f be a primitive of g. By (37) we have that $\mu_\pm = \lim_{x \to \pm\infty} f(x)/x$. Therefore we can apply Lemma 3, with the role of g in that lemma played by our f and $\phi = u_0$ (here it is important that f' is bounded), to conclude that

$$\lim_{\lambda \to \infty} \int_0^T g(\lambda u_0(t))\eta(t)\, dt = \int_0^T \chi(t)\eta(t)\, dt$$

for any $\eta \in L^1[0,T]$. This implies that $G_\lambda(0)$ converges *weakly* in $L^2[0,T]$ to χ. Therefore $N_\lambda(0) = -K(G_\lambda(0))$ converges *strongly* in C^1 to $u_1 = -K(\chi)$:

$$\lim_{\lambda \to \infty} \|N_\lambda(0) - u_1\|_{C^1} = 0. \tag{41}$$

Combining (39)–(41), we obtain the result. □

We note that the asymptotic behaviour of x_λ depends on g only through the limits $\mu_\pm$, on a, c because of the solutions of the linear equations involved, and on the forcing term p through the definitions of u_0, u_1. We also note that y_λ is simple to calculate, since it involves only the solutions of two linear equations. The linear equation defining u_0 is easy to solve if, e.g., p is a trigonometric polynomial, and the linear equation defining u_1 is always easy to solve, since the right-hand side is a step function.

It is worthwhile to single out the special case when $\mu_+ = \mu_- = \mu$. In this case u_1 does not depend on the forcing term p, in fact we have $\chi \equiv \mu$, so

$$y_\lambda = \lambda u_0 - \frac{\mu}{a}.$$

As was noted above this case includes the case of g almost periodic, and in particular of g periodic, in which μ is simply the average of g over a period.

Since unless $\mu_+ = \mu_-$, or u_0 has constant sign, u_1 will be C^1 but not C^2, we cannot replace the C^1-norm by the C^2-norm in (36).

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