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IN MEMORIAM JEAN LERAY (1906–1998)

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Jean Leray died at La Baule on November 10 1998, in this region of Bretagne where he was born, and where he had returned a few years ago. He was ninety-two since three days. With him we lose one of the most eminent mathematicians of our century, one of those who already belong to history. His mathematical work is astonishingly original, deep and diversified.

Born in Chantenay, near Nantes, on November 7 1906, the son of two teachers, a former student of the Ecole Normale Supérieure in Paris, Leray defends in 1933, at the Faculty of Science of Paris, a PhD thesis directed by Henri Villat [1]. It successfully deals with the global study of *nonlinear integral equations* and the stationary solutions of the *equations of hydrodynamics.* The corresponding evolution equations are masterly treated in two memoirs published in 1934 [2]. They introduce for the first time various fundamental concepts like those of *weak solutions,* called *turbulent solution* by Leray. Their contents is beautifully described by Leray himself, in his last publication [35]:

> The theoretical study of fluid motion with initial conditions leads thus in various cases to a same conclusion: the existence of at least one weak solution which is regular and unique near the initial time, and which exists for any further time. [...] In other words, in those cases: a fluid motion initially regular remains regular during some time interval; then he goes on indefinitely; but does it remain regular and uniquely determined? On still ignores the answer to this double question. It was asked, sixty

> years ago, in an extremely special case. Then H. Lebesgue, consulted, said: "Do not spend too much time to such a rebel question. Do something else!"[1].

We will see that Leray has followed the advice.

A meeting with the Polish mathematician Julius Schauder reveals to Leray the topological techniques required by his thesis. The consequence is a joint paper also published in 1934 [45], worked out in two weeks, the year before, in Luxembourg's garden of Paris. Topological degree theory in infinite-dimensional Banach spaces is born, as well as the global theory of *nonlinear elliptic partial differential equations.* The *Leray–Schauder degree* and the *Leray–Schauder continuation method* remain the model for the whole development of nonlinear functional analysis and of fixed point theory. The founding paper, whose style is astonishingly modern, is one of the most quoted and used mathematical works in this century. In the remaining years before the second World War, Leray initiates algebraic topology in infinite dimensional Banach spaces as an application of his *product formula* for degree [4], and successfully applies the new continuation method to the theory of *wakes and bows* [5], and to *fully nonlinear Dirichlet problems of Bernstein type* [6], [7].

The war distracts Leray, now professor at the Faculty of Science of Nancy, from his mathematical studies. The young reserve lieutenant remains alone to command his artillery battery, all professional officers having escaped. Made prisonner by the Germans on June 24 1940, Leray is confined in the Oflag XVII A in Austria, near Austerlitz. Among privations and difficulties of all sorts, he organizes and leads there a university for French prisonners, which delivers, in five years, five hundred diplomas, later validated by the University of Paris. Afraid that his expertise in fluid mechanics could lead the Germans to force him collaborating to their war effort, the French mathematician concentrates all his teaching in the Oflag on *algebraic topology,* a military harmless topics which underlines his work on nonlinear equations. With few documents at his disposal, Leray reconstructs this branch of mathematics through an original and personal approach, which provides a substantial generalization of his 1934 joint results with Schauder, avoids the finite-dimensional approximation, allows to work in nonlinear spaces and provides links with Lefschetz fixed point theory. In a series of substantial memoirs published just after the war and based upon his lectures in Austria [8]–[10], Leray introduces the fundamental concepts of *sheaves* and

[1] L'étude théorique d'écoulements fluides à donnée initiale aboutit donc dans des cas très divers à une même conclusion: l'existence d'au moins une solution faible qui est régulière et unique près de l'instant initial, et qui existe à toute période ultérieure. [. . .] Autrement dit, dans ces cas: un écoulement fluide initialement régulier reste régulier durant un certain intervalle de temps; ensuite il se poursuit indéfiniment; mais reste-t-il régulier et bien déterminé? On ignore la réponse à cette double question. Elle fut posée il y a soixante ans dans un cas extrêmement particulier. Alors H. Lebesgue, consulté, déclara: "Ne consacrez pas trop de temps à une question aussi rebelle. Faites autre chose!".

spectral sequences. In his hands and those of other mathematicians like Henri Cartan, Jean–Pierre Serre, Armand Borel, and Alexandre Grothendieck, those tools not only revolutionize algebraic topology, but also the theory of functions of several complex variables, homological algebra, algebraic geometry and, more recently, algebraic analysis.

Appointed Professor at the Collège de France in 1947, in the *Chair of differential and functional equations,* that he occupies till his *emeritus* in 1978, Leray developes till 1950 his ideas on the *cohomology of closed continuous maps, fiber spaces and Lie groups* (*Leray–Hirsch theorem*) [12], [13], and continues his work in fluid mechanics by contributing to the theory of *airplane wings* [11], in the line of Tchapliguine and Prandtl's work.

Leray then moves to *linear hyperbolic partial differential equations of arbitrary order,* masterly treated by Hadamard and Schauder in the second order case. When the coefficients are constants, Leray extends to several variables Heaviside symbolic calculus and applies Schwartz recent theory of distributions and algebraic geometry to study the *elementary solutions* (*Herglotz–Petrowsky–Leray formula*), extending and putting a new light on the classical work of Herglotz, Petrowsky, M. Riesz and Bureau. In the case of variable coefficients, besides introducing for the first time distribution solutions, Leray corrects and developes the results of Petrowsky based upon the *energy method.* The notes of the lectures delivered at Princeton in 1953 and Roma in 1956 [15], [16] are classical today.

Between 1955 and 1965, Leray works on *Cauchy problem for analytic partial differential equations* which are singular on the manifold carrying the initial data [17], [18], making the first important step since the work of Cauchy and Kovalewskaya in the XIXth century. His aim is to prove that the singularities of the solution belong to the characteristics issues from the singularities of the data, or tangent to the manifold carrying them. This vast program is not completely realized yet. The mathematical problems raised to solve this difficult question lead him to an important generalization of *Cauchy formula* and of the *residues theorem* to functions of several complex variables (*Cauchy–Fantappié–Leray formulas*) [19], to *continuations of the Laplace transform* [21], and to new insights on *asymptotic wave theory* (with Gårding and Kotake) [37]. This decade is also enriched by important work in *elasticity theory* (motivated by new techniques in the construction of bridges) [22], [23], [40], in *fixed point theory* in nonlinear spaces [20], simplifying and extending his work of the Oflag *(Leray trace)*, and in the theory of *monotone-like operators* [41] (with J. L. Lions), allowing the study of quasi-linear elliptic boundary value problems not monotone in their lower order terms (*Leray–Lions operators*).

The following five years are mainly devoted to the study of *non strict hyperbolic systems,* which are important in relativistic magnetohydrodynamics. They are solved in some Gevrey spaces, intermediate between the spaces of holomorphic and of smooth functions, in collaboration with Ohya and Waelbroeck [23], [42].

At the beginning of the seventies, Arnold calls Leray's attention on Maslov's work on *asymptotic solutions of partial differential equations,* connected to the WKB method in quantum mechanics. Leray introduces there techniques of pseudo-differential operators and a new structure based upon symplectic geometry, that he names *Lagrangian analysis.* His approach leads mathematically to a constant, which can be identified to Planck's one, and provides a new interpretation of the Schrödinger, Klein–Gordon and Dirac equations. The Seminar at the Collège de France which exposes this theory is translated in Russian in 1981 and in English in 1982 [26], [27].

Schrödinger equation for one electron can be solved using Fuch's theorem. Leray proposes, at the beginning of the eighties, an extension of Fuch's theorem for the case of an atom with several electrons, and explicits the behavior of the solutions near the atomic nucleus [28]–[30]. In the mean time, Leray had returned, together woth Hamada, Wagshal and Takeuchi, to various extensions of Cauchy–Kovalewskaya's theorem, with special emphasis upon the *Cauchy problem with ramified data* [24], [25], [39], and the *analytic continuation of the solutions* [31], [38], his ultimate research area. Motivated by soil mechanics, Leray has also contributed, around the turn of the nineties, to the *propagation of waves in an elastic half-plane,* through a new technics that he calls the *Laplace–d'Alembert transform* [32]–[34], [44].

The *Scientific Work of Leray* [36] is published in 1998, jointly by the French Mathematical Society and Springer–Verlag. It contains about half of his papers, distributed in three volumes, respectively devoted to topology and fixed point theorems, partial differential equations and fluid mechanics, functions of several complex variables and holomorphic partial differential equations, with corresponding introductions by Armand Borel, Peter Lax and Guennadi Henkin. They reproduce neither the monographs [20], [21], nor the papers [28]–[30], [31]–[34], [38], [44].

Such a diversity in mathematical creation is quite exceptional. Leray nicely describes it in the 1953 *Notice* [14] on his scientific work:

> The essential character of my publications is however their diversity: the problems which attracted me required techniques still unused in the specialty where they were classified; their solution required or suggested improvements of those techniques, such that, to develop them, I had to change my area[2].

[2]Le caractère essentiel de mes publications est cependant leur diversité: les problèmes qui m'attirèrent exigèrent des procédés encore inusités dans la spécialité où ils étaient catalogués;

Leray came back in a more pictured way to his diversity, when he wrote to me, more than thirty years later:

> Since 1934, more than one other virginal topics appeared to me, to which I did not resist. This intellectual donjuanism leaves me only rare periods of rest[3].

He had however to find some to marry, in 1932, a high school mathematical teacher, Marguerite Trumier, et to help in the education of their three children, Jean–Claude, Françoise and Denis, who respectively became engineer, biologist and medical doctor.

This diversity mostly reflects Leray's independence of mind. He never could alienate himself to a political party, a school, a group or an ideology. Present at the very first meetings of the future Bourbaki group in the early thirties, he leaves it soon. The spirit there must be too foreign to somebody writing in his *Notice* [14]:

> It seems to me that one must attribute an equal importance to technical problems, from which mathematics is born, and to theoretical constructions, which are their issue[4].

Fourty years laters, Leray was among the few important mathematicians to publicly react against the excesses related to introducing "modern mathematics" into primary and secondary schools programs.

With his caustical mind, which reminds Voltaire or Henri Poincaré, with his love of French language, whose severe elegance follows from a constant care for concision, Leray has given, in too scarce general articles, interesting and witty reflexions about the mathematical and teaching activity, and has written a few biographical notices revealing his culture, his humanism and his sense of friendship, specially for his former collaborator Schauder, murdered by the Nazis. Leray's lectures, without complaisance or embellishments, required a lot from the listeners.

The international fame of Leray's accomplishments provided him also many deserved honors: five times laureate of the French Academy of Science before becoming a member in 1953, elected to some fifteen foreign academies, Leray has been awarded the International Prices Malaxa (with Schauder) (1938), Feltrinelli (1971), Wolf (1979), and the Lomonossov medal (1988). Even if he was President of the International Congress of Mathematicians at Nice in 1970, it is somewhat strange not to find his name among the plenary speakers at any of the

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leurs solutions nécessitèrent ou suggérèrent des perfectionnements de ces procédés tels que pour les développer je dus changer de spécialité.

[3]Depuis 1934, plus d'un autre sujet vierge m'est apparu, à l'attrait duquel je n'ai pas résisté. Ce donjuanisme intellectuel ne me laisse que de rares moments de répit.

[4]Il me semble qu'on doive attribuer une importance égale aux problèmes techniques, dont sont nées les Mathématiques, et aux constructions théoriques, qui sont leur aboutissement.

fourteen International Congresses of Mathematicians held during his scientific career. Program committees could have been better inspired.

Leray has been an member of the editorial board of several mathematical journals. From 1946 till 1972, he has directed the famous *Journal de mathématiques pures et appliquées* created by Liouville. The *Topological Methods in Nonlinear Analysis* had the privilege to count Jean Leray in its editorial board from the creation of the journal, in 1993, till 1996, when the French mathematician decided to quit public scientific activity.

Jean Leray admired deeply Henri Poincaré and his mathematics, and has edited the part of his *Oeuvres* devoted to algebraic topology. He concludes as follows his biography of Poincaré in the French *Encyclopaedia Universalis*:

> The man and his work dazzeled his contemporaries. After a century of mathematical works, we can understand them more easily, speak about them in a more familiar way; but, the more we approach them, the more we admire and respect them[5].

No doubt that this conclusion will be entirely applicable, in the next century, to the personality and the mathematical work of Jean Leray.

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[5] L'homme et l'oeuvre éblouirent les contemporains. Après un siècle de travaux mathématiques, nous pouvons les comprendre avec plus d'aisance, parler plus familièrement d'eux; mais plus nous les approchons, plus nous les admirons et respectons.

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