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CONVERGENCE OF SOLUTIONS OF QUASI-VARIATIONAL INEQUALITIES AND APPLICATIONS

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1. Introduction

Let E be a Hausdorff topological vector space and let us consider, for any $n \in \mathbb{N}$, the following quasi-variational inequalities (in short q.v.i.) [1]:

$(1.1)_n$ find $u_n \in E$ such that $f_n(u_n, w) + \phi_n(u_n, u_n) \leq \phi_n(u_n, w)$ for any $w \in E$,
(1.2) find $u \in E$ such that $f(u, w) + \phi(u, u) \leq \phi(u, w)$ for any $w \in E$,

where $f_n : E \times E \to \mathbb{R}$, $f : E \times E \to \mathbb{R}$, $\phi_n : E \times E \to \mathbb{R} \cup \{\infty\}$, $\phi : E \times E \to \mathbb{R} \cup \{\infty\}$.

The aim of this paper is to give suitable conditions on the convergence of $(f_n)_n$ to f and $(\phi_n)_n$ to ϕ in order to obtain a convergence result for the solutions of $(1.1)_n$ to solutions of (1.2). This study was motivated by the increasing interest in the topic of generalized quasi-variational inequalites (in short g.q.v.i.), taking into account that a g.q.v.i. (see (3.2)) can be represented by a q.v.i. (1.2) with appropriate functions f and ϕ.

Moreover, while the problem of the existence of solutions of q.v.i. and g.q.v.i. has been investigated in many papers (see for example [5], [6], [14]), the problem of convergence of solutions of q.v.i. has been studied, in a particular setting, only in [3]. Finally, in Section 4, we consider Nash equilibria with dependent constraints (called in [7] generalized Nash equilibrium), which can be thought

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of as solutions of a q.v.i. ([1], [13]), and we obtain a convergence result which generalizes those contained in [4].

The paper, which is in line with a previous one concerning variational inequalities [10], is organized as follows. In Section 2 we establish conditions which guarantee that, for any sequence of solutions of $(1.1)_n$ converging to a point $u_0 \in E$, u_0 is a solution of (1.2). Such conditions are applied in Section 4 to the convergence of Nash equilibria with dependent constraints. Moreover, in Section 3 a convergence result for solutions of g.q.v.i. is given.

2. Convergence of solutions of quasi-variational inequalities

Let us assume that τ and σ are topologies on E with σ finer than τ. The main result of this section is the following:

Theorem 2.1. *We assume that the following conditions are satisfied:*

(2.1) *for any $u \in E$ $\phi(u, \cdot)$ is a proper τ-lower semicontinuous convex function on E,*

(2.2) *the sequence $(\phi_n)_n$ converges to ϕ in the following sense:*

(i) *for any $u \in E$, any $w \in E$ and any sequence $(u_n, w_n)_n$ τ-converging to (u, w) it results in*

$$\phi(u,w) \le \liminf_{n\to\infty} \phi_n(u_n, w_n),$$

(ii) *for any $u \in E$, any $w \in E$ and any sequence $(u_n)_n$ τ-converging to u there exists a sequence $(w'_n)_n$ σ-converging to w such that*

$$\phi(u,w) \ge \limsup_{n\to\infty} \phi_n(u_n, w'_n),$$

(2.3) *for any $u \in E$, $f(u, \cdot)$ is concave,*

(2.4) *for any $u \in E$, $f(u, u) = 0$,*

(2.5) *for any $w \in E$, $f(\cdot, w)$ is τ-lower semicontinuous on the segments of line of E,*

(2.6) *for any $u \in E$, any $w \in E$, any sequence $(u_n)_n$ τ-converging to u and any sequence $(w_n)_n$ σ-converging to w we have find that*

$$-f(w,u) \le \liminf_{n\to\infty} f_n(u_n, w_n).$$

Let $(u_n)_n$ be a sequence of solutions of $(1.1)_n$. If $(u_n)_n$ τ-converges in E to u_0, then u_0 is a solution of the quasi-variational inequality (1.2).

Proof. Let us prove that under assumptions (2.1), (2.3)–(2.5), any solution of the following problem

(2.7) find $u \in E$ such that $-f(w,u) + \phi(u,u) \le \phi(u,w)$ for any $w \in E$, is also a solution of (1.2). Indeed, let u_0 be a point of E satisfying (2.7) and let's assume that (1.2) is not verified by u_0. Then there exists a point w_0 in E such that

$$f(u_0, w_0) + \phi(u_0, u_0) > \phi(u_0, w_0).$$

Let us define, for any $t \in [0,1]$, $w_t = tu_0 + (1-t)w_0$ and note that from (2.1), (2.3) and (2.7) the following inequalities follow:

$$\begin{gathered} -f(w_t, u_0) + \phi(u_0, u_0) \le \phi(u_0, w_t) \le t\phi(u_0, u_0) + (1-t)\phi(u_0, w_0), \\ tf(w_t, u_0) + (1-t)f(w_t, w_0) \le f(w_t, w_t) = 0, \quad \text{for any } t \in [0,1]. \end{gathered}$$

Combining these inequalities we obtain

$$(1-t)\phi(u_0, w_t) + (1-t)[f(w_t, w_0) - \phi(u_0, w_0)] \le 0 \quad \text{for any } t \in [0,1].$$

But from lower semicontinuity of $\phi(u_0, \cdot\,)$ and (2.5) we infer that

$$\phi(u_0, w_0) < \liminf_{t \to 1} [f(w_t, w_0) + \phi(u_0, w_t)],$$

which implies that there exists t sufficiently near to 1 such that

$$-\phi(u_0, w_t) < f(w_t, w_0) - \phi(u_0, w_0),$$

and for such t it would result in $0 = (1-t)\phi(u_0, w_t) - (1-t)\phi(u_0, w_t) < 0$, which is a contradiction.

Now, if $(u_n)_n$ is a sequence of solutions of $(1.1)_n$ τ-converging to u_0, from (2.2) and (2.6) we infer that for any $w \in E$ there exists $(w'_n)_n$ σ-converging to w such that

$$\begin{aligned} -f(w, u_0) + \phi(u_0, u_0) &\le \liminf_{n \to \infty} (f_n(u_n, w'_n) + \phi_n(u_n, u_n)) \\ &\le \liminf_{n \to \infty} \phi_n(u_n, w'_n) = \phi(u_0, w), \end{aligned}$$

which proves that u_0 is a solution of (2.7) and therefore of (1.2). □

Remark 2.2. We wish to point out that Theorem 2.1 still holds if the functions involved satisfy only assumption (2.2) and the following one

(∗) for any $u \in E$, any $w \in E$, any sequence $(u_n)_n$ τ-converging to u and any sequence $(w_n)_n$ σ-converging to w we have

$$f(u, w) \le \liminf_{n \to \infty} f_n(u_n, w_n).$$

Such a result, which could appear simpler and more elegant than Theorem 2.1, actually can be applied only in particular situations. For example, let V be a Banach space with dual V^* and pairing $\langle \cdot\,, \cdot \rangle$, A and A_n, for any $n \in \mathbb{N}$, be operators between V and V^*. Let the topologies τ and σ be equal to the weak and to the strong topology on V respectively, and the sequence $(f_n)_n$ and the function f be defined by

$$f_n(u,v) = \langle A_n u, u-v\rangle \quad \text{and} \quad f(u,v) = \langle Au, u-v\rangle.$$

In order to satisfy condition $(*)$ one should assume that the sequence $(A_n)_n$ converges to A in the following sense

- for any $u \in E$, any sequence $(u_n)_n$, weakly converging to u, the sequence $(A_n u_n)_n$ strongly converges to Au.

Even if $A_n = A = I$, the identity operator of a Hilbert space, the previous condition is not satisfied! On the contrary, we will show in Corollary 2.4 that condition (2.6) holds under feasible assumptions.

From Theorem 2.1 we easily infer the following result.

Corollary 2.3. *Let $(S_n)_n$ be a sequence of non-empty valued multifunctions from E to E, and S be a multifunction from E to E such that $S(u)$ is a non-empty τ-closed convex set for any $u \in E$. We assume that f and $(f_n)_n$ satisfy condition* (2.6)*, and $(S_n)_n$ converges to S in the following sense*

(2.8)
- (i) *for any $u \in E$, any $w \in E$ and any sequence $(u_k, w_k)_k$ τ-converging to (u,w) such that $w_k \in S_{n_k}(u_{n_k})$ for a subsequence $(n_k)_k$, it results in $w \in S(u)$.*
- (ii) *for any $u \in E$, any sequence $(u_n)_n$ τ-converging to u and any $w \in S(u)$, there exists a sequence $(w'_n)_n$ σ-converging to w such that $w'_n \in S_n(u_n)$ for n sufficiently large.*

Let $(u_n)_n$ be a sequence of solutions of the q.v.i.

$(2.9)_n$ *find $u_n \in E$ such that $u_n \in S_n(u_n)$ and $f_n(u_n, w) \le 0$ for any $w \in S_n(u_n)$.*

If $(u_n)_n$ τ-converges to a point u_0, then u_0 is a solution of

(2.10) *find $u \in E$ such that $u \in S(u)$ and $-f(w,u) \le 0$ for any $w \in S(u)$.*

If, in addition, (2.3)–(2.5) *are satisfied, then u_0 is a solution of the q.v.i.*

(2.11) *find $u \in E$ such that $u \in S(u)$ and $f(u,w) \le 0$ for any $w \in S(u)$.*

Proof. It takes only to consider the function ϕ defined by:

$$\phi(u,w) = \psi_{S(u)}(w) \quad \text{and} \quad \phi_n(u,w) = \psi_{S_n(u)} w) \quad \text{for any } n \in \mathbb{N},$$

where ψ_K is, for any subset K of E, the indicator function of K, i.e. the function which takes the value 0 on K and ∞ otherwise. □

Remark 2.4. Let us note that, if τ and σ are respectively the weak and the strong topology on a Banach space V, assumption (2.8) amounts to saying that for any $u \in V$ and any sequence $(u_n)_n$ weakly converging to u, the sequence of sets $(S_n(u_n))_n$ Mosco converges to $S(u)$ [12].

The following two results deal with classical quasi-variational inequalities in Banach spaces, as considered, for example, in [3]. In the next results we consider a Banach space V with dual V^* and we denote by $\langle \cdot , \cdot \rangle$ the pairing between V and V^*.

Corollary 2.5. *Let A and A_n, $n \in \mathbb{N}$, be operators from V to V^* and ϕ and ϕ_n, $n \in \mathbb{N}$, be extended real-valued function on $V \times V$. We assume that the following assumptions are satisfied*

(2.12) *A is hemicontinuous on V,*

(2.13) *A_n are monotone operators uniformly bounded on V (see [8]),*

(2.14) *for any $u \in E$ $\phi(u, \cdot)$ is a proper lower semicontinuous convex function on V,*

(2.15) *the sequence $(\phi_n)_n$ converges to ϕ in the following sense*

(i) *for any $u \in E$, any $w \in E$ and any sequence $(u_n, w_n)_n$ weakly converging to (u, w) it results in*

$$\phi(u,w) \le \liminf_{n\to\infty} \phi_n(u_n, w_n),$$

(ii) *for any $u \in E$, any $w \in E$ and any sequence $(u_n)_n$ weakly converging to u there exists a sequence $(w'_n)_n$ strongly converging to w such that*

$$\phi(u,w) \ge \limsup_{n\to\infty} \phi_n(u_n, w'_n),$$

(2.16) *the sequence $(A_n)_n$ G-(s,s)-converges to A, that is for any $v \in V$ there exists a sequence $(v_n)_n$ strongly converging to v such that the sequence $(A_n v_n)$ strongly converges to Av in V^*.*

Let $(u_n)_n$ be a sequence of solutions of

$(2.17)_n$ *find $u \in V$ such that $\langle A_n u, u - v\rangle + \phi_n(u,u) \le \phi_n(u,v)$ for any $v \in V$.*

If $(u_n)_n$ is weakly convergent to a point u_0, u_0 is a solution of the q.v.i.

(2.18) *find $u \in V$ such that $\langle Au, u - v\rangle + \phi(u,u) \le \phi(u,v)$ for any $v \in V$.*

Proof. We see that $(2.17)_n$ and (2.18) are obtained respectively by $(1.1)_n$ and (1.2) taking

$$f_n(u,v) = \langle A_n u, u - v\rangle \quad \text{and} \quad f(u,v) = \langle Au, u - v\rangle.$$

Using the same arguments as in paper [10], we can prove that the assumptions of Theorem 2.1 are satisfied. More precisely, using (2.16), for any $w \in V$ we find a sequence $(w'_n)_n$ strongly converging to w such that $(A_n w'_n)_n$ strongly converges to Aw in V^*. Let u be a point of V, $(w_n)_n$ be a sequence strongly converging to w and $(u_n)_n$ be a sequence weakly converging to u. A_n being monotone, for any $n \in N$, we have

$$\langle A_n w_n', w_n' - u_n\rangle \geq \langle A_n u_n, w_n' - u_n\rangle = \langle A_n u_n, w_n' - w_n\rangle + \langle A_n u_n, w_n - u_n\rangle,$$

and let us note that, due to uniformly boundness, it results in

$$\lim_{n\to\infty} \langle A_n u_n, w_n' - w_n\rangle = 0.$$

Then we obtain

$$\langle Aw, w - u\rangle = \lim_{n\to\infty} \langle A_n w_n', w_n' - u_n\rangle \geq \limsup_{n\to\infty} \langle A_n u_n, w_n - u_n\rangle,$$

that is $f(w, u) \geq \limsup_{n\to\infty} -f_n(u_n, w_n)$, which is equivalent to the inequality (2.6). □

Corollary 2.6. *Let us assume that the sequence $(A_n)_n$ and let A be as in Corollary* 2.5 *and the sequence $(S_n)_n$ and S be as in Corollary* 2.3. *Let $(u_n)_n$ be a sequence of solutions of*

find $u_n \in V$ such that $u_n \in S_n(u_n)$ and $\langle A_n u_n, u_n - w\rangle \leq 0$ for any $w \in S_n(u_n)$.

If $(u_n)_n$ weakly converges to a point u_0, then u_0 is a solution of the q.v.i.

find $u \in V$ such that $u \in S(u)$ and $\langle Au, u - w\rangle \leq 0$ for any $w \in S(u)$.

Proof. As in Corollary 2.3. □

Remark 2.7. In [3] some results on the strong convergence of solutions of q.v.i. with strongly monotone operators are given. They can be easily derived from Corollary 2.6 adding the assumption that A is a strongly monotone operator.

If the function ϕ does not depend on the first variable, Theorem 2.1, as well as Corollary 2.5, imply two results of U. Mosco [12] and its generalizations given in [10].

Theorem 2.8 ([10, Theorem 2.2]). *Let ϕ and ϕ_n be functions from E to $\mathbb{R} \cup \{\infty\}$, f and f_n be as in Theorem* 2.1. *Let τ and σ be topologies on E with σ finer τ. We assume that* (2.3), (2.4), (2.5), (2.6) *and the following assumptions are satisfied:*

(2.19) *ϕ is a τ-proper lower semicontinuous convex function on E,*

(2.20) *the sequence $(\phi_n)_n$ converges to ϕ in the following sense:*

(i) *for any* $v \in E$ *and any sequence* $(v_n)_n$ τ*-converging to* v *it results in*

$$\phi(v) \le \liminf_{n\to\infty} \phi_n(v_n),$$

(ii) *for any* $v \in E$ *there exists a sequence* $(v'_n)_n$ σ*-converging to* v *such that*

$$\phi(v) \le \limsup_{n\to\infty} \phi_n(v'_n).$$

Let $(u_n)_n$ *be a sequence of solutions of*

$$\textit{find } u_n \in E \textit{ such that } f_n(u_n, w) + \phi_n(u_n) \le \phi_n(w) \textit{ for any } w \in E.$$

If $(u_n)_n$ *converges in* E *to a point* u_0*, then* u_0 *is a solution of the variational inequality*

$$\textit{find } u \in E \textit{ such that } f(u, w) + \phi(u) \le \phi(w) \textit{ for any } w \in E.$$

REMARK 2.9. Let us note that the convergences considered in (2.2), (2.6) and (2.20) are types of sequential Γ-convergences (see for example [2]).

COROLLARY 2.10 ([10, Corollary 2.1]). *Let* A *and* A_n, $n \in \mathbb{N}$, *be operators from* V *to* V^*, ϕ *and* ϕ_n, $n \in \mathbb{N}$, *be extended real-valued functions on* V. *We assume that* (2.12), (2.13), (2.16) *and the following assumption are satisfied*

(2.21) $(\phi_n)_n$ *Mosco converges to a function* ϕ*, proper, convex and lower semicontinuous on* V.

Let $(u_n)_n$ *be a sequence of solutions of*

$$\textit{find } u_n \in V \textit{ such that } \langle A_n u_n, u_n - v\rangle + \phi_n(u_n) \le \phi_n(v) \textit{ for any } v \in V.$$

If $(u_n)_n$ *weakly converges to a point* u_0*, then* u_0 *is a solution of the variational inequality*

$$\textit{find } u \in V \textit{ such that } \langle Au, u - v\rangle + \phi(u) \le \phi(v) \textit{ for any } v \in V.$$

REMARK 2.11. Corollary 2.10 generalizes part (a) of Theorem B in [12]; when the functions ϕ_n and ϕ are the indicator functions of non-empty closed convex sets K_n and K, it generalizes part (a) of Theorem A in [12] (see Remark 2.2 in [10]).

3. Generalized quasi-variational inequalities

Let us assume again that V is a Banach space with dual V^*. A multi-valued operator T is a map which associates to any point u of V a nonempty subset $T(u)$ of V^*. Let us recall some definitions of multi-valued operators. The operator T is *monotone* whenever, for any $u \in V$ and any $v \in V$, $\langle u^* - v^*, u - v\rangle \ge 0$ for any $u^* \in T(u)$ and any $v^* \in T(v)$. Let s and w^* be the strong topology on V and the weak topology on V^* respectively, T is (s,w^*)-*lower semi-continuous* if for any $u \in V$ and any sequence $(u_n)_n$ converging to u in the strong topology on V, there exists a sequence $(u_n^*)_n$ w^*-converging to u^* such that $u_n^* \in T(u_n)$ for n sufficiently large. In the following, T and T_n, for any $n \in \mathbb{N}$, will be multi-valued operators from V to V^*; S and S_n, for any $n \in \mathbb{N}$, will be non-empty valued multifunctions from V to V.

Now, let us consider the following g.q.v.i.:

$(3.1)_n$ find $u_n \in E$ such that $u_n \in S_n(u_n)$ and $\mathrm{Sup}_{u^* \in T_n(u_n)} \langle u^*, u_n - w\rangle \le 0$ for any $w \in S_n(u_n)$,

(3.2) find $u \in E$ such that $u \in S(u)$ and $\mathrm{Sup}_{u^* \in T(u)} \langle u^*, u - w\rangle \le 0$ for any $w \in S(u)$.

Using Corollary 2.2, we find a convergence result for solutions of these problems.

Theorem 3.1. *Let's assume that the following assumptions are satisfied*

(3.3) *$S(u)$ is non-empty, convex and closed for any $u \in V$,*

(3.4) *the sequence $(S_n)_n$ converges to S in the sense of* (2.8),

(3.5) *for any $n \in N$, T_n is a monotone multi-valued operator,*

(3.6) *for any sequence $(u_n)_n$ contained in a bounded set there exists a positive number k such that for any $u_n^* \in T_n(u_n)$ it results $\|u_n^*\| \le k$ for any $n \in \mathbb{N}$,*

(3.7) *the sequence $(T_n)_n$ converges to T in the following sense: for any $w \in V$, any $w^* \in T(w)$, there exists a sequence $(w_n', w_n^*)_n$ strongly converging to (w, w^*) such that $w_n^* \in T_n(w_n')$ for n sufficiently large (i.e. $(T_n)_n$ $G^-(s,s)$-converges to T),*

(3.8) *T is (s,w^*)-lower semicontinuous on the segments of line.*

Whenever a sequence of solutions of $(3.1)_n$ *weakly converges to a point u_0, then u_0 is a solution of* (3.2).

Proof. Let $(u_n)_n$ be a sequence of solutions of $(3.1)_n$ weakly converging to $u_0 \in V$ and let w be a point of $S(u_0)$. From (2.8)(ii) there exists a sequence $(w_n)_n$ strongly converging to w such that $w_n \in S_n(u_n)$ for n sufficiently large.

For any $w^* \in T(w)$ there exist sequences $(w_n^*)_n$ and $(w_n')_n$ strongly converging to w^* and w respectively such that $w_n^* \in T_n(w_n')$ for n sufficiently large.

From (3.5) and (3.6), for any $u_n^* \in T_n(u_n)$ we find that

$$\langle w_n^*, u_n - w_n'\rangle = \langle w_n^* - u_n^*, u_n - w_n'\rangle + \langle u_n^*, u_n - w_n'\rangle \le \langle u_n^*, u_n - w_n\rangle + \langle u_n^*, w_n - w_n'\rangle,$$

and

$$\lim_{n\to\infty} \langle u_n^*, w_n - w_n'\rangle = 0.$$

Therefore, for any $w^* \in T(w)$, it results in

$$\langle w^*, u_0 - w\rangle \le \liminf_{n\to\infty} \sup_{u_n^*\in T_n(u_n)} \langle u_n^*, u_n - w_n\rangle \le 0,$$

and this amounts to saying that

$$\sup_{w^*\in T(w)} \langle u^*, u_0 - w\rangle \le 0 \quad \forall w \in S(u_0). \tag{3.9}$$

Now it is sufficient to prove that any solution of problem (3.9) is also a solution of (3.2).

Let $w \in S(u_0)$, $(t_n)_n$ be a sequence of points of $]0,1]$ decreasing to 0 and consider $v_n = t_n w + (1-t_n)u_0$ for any $n \in \mathbb{N}$. From lower semicontinuity of T, for any $u^* \in T(u_0)$ there exists a sequence $(v_n^*)_n$ weakly converging to u^* with $v_n^* \in T(v_n)$ for n sufficiently large. Since $v_n \in S(u_0)$ for any $n \in \mathbb{N}$, and u_0 is a solution of (3.9), it results $\langle v_n^*, u_0 - v_n\rangle \le 0$ for n sufficiently large. We can conclude that $\langle v_n^*, u_0 - w\rangle \le 0$, which implies $\langle u^*, u_0 - w\rangle \le 0$. u^* being an arbitrary point of $T(u_0)$, we obtain that u_0 is a solution of (3.2). □

4. Application to convergence of Nash equilibria

Let us consider two Hausdorff topological vector spaces E_1 and E_2, two functionals J_1 and J_2 from $E = E_1 \times E_2$ to $\mathbb{R}$, and two multifunctions S_1 from E to E_1 and S_2 from E to E_2. We assume that τ_i and σ_i are topologies on E_i, for $i = 1, 2$, with σ_i finer than τ_i, and denote by τ and σ the product topologies on E.

Let us recall [7] that a point $u = (u_1, u_2)$ is a generalized Nash equilibrium if

$$u_1 \in S_1(u) \text{ and } J_1(u_1, u_2) \le J_1(v_1, u_2) \text{ for any } v_1 \in S_1(u),$$
$$u_2 \in S_2(u) \text{ and } J_2(u_1, u_2) \le J_2(u_1, v_2) \text{ for any } v_2 \in S_2(u).$$

Let $(J_{1,n})_n$ and $(J_{2,n})_n$ be sequences of real valued functionals on E, $(S_{1,n})_n$ and $(S_{2,n})_n$ be sequences of multifunctions from E to E_1 and to E_2 respectively. As a consequence of Theorem 2.1, we find a result about the convergence of generalized Nash equilibria.

Theorem 4.1. *Let us assume that the following assumptions are satisfied*

(4.1) *the sequences $(J_{1,n})_n$ and $(J_{2,n})_n$ continuously converge to J_1 and J_2 respectively, that is, for any $u \in E \times E$ and any sequence $(u_{1,n}, u_{2,n})$ τ-converging to $u = (u_1, u_2)$, it results in*

$$J_1(u_1, u_2) = \lim_{n\to\infty} J_{1,n}(u_{1,n}, u_{2,n}) \quad \textit{and} \quad J_2(u_1, u_2) = \lim_{n\to\infty} J_{2,n}(u_{1,n}, u_{2,n}),$$

(4.2) *the sequences $(S_{1,n})_n$ and $(S_{2,n})_n$ converge to S_1 and S_2 respectively in the sense of* (2.8).

Then, whenever a sequence $(u_n)_n$ of generalized Nash equilibria for $(J_{1,n}, J_{2,n})$, τ-converges to a point u_0, u_0 is a generalized Nash equilibrium for (J_1, J_2).

Proof. Let us define for any $u \in E$, any $w \in E$ and any $n \in \mathbb{N}$

$$\begin{aligned} f(u,w) &= f_n(u,w) = 0, \\ \phi(u,w) &= J_1(w_1,u_2) + J_2(u_1,w_2) + \psi_{S_1(u)\times S_2(u)}(w), \\ \phi_n(u,w) &= J_{1,n}(w_1,u_2) + J_{2,n}(u_1,w_2) + \psi_{S_{1,n}(u)\times S_{2,n}(u)}(w). \end{aligned}$$

Then Theorem 2.1 can be applied if we prove that (2.2) is satisfied. This could be done using results concerning variational convergences for functionals and convergences for multifunctions (see for example [8], [9]), but, for the sake of completeness, we prefer to prove it directly. In fact, if $(u_n)_n$ and $(w_n)_n$ are sequences converging in $E \times E$ to u and w respectively, in order to verify (i) in (2.2) it can be assumed that, for all k belonging to an infinite set $\mathbb{N}' \subset \mathbb{N}$

$$w_{1,k} \in S_{1,k}(u_k) \quad \text{and} \quad w_{2,k} \in S_{2,k}(u_k),$$

(if not there is nothing to prove). From (i) in (2.8) the point $w_1 \in S_1(u)$ and the point $w_2 \in S_2(u)$, so condition (i) in (2.2) follows from the continuous convergence of $(J_{i,n})_i$ to J_i, for $i = 1, 2$. In the same way, in order to prove (ii) in (2.2) it can be assumed that $w_1 \in S_1(u)$ and $w_2 \in S_2(u)$. The result then follows from (ii) in (2.8) and from (4.1). □

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