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NEW ASPECTS OF THE L-CONDITION FOR ELLIPTIC SYSTEMS

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Dedicated to O. Ladyzhenskaya

Introduction

In this paper, the L-condition for an elliptic system $(\mathcal{A}, \mathcal{B})$ in a bounded domain $\Omega \subset \mathbb{R}^n$ is reformulated in a new algebraic form. A square matrix function, $\Delta^+_{\mathcal{B}}$, defined on the unit cotangent bundle, $\mathrm{ST}^*(\partial\Omega)$, is constructed from the principal symbols of the coefficients of the boundary operator, $\mathcal{B}$, and a spectral pair for the family of matrix polynomials associated with the principal symbol of the elliptic operator, $\mathcal{A}$. The L-condition is equivalent to the condition that the function, $\Delta^+_{\mathcal{B}}$, have invertible values.

This paper is divided into three sections. In Section 1 we give the definition of elliptic systems and the L-condition. In Section 3 the L-condition is reformulated in various equivalent forms, which include (in addition to the new form indicated above) the Lopatinskiĭ condition, the complementing condition of Agmon–Douglis–Nirenberg and a condition of Fedosov. The purpose of Section 2 is to briefly state the definition of a spectral triple for a matrix polynomial, which is needed for the proof of the equivalence of all these conditions.

1. Elliptic systems and the L-condition

Let Ω be a bounded domain in $\mathbb{R}^n$, a *domain* being a connected, open set. We will consider systems of linear differential equations $\mathcal{A}(x, D)u(x) = f(x)$,

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$x \in \Omega$, where u and f are p-vector functions and

$$\mathcal{A}(x, D) = [A_{ij}(x, D)]_{p\times p}, \quad x \in \overline{\Omega},$$

is a $p \times p$ matrix such that the elements, A_{ij}, are linear differential operators

$$A_{ij}(x, D) = \sum_{|\alpha|\le\alpha_{ij}} a_{ij}^{(\alpha)}(x) D^\alpha$$

with smooth coefficients, $a_{ij}^{(\alpha)} \in C^\infty(\overline{\Omega})$. Here the usual multi-index notation is being used: $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, and for convenience when operating with the Fourier transformation, the basic derivatives include the factor $1/i$, that is, $D_j = i^{-1}\partial/\partial x_j$, where $i = \sqrt{-1}$. The boundary, $\partial\Omega$, of the domain Ω is assumed to be C^∞.

Let α_{ij} denote the order of A_{ij}. If $A_{ij} \equiv 0$, then we set $\alpha_{ij} = -\infty$. Now suppose that we have integers $s_1, \dots, s_p, t_1, \dots, t_p$ such that the operator A_{ij} has order

$$\alpha_{ij} \le s_i + t_j,$$

where it is to be understood that $A_{ij} \equiv 0$ if $s_i + t_j < 0$. Clearly, any given s_i, t_j may be replaced by s_i + constant, t_j − same constant.

Then we let $A'_{ij}(x, D)$ denote the sum of terms in $A_{ij}(x, D)$ which are exactly of the order $s_i + t_j$, with lower-order terms replaced by zeros. For arbitrary real $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ we define the DN principal part $\pi_D\mathcal{A}(x, \xi)$ as the polynomial $p \times p$ matrix

$$\pi_D\mathcal{A}(x, \xi) := [A'_{ij}(x, \xi)]_{p\times p} = [\pi_{s_i+t_j} A_{ij}(x, \xi)]_{p\times p}.$$

Definition 1.1. The operator $\mathcal{A}(x, D)$ is said to be *elliptic at* $x \in \overline{\Omega}$ if there exist DN numbers $s_1, \dots, s_p, t_1, \dots, t_p$ such that the characteristic polynomial

$$\chi(\xi) = \det \pi_D\mathcal{A}(x, \xi) \ne 0$$

for each $0 \ne \xi \in \mathbb{R}^n$.

Note that the choice of DN numbers (if they exist) is not unique. Nevertheless, from now on we write $\pi\mathcal{A}$ instead of $\pi_D\mathcal{A}$ for the principal part. The characteristic polynomial $\chi(\xi)$ is homogeneous in ξ of degree $m = \sum(s_i + t_i)$, that is,

$$\chi(c\xi) = c^m\chi(\xi), \quad c \in \mathbb{R}. \tag{1}$$

The differential operator $\mathcal{A}(x, D)$ is said to be *elliptic* on $\overline{\Omega}$ if it is elliptic at each point $x \in \overline{\Omega}$ with a fixed set of DN numbers.

Let $\nu : \partial\Omega \to \mathrm{T}(\mathbb{R}^n)$ be the inward-pointing unit normal along $\partial\Omega$, and let $n : \partial\Omega \to \mathrm{T}^*(\mathbb{R}^n)$ be the image of ν by the index-lowering operator $\mathrm{T}(\mathbb{R}^n) \to \mathrm{T}^*(\mathbb{R}^n)$ that maps $\partial/\partial x_i$ to dx_i. Because the space of conormal vectors at a point $x \in \partial\Omega$ is one-dimensional and $n(x) \neq 0$ for each x, every $\xi \in \mathrm{T}^*_x(\mathbb{R}^n)$ can be written uniquely in the form

$$\xi = \xi' + \xi_n \cdot n(x), \tag{2}$$

where $\xi' \in \mathrm{T}^*_x(\partial\Omega)$, $\xi_n \in \mathbb{R}$. This defines a vector bundle isomorphism $\mathrm{T}^*(\mathbb{R}^n) \simeq \mathrm{T}^*(\partial\Omega) \oplus (\partial\Omega \times \mathbb{R})$ and we are justified in writing for each $\xi \in \mathrm{T}^*_x\,\mathbb{R}^n$,

$$\xi = (\xi', \xi_n), \tag{2'}$$

where $\xi' \in \mathrm{T}^*_x(\partial\Omega)$, $\xi_n \in \mathbb{R}$.

The boundary, $\partial\Omega$, is a compact C^∞ manifold. When dealing with a boundary operator for an elliptic system it is important to be able to work with pseudo-differential operators on $\partial\Omega$; this makes it possible to modify the order of the boundary operator. For any compact C^∞ manifold M, let $\mathrm{OS}^m(M)$, $m \in \mathbb{R}$, be the set of pseudo-differential operators (p.d.o.'s) of order m on M as defined in [5, Chapter 18]. For each $m \in \mathbb{R}$, we define the "classical" p.d.o.'s to be those operators, $A \in \mathrm{OS}^m(M)$, that have a well-defined principal symbol $\pi A \in C^\infty(\mathrm{T}^*(M)\backslash 0)$, positively homogeneous of order m. The properties of classical p.d.o.'s are studied in detail in [8, Chapter 8].

We now turn to the formulation of boundary value problems for a (properly) elliptic operator $\mathcal{A}$, with DN numbers $s_1, \dots, s_p, t_1, \dots, t_p$. As usual, we let $D_n = i^{-1}\partial/\partial n$, where n is the inward-pointing unit (co)normal vector field on $\partial\Omega$. Points on the boundary will be denoted by $y \in \partial\Omega$. Let r be the number of roots of the polynomial $P(\lambda) = \det \pi\mathcal{A}(y, (\xi', \lambda))$ in the upper half-plane $\operatorname{Im}\lambda > 0$, i.e. half the order of the characteristic polynomial χ. In addition to the p equations, $\mathcal{A}(x, D)u(x) = f(x)$, in Ω, we consider r boundary conditions

$$\sum_{j=1}^{p} B_{kj}(y, D)u_j(y) = g_k(y), \quad k = 1, \dots, r,$$

that is, $\mathcal{B}(y, D)u(y) = g(y)$, where $\mathcal{B}(y, D)$ is the matrix operator $[B_{kj}(y, D)]_{r\times p}$. The boundary operators B_{kj} are taken in the form

$$B_{kj}(y, D) = \sum_{\kappa=0}^{\ell_{kj}} b^\kappa_{kj}(y, D')D^\kappa_n, \quad y \in \partial\Omega,$$

where $b^\kappa_{kj}(y, D')$ are (classical) pseudo-differential operators on the manifold $\partial\Omega$. The principal parts are denoted by $\pi b^\kappa_{kj}(y, \xi')$, $(y, \xi') \in \mathrm{T}^*(\partial\Omega)\backslash 0$, and we also write $\beta^\kappa_{kj} = \operatorname{ord} b^\kappa_{kj}$.

The DN principal part of the boundary operator $\mathcal{B}$ is defined as follows. Let

$$m_{kj} := \max_\kappa \operatorname{ord}(b^\kappa_{kj}D^\kappa_n) = \max_\kappa(\beta^\kappa_{kj} + \kappa)$$

(the numbers m_{kj} can be negative and also non-integer, i.e. $m_{kj} \in \mathbb{R}$) and then let

$$m_k := \max_{1\le j\le p} (m_{kj} - t_j), \quad k = 1, \dots, r,$$

so that $m_{kj} \le m_k + t_j$. The DN *principal part* of the boundary operator $\mathcal{B}(y, D)$ is defined as the $r \times p$ matrix

$$\pi_D \mathcal{B}(y, \xi) = [B'_{kj}(y, \xi)]_{r\times p}, \qquad B'_{kj}(y, \xi) = \sum_{\kappa}{}' \pi b^{\kappa}_{kj}(y, \xi')\xi^{\kappa}_n,$$

where $\sum'_\kappa$ denotes the sum over those terms with $\beta^{\kappa}_{kj} + \kappa = m_k + t_j$. In other words, $B'_{kj}(y, \xi)$ consists of the principal parts of the terms in B_{kj} which are just of order $m_k + t_j$, with the other terms replaced by 0. As usual, $\xi = (\xi', \xi_n), \xi' \in \mathrm{T}^*_y(\partial\Omega) \setminus 0$ and ξ_n is conormal at y. Normally we denote the DN principal part by $\pi\mathcal{B}$ rather than $\pi_D\mathcal{B}$.

Remark. The operators b^{κ}_{kj} can have negative order, $\beta^{\kappa}_{kj} < 0$.

Consider the decomposition (2), (2′) of the cotangent space $\mathrm{T}^*_{y_0}(\partial\Omega)$ at the boundary point $y_0 \in \partial\Omega$ where y_0 is fixed. We substitute ξ_n by $i^{-1}d/dt$ and fix $\xi' \ne 0$ in the DN principal part of $\mathcal{A}$ to obtain the system of ordinary differential equations (with constant coefficients)

$$\pi\mathcal{A}\bigg(y_0, \bigg(\xi', \frac{1}{i}\frac{d}{dt}\bigg)\bigg) w(t) = 0, \quad t > 0,\ \xi' \in \mathrm{T}^*_{y_0}(\partial\Omega)\backslash 0. \tag{3}$$

The solutions of this equation are p-columns of exponential polynomials of the form $\sum p_j(t)e^{i\lambda_j t}$ where the p_j's are polynomials in t and the λ_j's are eigenvalues of $L(\lambda)$. The solution space $\mathfrak{M} = \mathfrak{M}(\xi')$ of (3) decomposes directly into

$$\mathfrak{M} = \mathfrak{M}^- \oplus \mathfrak{M}^+,$$

where $\mathfrak{M}^+$ consists of all solutions $w(t)$ with $w(t) \to 0$ as $t \to \infty$.

Definition 1.2. The pair of operators

$$\mathcal{A}(y, D), \mathcal{B}(y, D), \quad y \in \partial\Omega,$$

is said to fulfill the L-*condition* if for all $y \in \partial\Omega$, $0 \ne \xi' \in \mathrm{T}^*_y(\partial\Omega)$, the zero initial value problem

$$\begin{aligned}\pi\mathcal{A}\bigg(y, \bigg(\xi', \frac{1}{i}\frac{d}{dt}\bigg)\bigg) w(t) &= 0, \quad t > 0,\\ \pi\mathcal{B}\bigg(y, \bigg(\xi', \frac{1}{i}\frac{d}{dt}\bigg)\bigg) w(t)|_{|t=0} &= 0,\end{aligned}$$

has in $\mathfrak{M}^+ = \mathfrak{M}^+(\xi')$ the unique solution $w(t) = 0$.

Now we state the definition of L-ellipticity of a boundary value problem.

Definition 1.3. The boundary value problem

$$\mathcal{A}(x,D)u(x) = f(x), \quad x \in \Omega, \quad \mathcal{B}(y,D)u(y) = g(y), \quad y \in \partial\Omega,$$

is said to be L-*elliptic* in $\overline{\Omega}$ if:

(i) the operator $\mathcal{A}(x,D)$ is elliptic for all $x \in \overline{\Omega}$ (see Definition 1.1);

(ii) $\mathcal{A}(y,D), \mathcal{B}(y,D)$ satisfies for all $y \in \partial\Omega$ the L-condition of Definition 1.2.

As an aside, we mention that the elliptic operator $\mathcal{A}(x,D)$ is said to be *proper* at $x \in \partial\Omega$ if for each $0 \neq \xi' \in \mathrm{T}^*_x(\partial\Omega)$ the polynomial in $\lambda \in \mathbb{C}$,

$$P(\lambda) = \chi(\xi',\lambda) = \det \pi\mathcal{A}(x,(\xi',\lambda)),$$

has as many roots, r, in the upper half-plane $\operatorname{Im}\lambda > 0$ as in the lower half-plane, $\operatorname{Im}\lambda < 0$, counting multiplicities. It is not hard to see that if $(\mathcal{A},\mathcal{B})$ is L-elliptic then $\mathcal{A}$ must be properly elliptic at each $x \in \partial\Omega$, due to the homogeneity (1) of χ with $c = -1$. In this case, the degree, m, of the polynomial P is necessarily even, i.e. $m = 2r$.

Recall that a *Fredholm operator* is a bounded linear operator between two Banach spaces whose kernel has finite dimension, α, and whose image has finite codimension, β; the *index* of this operator is defined to be $\alpha - \beta$. It is well known that L-ellipticity is the necessary and sufficient condition for the boundary value problem operator $(\mathcal{A},\mathcal{B})$ to define a Fredholm operator (in appropriate Sobolev spaces). For the proof, we refer the reader to [8, Chapter 9].

The next section develops a spectral theory of matrix polynomials which makes it possible to reformulate the L-condition in various equivalent algebraic forms in Section 3.

2. Spectral triples for matrix polynomials

For any finite-dimensional vector spaces $\mathfrak{M},\mathfrak{N}$, let $\mathcal{L}(\mathfrak{M},\mathfrak{N})$ denote the set of linear maps from $\mathfrak{M}$ to $\mathfrak{N}$. A triple of operators (X,T,Y) is called an *admissible triple* if $X \in \mathcal{L}(\mathfrak{M},\mathbb{C}^p), T \in \mathcal{L}(\mathfrak{M}) = \mathcal{L}(\mathfrak{M},\mathfrak{M})$ and $Y \in \mathcal{L}(\mathbb{C}^p,\mathfrak{M})$. The vector space $\mathfrak{M}$ is called the *base space* of the admissible triple. Two admissible triples, (X,T,Y) and (X',T',Y'), are called *similar* if there exists an invertible operator $M \in \mathcal{L}(\mathfrak{M}',\mathfrak{M})$ such that

$$X' = XM, \quad T' = M^{-1}TM, \quad \text{and} \quad Y' = M^{-1}Y.$$

The admissible pair (X,T) is referred to as a *right* admissible pair, while (T,Y) is called a *left* admissible pair. If $S_j \in \mathcal{L}(\mathfrak{M},\mathfrak{N}_j)$, $j = 1,\dots,n$, we define

$$\operatorname{col}(S_j)_{j=1}^n = \begin{pmatrix} S_1 \\ \vdots \\ S_n \end{pmatrix} \in \mathcal{L}(\mathfrak{M},\mathfrak{N}_1 \oplus \ldots \oplus \mathfrak{N}_n).$$

Similarly, if $T_j \in \mathcal{L}(\mathfrak{M}_j, \mathfrak{N})$, $j = 1, \ldots, n$, we define

$$\mathrm{row}(T_j)_{j=1}^{n} = [\,T_1, \ldots, T_n\,] \in \mathcal{L}(\mathfrak{M}_1 \oplus \ldots \oplus \mathfrak{M}_n, \mathfrak{N}).$$

Let $L(\lambda) = \sum_{j=0}^{\ell} A_j \lambda^j$ be a $p \times p$ matrix polynomial. A complex number λ_0 is called an *eigenvalue* of $L(\lambda)$ if $\det L(\lambda_0) = 0$ and the set of all eigenvalues is called the *spectrum* of $L(\lambda)$, denoted by $\mathrm{sp}(L)$, i.e.

$$\mathrm{sp}(L) = \{\lambda \in \mathbb{C} : \det L(\lambda) = 0\}.$$

Now let γ be a simple, closed (rectifiable) contour not intersecting $\mathrm{sp}(L)$. Also let G denote the region inside γ.

Definition 2.1. A *γ-spectral triple* for $L(\lambda)$ is defined to be an admissible triple (X_+, T_+, Y_+) with the following properties:

(i) $\mathrm{sp}(T_+) \subset G$ (i.e. inside γ),
(ii) $L^{-1}(\lambda) - X_+(I\lambda - T_+)^{-1} Y_+$ has an analytic continuation in G,
(iii) $\mathrm{col}(X_+ T_+^j)_{j=0}^{\ell-1}$ is injective,
(iv) $\mathrm{row}(T_+^j Y_+)_{j=0}^{\ell-1}$ is surjective.

Also, we say that (X_+, T_+) is a (*right*) *γ-spectral pair* for $L(\lambda)$ and (T_+, Y_+) is a *left γ-spectral pair* for $L(\lambda)$.

Note that property (ii) can be replaced by

$$\text{(ii}'\text{)} \qquad \frac{1}{2\pi i} \int_\gamma \lambda^j L^{-1}(\lambda)\, d\lambda = X_+ T_+^j Y_+, \quad j = 0, 1, \ldots$$

Indeed, property (ii) holds if and only if

$$\frac{1}{2\pi i} \int_\gamma \lambda^j \{L^{-1}(\lambda) - X_+(I\lambda - T_+)^{-1} Y_+\}\, d\lambda, \quad j = 0, 1, \ldots$$

Since $\mathrm{sp}(T_+) \subset G$, this is equivalent to (ii$'$). Let $\mathfrak{M}_L$ denote the set of solutions $u \in C^\infty(\mathbb{R}, \mathbb{C}^p)$ of the equation $L(d/dt)u(t) = 0$. Every $u \in \mathfrak{M}_L$ can be written in the form

$$\text{(4)} \qquad u(t) = \sum p_i(t) e^{t\lambda_i},$$

where the p_i are $\mathbb{C}^p$-valued polynomials, and the complex numbers λ_i are eigenvalues of $L(\lambda)$, i.e. roots of the polynomial equation $\det L(\lambda) = 0$. (Essentially, the coefficients of p_i form Jordan chains for the matrix polynomial $L(\lambda)$.) It is well known that the dimension of $\mathfrak{M}_L$ is equal to the degree, α, of $\det L(\lambda)$.

Let $\mathfrak{M}_L^+$ denote the subspace of $\mathfrak{M}_L$ consisting of solutions of the form (4) such that the eigenvalues λ_i lie inside γ. One can show that every $u \in \mathfrak{M}_L^+$ has the representation

$$\text{(5)} \qquad u(t) = \frac{1}{2\pi i} \int_\gamma e^{t\lambda} L^{-1}(\lambda) \cdot \sum_{j=0}^{\ell-1} L_{j+1}(\lambda) u^{(j)}(0)\, d\lambda$$

where $L_j(\lambda) = A_j + A_{j+1}\lambda + \ldots + A_\ell \lambda^{\ell-j}$, $j = 0, \ldots, \ell$. This expresses u in terms of its Cauchy data at $t = 0$.

Theorem 2.2. *Let γ be a simple, closed contour not intersecting* $\mathrm{sp}(L)$. *Then there exists a γ-spectral triple for $L(\lambda)$. Any two γ-spectral triples for $L(\lambda)$ are similar.*

Sketch of Proof. We define an admissible triple (X_+, T_+, Y_+) with base space $\mathfrak{M}_L^+$ as follows:

$$X_+ u = u(0), \quad T_+ u = \frac{du}{dt}, \quad (Y_+ x)(t) = \frac{1}{2\pi i} \int_\gamma e^{t\lambda} L^{-1}(\lambda) x \, d\lambda.$$

(Note that if $u \in \mathfrak{M}_L^+$ then $du/dt \in \mathfrak{M}_L^+$.) The fact that (ii′) of Definition 2.1 holds is an immediate consequence of the definition of (X_+, T_+, Y_+). Since $u^{(j)}(0) = X_+ T_+^j u$, the injectivity of $\mathrm{col}(X_+ T_+^j)_{j=0}^{\ell-1}$ follows from (5); this proves (iii). It is also easy to see that the representation (5) implies (iv), i.e. $\mathrm{row}(T_+^j Y_+)_{j=0}^{\ell-1}$ is surjective. The proof of (i) and of uniqueness of spectral triples requires somewhat more work.

Remark. If (X_+, T_+, Y_+) is a γ-spectral triple for $L(\lambda)$ then it can be shown that $\mathrm{sp}(T_+) = \mathrm{sp}(L) \cap G$.

The Calderón projector. For any $u \in \mathfrak{M}_L$, the column vector $\mathcal{U} \in \mathbb{C}^{p\ell}$ defined by

$$\mathcal{U} = \mathrm{col}(u^{(j)}(0))_{j=0}^{\ell-1}$$

is the *Cauchy data* (or initial conditions) of u at $t = 0$. Recall that every $u \in \mathfrak{M}_L^+$ has the representation

$$u(t) = \frac{1}{2\pi i} \int_\gamma e^{t\lambda} L^{-1}(\lambda) \, [\, L_1(\lambda), \ldots, L_\ell(\lambda) \,] \; d\lambda \cdot \mathcal{U}. \tag{5′}$$

By taking initial conditions on the left-hand side of this equation we obtain $\mathcal{U} = P_\gamma \cdot \mathcal{U}$, where

$$P_\gamma = \frac{1}{2\pi i} \int_\gamma \begin{pmatrix} I \\ \vdots \\ \lambda^{\ell-1} I \end{pmatrix} L^{-1}(\lambda) \cdot [\, L_1(\lambda), \ldots, L_\ell(\lambda) \,] \; d\lambda. \tag{6}$$

The following theorem shows that P_γ is a projector, which we call the *Calderón projector* (because of the reference to Calderón in [7]).

Theorem 2.3. *P_γ is a projector in $\mathbb{C}^{p\ell}$. The image of P_γ is equal to the set of all Cauchy data, $\mathcal{U}$, of functions $u \in \mathfrak{M}_L^+$.*

Proof. In view of the equation $\mathcal{U} = P_\gamma \cdot \mathcal{U}$, the set of Cauchy data of functions $u \in \mathfrak{M}_L^+$ is contained in the image of P_γ. On the other hand, the image of P_γ is contained in the set of Cauchy data for if $c = [\,c_0, \dots , c_{\ell-1}\,] \in \mathbb{C}^{p\ell}$ let

$$u(t) = \frac{1}{2\pi i}\int_\gamma e^{t\lambda} L^{-1}(\lambda) \cdot \sum_{j=0}^{\ell-1} L_{j+1}(\lambda) c_j \, d\lambda.$$

Then $u \in \mathfrak{M}_L^+$ and its Cauchy data is $\mathcal{U} = \operatorname{col}(u^{(j)}(0))_{j=0}^{\ell-1} = P_\gamma c$. The fact that P_γ is a projector is now clear, for the equation $\mathcal{U} = P_\gamma \cdot \mathcal{U}$ implies that P_γ is the identity on its image.

Corollary 2.4. *Let* (X_+, T_+) *be a* γ*-spectral pair of* $L(\lambda)$ (*Definition* 2.1). *Then* P_γ *and* $\operatorname{col}(X_+T_+^j)_{j=0}^{\ell-1}$ *have the same image. Hence every* $u \in \mathfrak{M}_L^+$ *has a representation*

$$u(t) = X_+ e^{tT_+} c$$

for a unique c *in the base space of* (X_+, T_+).

The "left" Calderón projector is

$$P'_\gamma = \frac{1}{2\pi i}\int_\gamma \begin{pmatrix} L_1(\lambda) \\ \vdots \\ L_\ell(\lambda) \end{pmatrix} L^{-1}(\lambda)\,[\,I, \dots , \lambda^{\ell-1} I\,]\; d\lambda. \tag{7}$$

Note that P'_γ is just the transpose of the Calderón projector for the transposed matrix polynomial $L^T(\lambda) = \sum A_j^T \lambda^j$.

Theorem 2.5. P'_γ *is a projector in* $\mathbb{C}^{p\ell}$. *Also, if* (T_+, Y_+) *is any left* γ*-spectral pair of* $L(\lambda)$ (*see Definition* 2.1) *then* $\ker P'_\gamma = \ker \operatorname{row}(T_+^j Y_+)_{j=0}^{\ell-1}$.

For details on the proofs and further information about properties of spectral triples, the reader is referred to the book [8]. One should also note that the spectral theory of matrix polynomials has been developed extensively in [4], and for rational matrix functions in [2].

3. Alternative versions of the L-condition

Let $\mathcal{A}$ be an elliptic operator with DN numbers $s_1, \dots , s_p, t_1, \dots , t_p$. Associated with the DN principal part of $\mathcal{A}$ on the boundary $\partial\Omega$, there is the $p \times p$ matrix polynomial

$$L_{y,\xi'}(\lambda) := \pi\mathcal{A}(y, (\xi', \lambda)) = \sum_{j=0}^{\ell} A_j(y, \xi')\lambda^j,$$

$y \in \partial\Omega$, $0 \neq \xi' \in \mathrm{T}_y^*(\partial\Omega)$, where $\ell \leq \max(s_i + t_j)$. We suppose that $\mathcal{A}$ is properly elliptic, so that $\det L_{y,\xi'}(\lambda)$ has r roots in the upper half-plane $\operatorname{Im}\lambda > 0$ and r in the lower half-plane.

Remark. Recall that (ξ', λ) is the short notation for $\xi' + \lambda \cdot n(y)$.

Let $\mathcal{B}$ be a boundary operator of the type considered in 1 with DN numbers $m_1, \dots, m_r, t_1, \dots, t_p$. We can write it in the form

$$\mathcal{B}(y, D) = \sum_{\kappa=0}^{\mu} \mathcal{B}_\kappa(y, D') D_n^\kappa \tag{8}$$

where the coefficient-operators, $\mathcal{B}_\kappa = [b_{kj}^\kappa]$, are $r \times p$ matrices of classical p.d.o.'s on $\partial\Omega$. Note that the order of b_{kj}^κ is $m_k + t_j - \kappa$. The integer $\mu \geq 0$ is the *transversal order* of the boundary operator and is the maximum of the orders of the normal derivatives that occur in the entries of $\mathcal{B}$. Associated with the DN principal part, $\pi\mathcal{B}$, there is the $r \times p$ matrix polynomial

$$B_{y,\xi'}(\lambda) := \pi\mathcal{B}(y, (\xi', \lambda)) = \sum_{j=0}^{\mu} B_j(y, \xi')\lambda^j,$$

$y \in \partial\Omega$, $0 \neq \xi' \in \mathrm{T}_y^*(\partial\Omega)$, where $B_j = \pi\mathcal{B}_j$ is the principal symbol of the j-th coefficient-operator, $\mathcal{B}_j$.

Remark. To simplify the notation, we often suppress the variables y, ξ' and write just $L(\lambda)$ rather than $L_{y,\xi'}(\lambda)$. The same is done for $B(\lambda)$.

The goal of this section is to reformulate the L-condition for elliptic boundary value problems in several equivalent versions. We start with an abstract, quite general formulation, and use the matrix notation of Section 2.
Let $L(\lambda) = \sum_{j=0}^{\ell} A_j \lambda^j$ be a $p \times p$ matrix polynomial. Suppose that $\det L(\lambda) \neq 0$ for $\lambda \in \mathbb{R}$ and let γ^+ be a simple, closed contour in the upper half-plane $\operatorname{Im}\lambda > 0$ that contains all the eigenvalues of $L(\lambda)$ there. By 2 there exists a γ^+-spectral triple (X_+, T_+, Y_+) for $L(\lambda)$, and the base space of the γ^+-spectral pair (X_+, T_+) has dimension r, where r is the number of roots of $\det L(\lambda) = 0$ inside γ^+. We let $\mathfrak{M}^+ = \mathfrak{M}^+_{L(i^{-1}d/dt)}$ denote the subspace of $C^\infty(\mathbb{R}, \mathbb{C}^p)$ consisting of the solutions of $L(i^{-1}d/dt)u = 0$ such that $u(t) \to 0$ as $t \to \infty$. Recall that $\dim \mathfrak{M}^+ = r$ and by Corollary 2.4 every $u \in \mathfrak{M}^+$ admits a representation of the form

$$u(t) = X_+ e^{itT_+} c,$$

for a unique $c \in \mathbb{C}^r$.

Remark. The matrix polynomial $L(i^{-1}\lambda)$ has the spectral pair (X_+, iT_+) with respect to the eigenvalues in the half-plane $\operatorname{Re}\lambda < 0$.

From now on we omit mention of the contour γ^+, i.e. we write $\int_+$ instead of $\int_{\gamma^+}$ because the value of the contour integral depends only on the residues in the upper half-plane $\operatorname{Im}\lambda > 0$, not on the particular contour. Recall that the (right) Calderón projector, i.e. $P_+ = P_{\gamma^+}$, is defined by (6); in this case the image of P_+ is equal to the set of initial conditions $\operatorname{col}((i^{-1}d/dt)^j u(0))_{j=0}^{\ell-1}$. Similarly, $P'_+ = P'_{\gamma^+}$ denotes the left Calderón projector defined by (7).

THEOREM 3.1. *Let* $B(\lambda) = \sum_{j=0}^{\mu} B_j\lambda^j$ *be an* $r \times p$ *matrix polynomial of degree* μ. *The following statements are equivalent:*

(a) *For any* $y \in \mathbb{C}^r$, *there is a unique* $u \in \mathfrak{M}^+$ *such that*

$$B\left(\frac{1}{i}\frac{d}{dt}\right)u\bigg|_{t=0} = y.$$

(b) *If* (X_+, T_+) *is a* γ*-spectral pair for* $L(\lambda)$, *where* X_+ *is an* $p \times r$ *matrix and* T_+ *is a* $r \times r$ *matrix, then the* $r \times r$ *matrix* $\Delta_B^+ = \sum_{j=0}^{\mu} B_j X_+ T_+^j$ *is invertible, i.e.*

$$\det \Delta_B^+ = \det\left(\sum_{j=0}^{\mu} B_j X_+ T_+^j\right) \neq 0. \tag{9}$$

PROOF. As we remarked above, any $u \in \mathfrak{M}^+$ can be represented in the form $u(t) = X_+ e^{itT_+}c$ for a unique $c \in \mathbb{C}^r$. The equivalence of (a) and (b) is clear since

$$B\left(\frac{1}{i}\frac{d}{dt}\right)u\bigg|_{t=0} = \left(\sum_{j=0}^{\mu} B_j X_+ T_+^j\right)\cdot c.$$

The matrix theory of 2 can now be applied to the L-condition for elliptic systems $(\mathcal{A}, \mathcal{B})$. In the following theorem, no assumptions are made concerning the transversal order, μ, of the boundary operator ($\mu \geq \ell$ is permitted). In part (v) of the theorem, $\widetilde{L}(\lambda)$ denotes the cofactor matrix of $L(\lambda)$, i.e. the matrix polynomial such that $L(\lambda)\widetilde{L}(\lambda) = \widetilde{L}(\lambda)L(\lambda) = \det L(\lambda)\cdot I$.

THEOREM 3.2. *Suppose that the operator* $\mathcal{A}(x, D)$ *is properly elliptic, let* $\mathcal{B}(y, D)$ *be a boundary operator as in* (8), *and fix* $y \in \partial\Omega$, $0 \neq \xi' \in \mathrm{T}_y^*(\partial\Omega)$. *The following statements are equivalent:*

(i) *The initial value problem*

$$\begin{aligned} &\pi\mathcal{A}\left(y, \left(\xi', \frac{1}{i}\frac{d}{dt}\right)\right)u(t) = 0, \quad t > 0,\\ &\pi\mathcal{B}\left(y, \left(\xi', \frac{1}{i}\frac{d}{dt}\right)\right)u(t)\bigg|_{t=0} = g, \end{aligned} \tag{10}$$

has for every choice of $g \in \mathbb{C}^r$ *a unique solution* $u \in \mathfrak{M}^+_{L(i^{-1}d/dt)}$. *As usual,* $\mathfrak{M}^+_{L(i^{-1}d/dt)}$ *is the space of solutions of* (10) *such that* $u(t) \to 0$ *as* $t \to \infty$ (*i.e. corresponding to the eigenvalues of* $L(\lambda)$ *with a positive imaginary part*).

(ii) *The* $r \times p\ell$ *matrix* $G = G_{y,\xi'}$ *defined by*

$$G = \int_+ \pi\mathcal{B}(y, (\xi', \lambda))\pi\mathcal{A}(y, (\xi', \lambda))^{-1}\,[\,I, \dots, \lambda^{\ell-1}I\,]\; d\lambda \tag{11}$$

has rank r, where $\int_+$ denotes the integral along a simple, closed contour γ^+ in the upper half-plane containing all roots of $\det L_{y,\xi'}(\lambda) = 0$ *with a positive imaginary part.*

(iii) *If* $(X_+(y,\xi'), T_+(y,\xi'), Y_+(y,\xi'))$ *is a γ^+-spectral triple for $L_{y,\xi'}(\lambda)$, where $X_+(y,\xi')$ is a $p \times r$ matrix, $T_+(y,\xi')$ a $r \times r$ matrix and $Y_+(y,\xi')$ an $r \times p$ matrix* (*existence by Theorem* 2.2), *then*

$$\det \Delta_B^+(y,\xi') = \det \left(\sum_{j=0}^{\mu} B_j(y,\xi') X_+(y,\xi') T_+^j(y,\xi') \right) \neq 0. \tag{12}$$

(iv) *There is a unique $p\ell \times r$ matrix $S = S_{y,\xi'}$ such that $GS = I_r$ and $SG = P'_+$, where $P'_+ = P'_{\gamma^+}$ is the left Calderón projector for $L_{y,\xi'}(\lambda)$.*

(v) *Let us factor the scalar polynomial,* $\det L_{y,\xi'}(\lambda)$, *as $\varrho^-(\lambda) \cdot \varrho^+(\lambda)$, where ϱ^+ contains all the roots above the real axis, and ϱ^- contains all roots below the real axis. If $\widetilde{L}(\lambda)$ denotes the matrix polynomial*

$$\widetilde{L}(\lambda) = \det L_{y,\xi'}(\lambda) \cdot L_{y,\xi'}^{-1}(\lambda),$$

then the rows of $B_{y,\xi'}(\lambda) \cdot \widetilde{L}(\lambda)$ are linearly independent modulo $\varrho^+(\lambda)$.

The first condition is of course the L-condition stated in Section 1. Condition (ii) is known as the *Lopatinskiĭ condition*; the matrix (11) is the Lopatinskiĭ matrix, and was introduced by Lopatinskiĭ in his paper [6]. Fedosov used condition (iv) in a series of papers beginning with [3] where he developed an index formula for elliptic boundary value problems. The last condition (v) is called the "covering" or "complementing condition", and was introduced and used by Agmon, Douglis and Nirenberg in their fundamental paper [1]. For the proof of the theorem above see [8, Chapter 10], where it is proved using spectral triples. Applications of our new form (12) of the L-condition (and the spectral theory of matrix polynomials) are also considered in that book; these applications include topics such as: theorems of Agranovič–Dynin type, construction of homotopies of elliptic boundary value problems, reduction of an elliptic boundary value problem to an elliptic system on the boundary $\partial\Omega$, and an index formula for elliptic systems in the plane.

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