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GRADIENT ESTIMATES FOR SOLUTIONS OF THE NAVIER–STOKES EQUATIONS

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Dedicated to O. A. Ladyzhenskaya on the occasion of her seventy-fifth birthday

The motivation for this paper derives from the classical property of solutions of the Laplace equation

$$\Delta w = 0 \tag{1}$$

in a bounded domain Ω, that the gradient at interior points is bounded in magnitude, depending only on the distance d to $S = \partial\Omega$ and on a bound for $|w|$ on S. This property was extended by Odqvist [7], under some smoothness hypotheses on S, to solutions of the Stokes equations

$$\Delta \vec{w} = \nabla p, \tag{2}$$

$$\operatorname{div} \vec{w} = 0, \tag{3}$$

for slow stationary viscous fluid flows. We ask whether an analogous property also holds for solutions $\vec{w}(x)$ of the full Navier–Stokes equations for stationary viscous fluid flows

$$\Delta \vec{w} - R\vec{w} \cdot \nabla \vec{w} = \nabla p, \tag{4}$$

$$\operatorname{div} \vec{w} = 0. \tag{5}$$

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We shall consider such solutions in a bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, with boundary $S \in C^{(2+\alpha)}$, $\alpha \in (0,1)$. Let $\vec{w}(x)$ be a solution of (4, 5) in Ω, twice differentiable on Ω and having trace $\vec{a}$ on S in L_1. We prove first:

THEOREM 1. *For any solution $\vec{w}(x)$ in Ω, if $p > n$ and $x \in \Omega$, then there is a bound on the magnitude $|D\vec{w}|$ of the first derivatives, depending only on $\|\vec{a}\|_{L_1(S)}$, on $\|\vec{w}\|_{L_{2p}(\Omega)}$, on R, and on the distance to S. The bound does not otherwise depend on the particular solution considered.*

PROOF. We represent $\vec{w}(x)$ in terms of Green's tensor

$$G(x,y) = X(x,y) + \mathcal{X}(x,y) \tag{6}$$

and the associated pressure tensor

$$P(x,y) = \Pi(x,y) + \pi(x,y) \tag{7}$$

of Odqvist [7], corresponding to the linearized equations (2, 3). Here

$$X = (X_{ij}) = \begin{cases} \dfrac{1}{4\pi}\left\{\delta_{ij}\log\dfrac{1}{|x-y|} + \dfrac{(x_i-y_i)(x_j-y_j)}{|x-y|^2}\right\} & \text{if } n=2, \\[2ex] \dfrac{1}{8\pi}\left\{\dfrac{\delta_{ij}}{|x-y|} + \dfrac{(x_i-y_i)(x_j-y_j)}{|x-y|^3}\right\} & \text{if } n=3, \end{cases} \tag{8}$$

denotes the "fundamental solution tensor" and

$$\Pi = (\Pi_i) = \frac{1}{\omega_{n-1}}\left\{\frac{x_i-y_i}{|x-y|^n}\right\} \tag{9}$$

the associated "fundamental pressure vector", with ω_n being the area of the unit n-sphere. We obtain the relations

$$\vec{w}(x) = \int_S \vec{a}\cdot\nu\cdot T_y G(x,y)\,d\omega_y + R\int_\Omega G\cdot(\vec{w}\cdot\nabla)\vec{w}\,dy \tag{10a}$$

$$= \vec{w}_0(x) - R\int_\Omega \vec{w}\cdot(\vec{w}\cdot\nabla_y)G\,dy \tag{10b}$$

in view of (5). Here ν is the unit exterior normal on S, and the operator Tu denotes the stress tensor,

$$(Tu)_{ij} = \left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i}\right) - \delta_{ij}p \tag{11}$$

(see [7] for details). The surface integral in (10) provides the (unique) solution $\vec{w}_0(x)$ of the Stokes linearization (2, 3) corresponding to the data $\vec{a}$. If x, y lie in the intersection with Ω of any fixed ball of radius $\varrho < 1/2$, then

$$\begin{gathered} |G(x,y)| < C\log\frac{1}{|x-y|} \quad \text{if } n=2, \quad |G(x,y)| < C\frac{1}{|x-y|} \quad \text{if } n=3, \\ |\nabla G(x,y)| < C|x-y|^{1-n}, \quad P(x,y) < C\,|x-y|^{1-n}, \end{gathered} \tag{12}$$

C being a generic constant that is uniform for all such balls. Further, for any two points $x,\ x'$ lying exterior to a ball of radius r_0 about y, we have for any $\varepsilon > 0$,

$$\begin{aligned} |G(x,y) - G(x',y)| &< C\frac{|x-x'|^{1-\varepsilon}}{r_0^{n-1}}, \\ |\nabla G(x,y) - \nabla G(x',y)| &< C\frac{|x-x'|^{1-\varepsilon}}{r_0^{n}}, \\ |P(x,y) - P(x',y)| &< C\frac{|x-x'|^{1-\varepsilon}}{r_0^{n}}, \end{aligned} \tag{13}$$

for some C depending only on Ω and on ε.

We find additionally, using these estimates, that

$$|D_x\nabla_y G(x,y)| < |D_x\nabla_y X(x-y)| + M(x,y) \tag{14}$$

where $M(x,y)$, a bound for derivatives of the regular part of Green's tensor, is bounded depending only on the distance of x to S.

We wish to use these results and the representation (10b) to estimate the pointwise magnitude of $|\vec{w}|$ and of the derivatives $|D\vec{w}|$ at interior points in terms of global averaging information on $\vec{w}$ in the closed domain. We begin with the observation, from (10b), that if $p > n$ and $1/p + 1/q = 1$, then $q < n/(n-1)$, so that by (12), $|\nabla G|^q$ is summable over Ω and

$$\begin{aligned} |\vec{w}(x)| &\le R\left(\int_\Omega |\vec{w}|^{2p}\,dy\right)^{1/p}\left(\int_\Omega |\nabla G|^q\,dy\right)^{1/q} + \max_S |TG| \int_S |\vec{a}|\,d\omega \\ &\le C_1 R\|\vec{w}\|^2_{L_{2p}(\Omega)} + C_2\|\vec{a}\|_{L_1(S)} \end{aligned} \tag{15}$$

with C_1, C_2 depending only on the distance of x from S.

We cannot differentiate (10b) directly, as the derivatives of ∇G are not integrable at the singularity. However, from (10b), if $1/p + 1/q = 1$ then

$$\begin{aligned} |\vec{w}(x) - \vec{w}(x')| \le{}& R\left|\int_\Omega \vec{w}\cdot(\vec{w}\cdot\nabla_y)(G(x,y) - G(x',y))\,dy\right| \\ &+ \left|\int_S \vec{a}\cdot(TG(x,y) - TG(x',y))\cdot\nu\,d\omega_y\right| \\ \le{}& R\left(\int_\Omega |\vec{w}|^{2p}\,dy\right)^{1/p}\left(\int_\Omega |\nabla_y G(x,y) - \nabla_y G(x',y)|^q\,dy\right)^{1/q} \\ &+ \max_S |TG(x,y) - TG(x',y)| \int_S |\vec{a}|\,d\omega_y. \end{aligned} \tag{16}$$

We choose $q < n/(n-1)$ and then ε small enough that $(n-1+\varepsilon)q < n$. Using the estimates (13) we verify that

$$\left(\int_\Omega |\nabla_y G(x,y) - \nabla_y G(x',y)|^q\,dy\right)^{1/q} < C|x-x'|^\alpha \tag{17}$$

for some $\alpha > 0$, depending only on q, on ε, and on Ω. The choice of ε can be made to depend only on q. *Thus, if for some $p > n$, $\|\vec{w}\|_{L_{2p}(\Omega)} < \infty$, then in any fixed compact subdomain of Ω, $\vec{w}$ satisfies a fixed Hölder condition*

$$|\vec{w}(x) - \vec{w}(x')| < (C_1 R\|\vec{w}\|^2_{L_{2p}(\Omega)} + C_2\|\vec{a}\|_{L_1(S)})\,|x - x'|^\alpha, \tag{18}$$

the positive constants C and α depending only on p and on the subdomain.

We now observe that for any fixed vector $\vec{w}'$,

$$\int_\Omega \vec{w}' \cdot (\vec{w}\cdot\nabla_y)G\,dy = 0. \tag{19}$$

Thus, setting $\vec{w}' = \vec{w}(x')$ for some fixed $x' \in \Omega$, we find

$$\vec{w}(x) = \int_S \vec{a}\cdot\nu\cdot TG\,d\omega_y - R\int_\Omega (\vec{w} - \vec{w}')\cdot(\vec{w}\cdot\nabla_y)G\,dy, \tag{20}$$

an expression that by (12) and (18) can be differentiated at $x = x'$ to yield

$$D\vec{w}(x) = \int_S \vec{a}\cdot\nu\cdot D_x TG\,d\omega_y - R\int_\Omega (\vec{w}(y) - \vec{w}(x))\cdot(\vec{w}\cdot D_x\nabla_y)G\,dy. \tag{21}$$

Let $d = \operatorname{dist}(x, S)$, and let $B_{d/2}$ be the ball of radius $d/2$ about x.

From (21) and the preceding estimates we find

$$\begin{aligned}
|D\vec{w}(x)| &< \max_S |D_x T_y G| \int_S |\vec{a}|\,d\omega_y \\
&\quad + R\max_{B_{d/2}}|\vec{w}| \int_{B_{d/2}} |\vec{w}(y) - \vec{w}(x)|\cdot|D_x\nabla_y G|\,dy \\
&\quad + R\int_{\Omega\setminus B_{d/2}} |\vec{w}(y) - \vec{w}(x)|\cdot|\vec{w}(y)|\cdot|D_x\nabla_y G|\,dy \\
&< Cd^{-n}\int_S |\vec{a}|\,d\omega_y + CR(d^\alpha + |\ln d|)\Big(R\|\vec{w}\|^2_{L_{2p}(\Omega)} + \int_S |\vec{a}|\,d\omega_y\Big)^2 \\
&\quad + 2R\Big(\int_\Omega |\vec{w}|^{2p}\,dy\Big)^{1/p}\Big(\int_{\Omega\setminus B_{d/2}} |D_x\nabla_y G|^q\,dy\Big)^{1/q} \\
&< C_1 d^{-n}\|\vec{a}\|_{L_1(\Omega)} + C_2 R(d^\alpha + |\ln d|)(R\|\vec{w}\|^2_{L_{2p}(\Omega)} + \|\vec{a}\|_{L_1(S)})^2 \\
&\quad + C_3 R d^{-1}\|\vec{w}\|^2_{L_{2p}(\Omega)}.
\end{aligned} \tag{22}$$

This completes the proof of Theorem 1.

Corollary 1. *Let $\vec{w}(x)$ be a solution of $(4,5)$ in Ω and suppose $|\vec{w}(x)| < M < \infty$ in Ω. Then $|D\vec{w}(x)|$ is bounded in Ω, depending only on M, on R and on the distance of x to the boundary S.*

Corollary 1 corresponds in general form to the classical gradient bound for harmonic functions in terms of a bound for the function over the domain and on the distance to the boundary. The statement following (1) above, of a gradient bound in terms of distance to the boundary S and on a bound for $|\vec{w}|$ on S, reduces for solutions of (1) to the maximum principle. No maximum principle or result that can replace it in such a context has, however, been established for solutions of $(4, 5)$. The question arises whether an a priori gradient bound of the type just described can nevertheless be valid for such solutions. Indeed, it follows from Odqvist's estimates (12) above that in a *given* smooth domain such a bound does hold for solutions of $(2, 3)$. That result can also be inferred from the Remark following the proof of Proposition 1 below, which has the appearance of a maximum principle. Nevertheless, the bound obtained by those procedures conceivably depends—as do the bounds of Theorem 1 and Corollary 1—on the particular region of definition, and not just on boundary magnitude of $|\vec{w}|$ and distance to the boundary.

The following example, which applies equally to the systems $(2, 3)$ and $(4, 5)$, illustrates the kind of behavior that can occur. We consider, in the annular plane domain $1/\ln^2 P < r < P$, the explicit solution $\vec{w} = (u, v)$ with

$$u = A\frac{y}{P}\ln\frac{r}{P}, \quad v = -A\frac{x}{P}\ln\frac{r}{P}. \tag{23}$$

This example is a particular case of a family of exact solutions given by Hamel in 1916 [3]. We observe that $\vec{w}$ vanishes on the outer boundary $r = P$. Choosing $A = P$, we find that $|\vec{w}|$ tends uniformly to zero on the inner boundary as $P \to \infty$. Nevertheless, both $|\vec{w}(\sqrt{P}, 0)|$ and $|\nabla\vec{w}(\sqrt{P}, 0)|$ increase unboundedly in P, with distance to the boundary also becoming unbounded. Thus, at arbitrarily large distances to the boundary, and with boundary data arbitrarily small, the velocity field and its gradient can simultaneously become arbitrarily large.

The example is not conclusive, as the associated pressure is not single-valued for circuits enclosing the origin; thus, the solution should best be considered as being periodic over a logarithmic Riemann surface. It should be of considerable interest to investigate the question numerically; one such possibility could be to calculate the solutions in a two-dimensional disk, corresponding to bounded and smooth, but successively more rapidly oscillating, boundary data. It is known that smooth solutions exist for such data subject only to an outflow condition (see [2], [5], [6]), although uniqueness and stability have been proved for them only under the hypothesis that the derivatives of the data are sufficiently small.

Interior gradient bounds for solutions of (4, 5) depending only on distance to the boundary and on bounds for the boundary data and their derivatives can be inferred from the material in [2], [5], [6]. In what follows, we restrict ourselves to the (physical) case $n = 3$, for which we intend to show in particular that such bounds can in fact be obtained using merely Hölder smoothness of the data. We shall prove:

Theorem 2. *For any solution $\vec{w}(x)$ of* (4, 5) *in* Ω*, there is a bound on* $|D\vec{w}(x)|$*, depending only on* R*, on* $\operatorname{dist}(x, S)$*, on* $\|\vec{a}\|_{L_p(S)}$*,* $p > 4$*, and on the* $W_2^{1/2}(S)$ *norm of* $\vec{a}$*.*

Remark. Since the $W_2^{1/2}$ norm can be estimated by a Hölder norm with exponent $\gamma > 1/2$, the bound mentioned in the statement of the theorem can be taken dependent on the $C^\gamma(S)$ norm of $\vec{a}$.

In virtue of Theorem 1, the proof of Theorem 2 reduces to the evaluation of the L_{2p}-norm of $\vec{w}$ ($p > 3$). As the first step, we estimate this norm by $\|\vec{w}\|_{L_6(\Omega)}$ and by the norms of the data.

We consider again the representation formula (10b) and, according to [7], write the solution $\vec{w}_0$ of the Stokes problem

$$\Delta \vec{w}_0 = \nabla p_0, \quad \nabla \cdot \vec{w}_0 = 0, \quad \vec{w}_0|_S = \vec{a}, \tag{24}$$

in the form of the double layer potential

$$\vec{w}_0(x) = \int_S P(x,y)\vec{h}(y)\, d\omega_y \tag{25}$$

where $P(x, y)$ is a matrix-valued kernel satisfying the estimates

$$|D^j P(x,y)| \le c(|j|)|x-y|^{-|j|-2}, \quad \int_S |P(x,y)|\, d\omega_y \le c, \tag{26}$$

and $\vec{h}(y)$ can be determined from the system of integral equations on S

$$\vec{h}(x) + \int_S K(x,y)\vec{h}(y)\, d\omega_y = \vec{a}(x)$$

with a weakly singular matrix-valued kernel $K(x, y)$, i.e.

$$|K(x,y)| \le c|x-y|^{-2+\beta}, \quad \beta \in (0,1),\ x, y \in S.$$

If $\int_S \vec{a} \cdot \vec{n}\, d\omega = 0$, then

$$\vec{h}(x) = \vec{a}(x) + \int_S R(x,y)\vec{a}(y)\, d\omega_y \tag{27}$$

where $R(x, y)$ is a resolvent kernel with the same properties as $K(x, y)$, so

$$\|\vec{h}\|_{L_r(S)} \le c\|\vec{a}\|_{L_r(S)}, \quad 1 \le r \le \infty. \tag{28}$$

From these estimates and from (26) the following proposition follows.

PROPOSITION 1. *We have the inequality*

$$\|\vec{w}_0\|_{L_q(\Omega)} \le c\|\vec{a}\|_{L_p(S)}, \quad q \ge p \ge 1,\ p > \tfrac{2}{3}q. \tag{29}$$

PROOF. We take arbitrary p and q satisfying the conditions in (29) and set

$$\frac{3}{2q} + 1 - \frac{1}{p} = \frac{1}{\vartheta}, \quad \vartheta \in (0,1).$$

In virtue of the Hölder inequality,

$$\begin{aligned}|\vec{w}_0(x)| &\le \left(\int_S |P(x,y)|^{3\vartheta/2} |\vec{h}(y)|^p \, d\omega_y\right)^{1/q} \\ &\le \left(\int_S |\vec{h}(y)|^p \, d\omega_y\right)^{1/p-1/q} \left(\int_S |P(x,y)|^{\vartheta} \, d\omega_y\right)^{1-1/p},\end{aligned}$$

from which (29) follows with

$$c = \left(\sup_{x\in\Omega} \int_S |P(x,y)|^{\vartheta} \, d\omega_y\right)^{1-1/p} \left(\sup_{y\in S} \int_\Omega |P(x,y)|^{3\vartheta/2} \, dx\right)^{1/q}.$$

REMARK. From (25)–(27) it also follows that

$$\sup_\Omega |\vec{w}_0(x)| \le c \sup_S |\vec{a}(x)|.$$

This inequality was announced in [9].

We call attention here to a paper by Kratz [4] in which an explicit estimate is given for $\max_\Omega |\vec{w}_0|$ in terms of $\max_S |\vec{a}|$, in the particular case in which Ω is an n-dimensional ball.

Consider the vector field

$$\vec{u}(x) = \int_\Omega G(x,y)(\vec{w}\cdot\nabla)\vec{w}(y)\, dy = -\int_\Omega \vec{w}(y)\cdot(\vec{w}\cdot\nabla_y)G(x,y)\, dy.$$

Making use of (13) and of the Hardy–Littlewood–Sobolev lemma on the estimates of potentials with weakly singular kernels, we obtain the following inequality:

$$\|\vec{u}\|_{L_q(\Omega)} \le c\|\vec{w}\|^2_{L_{2s_1}(\Omega)}, \quad \frac{1}{s_1} \le \frac{1}{q} + \frac{1}{3},\ q \ge s_1 > 1.$$

This inequality and Proposition 1 imply

$$\|\vec{w}\|_{L_q(\Omega)} \le c(\|\vec{a}\|_{L_p(S)} + R\|\vec{w}\|^2_{L_6(\Omega)}), \tag{30}$$

where $q \in (6, 3p/2)$, $p > 4$.

Further, in virtue of the Sobolev imbedding theorem,

$$\|\vec{w}\|_{L_6(\Omega)} \le c\|\vec{w}\|_{W_2^1(\Omega)}. \tag{31}$$

The final step of the proof is the application of estimates of the Dirichlet integral of $\vec{w}$ obtained in [2], [5], [6] under hypotheses of sufficient smoothness of the data

$\vec{a}$ and/or extendability of $\vec{a}$ into Ω in the form $\vec{a} = \operatorname{rot}\vec{b}$, $\vec{b} \in W_2^2(\Omega)$. We shall show that such an extension can be obtained for arbitrary $\vec{a} \in W_2^{1/2}(S)$ satisfying the necessary compatibility condition.

Proposition 2. *Every $\vec{a} \in W_2^{1/2}(S)$ such that*

$$\int_S \vec{a}\cdot\nu\,d\omega = 0 \tag{32}$$

can be extended into Ω in the form $\vec{a} = \operatorname{rot}\vec{b}$, $\vec{b} \in W_2^2(\Omega)$, satisfying the inequality

$$\|\vec{b}\|_{W_2^2(\Omega)} \le c\|\vec{a}\|_{W_2^{1/2}(S)}. \tag{33}$$

Proof. Following the ideas of the paper [2], we are looking for $\vec{b} = \vec{b}' + \vec{b}''$ with

$$\vec{\nu}\cdot\operatorname{rot}\vec{b}' = \vec{a}\cdot\vec{\nu} \equiv a', \quad \operatorname{rot}\vec{b}'' = \vec{a} - \operatorname{rot}\vec{b}' \equiv \vec{a}'', \quad \vec{a}''\cdot\vec{\nu} = 0 \tag{34}$$

and with both $\vec{b}'$ and $\vec{b}''$ satisfying (33).

We represent a' in the form

$$a' = \sum_{k=1}^{M} a_k'(x)$$

where $a_k' \in W_2^{1/2}(S)$ are functions with compact supports: $\operatorname{supp} a_k' \subset S_k$, satisfying the condition

$$\int_{S_k} a_k'\,d\omega = 0$$

and the estimate

$$\sum_k \|a_k'\|^2_{W_2^{1/2}(S_k)} \le c\|a'\|^2_{W_2^{1/2}(S)}.$$

We assume that $\operatorname{diam} S_k$ is small and that in a local cartesian coordinate system $\{y_1, y_2, y_3\}$ with y_3-axis directed along the interior normal $-\vec{\nu}(\xi_k)$, $\xi_k \in S$, S_k can be given by the equation

$$y_3 = F(y'), \quad y' = (y_1, y_2) \in \Sigma_d = \{|y'| \le d\}.$$

We construct $\vec{b}_k'(y)$ defined in a certain neighborhood of the origin and satisfying the boundary condition

$$\vec{\nu}\cdot\operatorname{rot}\vec{b}_k'|_S = a_k'$$

or

$$\nu_1\Big(\frac{\partial b_{k3}}{\partial y_2} - \frac{\partial b_{k2}}{\partial y_3}\Big) + \nu_2\Big(\frac{\partial b_{k1}}{\partial y_3} - \frac{\partial b_{k3}}{\partial y_1}\Big) + \nu_3\Big(\frac{\partial b_{k2}}{\partial y_1} - \frac{\partial b_{k1}}{\partial y_2}\Big) = a_k'. \tag{35}$$

We assume that

$$b_{k3}|_{S_k} = \frac{\partial b_{k3}}{\partial y_3}\Big|_{S_k} = \frac{\partial b_{k1}}{\partial y_3}\Big|_{S_k} = \frac{\partial b_{k2}}{\partial y_3}\Big|_{S_k} = 0. \tag{36}$$

Then the preceding equation reduces to

$$\frac{\partial b_{k2}}{\partial y_1} - \frac{\partial b_{k1}}{\partial y_2} = \frac{a'_k}{\nu_3} \equiv g_k,$$

or, if we "rectify" S_k introducing the coordinates

$$z_1 = y_1, \quad z_2 = y_2, \quad z_3 = y_3 - F(y'),$$

to

$$\frac{\partial u_1}{\partial z_1} + \frac{\partial u_2}{\partial z_2} = g_k(z', F(z')) \tag{37}$$

where

$$u_1(z') = b_{k2}(z', F(z')), \quad -u_2(z') = b_{k1}(z', F(z')). \tag{38}$$

Equation (37) was analyzed, in particular, in the paper [1] where the solution was constructed which had a compact support and satisfied the inequalities

$$\|u_1\|^2_{W_2^m(\Sigma_d)} + \|u_2\|^2_{W_2^m(\Sigma_d)} \le c\|g_k\|^2_{W_2^{m-1}(\Sigma_d)}$$

for $m = 1, 2$ and (by interpolation) for $m = 3/2$. We can extend u_k into $\mathbb{R}^2\backslash\Sigma_d$ by zero and construct a vector field $\vec{b}_k \in W_2^2(\mathbb{R}^3_+)$ satisfying the boundary conditions (36) and (38) (which we interpret as conditions on Σ_d) and the inequality

$$\|\vec{b}_k\|^2_{W_2^2(\mathbb{R}^3_+)} \le c\|\vec{u}\|^2_{W_2^{3/2}(\Sigma_d)} \le c\|g_k\|^2_{W_2^{1/2}(\Sigma_d)}.$$

This can be done by standard methods of the theory of Sobolev spaces (see, for example, [8]). Multiplying b_k by an appropriate cut-off function we can make the support of $\vec{b}_k$ compact without violating of equation (27). In the initial coordinates $\{x_1, x_2, x_3\}$, $\vec{b}_k(x)$ is a vector field defined in Ω and different from zero in a certain neighborhood of ξ_k. Clearly, $\vec{b}' = \sum_{k=1}^M \vec{b}_k$ satisfies equation (34) and inequality (33).

Now we turn to the determination of $\vec{b}''(x)$. We define it as a vector field from $W_2^2(\Omega)$ satisfying the boundary conditions

$$\vec{b}''|_S = 0, \quad \left.\frac{\partial \vec{b}''}{\partial \nu}\right|_S = \vec{a}'' \times \vec{\nu} \tag{39}$$

and the inequality

$$\|\vec{b}''\|_{W_2^2(\Omega)} \le c\|\vec{a}'' \times \vec{\nu}\|_{W_2^{1/2}(S)} \le c\|\vec{a}''\|_{W_2^{1/2}(S)} \le \|\vec{a}\|_{W_2^{1/2}(S)}.$$

In virtue of (39),

$$\operatorname{rot} \vec{b}''|_S = \vec{\nu} \times \left.\frac{\partial \vec{b}''}{\partial \nu}\right|_S = \vec{\nu} \times [\vec{a}'' \times \vec{\nu}\,] = \vec{a}''.$$

The proposition is proved.

This proposition leads to the following improved result:

Theorem 3. *For every $\vec{a} \in W_2^{1/2}(S)$ satisfying the necessary condition*

$$\int_S \vec{a} \cdot \nu \, d\omega = 0$$

the system (4, 5) *has at least one solution $\vec{w} \in W_2^1(\Omega)$ satisfying the inequality*

$$\|\vec{w}\|_{W_2^1(\Omega)} \le C(\|\vec{a}\|_{W_2^{1/2}(S)} + \|\vec{a}\|^2_{W_2^{1/2}(S)}) \tag{40}$$

for some $C > 0$ depending only on Ω.

Now, (31) and (40) imply

$$\|\vec{w}\|_{L_6(\Omega)} \le C(\|\vec{a}\|_{W_2^{1/2}(S)} + \|\vec{a}\|^2_{W_2^{1/2}(S)}),$$

and the proof of Theorem 2 is complete.

Similar estimates can be obtained for $|p(x)|$, as well as for the Hölder norms of $D\vec{w}$ and p in an arbitrary domain $\omega \subset \Omega$ bounded away from S. One can also find bounds for higher order derivatives of $\vec{w}$ and p making use of well known "local estimates". For instance, if $\operatorname{dist}(\omega, S) \ge d$ and ω' is a strictly interior subdomain of ω, then

$$\|\vec{w}\|_{C^{2+\alpha}(\omega')} + \|p\|_{C^{1+\alpha}(\omega')} \le c(\|(\vec{w} \cdot \nabla)\vec{w}\|_{C^{\alpha}(\omega)} + \sup_{\omega} |\vec{w}(x)|).$$

Since the right-hand side is already evaluated, we obtain estimates of $D^2\vec{w}, \nabla p$ etc.

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