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SYMMETRY RESULTS FOR PERTURBED PROBLEMS AND RELATED QUESTIONS

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Abstract. In this paper we prove a symmetry result for positive solutions of the Dirichlet problem

$$(0.1)\qquad \begin{cases} -\Delta u = f(u) & \text{in } D,\\ u = 0 & \text{on } \partial D,\end{cases}$$

when f satisfies suitable assumptions and D is a small symmetric perturbation of a domain Ω for which the Gidas–Ni–Nirenberg symmetry theorem applies. We consider both the case when f has subcritical growth and $f(s) = s^{(N+2)/(N-2)} + \lambda s$, $N \geq 3$, λ suitable positive constant.

1. Introduction

Let us consider the following problem

$$(1.1)\qquad \begin{cases} -\Delta u = f(u) & \text{in } \Omega,\\ u > 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\end{cases}$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is a C^1-function and Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 2$, which contains the origin and is symmetric with respect to the hyperplane

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$T_0 = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N,\ x_1 = 0\}$ and convex in the x_1-direction. Under this hypothesis it is well known that classical $C^2(\Omega) \cap C^1(\overline{\Omega})$ solutions are even in x_1 and strictly increasing in the x_1-variable in the cap $\Omega^- = \{x \in \Omega,\ x_1 < 0\}$. This is the famous symmetry result of Gidas, Ni and Nirenberg (see [7]) which is based on the method of moving planes which goes back to Alexandrov and Serrin in [11].

Now we would like to consider the same problem in some domains Ω_n which are suitable approximations of Ω but are not any more convex in the x_1-direction. The typical example is obtained by making one or more holes in Ω, i.e. $\Omega_n = \Omega \setminus \bigcup_{i=1}^k B_i$ where B_i are small balls in Ω whose radius tends to zero, as $n \to \infty$.

The question we address in this paper is whether the symmetry of the solutions of the same problem as in (1.1), but with Ω replaced by Ω_n, is preserved. Let us note immediately that the moving planes method cannot be applied in Ω_n if the convexity in the x_1-direction is destroyed but can only give information on the monotonicity of the solutions in x_1, in some subsets of Ω_n.

Nevertheless in an interesting paper ([3]) Dancer proved, among other results, that for some subcritical nonlinearities, as for example $f(s) = s^p$, $p > 1$ if $N = 2$, $1 < p < (N+2)/(N-2)$ if $N \geq 3$, if the solution u is unique and nondegenerate in Ω then also the approximating problems in Ω_n have only one solution which is necessarily symmetric, if Ω_n is symmetric. So in this case the symmetry comes from the uniqueness of the solution. Thus the question is whether the symmetry is preserved even if the solution is not unique.

Here we answer positively this question in two different cases: first we consider nonlinear terms $f(s)$ with subcritical growth and then we take $f(s) = s^{(N+2)/(N-2)} + \lambda s$, with $\lambda \in (0, \lambda_1)$ if $N \geq 4$, λ_1 being the first eigenvalue of the Laplace operator with zero Dirichlet boundary condition, or $\lambda \in (\lambda^*, \lambda_1)$ if $N = 3$, for a certain number $\lambda^* > 0$.

In the subcritical case we prove that all solutions of the approximating problems are symmetric if Ω_n is sufficiently close to Ω. In particular they are radial if Ω_n is an annulus with a small hole. In the critical case we prove the same result for least energy solutions which exist by the well-known Brezis–Nirenberg result ([2]). We also prove that the least energy solution is unique and nondegenerate if the domains Ω_n are approximations of Ω which is a ball. This last result is not contained in the uniqueness theorem of Dancer.

Let us note that in the recent paper [10] it is shown, among other results, that if f is strictly convex, A is an annulus and u is a solution of (1.1) in A of index one, then u is axially symmetric. Thus, in the case when Ω_n is an annulus and f is convex our result extend that of [10] because it states that if the hole is sufficiently small the solution of index one are not only axially symmetric but actually radially symmetric.

To prove our result the main idea is to exploit the sign of the first eigenvalue of the linearized operator $L = -\Delta - f'(u)I$ at a solution u in the cap $\Omega^- = \{x \in \mathbb{R}^N,\ x_1 < 0\}$ or $\Omega^+ = \{x \in \mathbb{R}^N,\ x_1 > 0\}$. In fact, if $f(0) \geq 0$ it was shown in a lecture by L. Nirenberg (see [6]) that this eigenvalue is indeed positive; in particular zero is not an eigenvalue of L in Ω^- or Ω^+. From this and some results on the convergence of the solutions, we deduce the symmetry of the solutions u_n in Ω_n, with respect to the x_1-variable when Ω_n is symmetric with respect to the hyperplane $T_0 = \{x_1 = 0\}$.

We also get the radial symmetry of the solutions when the domains Ω_n are annuli with a small hole.

The outline of the paper is the following. In Section 2 we prove some preliminary theorems on the convergence of the eigenvalues of linear operators and we also show the uniform L^∞-estimates in the subcritical case, already proved by Dancer (see [5]). In Section 3 we study the subcritical case while in Section 4 we deal with the critical case.

Finally let us remark that the perturbed domains we consider are obtained by the initial domain Ω by removing from it a finite number of symmetric subdomains, i.e. making finitely many holes whose size tends to zero. It could be possible analyze other kind of perturbations but, for simplicity, we will not consider them.

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2. Preliminary results

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$ with smooth C^2-boundary and D^i, $i = 1, \ldots, k$ smooth open subsets of Ω, star shaped with respect to an interior point y_i and such that $D_i \cap D_j = \emptyset$, for $i \neq j$. Then we define the homothetic domains $D^i_n = [\varepsilon^i_n(D^i - y_i)] + y_i$ with respect to y_i, with the sequences ε^i_n converging to zero as $n \to \infty$. Our approximating domains will be

$$\Omega_n = \Omega \setminus \bigcup_{i=1}^{k} D^i_n. \tag{2.1}$$

In Ω_n we consider the linear operators $L_n = -\Delta - a_n(x)I$ where Δ is the Laplace operator, I is the identity and $a_n \in L^\infty(\Omega_n)$. Analogously we consider in Ω the linear operator $L = -\Delta - a(x)I$ with $a \in L^\infty(\Omega)$.

For these operators we would like to prove some results about the convergence of the first and second eigenvalue.

Proposition 2.1. *Assume that a_n converges to the function a in $L^{N/2}(\Omega)$, $N \geq 3$. Then the first eigenvalues $\lambda_1(L_n, \Omega_n)$ in Ω_n with zero Dirichlet boundary conditions, converge to the first eigenvalue $\lambda_1(L, \Omega)$ analogously defined.*

Proof. Let us set $\lambda_{1,n} = \lambda_1(L_n, \Omega_n)$, $\lambda_1 = \lambda_1(L, \Omega)$ and show that the sequence $\{\lambda_{1,n}\}$ is bounded. By the variational characterization we have that

$$\lambda_{1,n} \leq \int_{\Omega_n} |\nabla \phi|^2 \, dx - \int_{\Omega_n} a_n \phi^2 \, dx \tag{2.2}$$

for a function $\phi \in C_0^\infty(B)$, $\int_B \phi^2 \, dx = 1$, where B is a ball contained in any Ω_n. Hence by (2.2)

$$\begin{aligned} \lambda_{1,n} &\leq \int_B |\nabla \phi^2| \, dx + \int_B (a - a_n) \phi^2 \, dx - \int_B a \phi^2 \, dx \\ &\leq \int_B |\nabla \phi^2| \, dx + \bigg(\int_B |a - a_n|^{N/2} \, dx \bigg)^{2/N} \bigg(\int_B \phi^{2^*} \, dx \bigg)^{2/2^*} \\ &\quad + \int_B |a| \phi^2 \, dx \leq C \end{aligned} \tag{2.3}$$

for a suitable constant C, having denoted by 2^* the critical Sobolev exponent, $2^* = 2N/(N-2)$. So $\{\lambda_{1,n}\}$ is bounded from above.

Let $\phi_n > 0$ be a first eigenfunction of L_n in Ω_n with $\int_{\Omega_n} |\nabla \phi_n^2| \, dx = 1$. Then, extending ϕ_n by zero to the whole domain Ω we get

$$\lambda_{1,n} = \frac{\int_\Omega |\nabla \phi_n^2| \, dx - \int_\Omega a_n \phi_n^2 \, dx}{\int_\Omega \phi_n^2 \, dx} \tag{2.4}$$

The sequence ϕ_n converges weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$ to a function $\phi \in H_0^1(\Omega)$ which cannot be zero otherwise from (2.4) and the convergence of a_n to a in $L^{N/2}(\Omega)$ we would get that $\lambda_{1,n} \to \infty$ against what we proved. Then, from (2.4) we deduce

$$\lambda_{1,n} \geq \frac{1 + (\int_\Omega |a_n|^{N/2} \, dx)^{2/N} (\int_\Omega \phi_n^{2^*} \, dx)^{2/2^*}}{\int_\Omega \phi_n^2 \, dx} \geq C \tag{2.5}$$

for a suitable constant C. Hence $\lambda_{1,n}$ is bounded from below and thus converges, up to a subsequence, to a number $\overline{\lambda}$.

Moreover, from the weak convergence of ϕ_n to ϕ in $H_0^1(\Omega)$ we get that ϕ is nonnegative and solves the problem

$$\begin{cases} -\Delta \phi - a\phi = \overline{\lambda} \phi & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.6}$$

Since we already proved that $\phi \not\equiv 0$, by the strong maximum principle we get that $\phi > 0$ and hence is a first eigenfunction of L in Ω, i.e. $\overline{\lambda} = \lambda_1$. □

In the sequel we shall need to estimate the sign of the second eigenvalue of L_n in Ω_n. Therefore, with the same notations as in Proposition 2.1 we prove

Proposition 2.2. *If a_n converges to a in $L^{N/2}(\Omega)$, $N \geq 3$ and $\lambda_2 = \lambda_2(L,\Omega) > 0$ then also $\lambda_{2,n} = \lambda_2(L_n,\Omega_n)$ is positive for n sufficiently large.*

Proof. Arguing by contradiction let us assume that for a subsequence that we still denote in the same way, $\lambda_{2,n} \leq 0$. Then, since $\lambda_{1,n} < \lambda_{2,n}$ and the sequence $\{\lambda_{1,n}\}$ is bounded by Proposition 2.1, the same holds for $\{\lambda_{2,n}\}$ and hence it converges, up to another subsequence, to a number $\widetilde{\lambda} \leq 0$. Then, considering a sequence of second eigenfunctions $\phi_{2,n}$ in Ω_n with $\int_\Omega |\nabla\phi_{2,n}^2|\,dx = 1$ and extending them to zero to the whole domain Ω, we have that $\phi_{2,n} \rightharpoonup \widetilde{\phi}$ weakly in $H_0^1(\Omega)$. As in the previous proposition, using the variational characterization of $\lambda_{2,n}$ we get that $\widetilde{\phi} \not\equiv 0$ and is a solution of

$$\begin{cases} -\Delta\widetilde{\phi} - a\widetilde{\phi} = \widetilde{\lambda}\widetilde{\phi} & \text{in } \Omega, \\ \widetilde{\phi} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

So $\widetilde{\phi}$ is an eigenfunction of L corresponding to the eigenvalue $\widetilde{\lambda} \leq 0$. Since, by hypothesis, $\lambda_2 > 0$, the only possibility is that $\widetilde{\lambda} = \lambda_1(L,\Omega)$ and hence $\widetilde{\phi}$ is a first eigenfunction of L in Ω. If $\phi_{1,n}$ is a sequence of first eigenfunctions of L_n in Ω_n we have the orthogonality condition

$$\int_\Omega \phi_{1,n}\phi_{2,n}\,dx = 0. \tag{2.8}$$

Since, by the previous proposition, we have that also $\phi_{1,n} \rightharpoonup \widetilde{\phi}$ weakly in $H_0^1(\Omega)$, from (2.8), passing to the limit we get

$$\int_\Omega \widetilde{\phi}^2\,dx = 0 \tag{2.9}$$

which is impossible. Hence $\lambda_{2,n} > 0$ for n sufficiently large. □

Now we consider the following semilinear elliptic problem in Ω_n

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega_n, \\ u > 0 & \text{on } \Omega_n, \\ u = 0 & \text{on } \partial\Omega_n, \end{cases} \tag{2.10}$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is of class C^1.

We would like to get some uniform L^∞-estimates for the solutions of (2.10) when f is subcritical. This was already shown by Dancer in [5] using the Gidas–Spruck approach (see [8]). For the reader's convenience we sketch the proof here.

Proposition 2.3. *Let* $u \in C^2(\overline{\Omega}_n)$ *be a classical solution of* (2.10) *with* f *satisfying*

$$\lim_{s\to\infty} \frac{f(s)}{s^p} = a > 0, \tag{2.11}$$

where $1 < p < (N+2)/(N-2)$ *if* $N \geq 3$, $p > 1$ *if* $N = 2$. *Then there exists a number* $C > 0$, *independent of* u *and* n *such that*

$$||u||_{L^\infty(\Omega_n)} \leq C \tag{2.12}$$

Proof. Arguing by contradiction we assume that for a sequence $\{u_n\}$ of solutions of (2.10) we have

$$||u_n||_{L^\infty(\Omega_n)} = u_n(x_n) \to \infty \tag{2.13}$$

for some sequence of points $x_n \in \Omega_n$. Let us set

$$v_n(x) = \frac{1}{||u_n||_\infty} u_n\bigg(\frac{x}{||u_n||_\infty^{(p-1)/2}} + x_n\bigg).$$

These functions satisfy

$$\begin{cases} -\Delta u = \dfrac{1}{||u_n||_\infty^p} f(||u_n||_\infty v_n) & \text{in } \widetilde{\Omega}_n, \\ v_n(0) = 1 & \text{on } \widetilde{\Omega}_n, \\ 0 < v_n \leq 1 & \text{on } \partial\widetilde{\Omega}_n, \end{cases} \tag{2.14}$$

where $\widetilde{\Omega}_n = (\Omega_n - x_n)||u_n||_\infty^{(p-1)/2}$. From (2.11) we get

$$\frac{f(||u_n||_\infty v_n)}{||u_n||_\infty^p} = \frac{v_n^p f(||u_n||_\infty v_n)}{(||u_n||_\infty v_n)^p} \leq C \tag{2.15}$$

for some positive constant C.

Now let us fix any compact set $K \subset \widetilde{\Omega}_n$, such that

$$d(K, \partial\widetilde{\Omega}_n) \geq \alpha > 0 \tag{2.16}$$

where $d(K, \partial\widetilde{\Omega}_n)$ is the distance of K from $\partial\widetilde{\Omega}_n$. Since the right hand side in the equation (2.14) is uniformly bounded, by the standard regularity theory we have that v_n converges to a function v in $C^1(K)$. Therefore, by (2.11) and (2.14) we get

$$\begin{cases} -\Delta v = av^p & \text{in } D, \\ 0 < v \leq 1 & \text{in } D, \end{cases} \tag{2.17}$$

where D is the "limit" domain whose shape depends on the comparison between the rate of divergence of $||u_n||_\infty^{(p-1)/2}$ and of convergence to zero of the parameters ε_n^i (which define the size of the holes). As shown in [5] there are four possibilities

$$
\begin{aligned}
&\text{(i)} && D = \mathbb{R}^N, \\
&\text{(ii)} && D = \mathbb{R}^N_+ = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N,\ x_1 > 0\}, \\
&\text{(iii)} && D = \mathbb{R}^N \setminus \{0\}, \\
&\text{(iv)} && D = \mathbb{R}^N \setminus \alpha D^i \text{ for some } i \in \{1, 2, \ldots, k\} \\
& && \qquad\qquad \text{and } \alpha = \lim_{n\to\infty} \varepsilon_n^i ||u_n||_\infty^{(p-1)/2}.
\end{aligned}
\tag{2.18}
$$

The first two cases (i) and (ii) are excluded as in [8] because of the nonexistence of nontrivial solutions of (2.17) in $\mathbb{R}^N$ or $\mathbb{R}^N_+$.

The case (iii) is excluded because if $D = \mathbb{R}^N \setminus \{0\}$ only singular solutions at zero can exist while v is bounded (see [9] and [5]). Hence, the only possibility is that $v \equiv 0$ a.e. in $\mathbb{R}^N$, but, this can be excluded arguing exactly as in [5] (see Theorem 2 and proof of (ii) of Theorem 1 therein).

Finally, by Theorem 1 of [5], also case (iv) is excluded, since the subdomains D^i are star shaped. Hence (2.12) holds. □

Remark 2.4. The hypothesis that the subdomains D^i are star shaped is only used in the proof of (iv) of the previous proposition. We would like to point out that it is not needed when $p \le N/(N-2)$ (see Theorem 1 of [5]).

Now we assume that Ω contains the origin and is symmetric with respect to the hyperplane $T_0 = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N,\ x_1 = 0\}$ and convex in the x_1-direction. We also define the caps $\Omega^- = \{x \in \Omega,\ x_1 < 0\}$ and $\Omega^+ = \{x \in \Omega,\ x_1 > 0\}$. We end this section by recalling the following result.

Proposition 2.5. *Let u be a positive solution of the semilinear problem*

$$
\begin{cases}
-\Delta u = f(u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{2.19}
$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is a C^1-function with $f(0) \ge 0$. Then the first eigenvalue of the linearized operator $L_u = -\Delta - f'(u)I$ in Ω^- (or Ω^+) with zero Dirichlet boundary conditions is positive.

Proof. The statement was proved in a lecture by L. Nirenberg (see also [6, Theorem 2.1]). We recall here the simple proof.

By the symmetry result of [7] any positive solution of (2.19) is symmetric and $\partial u/\partial x_1 > 0$ in Ω^-. Deriving (2.19) we have that the function $\partial u/\partial x_1$ solves the linearized equation, i.e.

$$-\Delta\left(\frac{\partial u}{\partial x_1}\right) - f'(u)\frac{\partial u}{\partial x_1} = 0 \quad \text{in } \Omega^- \tag{2.20}$$

and, by the condition $f(0) \geq 0$, the Hopf's Lemma applied to the solution u implies that $\partial u/\partial x_1 \not\equiv 0$ on $\partial\Omega^-$. Since $\partial u/\partial x_1 > 0$ in Ω^- this yelds the validity of the maximum principle in Ω^- which, in turns, is equivalent to claim that the first eigenvalue of L_u is positive in Ω^-. Since u is symmetric the same holds in Ω^+. □

3. The subcritical case

As at the beginning of the previous section we assume that Ω *is a smooth bounded domain containing the origin, symmetric with respect to the hyperplane* T_0 *and convex in the* x_1*-direction.* We also take the *smooth star shaped subdomains* D^i, $i = 1, \ldots, k$ *in such a way that the domains* $\Omega_n = \Omega \setminus \bigcup_{i=1}^k D_n^i$, $\varepsilon_n^i \to 0$ (see Section 2 for the precise definition) are also *symmetric with respect to* T_0, but, of course, they are not any more convex in the x_1-direction. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a C^1-function such that

$$\lim_{s\to\infty} \frac{f(s)}{s^p} = a > 0, \tag{3.1}$$

where $1 < p < (N+2)/(N-2)$ if $N \geq 3$, $p > 1$ if $N = 2$ and

$$f(0) \geq 0, \quad f'(0) \neq \lambda_1 \tag{3.2}$$

where λ_1 is the first eigenvalue of the Laplace operator, with zero Dirichlet boundary conditions in Ω.

With this nonlinearity we consider the following semilinear problem

$$\begin{cases} -\Delta u = f(u) & \text{in } D, \\ u > 0 & \text{on } D, \\ u = 0 & \text{on } \partial D, \end{cases} \tag{3.3}$$

where D is either Ω_n or Ω.

Remark 3.1. If f is convex at zero and $f(0) = 0$ it is easy to see that a necessary condition to have positive solutions of (3.3) in Ω is to require $f'(0) < \lambda_1$ and the same is true in Ω_n. Since the first eigenvalue $\lambda_{1,n}$ of the Laplace operator in $H_0^1(\Omega_n)$ converges to λ_1 if we have $f'(0) < \lambda_1$ then also $f'(0) < \lambda_{1,n}$ for sufficiently large n. Hence the requirement $f'(0) \neq \lambda_1$ in (3.2) is often not a real hypothesis.

Now we show the convergence of the solutions of (3.3) in Ω_n to the solution of (3.3) in Ω. This was already proved by Dancer ([3]); since some steps of the proof will be also used later we give all details here.

THEOREM 3.2. *Let u_n be a solution of* (3.3) *in Ω_n. Then the sequence $\{u_n\}$ converges weakly in $H_0^1(\Omega)$ to a solution u_0 of* (3.3) *in Ω.*

PROOF. Let us extend the functions u_n to the whole domain Ω giving the value zero outside of Ω_n. By Proposition 2.3 we know that the functions $\{u_n\}$ are uniformly bounded in the L^∞-norm and by (3.3) they are bounded in $H_0^1(\Omega)$, so that, up to a subsequence that we still denote by u_n, we have that $u_n \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$. Let us show that u_0 is a weak solution of (3.3) in Ω. To do this, it is enough to use as test functions those belonging to the set $V = \{\psi \in H_0^1(\Omega)$ such that $\psi \in H_0^1(\Omega_n)$, for some $n \in \mathbb{N}\}$. In fact the set V is dense in $H_0^1(\Omega)$, because the subdomains D_n^i reduce to a finite number of points, as $n \to \infty$. So, fixed $\psi \in V$ we have that there exists $\overline{n} \in \mathbb{N}$ such that $\psi \in H_0^1(\Omega_{\overline{n}})$ and hence $\psi \in H_0^1(\Omega_n)$, for any $n \geq \overline{n}$. Then, by (3.3), we have

$$\int_\Omega \nabla u_n \nabla \psi \, dx = \int_\Omega f(u_n)\psi \, dx \quad \text{for all } n \geq \overline{n}. \tag{3.4}$$

Since $u_n \rightharpoonup u_0$ in $H_0^1(\Omega)$ we have that $\int_\Omega \nabla u_n \nabla \psi \, dx \to \int_\Omega \nabla u_0 \nabla \psi \, dx$ and also $f(u_n) \to f(u_0)$ a.e. in Ω. Thus, since the sequence $\{u_n\}$ is uniformly bounded, using the dominated convergence theorem, we get $\int_\Omega f(u_n)\psi \, dx \to \int_\Omega f(u_0)\psi \, dx$. By this and (3.4) we deduce that u_0 is a weak solution of (3.3) in Ω and hence, by standard regularity theorems, u_0 is also a classical $C^2(\overline{\Omega})$ solution.

Obviously $u_0 \geq 0$ so that, by the hypothesis $f(0) \geq 0$ we can apply the strong maximum principle claiming that either $u_0 > 0$, as we wanted prove, or $u \equiv 0$. In the last case $f(0)$ must be zero. So, considering the functions $v_n = u_n/||\nabla u_n||_{L^2(\Omega)}$, we have that $v_n > 0$ and

$$-\Delta v_n = \left(\int_0^1 f'(tu_n) \, dt \right) v_n \quad \text{in } \Omega_n. \tag{3.5}$$

Moreover, up to a subsequence, $v_n \rightharpoonup v_0$ in $H_0^1(\Omega)$ while $v_n \to v_0$ strongly in $L^2(\Omega)$. The limit function v_0 cannot be zero because we have

$$1 = \int_\Omega |\nabla v_n^2| \, dx = \int_{\Omega_n} \left(\int_0^1 f'(tu_n) \, dt \right) v_n^2 \, dx \leq C_1 \int_{\Omega_n} v_n^2 \, dx \tag{3.6}$$

where $C_1 = ||f'(s)||_{L^\infty([0,C])}$, C being the constant which appears in (2.12). Passing to the limit in (3.5), arguing as for (3.4), we get that v_0 is a solution of

$$\begin{cases} -\Delta v_0 = f'(0) v_0 & \text{in } \Omega, \\ v_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

Since $v_0 \geq 0$ and $v_0 \not\equiv 0$, by the strong maximum principle we have $v_0 > 0$ which implies that $f'(0) = \lambda_1$, against the hypothesis. So u_0 is a positive solution of (3.3) in Ω. □

Now we show the symmetry of the solutions of (3.3) when the domains D_n^i are sufficiently small.

Theorem 3.3. *For any nonlinearity f satisfying* (3.1) *and* (3.2) *there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ all solutions of problem* (3.3) *in Ω_n are even in the x_1-variable.*

Proof. Arguing by contradiction let us assume that there exists a sequence $\{u_n\}$ of solutions of (3.3) in Ω_n such that u_n are not even in x_1. This implies that, denoting by Ω_n^- the set $\{x \in \Omega_n$, such that $x_1 < 0\}$ the functions $w_n(x) = u_n(x_1, \ldots, x_N) - u_n(-x_1, x_2, \ldots, x_N)$ are not identically zero in Ω_n^- and actually, since they are continuous, they are not zero in a set of positive measure. Therefore, extending w_n by zero to the whole set $\Omega^- = \{x \in \Omega,\ x_1 < 0\}$ we can define in Ω^- the functions $v_n = w_n / ||\nabla w_n||_{L^2(\Omega^-)}$ which belong to $H_0^1(\Omega^-)$ since they are zero on $\partial\Omega^-$.

It is easy to see that v_n satisfy

$$(3.8) \qquad -\Delta v_n = \bigg(\int_0^1 f'(tu_n(x) + (1-t)u_n(-x_1, x_2, \ldots, x_N))\, dt \bigg) v_n \quad \text{in } \Omega^-$$

and converge weakly in $H_0^1(\Omega^-)$ to a function v_0, while $v_n \to v_0$ in $L^2(\Omega^-)$. Exactly as in Theorem 3.2 we prove that $v_0 \not\equiv 0$ and v_0 is a solution of

$$(3.9) \qquad \begin{cases} -\Delta v_0 = f'(u_0)v_0 & \text{in } \Omega, \\ v_0 = 0 & \text{on } \partial\Omega, \\ v_0 \not\equiv 0 & \text{in } \Omega, \end{cases}$$

where u_0 is the limit of the solutions u_n that, by Theorem 3.2, is a positive solution of (3.3) in Ω. Then, by Proposition 2.5, we know that the first eigenvalue of the linearized operator $-\Delta - f'(u_0)I$ in Ω^-, with zero Dirichlet boundary conditions, is positive. This contradicts (3.9) which implies that zero is an eigenvalue of $-\Delta - f'(u_0)I$ in Ω^-. Thus the assertion holds □

We also get the radial symmetry of the solutions when Ω_n are annuli and n is sufficiently large.

Corollary 3.4. *Let Ω_n be annuli. Then for any nonlinearity f satisfying* (3.1) *and* (3.2) *there exists $\overline{n} \in \mathbb{N}$ such that for any $n \geq \overline{n}$ all solutions of problem* (3.3) *in Ω_n are radial.*

Proof. Since Ω_n is symmetric with respect to any hyperplane T_ν passing through the origin and orthogonal to a direction $\nu \in S^{N-1}$, (S^{N-1} being the unit sphere in $\mathbb{R}^N$), to each T_ν the previous theorem applies and gives an integer n_ν such that, for any $n \geq \nu$, all solutions of (3.3) in Ω_n are symmetric with respect to the hyperplane T_ν. To prove the statement we need to show that there exists a positive integer $\overline{n}$ such that $n_\nu \leq \overline{n}$, for any $\nu \in S^{N-1}$.

Arguing by contradiction we construct a sequence of directions ν_k such that $n_{\nu_k} \to \infty$ and, up to a subsequence $\nu_k \to \nu_0 \in S^{N-1}$. This means that we have a sequence of solutions $\{u_k\}$ in Ω_k, $k \to \infty$ which are not symmetric with respect to the hyperplane $T_{\nu_k} = \{x \in \mathbb{R}^N : x \cdot \nu_k = 0\}$. This implies that the functions

$$w_k = u_k(x) - u_k(x_{\nu_k}), \quad x \in \Omega_k^- = \{x \in \Omega_k : x \cdot \nu_k < 0\} \tag{3.10}$$

where x_{ν_k} is the reflected point of x with respect to T_{ν_k}, are not zero in a set of positive measure.

Hence considering the limit symmetry hyperplane $T_{\nu_0} = \{x \in \mathbb{R}^N : x \cdot \nu_0 = 0\}$ and extending w_k by zero to the domain Ω we can define the functions

$$v_k = \frac{w_k}{||\nabla w_k||_{L^2(\Omega)}}$$

which satisfy the equation

$$-\Delta v_k = \left(\int_0^1 f'(tu_k(x) + (1-t)u_k(x_{\nu_k}))\,dt \right) v_k \quad \text{in } \Omega_k^-.$$

Exactly, as in Theorems 3.2 and 3.3, we prove that v_k converges weakly in $H_0^1(\Omega)$ to a function $v_0 \not\equiv 0$ such that

$$\begin{cases} -\Delta v_0 = f'(u_0)v_0 & \text{in } \Omega_0^-, \\ v_0 = 0 & \text{on } \partial\Omega_0^-, \end{cases} \tag{3.11}$$

where u_0 is the limit of the solutions u_k, which is symmetric with respect to T_{ν_0}. Therefore, by Proposition 2.5, we reach the same contradiction as in Theorem 3.3. □

4. The critical case

Keeping the previous notation we consider a bounded smooth domain Ω containing the origin, symmetric with respect to the hyperplane T_0 and convex in the x_1-direction and denote by Ω_n the approximating domains as defined in Sections 2 and 3.

We study the following "critical" semilinear problem

$$\begin{cases} -\Delta u = u^{(N+2)/(N-2)} + \lambda u & \text{in } D, \\ u > 0 & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases} \tag{4.1}$$

where D is either Ω or Ω_n, $N \geq 4$ and $\lambda \in (0, \lambda_1)$ where λ_1 is the first eigenvalue of the Laplace operator in Ω with Dirichlet boundary conditions. Note that, by the continuity of this eigenvalue with respect to the domain, we have that $\lambda \in (0, \lambda_{1,n})$ for n sufficiently large, having denoted by $\lambda_{1,n}$ the first eigenvalue of $-\Delta$ in Ω_n with the same boundary conditions.

Thus by a famous result of Brezis and Nirenberg, we have that, for every $\lambda \in (0, \lambda_1)$, there exists a solution of (4.1), both in Ω or Ω_n which, up to a multiplier, minimizes the functional

$$Q_\lambda(u) = \int_D |\nabla u|^2 - \lambda \int_D u^2 \tag{4.2}$$

among the functions $u \in H_0^1(D)$ with $\int_D u^{2^*} = 1$ where $2^* = 2N/(N-2) = (N+2)/(N-2) + 1$ and D is either Ω or Ω_n.

If $N = 3$, the same existence result holds in a suitable neighbourhood of λ_1, say (λ^*, λ_1), (see [2]). As a consequence for $N = 3$, it will understood that we will take $\lambda \in (\lambda^*, \lambda_1)$. In particular, if Ω is a ball $\lambda^* = \lambda_1/4$.

Let us denote by S^λ and S_n^λ the infimum of (4.2) in $H_0^1(\Omega)$ and $H_0^1(\Omega_n)$ respectively.

We start by proving the following result.

Proposition 4.1. *$S_n^\lambda \to S^\lambda$ as $n \to \infty$.*

Proof. Since $\Omega_n \subset \Omega$ we have that $S^\lambda \le S_n^\lambda$ for any $n \ge 1$. So if we prove that

$$\lim_{n\to\infty} S_n^\lambda \le S^\lambda \tag{4.3}$$

the claim follows. Let us consider the function $\eta_r : B(0, 2r) \to \mathbb{R}$ defined as

$$\eta_r(x) = \begin{cases} \dfrac{2^{n-2}}{1-2^{n-2}}\left(\dfrac{r^{n-2}}{|x|^{n-2}} - 1\right) & \text{if } r < |x| < 2r, \\ 0 & \text{if } |x| \le r. \end{cases} \tag{4.4}$$

Recalling the definition of D_n^i let us consider the smallest number r_n^i such that $D_n^i \subset B(y_i, r_n^i)$ and set

$$\zeta_n(x) = \begin{cases} \eta_{r_n^i}(x - y_i) & \text{if } |x - y_i| < 2r_n^i, \\ 1 & \text{elsewhere.} \end{cases} \tag{4.5}$$

Finally we consider the function $v_n \in H_0^1(\Omega_n)$,

$$v_n = \frac{\zeta_n v_0}{||\zeta_n v_0||_{L^{p+1}(\Omega)}},$$

where v_0 is the function which minimizes $Q_\lambda(u)$ in Ω. Let us compute $Q_\lambda(v_n)$.

By Lebesgue dominated convergence theorem we get

$$\int_\Omega v_n^2\,dx = \frac{\int_\Omega \zeta_n^2 v_0^2\,dx}{||\zeta_n v_0||^2_{L^{p+1}(\Omega)}} \to \frac{\int_\Omega v_0^2\,dx}{(\int_\Omega v_0^{p+1})^{2/(p+1)}}. \tag{4.6}$$

Concerning the integral $\int_\Omega |\nabla v_n|^2\,dx$ we have the following estimate

$$\begin{aligned}(4.7)\qquad \int_\Omega |\nabla(\zeta_n v_0)|^2\,dx &= \int_\Omega |\nabla v_0|^2 \zeta_n^2\,dx \\ &\quad + \int_\Omega |\nabla \zeta_n^2|^2 v_0^2\,dx + 2\int_\Omega \nabla v_0\cdot\nabla\zeta_n v_0\zeta_n\,dx \\ &= \int_\Omega |\nabla v_0|^2\,dx + O\bigg(\sum_{i=0}^k r_n^{i^{(N-2)}}\bigg).\end{aligned}$$

The estimates (4.6) and (4.7) imply that

$$(4.8)\qquad \lim_{n\to\infty} Q_\lambda(v_n) = Q_\lambda(v_0) = S_\lambda$$

which proves (4.3). □

Let v_n be a sequence of minima of (4.2) in $H_0^1(\Omega_n)$ corresponding to a fixed value of λ belonging to $(0,\lambda_1)$ if $N\ge 4$ or to (λ^*,λ_1) if $N=3$. We extend v_n by zero to the whole domain Ω and we have

Proposition 4.2. *The sequence v_n converges strongly in $H_0^1(\Omega)$ to a function v_0 which is a minimizer of* (4.2) *in $H_0^1(\Omega)$.*

Proof. By definition we have that

$$(4.9)\qquad \int_\Omega v_n^{2^*}\,dx = 1 \quad\text{and}\quad \int_\Omega |\nabla v_n|^2\,dx - \lambda\int_\Omega v_n^2 = S_n^\lambda.$$

Since by the previous proposition $S_n^\lambda \to S^\lambda$ we have that $\{v_n\}$ is a minimizing sequence for the functional Q_λ defined in (4.2), in the space $H_0^1(\Omega)$. By the result of Brezis and Nirenberg ([2]) we know that for the value of λ considered, S^λ is smaller than S which is the best Sobolev constant for the imbedding of $H_0^1(\Omega)$ into $L^{2^*}(\Omega)$. Again by a result of [2] we have that this implies that v_n converges strongly in $H_0^1(\Omega)$ to a function v_0 which is a minimizer of (4.2) in $H_0^1(\Omega)$. □

The minimizer v_0 (respectively v_n) of (4.2) are obviously solutions of the problem

$$(4.10)\qquad \begin{cases} -\Delta v = \mu v^{(N+2)/(N-2)} + \lambda v & \text{in } D,\\ v > 0 & \text{in } D,\\ v = 0 & \text{on } \partial D,\end{cases}$$

where D is either Ω or Ω_n and μ is a suitable Lagrange multiplier, namely $\mu = S^\lambda$ or $\mu = S_n^\lambda$. Then it is easy to see that the function $u = \mu^{1/(p-1)}v$, $p=(N+2)/(N-2)$ is a solution of (4.1). These solutions, obtained through the minimization procedure, will be called least-energy solutions of (4.1). To prove the symmetry of these solutions we can either argue as in the subcritical case or, since the nonlinearity $f(s) = s^{(N+2)/(N-2)} + \lambda s$ is strictly convex, exploit the following proposition which is proved in [10].

Proposition 4.3. *Let us denote by $\lambda_1(L_n, \Omega_n^-)$ and $\lambda_1(L_n, \Omega_n^+)$ the first eigenvalues of the linearized operators $L_n = -\Delta - f'(u_n)I$ in the domains $\Omega_n^- = \{x \in \Omega_n,\ x_1 < 0\}$, $\Omega_n^+ = \{x \in \Omega_n,\ x_1 > 0\}$. If they are both nonnegative then u_n is even in x_1.*

Proof. See Proposition 1.1 in [10]. □

Theorem 4.4. *For every $\lambda \in (0, \lambda_1)$ if $N \geq 4$, (λ^*, λ_1) if $N = 3$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ all least-energy solutions of* (4.1) *in Ω_n are even in the x_1-variable.*

Proof. Arguing by contradiction let us assume that there is a sequence $\{u_n\}$ of least-energy solutions of (4.1) in Ω_n which are not even in x_1.

By Proposition 4.2 $\{u_n\}$ converges strongly in $H_0^1(\Omega)$ to a least energy solution u_0 of (4.1) in Ω and hence $a_n(x) = (pu_n^{p-1} + \lambda)x$, $p = (N+2)/(N-2)$ converges in the space $L^{N/2}(\Omega)$ to the function $a(x) = (pu_0^{p-1} + \lambda)x$. By Proposition 2.1 applied to the linearized operators $L_n = -\Delta - a_n(x)I$ and $L = -\Delta - a(x)I$ we have that the eigenvalues $\lambda_1(L_n, \Omega_n^-) \to \lambda_1(L, \Omega^-)$ and $\lambda_1(L_n, \Omega_n^+) \to \lambda_1(L, \Omega^+)$. By Proposition 2.5 we know that $\lambda_1(L, \Omega^-)$ and $\lambda_1(L, \Omega^+)$ are both positive so that also $\lambda_1(L_n, \Omega_n^-)$ and $\lambda_1(L_n, \Omega_n^+)$ are positive for n sufficiently large. This, in turns, implies, by Proposition 4.3, that the functions u_n are even in x_1 against what we assumed.

Alternatively, arguing as in the proof of Theorem 3.3 we could consider the functions

$$w_n(x) = u_n(x_1, x_2, \ldots, x_N) - u_n(-x_1, x_2, \ldots, x_N) \quad \text{in } \Omega_n^-$$

and then set $v_n = w_n / ||\nabla w_n||_{L^2(\Omega^-)}$. The functions v_n satisfy (3.8) in Ω_n^- with $f'(s) = s^{4/(N-2)} - \lambda$ and converge weakly in $H_0^1(\Omega^-)$ to a function v_0. To prove that $v_0 \not\equiv 0$, in Theorem 3.3 we used the uniform L^∞-estimates which are not known when the nonlinearity has a critical growth. In our case we observe that, by Proposition 4.2, $u_n \to u_0$ in $H_0^1(\Omega)$ and hence in $L^{2N/(N-2)}(\Omega)$. Therefore

$$u_n^{4/(N-2)} \to u_0^{4/(N-2)} \quad \text{in } L^{N/2}(\Omega) \tag{4.11}$$

and this implies that v_0 is a solution of (3.11) and $v_0 \not\equiv 0$. In fact

$$\begin{aligned} 1 &= \int_\Omega |\nabla v_n^2|\,dx \\ &= \int_{\Omega_n} \bigg(\int_0^1 (tu_n(x) + (1-t)u_n(-x_1, x_2, \ldots, x_N))^{4/(N-2)}\,dt \bigg) v_n^2\,dx. \end{aligned} \tag{4.12}$$

If $v_0 \equiv 0$ then $v_n \rightharpoonup 0$ weakly in $H_0^1(\Omega)$ and using (4.11) we have that

$$\int_{\Omega_n} \bigg(\int_0^1 (tu_n(x) + (1-t)u_n(-x_1, x_2, \ldots, x_N))^{4/(N-2)}\,dt \bigg) v_n^2\,dx \to 0 \tag{4.13}$$

and it gives a contradiction with (4.12). So $v_0 \not\equiv 0$.

After this we can repeat exactly the same proof as in Theorem 3.3 and then the same contradiction arises. □

Corollary 4.5. *Let Ω_n be annuli. Then for every λ in the intervals considered there exists $\overline{n} \in \mathbb{N}$ such that for any $n \geq \overline{n}$ all solutions of problem* (4.1) *in Ω_n are radial.*

Proof. It is similar to that of Corollary 3.4, using (4.11) instead of the uniform L^∞-estimates. □

If Ω is a ball we know that problem (4.1) has only one solution which is also non degenerate (see [1] or [12]). We end by showing that the same is true for the approximating domains Ω_n.

Theorem 4.6. *Let Ω be a ball. Then, for every λ in the intervals considered there exists $\overline{n} \in \mathbb{N}$ such that, for any $n \geq \overline{n}$, problem* (4.1) *in Ω_n has only one least-energy solution which is also nondegenerate.*

Proof. By contradiction let us assume that there exist two sequences of least-energy solutions of (4.11) in Ω_n, say $\{u_{1,n}\}$, $\{u_{2,n}\}$, with $\{u_{1,n}\} \not\equiv \{u_{2,n}\}$. As usual we extend them by zero in Ω and define the functions

$$v_n = \frac{u_{1,n} - u_{2,n}}{||\nabla(u_{1,n} - u_{2,n})||_{L^2(\Omega)}} \tag{4.14}$$

which satisfy

$$\begin{cases} -\Delta v_n = p\left(\displaystyle\int_0^1 (tu_{1,n} + (1-t)u_{2,n})^{p-1}\, dt\right) v_n + \lambda v_n & \text{in } \Omega_n, \\ v_n = 0 & \text{on } \partial\Omega_n, \end{cases} \tag{4.15}$$

where $p = (N+2)/(N-2)$. Since $\int_\Omega |\nabla v_n|^2\, dx = 1$ we get that $v_n \rightharpoonup v_0$ in $H_0^1(\Omega)$. By Proposition 4.2 both $\{u_{1,n}\}$ and $\{u_{2,n}\}$ converge strongly in $H_0^1(\Omega)$ to the unique least-energy solution u_0 of (4.1) in the ball Ω. Using this and arguing as in the second part of the Proof of Theorem 4.4 we deduce that $v_0 \not\equiv 0$. Then, passing to the limit in (4.15), we get that v_0 is a solution of the linearized equation at u_0 in Ω, i.e.

$$\begin{cases} -\Delta v_0 - p u_0^{p-1} v_0 - \lambda v_0 = 0 & \text{in } \Omega, \\ v_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.16}$$

Since u_0 is a nondegenerate solution of (4.1) (see [1], [12]) we reach a contradiction which shows that $\{u_{1,n}\} \equiv \{u_{2,n}\}$ for n sufficiently large.

Finally, since u_0 is a least-energy solution of (4.1) in the ball Ω we also have that it has index one, i.e. the second eigenvalue of the linearized operator at u_0 is positive in Ω. Therefore, using the strong convergence of the least-energy solutions u_n of (4.1) in Ω_n and Proposition 2.2, we get that the second eigenvalue of the linearized operators at u_n in Ω_n are also positive. This implies that the solutions u_n are nondegenerate. □

From the proof of Theorem 4.6 it is obvious that the same result holds if Ω is not a ball but any domain where (4.1) has only one non degenerate least-energy solution.

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