Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 21, 2003, 81–100

A NEW MORSE THEORY AND STRONG RESONANCE PROBLEMS

Shujie Li

Abstract. Is it possible to establish a new Morse theory if the function f losses the (PS) condition at some isolated values? Yes, it is! In this paper we will recall a such a theory. One of the purposes of establishing such a theory is to consider multiplicity results for strong resonance problems and to deal with multiple resonant energy levels. Both of these questions were not studied much in the past because of the limitation of methods. Using the new Morse theory we can deal with these problems.

1. Introduction

The Morse theory was established in the 20s by M. Morse (see [12]). Its object is the relation between the topological type of critical points of a function f and the topological structure of the manifold on which the function f is defined. The Morse theory of functional defined on an infinite dimensional Hilbert space (or manifold) was given by R. S. Palais, S. Smale, E. Rothe, D. Gromoll and W. Meyer in the 60s (see [14], [15] and [9]). For the equivariant Morse theory, which was first studied by R. Bott (see [3], [4]). For the Finsler manifolds modelled on Banach space, it was given by K. Uhlenbeck ([17]), K. C. Chang ([5]) and T. Tromba ([16]) starting from the 70s. The tool in this study is the deformation theorem. Since the space (or manifold) X is infinite dimensional one

2000 *Mathematics Subject Classification.* 58E05, 35J65.
Key words and phrases. Morse theory, strong resonance problems, elliptic boundary value problems.
Support by Chinese National Science Foundation.

can always assume that f satisfies some compactness conditions. A well known condition was called the Palais–Smale condition: $f \in C^1(X, \mathbb{R}^1)$, if any sequence $\{x_n\} \subset X$, along which $|f(x_n)|$ is bounded and $df(x_n) \to 0$ possesses a convergent subsequence. We denote this condition by (PS) for simplicity. Without this condition at some isolated values it means that the deformation theorem fails, so does the usual Morse theory.

Is it possible to establish a new Morse theory if the function f looses the (PS) condition at some isolated values? In this paper we will recall a such a theory which was first introduced by N. Hirano, Shujie Li and Z. Q. Wang for global case in [10], and by T. Bartsch and Shujie Li for local behavior of f near infinity in [2].

First, let us recall the usual Morse inequalities. Let X be a Hilbert space. $f\colon X \to R^1$ be of class C^1. We write $K = \{x \in X \mid f'(x) = 0\}$ for the set of critical points of f, K is finite, and $f^c = \{x \in X \mid f(x) \leq c\}$, the level set of f at c. Let $x_0 \in K$ be an isolated critical point with value $c = f(x_0)$. Then the critical groups of f at x_0 are well defined (see [6] and [13])

$$C_k(f, x_0) = H_k(f^c, f^c \setminus \{x_0\}; G), \quad k \in \mathbb{Z}.$$

Here $H_*(\,\cdot\,; G)$ denotes the singular homology group with coefficients in a commutative ring G. Suppose that f satisfies the (PS) condition, then the usual Morse inequalities read as

$$\sum_{x \in K} P(f, x) = P(f, \infty) + (1 + t)Q(t) \tag{1.1}$$

where

$$\begin{aligned} P(f, x) &= \sum_{k=0}^{\infty} \beta_k(f, x) t^k && \text{for } x \in K, \\ P(f, \infty) &= \sum_{k=0}^{\infty} \dim H_k(X, f^a) t^k, \\ \beta_k(f, x) &= \dim C_k(f, x) && \text{for all } x \in K, \end{aligned}$$

$Q(t)$ is a formal series with nonnegative coefficients, $a < 0$ is such that $a < \inf_{x \in K} f(x)$. We call $\sum_{x \in K} P(f, x)$ the Morse polynomial, and $P(f, \infty)$ the Poincare polynomial. (1.1) is a very important tool in critical point theory. (1.1) establishes the relation between the topological type of critical points of f and the topological structure of X.

What will happen if f looses the (PS) condition? What is the relation between the topological type of critical points of f and the topological structure of X?

It is still possible to establish the Morse inequalities in some cases, for instance, if f looses the (PS) in a set $C_\infty \subset \mathbb{R}^1$ and there are only finite values of f in $\mathbb{R}^1$. The case of

$$P(f,c) = \sum_{k=0}^{\infty} \beta_k(f,c) t^k \quad \text{for } c \in C_\infty,$$

where $\beta_k(f,c) = \dim H_k(f^{c+\varepsilon} \cap \widetilde{C}_{R,M}, f^{c-\varepsilon} \cap \widetilde{C}_{R,M})$ for $c \in C_\infty$ and $\widetilde{C}_{R.M}$ is a special set will be given in the next section. Though in our setting f looses the (PS) condition we can still establish the following inequalities

$$\sum_{x\in K} P(f,x) + \sum_{c\in C_\infty} P(f,c) = P(f,\infty) + (1+t)Q(t) \tag{1.2}$$

where $a < \min\{\inf_{x\in K} f(x), \inf_{f(x)\in C_\infty} f(x)\}$ in $P(f,\infty)$. When comparing the new inequalities (1.2) with (1.1) there is a new polynomial $\sum_{c\in C_\infty} P(f,c)$ on the left hand side. This new polynomial was determined by the critical groups at infinity and characterized the topological changes of the level set of f at such isolated values. It is a very delicate task to compute these critical groups at infinity. In fact, we need a splitting theorem at infinity which was given by T. Bartsch and Shujie Li in [2]. When f satisfies the (PS) condition the new polynomial is trivial and we obtain the usual Morse inequalities.

One of the purposes of establishing such a theory is to consider multiplicity results for strong resonance problems and to deal with multiple resonant energy levels. Both of these questions were not studied much in the past because of the limitation of methods. Using the new Morse theory we can deal with these problems.

2. A new Morse theory

We consider the following functional:

$$f(x) = \frac{1}{2}(Ax,x) + g(x)$$

where $A\colon X \to X$ is a self-adjoint linear operator such that 0 is isolated in the spectrum of A.

Set $V = \operatorname{Ker} A$, $W = V^\perp$. W splits as $W = W^+ \bigoplus W^-$ with $W^\pm$ invariant under A and $A|_{W^+}$ is positive definite, $A|_{W^-}$ is negative definite.

Let $x = v + \omega$ where $v \in V$, $\omega \in W$. There exists $\alpha > 0$ such that $\pm 1/2\langle A\omega, \omega\rangle \geq \alpha\|\omega\|^2$ for $\omega \in W^\pm$. We denote $\mu = \dim W^-$, and $\nu = \dim V$.

We impose the following condition on f:

(A_∞) $g \in C^2(X,R)$, $\|g'(x)\|$ is bounded. For any $M > 0$, uniformly in $\omega \in \{\|\omega\| \leq M\}$, $\|g''(\omega+\nu)\| < \alpha$, $g'(\omega+\nu) \to 0$ as $\|v\| \to \infty$. Moreover, g is assumed to be bounded on any bounded set.

In applications g' usually has to be compact and $\dim \operatorname{Ker} A$ is finite. In this case f satisfies the bounded Palais–Smale condition $(\mathrm{BPS})_c$: any bounded sequence $\{x_n\} \subset X$ such that $f(x_n) \to c$ and $f'(x_n) \to 0$ has a convergent subsequence.

To study the (PS) condition for f, we define

$$C_\infty := \{c \in R \mid \exists v_n \in V,\ \omega_n \in W \text{ with } \|v_n\| \to \infty,\\ \|\omega_n\| \to 0 \text{ such that } g(v_n + \omega_n) \to c\}.$$

Clearly, C_∞ is a closed set. Let $C_{R,M} = \{x = v + \omega \mid \|v\| > R,\ \|\omega\| < M\}$.

Lemma 2.1. *Let (A_∞) hold and assume g' is compact and $\nu < \infty$. Then for any fixed $R, M > 0$, f satisfies* (PS) *condition in $X \setminus C_{R,M}$.*

Proof. Let $\{x_n\}$ be a $(\mathrm{PS})_c$ sequence of f, $\{x_n\} \notin C_{R,M}$, i.e.

$$\frac{1}{2}\langle Ax_n, x_n\rangle + g(x_n) = c + o(1), \tag{2.1}$$

$$Ax_n = -g'(x_n) + o(1). \tag{2.2}$$

Since $\|Ax_n\| = \|A\omega_n\| \geq 2\alpha\|\omega_n\|$. From (A_∞) and (2.2) we have that $\|\omega_n\|$ is bounded. If $\|v_n\| \to \infty$ then $\|\omega_n\| \to 0$. It implies $\{x_n\} \subset C_{R,M}$, a contradiction. So $\|x_n\|$ is bounded, and by a standard argument we get the lemma. □

Corollary 2.2. *If $\{x_n\}$ is a $(\mathrm{PS})_c$ sequence, then either*

(a) *$\{x_n\}$ has a bounded subsequence, or*
(b) *$c \in C_\infty$ is such that up to a subsequence, $\|v_n\| \to \infty$, $\|\omega_n\| \to 0$ and $g(v_n + \omega_n) \to c$.*

Remark 2.3. f satisfies the $(\mathrm{PS})_c$ condition if $c \notin C_\infty$. Especially, when $C_\infty = \phi$, f satisfies the $(\mathrm{PS})_c$ condition for all $c \in R$.

Let $K = \{x \mid f'(x) = 0\}$, $K_c = \{x \mid f'(x) = 0,\ f(x) = c\}$. From Lemma 2.1 we know that K is bounded in $X \setminus C_{R,M}$. Now, we discuss the deformation condition.

Definition 2.4. We say that f satisfies the deformation condition $(\mathrm{D})_c$ at $c \in R$, if for any $\overline{\varepsilon} > 0$ and any neighbuorhood N of K_c there exist $\overline{\varepsilon} > \varepsilon > 0$ and a continuous deformation η: $[0,1] \times X \to X$ such that

(i) $\eta(0, \cdot) = \mathrm{id}_X$,
(ii) $\eta(t, x) = x$ if $x \notin f^{-1}([c - \overline{\varepsilon}, c + \overline{\varepsilon}])$,
(iii) $f(\eta(s, x)) \leq f(\eta(t, x))$ if $s \geq t$,
(iv) $\eta(1, f^{c+\varepsilon} \setminus N) \subset f^{c-\varepsilon}$.

The following is well known (see [2], [6]).

Corollary 2.5.

(a) *If f satisfies the* $(\mathrm{PS})_c$ *condition, then* $(\mathrm{D})_c$ *holds.*
(b) *If f satisfies* $(\mathrm{D})_c$ *for all $c \in [a,b]$ and if $K_c = \phi$ for $c \in [a,b]$ then there exists a deformation $\eta(t,\cdot)\colon X \to X$ such that $\eta(0,\cdot) = \mathrm{id}$, $\eta(t,x) = x$ if $x \notin f^{-1}([a-1,b+1])$, $f(\eta(t,x))$ is decreasing in t and $\eta(1,f^b) \subset f^a$.*
(c) *If f satisfies* $(\mathrm{D})_c$ *for all $c \geq a$ and if $K_c = \phi$ for $c \geq a$ then there exists a deformation $\eta(t,\cdot)\colon X \to X$ with $\eta(0,\cdot) = \mathrm{id}$, $\eta(t,x) = x$ if $f(x) \leq a-1$, $f(\eta(t,x))$ is decreasing in t and $\eta(1,X) \subset f^a$.*

Corollary 2.6. *Let (A_∞) hold and assume that g' is compact and $\nu < \infty$. Then, for any $c \notin C_\infty$,* $(\mathrm{D})_c$ *holds.*

Now we consider the computation of $H_q(f^{c+\varepsilon}, f^{c-\varepsilon})$, where c is an isolated value in C_∞. Let us fix some notation first. $f_{c-\varepsilon}^{c+\varepsilon} = \{x \mid c-\varepsilon \leq f(x) \leq c+\varepsilon\}$ and $K_{c-\varepsilon}^{c+\varepsilon} = K \cap f_{c-\varepsilon}^{c+\varepsilon}$. Define a normalized negative gradient flow for f

$$\begin{cases} \dot\eta(t,x) = -\dfrac{f'(\eta(t,x))}{\|f'(\eta(t,x))\|}, \\ \eta(0,x) = x. \end{cases} \tag{2.3}$$

In the following, for a subset $F \subset X$, we denote

$$\widetilde{F} = \bigcup_{t \in R} \eta(t,F). \tag{2.4}$$

In this paper we assume that f has only isolated critical points so there is an $\varepsilon_0 > 0$ such that $K_{c-\varepsilon_0}^{c+\varepsilon_0} = K_c$ and K_c is compact. Define

$$\begin{aligned} U_{R,M} &= \{x = v+\omega \mid \|v\| \leq \mathbb{R}\} \cup \{x = v+\omega \mid \|v\| > R,\ \|\omega\| \geq M\}, \\ U_{R,M}^{c+\omega} &= U_{R,M} \cap f^{c+\omega}, \\ C_{R,M} &= \{x = v+\omega \mid \|v\| > R,\ \|\omega\| < M\} = X \setminus U_{R,M}, \\ C_{R,M}^{c+\varepsilon} &= C_{R,M} \cap f^{c+\varepsilon}, \\ A_{R,M}^{c+\varepsilon} &= U_{2R,M/2}^{c+\varepsilon} \cap C_{R,M}^{c+\varepsilon}. \end{aligned}$$

Lemma 2.7. *For R large and $R > M > 0$, there exists $\varepsilon_1 > 0$ such that for all $0 < \varepsilon < \varepsilon_1$*

(a) $(f_{c-\varepsilon}^{c+\varepsilon} \cap \widetilde{U}_{R,M}^{c+\varepsilon}) \cap (f_{c-\varepsilon}^{c+\varepsilon} \cap \widetilde{C}_{2R,M/4}^{c+\varepsilon}) = \phi$,
(b) $(f^{c+\varepsilon} \cap \widetilde{A}_{R,M}^{c+\varepsilon}) \cong (f^{c-\varepsilon} \cap \widetilde{A}_{R,M}^{c+\varepsilon})$.

Proof. Choose R large, $R > M > 0$ such that $K \subset B(0,R/2) \cup C_{3R,M/8}$. By Lemma 2.1. f satisfies the (PS) condition in $U_{3R,M/8}^{c+\varepsilon}$. Then there exists an $\varepsilon' > 0$ such that $\|f'(x)\| \geq \varepsilon'$, for all $x \in f_{c-\varepsilon_0}^{c+\varepsilon_0} \cap (U_{3R,M/8} \setminus B(0,R/2))$. Let $0 < \varepsilon < \min\{\varepsilon_0, M/8\varepsilon'\}$. If for some $x \in f_{c-\varepsilon}^{c+\varepsilon} \cap U_{R,M}$, $\eta(t,x)$ ranges from $f_{c-\varepsilon}^{c+\varepsilon} \cap U_{R,M}$ to $f_{c-\varepsilon}^{c+\varepsilon} \cap C_{R+M/4,3M/4}$ then there exist $t_1 < t_2$ such that $\eta(t_1,x) \in f_{c-\varepsilon}^{c+\varepsilon} \cap \partial U_{R,M}$,

$\eta(t_2,x)\in f_{c-\varepsilon}^{c+\varepsilon}\cap\partial C_{R+M/4,3M/4}\subset f_{c-\varepsilon}^{c+\varepsilon}\cap U_{2R,M/2}, \eta(t,x)\in\overline{C}_{R,M}\cap U_{R+M/4,3M/4}$ for all $t\in[t_1,t_2]$, so that

$$\frac{M}{4}\le\|\eta(t_1,x)-\eta(t_2,x)\|\le\int_{t_1}^{t_2}\|\dot\eta(s,x)\|\,ds\le|t_2-t_1|.$$

On the other hand

$$\begin{aligned}f(\eta(t_2,x))-f(\eta(t_1,x)) &= \int_{t_1}^{t_2}\frac{d}{ds}f(\eta(s,x))\,ds\\ &= \int_{t_1}^{t_2}-\|f'(\eta(s,x))\|ds\le-\varepsilon'|t_2-t_1|.\end{aligned}$$

Then

$$\varepsilon'\frac{M}{4}\le\varepsilon'|t_2-t_1|\le f(\eta(t_1,x))-f(\eta(t_2,x))\le 2\varepsilon.$$

We get a contradiction with the choice of ε. Therefore

$$(\widetilde{f_{c-\varepsilon}^{c+\varepsilon}\cap U_{R,M}})\cap(f_{c-\varepsilon}^{c+\varepsilon}\cap C_{R+M/4,3M/4})=\phi.$$

Since

$$f_{c-\varepsilon}^{c+\varepsilon}\cap\widetilde{U}_{R,M}^{c+\varepsilon}\subset(\widetilde{f_{c-\varepsilon}^{c+\varepsilon}\cap U_{R,M}}),$$

we get

$$(f_{c-\varepsilon}^{c+\varepsilon}\cap\widetilde{U}_{R,M}^{c+\varepsilon})\cap(f_{c-\varepsilon}^{c+\varepsilon}\cap C_{R+M/4,3M/4})=\phi. \tag{2.5}$$

Similarly, if $0<\varepsilon<\min\{\varepsilon_0,M\varepsilon'/8\}$, we have

$$(\widetilde{f_{c-\varepsilon}^{c+\varepsilon}\cap C_{2R,M/4}})\cap(f_{c-\varepsilon}^{c+\varepsilon}\cap U_{2R-M/4,\frac{M}{2}})=\phi.$$

Since

$$f_{c-\varepsilon}^{c+\varepsilon}\cap\widetilde{U}_{2R,M/4}^{c+\varepsilon}\subset(\widetilde{f_{c-\varepsilon}^{c+\varepsilon}\cap U_{2R,M/4}^{c+\varepsilon}}),$$

we have

$$(f_{c-\varepsilon}^{c+\varepsilon}\cap\widetilde{C}_{2R,M/4}^{c+\varepsilon})\cap(f_{c-\varepsilon}^{c+\varepsilon}\cap U_{2R-M/4,M/2})=\phi. \tag{2.6}$$

Combining (2.5) and (2.6) we get (a). Finally, from the proof of (a) we have

$$f_{c-\varepsilon}^{c+\varepsilon}\cap\widetilde{A}_{R,M}^{c+\varepsilon}\subset U_{3R,M/4}^{c+\varepsilon}$$

and f satisfies the (PS) condition in $f_{c-\varepsilon}^{c+\varepsilon}\cap\widetilde{A}_{R,M}^{c+\varepsilon}$. Since R is large and $K\cap\widetilde{A}_{R,M}^{c+\varepsilon}=\phi$, from the deformation theorem we immediately get (b). □

Let S be an open subset of X, $K(\widetilde{S})=K\cap\widetilde{S}, K_c(\widetilde{S})=K_c\cap\widetilde{S}$ where $\widetilde{S}$ was given by (2.3) and (2.4).

Lemma 2.8. *Let* $f \in C^1$ *and* f *satisfy the* (PS) *condition in* $\widetilde{S} \cap f_{c-\varepsilon}^{c+\varepsilon}$. *Assume that* c *is an isolated critical value and* $K_c(\widetilde{S})$ *is finite. Then for* $\varepsilon > 0$ *small enough*

$$H_k(f^{c+\varepsilon} \cap \widetilde{S}, f^{c-\varepsilon} \cap \widetilde{S}) \cong \bigoplus_{x \in K_c(\widetilde{S})} C_k(f, x).$$

Proof. By the deformation theorem and the homotopy invariance of singular homology groups, we have

$$H_k(f^{c+\varepsilon} \cap \widetilde{S}, f^{c-\varepsilon} \cap \widetilde{S}) \cong H_k(f^{c} \cap \widetilde{S}, f^{c-\varepsilon} \cap \widetilde{S})$$

and

$$H_k((f^c \setminus K_c(\widetilde{S})) \cap \widetilde{S}, f^{c-\varepsilon} \cap \widetilde{S}) \cong H_k(f^{c-\varepsilon} \cap \widetilde{S}, f^{c-\varepsilon} \cap \widetilde{S}) \cong 0.$$

Applying the exactness of singular homology groups to the triple $(f^c \cap \widetilde{S}, (f^c \setminus K_c(\widetilde{S})) \cap \widetilde{S}, f^{c-\varepsilon} \cap \widetilde{S})$:

$$\begin{aligned} \cdots &\to H_k((f^c \setminus K_c(\widetilde{S})) \cap \widetilde{S}, f^{c-\varepsilon} \cap \widetilde{S}) \to H_k(f^c \cap \widetilde{S}, f^{c-\varepsilon} \cap \widetilde{S}) \\ &\to H_k(f^c \cap \widetilde{S}, (f^c \setminus K_c(\widetilde{S})) \cap \widetilde{S}) \to H_{k-1}((f^c \setminus K_c(\widetilde{S})) \cap \widetilde{S}, f^{c-\varepsilon} \cap \widetilde{S}) \to \cdot s \end{aligned}$$

we have

$$0 \to H_k(f^c \cap \widetilde{S}, f^{c-\varepsilon} \cap \widetilde{S}) \to H_k(f^c \cap \widetilde{S}, (f^c \setminus K_c(\widetilde{S})) \cap \widetilde{S}) \to 0,$$

i.e.

$$H_k(f^c \cap \widetilde{S}, f^{c-\varepsilon} \cap \widetilde{S}) \cong H_k(f^c \cap \widetilde{S}, (f^c \setminus K_c(\widetilde{S})) \cap \widetilde{S}).$$

Let $K_c(\widetilde{S}) = \{x_1, \ldots, x_n\}$. Using the excision property we have

$$\begin{aligned} &H_k(f^c \cap \widetilde{S}, (f^c \setminus K_c(\widetilde{S})) \cap \widetilde{S}) \\ &\quad \cong H_k\bigg(f^c \cap \bigcup_{j=1}^{n} B(x_j, \varepsilon), f^c \cap \bigcup_{j=1}^{m} (B(x_j, \varepsilon) \setminus \{x_j\})\bigg) \cong \bigoplus_{x \in K_c(\widetilde{S})} C_k(f, x), \end{aligned}$$

for $\varepsilon > 0$ small enough, where $B(x, \varepsilon)$ is the ball centered at x with radius ε. □

Theorem 2.9. *Let* (A_∞) *hold and assume that* g' *is compact and* $\nu < \infty$, *assume further that* K_c *is finite, then for* R *large and* $R > M > 0$ *there exists* $\varepsilon_1 > 0$, *such that for all* $0 < \varepsilon < \varepsilon_1$,

$$\begin{aligned} H_q(f^{c+\varepsilon}, f^{c-\varepsilon}) &\cong H_q(f^{c+\varepsilon} \cap \widetilde{U}_{2R,M/2}^{c+\varepsilon}, f^{c-\varepsilon} \cap \widetilde{U}_{2R,M/2}^{c+\varepsilon}) \\ &\quad \oplus H_q(f^{c+\varepsilon} \cap \widetilde{C}_{R,M}^{c+\varepsilon}, f^{c-\varepsilon} \cap \widetilde{C}_{R,M}^{c+\varepsilon}), \quad \textit{for all } q = 0, 1, \ldots \end{aligned}$$

Proof. Since K_c is finite, so the left hand side of the above formula is independent of $\varepsilon > 0$ small. By (b) of Lemma 2.7

$$H_q(f^{c+\varepsilon} \cap (\widetilde{A}_{R,M}^{c+\varepsilon})) \cong H_q(f^{c-\varepsilon} \cap (\widetilde{A}_{R,M}^{c+\varepsilon})),$$

i.e.

$$H_q(f^{c+\varepsilon}\cap(\widetilde{A}^{c+\varepsilon}_{R,M}), f^{c-\varepsilon}\cap(\widetilde{A}^{c+\varepsilon}_{R,M}))\cong 0. \tag{2.7}$$

By the following Mayer–Vietoris sequence, (2.7) and Lemma 2.7(a)

$$\begin{aligned}\cdots H_q(f^{c+\varepsilon}\cap\widetilde{U}^{c+\varepsilon}_{2R,M/2}, f^{c-\varepsilon}\cap\widetilde{U}^{c+\varepsilon}_{2R,M/2})\oplus H_q(f^{c+\varepsilon}\cap\widetilde{C}^{c+\varepsilon}_{R,M}, f^{c-\varepsilon}\cap\widetilde{C}^{c+\varepsilon}_{R,M})\\ \to H_q(f^{c+\varepsilon}, f^{c-\varepsilon})\to H_{q-1}(f^{c+\varepsilon}\cap(\widetilde{A}^{c+\varepsilon}_{R,M}), f^{c-\varepsilon}\cap(\widetilde{A}^{c+\varepsilon}_{R,M}))\to\cdots\end{aligned}$$

we get the conclusion. □

Though in our setting, we do not have the (PS) condition at all levels, we shall still establish Morse inequalities. Assume K is finite and C_∞ is finite. Let

$$\beta_k(f,x)=\dim C_k(f,x)\quad\text{for all } x\in K$$

be the Betti numbers of f at $x\in K$, and

$$\beta_k(f,c)=\dim H_k(f^{c+\varepsilon}\cap\widetilde{C}_{R,M}, f^{c-\varepsilon}\cap\widetilde{C}_{R,M}),\quad\text{for } c\in C_\infty$$

be the Betti number of f at $c\in C_\infty$, where M,R are given in Theorem 2.9. Let

$$\begin{aligned}P(f,x)&=\sum_{k=0}^{\infty}\beta_k(f,x)t^k &&\text{for } x\in K,\\ P(f,c)&=\sum_{k=0}^{\infty}\beta_k(f,c)t^k &&\text{for } c\in C_\infty,\\ P(f,\infty)&=\sum_{k=0}^{\infty}\dim H_K(X,f^a)t^k,\end{aligned}$$

be the Morse polynomials for f at $x\in K, c\in C_\infty$, and ∞, where $a<0$ is such that

$$a<\min\{\inf_{x\in K}f(x),\inf_{f(x)\in C_\infty}f(x)\}.$$

Theorem 2.10. *There exists a polynomial $Q(t)$ with nonnegative integer coefficients such that*

$$P(f,\infty)+(1+t)Q(t)=\sum_{x\in K}P(f,x)+\sum_{c\in C_\infty}P(f,c).$$

Proof. With the aid of Theorem 2.9, we can follow the proof of the usual Morse inequalities (cf. [6], [13]). □

Remark 2.11. Theorem 2.10 was proved in [10]. If $C_\infty=\phi$, then we recover the usual Morse inequalities. When $c\in C_\infty$, or say, without the $(\mathrm{PS})_c$ condition, we may understand that there is a critical point at infinity with value the c. We can replace X by f^b where $a<b$ neither are critical values nor are in C_∞ such that $f^b_a\cap K$ is finite and $[a,b]\cap C_\infty$ is finite.

The usefulness of Theorem 2.10 depends upon the computation of $P(f,c)$ for $c \in C_\infty$. The following splitting theorem is very crucial for the computation of $P(f,c)$.

Theorem 2.12. *Let f satisfy (A_∞). Then for any $M > 0$ there exist $R_0 > 0$, $\delta > 0$, a C^1-diffeomorphism $\psi\colon C_{R_0,M} \to C_{R_0,2M}$ and C^1-map $\omega\colon \{v \in V \mid \|v\| > R_0\} \to W^\delta = \{\omega \in W \mid \|\omega\| \le \delta\}$ such that*

$$f(\psi(u)) = \frac{1}{2}\langle A\omega, \omega\rangle + h(v) \quad \textit{for all } u \in C_{R_0,M}$$

where $h(v) = f(v + \omega(v))$, δ can be chosen as small as we please, if we choose R_0 large, and $\omega = \omega(v)$ is the unique solution of

$$P_W f'(v + \omega) = 0$$

with $P_W\colon X \to W$ being the linear projection. Furthermore, for any $\theta \in V$, we have:

$$\langle h'(v), \theta\rangle = \langle g'(v + \omega(v)), \theta\rangle.$$

Remark 2.13. Theorem 2.12 is the generalization of the Morse lemma at infinity. It was given in [2], and here is a slightly different version.

Next, using examples of nonlinear elliptic BVPs with strong resonance, we give some results for computation of $P(f,c)$ and then deal with multiple solutions problems with multiple resonant energy levels. Consider

$$\begin{cases} -\Delta u = \lambda u + q(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.8}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open subset, and $\lambda \in \delta(-\Delta) = \{0 < \lambda_1 < \lambda_2 \le \dots\}$, the set of eigenvalues of the Laplacian $-\Delta$ on Ω with zero boundary conditions, counted with multiplicity. Define

$$f(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{\lambda}{2}\int_\Omega u^2\,dx - \int_\Omega Q(x,u)\,dx,$$

where $u \in X := H_0^1(\Omega)$, $Q(x,t) = \int_0^t q(x,s)\,ds$. Then critical points of f on X correspond to classical solutions of (2.8) when we assume $q \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$. Let

$$\langle Au, u\rangle := \int_\Omega |\nabla u|^2 - \lambda \int_\Omega u^2.$$

A is a self-adjoint linear operator. According to the spectral decomposition of A we have

$$X = V \oplus W^- \oplus W^+$$

where $V = \operatorname{Ker}(-\Delta - \lambda)$ and W^- (W^+, resp.) corresponding to the eigenvalues less than (greater than, resp.) λ. We impose the following assumptions on q.

(q_1) $q \in C^1(\Omega \times R, R)$ and uniformly in $x \in \Omega$

$$\begin{cases} q(x,t) \to 0 \\ \dfrac{\partial}{\partial t} q(x,t) \to 0 \end{cases} \quad \text{as } |t| \to \infty,$$

under this condition (2.8) is called a resonant problem.

(q_2) There exists $M > 0$ such that for all $(x,t) \in \Omega \times R$, $Q(x,t) - q(x,t)t/2 \leq M$ and $Q(x,t) \leq M$.

(q_3) $q(x,0) = 0$ and $a_0 = \lim_{t\to 0} q(x,t)/t$ exists uniformly in $x \in \Omega$.

(q_4) $Q_{\pm\infty} = \lim_{t\to\pm\infty} Q(x,t)$ exists uniformly in $x \in \Omega$ with $Q_\pm \in (-\infty, \infty)$.

We call (2.8) a strong resonant problem if the following set Λ is nonempty and bounded.

$$\Lambda := \left\{ c \in R \;\middle|\; -\int_\Omega Q(x,tv)\,dx \to c \text{ as } t \to \infty \text{ for some } v \in \operatorname{Ker}(-\Delta - \lambda) \right\}.$$

Strong resonant problem is more delicate to deal with because the energy functional fails the $(\mathrm{D})_c$ condition.

Next theorem is about the computation of $P(f, \infty)$ in the strong resonant case. In [2] a notion of critical groups at infinity was introduced. If f has no critical point in f^{b_0} for some b_0 and satisfies the deformation property for $c \leq b_0$, then $H_q(X, f^c)$ is independent of $c \leq b_0$ and is defined as the critical groups of f at infinity, denoted by $C_q(f, \infty)$.

Theorem 2.14. *Let* (q_2) *hold, then* $C_q(f,\infty) \cong \delta_{q\mu} G$ *for all* $q = 0, 1, \ldots,$ *where* $\mu = \dim W^-$.

Proof. (Main idea, see [10] for details.)

(a) $f(tu) \leq -b$ for b very large. By (q_2) we have

$$\frac{d}{dt} f(tu) < 0,$$

where $u \in S = \{u \in X \mid \|u\| = 1\}$.

(b) By the implicit function theorem, there exists a unique $T(u) \in C(Y, R)$ such that $f(T(u)u) = -b$, where $Y = \{u \in X \mid \|u^+\| < \|u^-\|\}$, $\|u^+\|$ and $\|u^-\|$ are equivalent norms in W^+ and W^- respectively, $u = u^0 + u^- + u^+ \in V \bigoplus W^- \bigoplus W^+$.

(c) $T(u)$ has a positive lower bound $\varepsilon_0 > 0$. We can define a deformation retract η: $[0,1] \times (Y \setminus B_{\varepsilon_0}(0)) \to Y \setminus B_{\varepsilon_0}(0)$ with $B_{\varepsilon_0}(0)$ being the ε_0-ball centered at 0, by

$$\eta(s,u) = (1-s)u + sT(u)u \quad \text{for all } (s,u) \in [0,1] \times (Y \setminus B_{\varepsilon_0}(0)).$$

This implies that $Y \setminus B_{\varepsilon_0}(0) \cong f^{-b}$ and $Y \setminus B_{\varepsilon_0}(0) \cong S^{\mu-1}$. $\square$

Next, we consider the computation of $P(f,c)$ for $c \in C_\infty$. For $v \in V$, we define

$$\Omega_\pm(v) = \{x \in \Omega \mid \pm v(x) > 0\}.$$

First we characterize C_∞. In the following $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^N$.

Lemma 2.15. *Let* (q_1) *and* (q_4) *hold. Then*

(a) $C_\infty = \{-(Q_\infty|\Omega_+(v)| + Q_{-\infty}|\Omega_-(v)|) \mid v \in V,\ \|v\| = 1\}$,
(b) $C_\infty = \{-(Q_\infty|\Omega_+(v)|+Q_{-\infty}|\Omega_-(v)|), -(Q_{+\infty}|\Omega_-(v)|+Q_{-\infty}|\Omega_+(v)|)\}$, *if* $\dim V = 1$, $\|v\| = 1$,
(c) $C_\infty = \{-Q_\infty|\Omega|, -Q_{-\infty}|\Omega|\}$, *if* $\lambda = \lambda_1$,
(d) $C_\infty = \{-Q_\infty|\Omega|\}$ *if* $Q_\infty = Q_\infty = Q_{-\infty}$.

Concerning the solution $\omega(v)$ given in Theorem 2.12 we have the following estimation.

Lemma 2.16. *Let* (q_1) *hold. Then we may apply Theorem* 2.12 *to* f. *Moreover, we have that* $\omega = \omega(v) \in C(V, C_0^1(\Omega))$ *satisfies*

$$\|\omega(v)\|_{C_0^1(\Omega)} \to 0 \quad \text{as } \|v\| \to \infty$$

and that, for any $\theta \in V$,

$$\langle h'(v), \theta\rangle = -\int_\Omega q(x, v + \omega(v))\theta\, dx.$$

From (q_1) to get (A_∞) we need the following

Lemma 2.17 ([1]). *Let* V *be a finite dimensional subspace of* $C(\overline{\Omega})$ *such that every* $u \in V \setminus \{0\}$ *is different from zero a.e. in* Ω. *Let* $h \in L^\infty(\mathbb{R})$ *such that*

$$h(t) \to 0 \quad \text{as } |t| \to \infty.$$

Moreover, consider a compact subset K *of* $L^p(\Omega) (p \geq 1)$. *Then*

$$\lim_{|t|\to\infty} \int_\Omega |h(tu(x) + v(x))|\, dx = 0$$

uniformly as $v \in K$ *and* $u \in S$ *where* $S = \{u \in V \mid \|u\|_C = 1\}$ *and* $\|u\|_C = \sup_{x\in\Omega} |u(x)|$.

The proof of Lemma 2.16 needs the L^p-theory and bootstrap argument, see [10] for details.

We will introduce a technical condition here. It is easy to be checked in applications.

$(\mathrm{q}_5)_\pm$ For any $\omega\colon \{v \in V \mid \|v\| > R\} \to W \cap C_0^1(\Omega)$ with $\|\omega(v)\|_{C_0^1(\Omega)} \to 0$ as $\|v\| \to \infty$, it holds

$$\pm \int_\Omega q(x, v + \omega(v))v\,dx > 0 \quad \text{for } \|v\| \text{ large.}$$

The following theorems give the computation of $P(f, c)$ for $c \in C_\infty$.

THEOREM 2.18. *Assume* (q_1), (q_4) *and* $(\mathrm{q}_5)_+$ *hold. Assume* $\dim V = 1$.

(a) *If* C_∞ *contains two different values* $c_+ \neq c_-$ *with* $c_+ = -(Q_{+\infty}|\Omega_+(v)| + Q_{-\infty}|\Omega_-(v)|)$, $c_- = -(Q_{+\infty}|\Omega_-(v)| + Q_{-\infty}|\Omega_+(v)|)$, *then for* $M > 0$, $R_1 > 0$ *large, there exists* $\varepsilon_1 > 0$ *for all* $0 < \varepsilon < \varepsilon_1$

$$H_q(f^{c_\pm+\varepsilon} \cap \widetilde{C}_{R_1,M}^{c_\pm+\varepsilon}, f^{c_\pm-\varepsilon} \cap \widetilde{C}_{R_1,M}^{c_\pm+\varepsilon}) \cong \delta_{q\mu} G.$$

(b) *If* C_∞ *contains only one value* c, *then for* $M > 0$, $R_1 > 0$ *large enough there exists* $\varepsilon_1 > 0$, *for all* $0 < \varepsilon < \varepsilon_1$

$$H_q(f^{c+\varepsilon} \cap \widetilde{C}_{R_1,M}^{c+\varepsilon}, f^{c-\varepsilon} \cap \widetilde{C}_{R_1,M}^{c+\varepsilon}) \cong \delta_{q\mu} G \oplus G,$$

where $\widetilde{C}_{R_1,M}$ *was given before.*

THEOREM 2.19. *Assume* (q_1), (q_4) *and* $(\mathrm{q}_5)_+$ *hold with* $C_\infty = \{c\}$ *containing only one value. Then for* $M > 0$, $R_1 > 0$ *large, there exists* $\varepsilon_1 > 0$ *for all* $0 < \varepsilon < \varepsilon_1$,

$$H_q(f^{c+\varepsilon} \cap \widetilde{C}_{R_1,M}^{c+\varepsilon}, f^{c-\varepsilon} \cap \widetilde{C}_{R_1,M}^{c+\varepsilon}) \cong \begin{cases} G & \text{for } q = \mu, \\ G & \text{for } q = \mu + \nu - 1, \\ 0 & \text{otherwise,} \end{cases}$$

where c *is the only value in* C_∞, *and it is understood that when* $\nu = 1$, *at the level* μ, *there are two* G.

THEOREM 2.20. *Assume* (q_1), (q_4) *and* $(\mathrm{q}_5)_-$ *hold. Assume* $\dim V = 1$. *Then* C_∞ *contains either two values* $c_+ \neq c_-$ *or one value. In any case, for* $M > 0$, $R_1 > 0$ *large there exists* $\varepsilon_1 > 0$, *for all* $0 < \varepsilon < \varepsilon_1$

$$H_q(f^{c+\varepsilon} \cap \widetilde{C}_{R_1,M}^{c+\varepsilon}, f^{c-\varepsilon} \cap \widetilde{C}_{R_1,M}^{c+\varepsilon}) \cong 0 \quad \text{for all } q,$$

where $c = c_+$, *or* $c = c_-$, *or* $c = c_+ = c_-$.

Theorem 2.21. *Assume* (q_1), (q_4) *and* (q_5)$_-$ *hold. Assume* $C_\infty = \{c\}$ *contains only one value. Then for* $M > 0$, $R_1 > 0$ *large, there exists* $\varepsilon_1 > 0$ *for all* $0 < \varepsilon < \varepsilon_1$

$$H_q(f^{c+\varepsilon} \cap \widetilde{C}^{c+\varepsilon}_{R_1,M}, f^{c-\varepsilon} \cap \widetilde{C}^{c+\varepsilon}_{R_1,M}) \cong 0 \quad \textit{for all } q.$$

The proofs of these four theorems are similar, and we prove Theorem 2.19 only, see [10] for more details.

Proof of Theorem 2.19. From Lemma 2.7 for $R > M > 0$ large there is $\varepsilon_1 > 0$ such that for $\varepsilon_1 > \varepsilon > 0$

$$f^{c+\varepsilon}_{c-\varepsilon} \cap \widetilde{C}^{c+\varepsilon}_{2R,M} \subset f^{c+\varepsilon}_{c-\varepsilon} \cap C_{R,2M}.$$

By (q_1) and Theorem 2.12 with $R > M > 0$ large, $f(u)$ can be written as

$$f(\psi(u)) = \frac{1}{2}\langle A\omega, \omega\rangle + h(v) \quad \text{for all } u \in C_{R,2M}.$$

By (q_5)$_+$ we have for $\|v\|$ large

$$\langle h'(v), v\rangle = -\int_\Omega q(x, v + \omega(v))v\,dx < 0.$$

Thus $h(tv)$ decreases to c as $t \to \infty$ for any $v \in V \setminus \{0\}$. Define $\widetilde{h}(v) = h(v) - c$, then $\widetilde{h}(v)$ decreases to 0 as $\|v\| \to \infty$. Then

$$\begin{aligned} A_1 &\triangleq f^{c+\varepsilon} \cap \widetilde{C}^{c+\varepsilon}_{2R,M} \\ &= \{u = v + \omega^+ + \omega^- \in \widetilde{C}^{c+\varepsilon}_{2R,M} \mid \widetilde{h}(v) > 0,\ (\|\omega^+\|^2 - \|\omega^-\|^2)/2 \le \varepsilon - \widetilde{h}(v)\} \end{aligned}$$

and

$$\begin{aligned} B_1 \triangleq f^{c-\varepsilon} \cap \widetilde{C}^{c+\varepsilon}_{2R,M} = \{u = v + \omega^+ + \omega^- \in \widetilde{C}^{c+\varepsilon}_{2R,M} \mid \widetilde{h}(v) > 0,& \\ (\|\omega^+\|^2 - \|\omega^-\|^2)/2 \le -\varepsilon - \widetilde{h}(v)\}&. \end{aligned}$$

We first define a deformation retract from (A_1, B_1) to (A_2, B_2), where

$$A_2 = \{u \in A_1 \mid \|\omega^-\|^2/2 \le \varepsilon + \widetilde{h}(v)\}, \quad B_2 = \{u \in B_1 \mid \|\omega^-\|^2/2 \le \varepsilon + \widetilde{h}(v)\},$$

for $\|v\|$ large $\varepsilon - \widetilde{h}(v) > 0$. It is easy to see that $B_2 = \{u = v + \omega^+ + \omega^- \mid \omega^+ = 0,\ \|\omega^-\|^2 = 2(\varepsilon + \widetilde{h}(v))\}$ and we can get $\eta_1\colon [0,1] \times (A_1, B_1) \to (A_1, B_1)$ deforming (A_1, B_1) to (A_2, B_2).

Next, one has a simple deformation transforming (A_2, B_2) to (A_3, B_3) with $B_3 = B_2$ and

$$A_3 = \{u = v + \omega^- + \omega^+ \in A_2 \mid \omega^+ = 0\}.$$

In fact, $\eta_2(t, u) = v + \omega^- + t\omega^+$ suffices.

Note now that

$$\begin{aligned} A_3 &= \{v + \omega^- + \omega^+ \mid \omega^+ = 0,\ \widetilde{h}(v) - \varepsilon \le \|\omega^-\|^2/2 \le \varepsilon + \widetilde{h}(v)\}, \\ B_3 &= \{v + \omega^- + \omega^+ \mid \omega^+ = 0,\ \|\omega^-\|^2/2 = \varepsilon + \widetilde{h}(v)\}. \end{aligned}$$

Since $\widetilde{h}(v)$ decreases monotonically to zero, we can find $R_0 > 0$ large, such that (A_3, B_3) is deformed to (A_4, B_4) with

$$\begin{aligned} A_4 &= \{v + \omega^- + \omega^+ \mid \|v\| = R_0,\ \omega^+ = 0,\ \|\omega^-\|^2/2 \le \varepsilon + \widetilde{h}(v)\}, \\ B_4 &= \{v + \omega^- + \omega^+ \mid \|v\| = R_0,\ \omega^+ = 0,\ \|\omega^-\|^2/2 = \varepsilon + \widetilde{h}(v)\}. \end{aligned}$$

Then it is easy to see (A_4, B_4) is topologically equivalent to

$$(S^{\nu-1} \times B^{\mu}, S^{\nu-1} \times S^{\mu-1}),$$

where $S^{\nu-1}$ is a $\nu-1$-dimensional sphere and B^{μ} is a μ-dimensional ball. Therefore

$$\begin{aligned} H_q(f^{c+\varepsilon} \cap \widetilde{C}^{c+\varepsilon}_{2R,M}, f^{c-\varepsilon} \cap \widetilde{C}^{c+\varepsilon}_{2R,M}) &\cong H_q(S^{\nu-1} \times B^{\mu}, S^{\nu-1} \times S^{\mu-1}) \\ &\cong \begin{cases} G & \text{for } q = \mu, \\ G & \text{for } q = \mu + \nu - 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

□

3. Applications to strong resonant problems of elliptic BVPs

Note that $(\mathrm{q}_5)_\pm$ is an abstract condition, but it is easy to be checked. Under this condition many existence and multiplicity results for (2.8) were given in [10]. Let μ_0 denote the Morse index of f at 0. Assume $\dim \operatorname{Ker} f''(0) = 0$, i.e. 0 is a nondegenerate critical point of f.

Theorem 3.1. *Let* $\lambda = \lambda_1$. *Assume* $C_\infty = \{c_+, c_-\}$ *or* $C_\infty = \{c_0\}$.

(a) *Under the assumptions* (q_1), (q_4), $(\mathrm{q}_5)_+$ *and* (q_3) *with* $\mu_0 \ne 1$, (2.8) *has at least two nontrivial solutions.*

(b) *Under the assumptions* (q_1), (q_4), $(\mathrm{q}_5)_-$ *and* (q_3), *in which* $a_0 < 0$, $\max\{c_-, c_+\} \le 0$, (2,8) *has at least three nontrivial solutions, including one positive and one negative. Moreover, if the third solution* u_3 *with* $\dim \operatorname{Ker} f''(u_3) = 0$, *then* (2.8) *has at least four nontrivial solutions.*

(c) *Under the assumptions* (q_1), (q_4), $(\mathrm{q}_5)_-$ *and* (q_3) *with* $\mu_0 \ge 2$, (2.8) *has at least three nontrivial solutions, including one positive and one negative. Moreover, if the third solution* u_3 *with* $\dim \operatorname{Ker} f''(u_3) \le \mu_0 - 1$ *then* (2.8) *has at least four nontrivial solutions.*

Theorem 3.2. *Let* $\lambda = \lambda_2$. *Assume* $C_\infty = \{c_0\}$ *if* $\dim \operatorname{Ker}(-\Delta - \lambda_2) \geq 2$ *or* $C_\infty = \{c_+, c_-\}$ *if* $\dim \operatorname{Ker}(-\Delta - \lambda_2) = 1$. *Assume* (q_1), (q_2), (q_4), $(q_5)_-$ *and* (q_3) *with* $\mu_0 \neq 1$, *then* (2.8) *has at least two nontrivial solutions* u_1, u_2. *Moreover, if the Morse index of* u_2 *is greater than* $\mu_0 + 1$ *then* (2.8) *has at least three nontrivial solutions.*

Theorem 3.3. *Let* $\lambda = \lambda_k$ *with* $k \geq 3$. *Assume* $C_\infty = \{c_0\}$ *if* $\dim \operatorname{Ker}(-\Delta - \lambda_k) \geq 2$ *or* $C_\infty = \{c_+, c_-\}$ *if* $\dim \operatorname{Ker}(-\Delta - \lambda_k) = 1$. *Assume* (q_1), (q_2), (q_4), $(q_5)_-$ *and* (q_3) *with* $\mu_0 \neq \mu$, *then* (2.8) *has at least one nontrivial solution* u_1. *Moreover, if* $\dim \operatorname{Ker} f''(u_1) \leq |\mu - \mu_0|$, *then* (2.8) *has at least two nontrivial solutions.*

Theorem 3.4. *Let* $\lambda = \lambda_k$ *with* $k \geq 3$. *Assume* $C_\infty = \{c_0\}$ *if* $\dim \operatorname{Ker}(-\Delta - \lambda_k) \geq 2$ *or* $C_\infty = \{c_+, c_-\}$ *if* $\dim \operatorname{Ker}(-\Delta - \lambda_k) = 1$. *Assume* (q_1), (q_2), (q_4), $(q_5)_+$ *and* (q_3) *with* $\mu_0 \neq \mu$, *then* (2.8) *has at least one nontrivial solution* u_1. *Moreover, if* $\dim \operatorname{Ker} f''(u_1) \leq |\mu + \nu - \mu_0|$, *then* (2.8) *has at least two nontrivial solutions.*

Proof of Theorem 3.1. Without loss of generality we assume $c_+ \geq c_-$.

(a) From Theorem 2.12 we have

$$f(\psi(v, \omega)) = \frac{1}{2}\langle A\omega, \omega\rangle + h(v) \quad \text{for } \|v\| \text{ large}$$

and, by $(q_5)_+$, $h(tv)$ is monotonic decreasing to c_+ as $t \to \infty$ and to c_- as $t \to -\infty$, where $v \geq 0$ is the eigenfunction corresponding to λ_1. Note that $\lambda = \lambda_1$ implies $W = W^+$, $\mu = 0$. Take $t' > t > 0$ large and consider

$$c = \inf_{r \in \Gamma} \sup_{s \in [0,1]} f(r(s)),$$

where

$$\Gamma = \{r \in C([0,1], X) \mid r(0) = -t'v, r(1) = t'v\}.$$

From

$$r([0,1]) \cap (tv + W) \neq \phi \quad \text{for all } r \in \Gamma,$$

we have

$$c \geq \inf_{u \in tv + W} f(\psi(u)) \geq h(tv) > \max\{b | b \in C_\infty\}.$$

We also have $h(tv) > \max\{h(-t'v), h(t'v)\}$. Then a standard argument shows that c is a critical value of f because f satisfies $(PS)_c$ condition for $c \notin C_\infty$. Thus, there exists $u_1 \in X$ such that $f(u_1) = c$, $f'(u_1) = 0$. It is well-known that $C_q(f, u_1) = \delta_{q1} G$, for all q (see [6]). Since $\mu_0 \neq 1$ we get $u_1 \neq 0$. If 0 and u_1 are the only critical points, then will get a contradiction. In fact, computing directly, we have

$$\sum_{u \in K} P(f, u) = t^{\mu_0} + t, \qquad \sum_{c \in C_\infty = \{c_+, c_-\}} P(f, c) = 2$$

(by Theorem 2.18), and $P(f,\infty)=1$.

From Theorem 2.10 we have

$$1+(1+t)Q(t)=t^{\mu_0}+t+2.$$

Setting $t=-1$ we get $1=(-1)^{\mu_0}+1$, a contradiction. Thus, f has at least another nontrivial critical point.

(b) Consider the negative gradient flow $\eta(t,u)$ of f on $H_0^1(\Omega)$ which satisfies

$$\begin{cases} \dfrac{d\eta(t,u)}{dt} = -\nabla f(\eta(t,u)), \\ \eta(0,u)=u. \end{cases} \tag{3.1}$$

It is well-known that $\eta(t,u)\in C_0^1(\Omega)$ if $u\in C_0^1(\Omega)$ and $\eta(t,u)$ satisfies the deformation property for f. Let P be the positive cone in $C_0^1(\Omega)$. Then from the maximal principle we know that $P,-P$ are positively invariant under the negative flow $\eta(t,u)$. Since 0 is a minimizer of f and $\max\{c_+,c_-\}\le 0$ then we can use the mountain pass theorem in cone. We have two mountain pass critical points $u_\pm$ and

$$C_q(f,u_\pm)\cong\delta_{q1}G \quad \text{for all } q.$$

Now, if f has only three critical points: 0, u_+, u_-, we shall get a contradiction. In this case

$$\sum_{u\in K}P(f,u)=1+2t,\quad \sum_{c\in C_\infty}P(f,c)=0,\quad P(f,\infty)=1.$$

Thus $1+(1+t)Q(t)=1+2t+0$, a contradiction. Thus (2.8) has a third solution $u_3\ne 0$. If u_3 is nondegenerate, then by Theorem 2.10 we can get a fourth nontrivial solution.

(c) Since P and $-P$ are positively invariant under the flow $\eta(t,u)$, from $a_0+\lambda_1>\lambda_2$ we have

$$\inf_{u\in P}f(u)<0,\quad \inf_{u\in -P}f(u)<0.$$

By $(\mathrm{q}_5)_-$, neither c_+ nor c_- can be the infimum of f. Thus $\inf_P f$ is achieved at $u_+\in\overset{\circ}{P}$ by the maximum principle. Similarly, one gets a negative solution $u_-\in -\overset{\circ}{P}$. Both u_+ and u_- are local minimum points of f. If f has only u_+, u_-, 0 as its critical points, we have

$$\sum_{u\in K}P(f,u)=2+t^{\mu_0},\quad \sum_{c\in C_\infty}P(f,u)=0,\quad P(f,\infty)=1.$$

Since $\mu_0\ge 2$, this gives $0-2=M_1-M_0\ge\beta_1-\beta_0=0-1$, a contradiction. So (2.8) has a third solution u_3. If $\dim\operatorname{Ker} f''(u_3)\le\mu_0-1$, then (2.8) has a fourth nontrivial solution. Otherwise we have

$$1+(1+t)Q(t)=2+t^{\mu_0}+P(f,u_3)$$

It implies that $Q(t)$ includes at least $1+t^{\mu_0-1}$ (or $1+t^{\mu_0}$) term. Therefore $P(f,u_3)$ includes at least $t+t^{\mu_0-1}$ (or $t+t^{\mu_0+1}$) term. But from the shifting theorem, see Corollary 5.1 of [6], we have $\dim \operatorname{Ker} f''(u_3) > \mu_0 - 1$, a contradiction. □

Using Theorems 2.10, 2.14, 2.18, 2.19, 2.20 and 2.21, we can prove Theorem 3.2, 3.3 and 3.4.

Next, we give some concrete conditions which imply that $(\mathrm{q}_5)_\pm$ hold.

Consider first (2.8) with $\lambda = \lambda_1$, the first eigenvalue of the Laplacian operator on Ω. Let us make the following assumption

$(\mathrm{q}_6)_\pm$ $\pm q(x,t)\cdot t > 0$ for $t \in R \setminus \{0\}$ and $x \in \Omega$.

Lemma 3.5 ([11]). *Let $\lambda = \lambda_1$ and let q satisfy $(\mathrm{q}_6)_+$ ($(\mathrm{q}_6)_-$, resp.), then $(\mathrm{q}_5)_+$ ($(\mathrm{q}_5)_-$, resp.) is satisfied.*

When λ equals the higher eigenvalues, the following conditions were introduced. Let $\Omega_0(v) = \{x \in \overline{\Omega} \mid v(x) = 0\}$ be the nodal set of v.

(V_1) For any $v \in V$, $\Omega_0(v)$ is a union of $(N-1)$-dimensional manifolds.

(V_2) For any $v \in V$ there exists $c > 0$ such that for t large

$$\mu(\{x \in \Omega \mid |v(x)| \le 1/t\}) \le c/t,$$

here $\mu(\{\,\cdot\,\})$ is the Lebesgue measure.

(V_3) For any $v \in V$, $\Omega_0(v)$ is a union of disjoint closed manifolds.

Note that (V_3) implies (V_1) and (V_2). If the nodal set of v is a manifold, then (V_2) is satisfied automatically. Let us make further assumptions on q. The first one is the following

$(\mathrm{A}_1)_\pm$ $\pm q(x,t) \ge 0$, for all $(x,t) \in \Omega \times \mathbb{R}$ and

$$\pm q_0 = \pm \frac{\partial q}{\partial t}(x,0) > 0, \quad \text{for all } x \in \overline{\Omega}.$$

Lemma 3.6 ([11]). *Suppose that (V_1), (V_2), $(\mathrm{A}_1)_\pm$ hold. Then $(\mathrm{q}_5)_\pm$ is satisfied.*

Now, we consider the following conditions

$(\mathrm{A}_2)_\pm$ For some $2 > \alpha > 0$, there exists $c_0 > 0$ such that

$$\pm \lim_{|t|\to\infty} \frac{q(x,t)}{|t|^{-(\alpha+1)}t} \ge c_0.$$

$(\mathrm{A}_3)_\pm$ For some $\alpha > 2$, there exist $c_\alpha > 0$ and $a > 0$ such that

$$\overline{\lim_{|t|\to\infty}} \frac{|q(x,t)|}{|t|^{-\alpha}} \le c_\alpha, \quad \pm q_0 = \frac{\partial q}{\partial t}(x,0) > 0, \quad \pm q(x,t)t \ge 0, \quad |t| \le a.$$

This condition means that q decays faster than $1/t^2$ at infinity.

Lemma 3.7 ([11]). *Suppose that* (V$_1$), (V$_2$), (A$_2$)$_\pm$ *hold. Then* (q$_5$)$_\pm$ *is satisfied.*

Lemma 3.8 ([11]). *Let* (V$_3$) *hold. Assume* (A$_3$)$_\pm$, *then if* $C_\infty/a^{\alpha-2}$ *is small enough,* (q$_5$)$_\pm$ *is satisfied.*

From (A$_1$)$_\pm$, (A$_2$)$_\pm$ and (A$_3$)$_\pm$ we see that it seems the decay rate $\alpha = 2$ is critical in the following sense that in (A$_2$)$_\pm$ the decay is slower than $\alpha = 2$ and in this case the behavior of q at infinity dominates, on the other hand, in (A$_3$)$_\pm$ the decay is faster than $\alpha = 2$ at infinity for q and in this case the oscillation of q at infinity does not effect the problem.

Using the new Morse inequalities given in Theorem 2.10 one can prove the following theorems which can be found in [11].

Theorem 3.9. *Assume* (q$_1$), (q$_4$), *and* (A$_2$)$_+$ *or* (A$_2$)$_-$ *hold. Then* (2.8) *has a solution.*

Theorem 3.10. *Assume* (q$_1$) *and* (q$_4$) *hold. Let* (A$_2$)$_+$ *hold for* $t > 0$ ($t < 0$, *resp.*) *and suppose one of* (A$_1$)$_+$ *and* (A$_3$)$_+$ *holds for* $t < 0$ ($t > 0$, *resp.*). *Then* (2.8) *has a solution.*

Let μ_0 denote the Morse index of f at 0. We have

Theorem 3.11. *Assume* (q$_1$), (q$_2$) *and* (q$_4$), *and assume* (q$_3$) *with* $\mu_0 \neq \mu$.

(i) *Assume one of the three conditions* (A$_1$)$_+$–(A$_3$)$_+$ *holds for* $t > 0$ *and one of the three holds for* $t < 0$. *Then* (2.8) *has a nontrivial solution* u_1. *Moreover, if* $\dim \operatorname{Ker} f''(u_1) \leq |\mu - \mu_0|$, *then* (2.8) *has at least two nontrivial solutions.*

(ii) *If* $+$ *is replaced by* $-$ *in* (i), *then* (2.8) *has at least two nontrivial solutions.*

Theorem 3.12. *Let* $\lambda = \lambda_1$. *Assume* (q$_1$), (q$_2$), (q$_4$) *and* (q$_3$) *with* $\mu_0 \neq 1$ *and assume* (q$_5$)$_+$, *then* (2.8) *has at least two nontrivial solutions. Furthermore, in case* $\mu_0 = 0$, *for the two solutions, one is positive and one is negative; in case* $\mu_0 \geq 2$, *one of the two nontrivial solution is sign-changing. If in addition we assume* $\mu_0 \geq 2$ *and* $\inf_{\pm P} f < \min_{C_\infty}\{c\}$, *where* P *is the positive cone in* $C_0^1(\Omega)$, *then* (2.8) *has at least two positive solutions* u_1^+, u_2^+, *and two negative solutions* u_1^-, u_2^- *and two sign-changing solutions* u_3, u_4, *where* u_1^+, u_1^- *are local minimizers,* u_2^+, u_2^- *and* u_3 *are mountain pass solution.*

Theorem 3.13. *Let* $\lambda = \lambda_1$. *Assume* (q$_1$), (q$_2$), (q$_4$), *and* (q$_3$) *with* $\mu_0 \neq 1$. *Suppose* (q$_5$)$_-$ *holds. Suppose also* $\max\{b \mid b \in C_\infty\} \leq 0$. *If* $\mu_0 = 0$ *then* (2.8) *has two pairs of positive and negative solutions, one are local minimizers and the other are mountain pass solutions. If* $\mu_0 \geq 2$, (2.8) *has at least four nontrivial* *solutions, including a pair of positive and negative solutions* u_1^+, u_1^-, *which are local minimizers, and a pair of sign-changing solutions* u_3, u_4, *one of which is a mountain pass solution, such that* $u_1^- < u_3 < u_1^+$, $\quad u_1^- < u_4 < u_1^+$.

Theorem 3.14. *Let* $\lambda = \lambda_2$. *Assume* (q_1), (q_2), (q_4) *and* (q_3) *with* $\mu_0 \neq 1$. *Suppose* $(\mathrm{q}_5)_-$ *holds. If* $\mu_0 = 0$, *then* (2.8) *has solutions: a positive and a negative one. If* $\mu_0 \geq 2$, *then* (2.8) *has at least two nontrivial solutions.*

In the literature, strong resonant problems have been considered for the case of $\Lambda = \{c_0\}$, a singleton (see [1], [7], [8]). The existence results have been given in these papers. Linking methods were used in [1]. A compactification methods was used in [7] and [8] to reduce the problem to a nonresonant problem. However, the methods in these papers seem to be not applicable to the case when Λ contains more (than one) finite values. Furthermore, so far few multiplicity results have been obtained, if any.

References

[1] P. Bartolo, V. Benci and D. Fortunato, *Abstract critical point theorems and applications to some nonlinear problems with "strong" resonance at infinity*, Nonlinear Anal. **7** (1983), 981–1012.

[2] T. Bartsch and S. J. Li, *Critical point theory for asymptotically quadratic functionals and applications to problems with resonance*, Nonlinear Anal. **28** (1997), 419–441.

[3] R. Bott, *Nondegenerate critical manifolds*, Ann. of Math. **60** (1954), 248–261.

[4] ______, *Lectures on Morse theory, old and new*, Bull. Amer. Math. Soc. **7** (1982), 331–358.

[5] K. C. Chang, *Morse theory on Banach spaces and its applications*, Chinese Ann. Math. Ser. **64** (1983), 381–399.

[6] *Infinite Dimensional Morse Theory and Multiple Solution Problems*, Birkhöuser, 1993.

[7] K. C. Chang and J. Q. Liu, *A strong resonance problem*, Chinese Ann. Math. Ser. B **11** (1990), 191–210.

[8] K. C. Chang, J. Q. Liu and M. J. Liu, *Nontrivial periodic solution for strong resonance Hamiltonian systems*, Ann Inst. H. Poincaré Anal. Non Linéaire **14** (1997), 103–117.

[9] D. Gromoll and W. Meyer, *On differentiable functions with isolated critical points*, Topology **8** (1969), 361–369.

[10] N. Hirano, S. J. Li and Z. Q. Wang, *Morse theory without* (PS) *condition at isolated values and strong resonance problems*, Calc. Var. Partial Differential Equations (to appear).

[11] S. J. Li and Z. Q. Wang, *Dirichlet problem of elliptic equations with strong resonances.*

[12] M. Morse, *The calculus of variations in the large*, Amer. Math. Soc. Colloq. Publ. **18** (1934).

[13] J. Mawhin and M. Willem, *Critical point theory and Hamiltonian systems*, Appl. Math. Sci. **74** (1989).

[14] R. S. Palais and S. Smale, *A generalized Morse theory*, Bull. Amer. Math. Soc. **70** (1964), 165–171.

[15] E. Rothe, *Critical point theory in Hilbert space*, Rocky Mountain J. Math. **3** (1973), 251–274.

[16] A. Tromba, *A general approach to Morse theory*, J. Differential Geom. **12** (1997), 47–85.

[17] K. Uhlenbeck, *Morse theory on Banach manifolds*, J. Funct. Anal. **10** (1972), 430–445.

Manuscript received September 13, 2001

Shujie Li
Academy of Mathematics and Systems Sciences
Institute of Mathematics
Chinese Academy of Sciences
Beijing 100080, P. R. CHINA

E-mail address: lisj@math.ac.cn