Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 13, 1999, 251–271

A SEMILINEAR ELLIPTIC EQUATION WITH CONVEX AND CONCAVE NONLINEARITIES

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Abstract. In this paper we establish the existence of multiple solutions for a semilinear elliptic equation with competing convex and concave nonlinearities. With either a subcritical or critical exponent in the nonlinearity, the existence of solutions is determined with critical point theorems based on the symmetric mountain pass theorem.

1. Introduction

We consider the problem

$$(1)\qquad \begin{cases} -\Delta u - \lambda g(x)u = k(x)|u|^{q-2}u - h(x)|u|^{p-2}u & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N,\ u \in D^{1,2}(\mathbb{R}^N), \end{cases}$$

where $N \geq 3$ and $1 < q < 2 < p \leq 2^* = 2N/(N-2)$ and with integrability and sign conditions on $g(x)$, $k(x)$ and $h(x)$. Throughout this paper, we assume the following hold:

- (G1) $g(x) \in L^{N/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is indefinite in sign,
- (H1) for $p < 2^*$, $h(x) \in L^{p_0}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, where $p_0 = 2N/(2N - pN + 2p)$ while for $p = 2^*$, $h(x) \in L^\infty(\mathbb{R}^N)$,
- (K1) $k(x) \in L^{q_0}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, where $q_0 = 2N/(2N - qN + 2q)$.

1991 *Mathematics Subject Classification.* 58E05, 35J20, 58G10.
Key words and phrases. Fountain theorem, dual fountain theorem, symmetric mountain pass.

The requirements on $g(x)$ are used in [16] to analyse the associated linear eigenvalue problem. The integrability requirements provide a compactness condition.

Problems such as (1) have captured interest since the seminal work of Brézis and Nirenberg [14]. In particular, the difficulties associated with overcoming a critical Sobolev exponent are discussed there.

The fountain theorem [7] demands that, with a sequential decomposition of the underlying space, the geometry of a functional is appropriate to recycle the symmetric mountain pass theorem at an unbounded sequence of energies. With a Palais–Smale condition, this confirms an infinitude of solutions.

The dual fountain theorem [11] uses a stronger compactness condition in the form of the dual Palais–Smale condition [6], [22] or a localised version, [21]. This stronger condition permits a Galerkin technique to be applied to the decomposition of the underlying space. A sequence of critical energies is verified, monotonically rising to zero.

In this paper, we extend and apply both of these theorems. In essence, the underlying space is decomposed into a subspace upon which the minimax theorem is applicable, and a complementary subspace where the functional remains positive definite and hence does not impede the existence of critical points. The concept of extracting a subspace and performing a successive decomposition and mountain pass techniques on the conjugate space is also undertaken in [8]. When $k(x) \geq 0$, the extracted positive definite subspace is associated with $k(x)$ assuming the value zero on some set in $\mathbb{R}^N$. For the case that $h(x) \leq 0$ and $k(x)$ is indefinite, a subspace associated with the set where $h(x) = 0$ is identified.

The fountain theorems rely on symmetry of the functional in the sense of admissibility posed by Bartsch [7], of which even functionals are an example. In fact, when symmetry is destroyed the method collapses and a limited number of solutions may be discerned from the remnants: [23], [20], [25], [9]. It appears possible that other methods may overcome restrictions on symmetry, given the evidence in [5].

Ruppen [27] has advocated that terms of the form $k(x)|u|^q$ encourage the existence of solutions while $-h(x)|u|^p$ tend to prohibit solutions.

For $g(x) \equiv 0$, and $k(x)$ and $h(x)$ replaced with real parameters, Ambrosetti *et al* [3] contemplated problems like (1) on a bounded domain. Results there were extended by Bartsch and Willem [12] using a fountain theorem. Further uses are documented in [31], [7], [10], [9].

In [30], Tshinanga considered a similarly structured subcritical problem on a unbounded domain, with functions $h(x)$ and $k(x)$ fixed to the form $(1 + |x|)^{-b}$ (related to Hardy's inequality) with two multiplicative parameters, μ and λ, respectively taking the role of h and k. For $\mu > 0$ fixed the geometry is valid for the fountain theorem while for fixed $\lambda > 0$, the dual fountain theorem may be applied.

For $k(x) \geq 0$, our results closely resemble and extend those achieved in [20]. However, our formulation does not seem capable of reproducing extensions to supercritical growth.

These methods expose an infinite number of solutions. Our notation shall enumerate the solutions as u^m, while a sequence approaching a solution will be expressed as u_n. Eigenvalues λ_1 and λ_{-1} are defined in the next section.

The main results derived are the following:

Theorem 1.1. *Assume $1 < q < 2 < p < 2^*$, and $h(x) \geq 0$, $k(x) \geq 0$ are not identically zero. Let $\lambda \in (-\lambda_{-1}, \lambda_1)$. Then problem* (1) *admits infinitely many solutions at negative levels. A labelling of solutions $\{u^m\}$ gives that $\|\nabla u^m\| \to 0$ as $m \to \infty$.*

Theorem 1.2. *Assume $1 < q < 2 < p < 2^*$, $k(x) \geq 0$ is not identically zero and $h(x)$ changes sign. Let $\lambda \in (-\lambda_{-1}, \lambda_1)$. Then problem* (1) *admits infinitely many solutions at negative levels.*

Theorem 1.3. *Assume that $1 < q < 2 < p < 2^*$, and $h(x) \leq 0$ is not identically zero. Let $k(x) \not\equiv 0$ and $\lambda \in (-\lambda_{-1}, \lambda_1)$. Then problem* (1) *possesses an unbounded sequence of solutions $\{u^m\} \subset D^{1,2}$.*

Theorem 1.4. *Suppose $p = 2^*$ and $k(x) \geq 0$, $h(x) \geq 0$ are not identically zero. For each $\lambda \in (-\lambda_{-1}, \lambda_1)$ there exists an infinite number of solutions to* (1) *with negative energy. Labelling these solutions gives $\|\nabla u^m\| \to 0$ as $m \to \infty$.*

Theorem 1.5. *Let $p = 2^*$, suppose $h(x)$ is indefinite in sign and $k(x) \geq 0$. Suppose $-\lambda_{-1} < \lambda < \lambda_1$. If $\|k(x)\|_{q_0} > 0$ is sufficiently small then the problem* (1) *possesses an infinite number of solutions at negative energy.*

2. Preliminaries

We seek weak solutions to (1) in the space $D^{1,2}(\mathbb{R}^N)$, defined as the completion of C_0^∞ with respect to the norm $\|\nabla u\|^2 = \int_{\mathbb{R}^N} |\nabla u|^2$. A continuous imbedding exists $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$. By S we refer to the best Sobolev constant for the imbedding:

$$S = \inf_{\substack{u \in D^{1,2} \\ \|u\|_{2^*} = 1}} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx.$$

If γ is a measurable set in $\mathbb{R}^N$, then we denote

$$D_0^1(\gamma) = \{u \in D^{1,2}(\mathbb{R}^N) : u(x) = 0 \text{ a.e. } x \in \mathbb{R}^N \setminus \gamma\}.$$

If a domain is omitted, by default it shall be $\mathbb{R}^N$. Weak convergence shall be denoted "$\rightharpoonup$" while strong convergence is represented by "$\to$". For a nonnegative measurable function $l(x)$, define the weighted Lebesgue space L_l^p by all measurable functions u which satisfy $\int_{\mathbb{R}^N} l(x)|u|^p < \infty$, and associate with it the seminorm $\|u\|_{p,l}^p = \int_{\mathbb{R}^N} l(x)|u|^p$.

Critical points of the C^1 functional

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \frac{\lambda}{2}\int_{\mathbb{R}^N} gu^2\,dx - \int_{\mathbb{R}^N} k(x)|u|^q\,dx - \int_{\mathbb{R}^N} h(x)|u|^p\,dx$$

correspond with weak solutions to the Euler equations (1). In the sequel, we may omit the notation for the domain of integration which shall by default be $\mathbb{R}^N$.

Consider the eigenvalue problem

$$-\Delta u = \lambda g(x)u, \quad x \in \mathbb{R}^N, \qquad \lim_{|x|\to\infty} u(x) = 0. \tag{2}$$

A lemma from [16] which is important for eigenvalue results is included below. Assuming (G1), define a linear form on $D^{1,2}$ by

$$\beta(u,v) = \int_{\mathbb{R}^N} g(x)uv\,dx.$$

By the Riesz representation theorem, there is a bounded linear operator L such that

$$\beta(u,v) = \langle Lu, v\rangle \quad \text{for all } u,v \in D^{1,2}(\mathbb{R}^N).$$

Lemma 2.1. *L is compact.*

From this, the existence of a principal positive eigenvalue results.

Assume (G1) and $g^+ \not\equiv 0$. We set

$$0 < \lambda_1(g) = \lambda_1 = \left[\sup_{v\in D^{1,2}\setminus\{0\}} \frac{\int g(x)v^2}{\int |\nabla v|^2}\right]^{-1}. \tag{3}$$

Then λ_1 is the principal positive eigenvalue of the eigenvalue problem (2) and the associated eigenfunction φ_1 is strictly positive by construction in [15]. Similarly, for $g^- \not\equiv 0$,

$$0 < \lambda_1(-g) = \lambda_{-1} = \left[\sup_{v\in D^{1,2}\setminus\{0\}} \frac{\int (-g(x))v^2}{\int |\nabla v|^2}\right]^{-1} \tag{4}$$

is the principal negative eigenvalue. If $g^+ \equiv 0$ ($g^- \equiv 0$) then it is natural to denote $\lambda_1 = \infty$ ($\lambda_{-1} = \infty$). Tertikas [29] has gained eigenvalue results similar to these with relaxed conditions on $g(x)$. However, we maintain the condition (G1) to take advantage of compactness properties.

For a nonempty set γ, we may define

$$\lambda_1(\gamma) = \left[\sup_{D_0^1(\gamma)\setminus\{0\}} \frac{\int g(x)v^2}{\int |\nabla v|^2}\right]^{-1} \tag{5}$$

with $\lambda_{-1}(\gamma)$ defined analogously to (4) provided $g^- \not\equiv 0$ on γ. Cingolini and Gámez [17] show that a maximum is attained in (5). Accordingly, suprema in (3) and (5) are distinct since φ_1 is positive everywhere in $\mathbb{R}^N$. This is expressible as

$$-\lambda_{-1}(\gamma) \le \lambda_{-1} < 0 < \lambda_1 \le \lambda_1(\gamma).$$

The following lemma from [17] shows that for $2 \le s < 2^*$ and $l(x) \in L^{2N/(2N-sN+2s)}$, the imbedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^s_l(\mathbb{R}^N)$ holds compactly. Indeed, we remark that the lemma remains valid with no alteration in the proof for $1 < s < 2$.

Lemma 2.2. *Given $l(x) \in L^{2N/(2N-sN+2s)}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $1 < s < 2^*$, the problem*

$$-\Delta w = l(x)|u|^{s-2}u \quad \text{in } \mathbb{R}^N$$

admits a unique solution for each $u \in D^{1,2}(\mathbb{R}^N)$. Further, the operator K^s_l : $D^{1,2}(\mathbb{R}^N) \mapsto D^{1,2}(\mathbb{R}^N)$ defined by $K^s_l(u) = w$ is compact.

Remark 2.3. In particular, this implies weak continuity of a subcritical functional: if $u_n \rightharpoonup u_0$ in $D^{1,2}$ then

$$\int_{\mathbb{R}^N} l(x)|u_n|^s \to \int_{\mathbb{R}^N} l(x)|u_0|^s.$$

Lemma 2.2 does not extend to the case $p = 2^*$. Drábek and Huang [18] developed the following weak case:

Lemma 2.4. *Let u_n be a bounded sequence in $D^{1,2}$ and $\phi \in C^\infty_0(\mathbb{R}^N)$. Let $h(x) \in L^\infty(\mathbb{R}^N)$. Then*

$$\int h(x)|u_n|^{2^*-2}u_n\phi \to \int h(x)|u_0|^{2^*-2}u_0\phi.$$

Remark 2.5. It is trivial to extend this result to account for $\phi \in L^{2^*}$. For choosing R sufficiently large, partition the integral

$$\int_{\mathbb{R}^N} h(x)|u_n|^{p-2}u_n\phi = \int_{B_R} h(x)|u_n|^{p-2}u_n\phi + \int_{B^c_R} h(x)|u_n|^{p-2}u_n\phi$$

and estimate the last term using Hölder's inequality

$$\begin{aligned}&\int_{B^c_R} h(x)|u_n|^{p-2}u_n\phi \\ &\le \|h(x)\|_\infty \biggl(\int_{B^c_R} u_n^{2N/(N-2)}\biggr)^{(N+2)/2N} \biggl(\int_{B^c_R} \phi^{2^*}\biggr)^{1/2^*} \le C\biggl(\int_{B^c_R} \phi^{2^*}\biggr)^{1/2^*},\end{aligned}$$

which can be made arbitrarily small with large R.

For nonnegative $k(x)$, bounds exist on the values of λ for which solutions may exist. Define $W^- = \{x \in \mathbb{R}^N : h(x) < 0\}$.

Lemma 2.6. *Suppose W^- is nonempty, $g(x)$ changes sign in W^-, $k(x)$ is nonnegative, $h(x) \in L^\infty$ and $1 < q < 2 < p$. Then for every positive solution $(u,\lambda) \in D^{1,2}(\mathbb{R}^N) \times \mathbb{R}^N$, one has $-\lambda_{-1}(W^-) < \lambda < \lambda_1(W^-)$.*

Proof. Firstly note that due to the compactness Lemma 2.1, a maximiser u_0 exists for the eigenvalue problem:

$$\lambda_1(W^-) = \bigg(\sup_{u \in D_0^1(W^-)\setminus\{0\}} \frac{\int_{W^-} g(x)u^2}{\int_{W^-} |\nabla u|^2} \bigg)^{-1} = \frac{\int_{W^-} |\nabla u_0|^2}{\int_{W^-} g(x)u_0^2},$$

for some $u_0 \in D_0^1(W^-)$. If u is a positive solution to (1), it must hold that

$$\begin{aligned}
\lambda &= \bigg(\sup_{v \in D^{1,2}\setminus\{0\}} \frac{\int (g(x) + k(x)|u|^{q-2}/\lambda - h(x)|u|^{p-2}/\lambda) v^2\, dx}{\int |\nabla v|^2 dx} \bigg)^{-1} \\
&< \bigg(\frac{\int_{W^-} (g(x) + k(x)|u|^{q-2}/\lambda - h(x)|u|^{p-2}/\lambda) u_0^2\, dx}{\int_{W^-} |\nabla u_0|^2 dx} \bigg)^{-1} \\
&= \bigg(\frac{1}{\lambda_1(W^-)} + \frac{\int_{W^-} k(x)|u|^{q-2}u_0^2}{\lambda \int_{W^-} |\nabla u_0|^2} - \frac{\int_{W^-} h(x)|u|^{p-2}u_0^2}{\lambda \int |\nabla u_0|^2} \bigg)^{-1} \le \lambda_1(W^-).
\end{aligned}$$

The case of $\lambda < 0$ follows symmetrically. If $g(x)$ does not change sign in W^-, then $\lambda_{\pm 1}(W^-)$ may become infinite. □

3. Critical point theorems

3.1. Dual fountain theorem. A variation on an abstract critical point theorem by Bartsch and Willem [12] is utilised to guarantee an infinite number of solutions to equation (1). The alteration is required to widen the allowable function class for $k(x)$ from positive to nonnegative functions. The result from [12] requires symmetry of the functional to be expressible in terms of a compact group action which includes even functionals as a simple example. In essence, the theorem is based on a symmetric mountain pass theorem by Ambrosetti and Rabinowitz [4], but with a sequence of decompositions providing an infinite number of critical values. The Galerkin technique construction of each linking result within the fountain theorem dictates that the usual (PS)-condition is inadequate and a dual formulation of the condition, the (PS)*-condition is introduced.

Let X be an infinite dimensional separable Banach space with norm $\|\cdot\|$. Suppose that X may be decomposed as two subspaces, $X = X^1 \oplus X^2$. Define orthonormal bases for the subspaces as $X^1 = \mathrm{sp}\{e_j^1\}_{j\in\mathbb{N}}$ and $X^2 = \mathrm{sp}\{e_j^2\}_{j\in\mathbb{N}}$. Define

$$\begin{aligned}
X^1(j) &= \mathrm{sp}\{e_j^1\}, & Y_m &= \bigoplus_{j\ge m} X^1(j), & Z_m &= \bigoplus_{j \ge m} X^2(j), \\
X^2(j) &= \mathrm{sp}\{e_j^2\}, & Y^m &= \bigoplus_{0\le j\le m} X^1(j), & Z^m &= \bigoplus_{0 \le j \le m} X^2(j).
\end{aligned}$$

We assume that the functional displays a group symmetry similar to [12]. Let G be a compact Lie group, and V a finite dimensional orthogonal representation of G.

Definition 3.1. The action of G is said to be *admissible* if every continuous equivariant map $\gamma : \overline{\mathcal{O}} \mapsto V^m$ where $\mathcal{O}$ is an open bounded and invariant neighbourhood of 0 in V^{m+1}, $m \geq 1$, has a zero in $\partial\mathcal{O}$.

Here $\mathcal{O}$ is invariant if $gv = (gv_1, \dots, gv_{m+1}) \in \mathcal{O}$ for every $g \in G$ and $v = (v_1, \dots, v_{m+1}) \in \mathcal{O}$. The map γ is equivariant if $\gamma(gv) = g\gamma(v)$. We remark that an even functional has an admissible representation.

Compactness of the functional is expressed in a dual formulation of the usual Palais–Smale condition, the $(\mathrm{PS})_c^*$ condition. We remark that the $(\mathrm{PS})_c^*$-condition implies the usual $(\mathrm{PS})_c$-condition (Definition 3.2) but the converse has not been proven [31].

Definition 3.2. Let $\Phi \in C^1(X, \mathbb{R}^N)$ and $c \in \mathbb{R}^N$. The functional Φ satisfies the $(\mathrm{PS})_c$-condition if any sequence $\{u_n\} \subset X$ such that

$$n \to \infty, \quad \Phi(u_n) \to c, \quad \Phi'(u_n) \to 0 \quad \text{in } D^{-1,2}$$

(denoted a $(\mathrm{PS})_c$-sequence) contains a subsequence converging to a critical point of Φ.

Definition 3.3. Let $\Phi \in C^1(X, \mathbb{R}^N)$ and $c \in \mathbb{R}^N$. The function Φ satisfies the $(\mathrm{PS})_c^*$-condition (with respect to $Y^n \oplus Z^n$) if any sequence $\{u_{n_j}\} \subset X$ such that

$$n_j \to \infty, \quad u_{n_j} \in Y^{n_j} \oplus Z^{n_j}, \quad \Phi(u_{n_j}) \to c, \quad \Phi'|_{Y^{n_j} \oplus Z^{n_j}}(u_{n_j}) \to 0$$

(denoted a $(\mathrm{PS})_c^*$-sequence) contains a subsequence converging to a critical point of Φ.

In a slightly less restricted form than [12], sufficient conditions on a functional Φ are expressed in (A1)–(A5).

- (A1) The compact group G acts isometrically on the Banach space $X = X^1 \oplus X^2 = \overline{\bigoplus_{j\in\mathbb{N}} X^1(j) \oplus \bigoplus_{j\in\mathbb{N}} X^2(j)}$, the spaces $X^1(j)$ are invariant and there exists a finite dimensional space V such that for every $j \in \mathbb{N}$, $X^1(j) \cong V$ and the action of G on V is admissible.
- (A2) There exists $m_0 \in N$ such that for all $m \geq m_0$ there exists $R_m > 0$ such that $\Phi(w) \geq 0$ for every $w \in Y_m \oplus X^2$ with $\|w\| = R_m$.
- (A3) Suppose $b_m = \inf_{B_m} \Phi(u) \to 0$ as $m \to \infty$ where $B_m = \{u \in Y_m \oplus X^2 : \|u\| \leq R_m\}$.
- (A4) For all $m \geq m_0$ there exists $r_m \in (0, R_m)$ and $d_m < 0$ such that $\Phi(u) \leq d_m$ for every $u \in Y^m$ with $\|u\| = r_m$.
- (A5) The $(\mathrm{PS})_c^*$-condition holds for Φ for every $c \in [b_{m_0}, 0)$.

Remark 3.4. We may remark that if Φ is positive definite on subspace X^2, then to guarantee condition (A2), it is sufficient to check nonnegativity of Φ on a sphere in Y_m and to verify (A3) we may replace the definition of B_m with $\{u \in Y_m : \|u\| \le R_m\}$.

Theorem 3.5. *If Φ satisfies* (A1)–(A5)*, then for each $m \ge m_0$, Φ has a critical value $c_m \in [b_m, d_m]$, hence $c_m \to 0$ as $m \to \infty$.*

The proof for this theorem is based on the deformation lemma [7] which has been weakened slightly by broadening the restriction (A1).

Lemma 3.6. *Assume Φ satisfies* (A1)*. Let $B^m = \{u \in Y^m : \|u\| \le R_m\}$ (in contrast to B_m in* (A3)*) and suppose $0 < r_m < R_m$. Define, for $m \ge 2$,*

$$\begin{aligned} c_m &= \inf_{\gamma \in \Gamma_m} \max_{u \in B^m} \Phi(\gamma(u)), \\ \Gamma_m &= \{\gamma \in C(B^m, X) : \gamma \text{ is equivariant and } \gamma|_{\partial B^m} = \mathrm{id}\}. \end{aligned}$$

If

$$d_m = \inf_{u \in Y_m \oplus X^2,\ \|u\| = r_m} \Phi(u) > a_m = \max_{u \in Y^m,\ \|u\| = R_m} \Phi(u),$$

then $c_m \ge d_m$ and, for every $\varepsilon \in (0, (c_m - a_m)/2)$, $\delta > 0$ and $\gamma \in \Gamma_m$ such that $\max_{B^m} \Phi \circ \gamma \le c_m + \varepsilon$, there exists $u \in X$ such that

(a) $c_m - 2\varepsilon \le \Phi(u) \le c_m + 2\varepsilon$,
(b) $\mathrm{dist}(u, \gamma(B^m)) \le 2\delta$,
(c) $\|\Phi'(u)\| \le 8\varepsilon/\delta$.

Proof of Theorem 3.5. Fix $n \ge m \ge m_0$ and define

$$\begin{aligned} Z_m^n &= \bigoplus_{j=m}^{n} X^1(j) \oplus Z^n, \\ B_m^n &= \{u \in Z_m^n : \|u\| \le R_m\}, \\ \Gamma_m^n &= \{\gamma \in C(B_m^n, Y^n \oplus Z^n) : \gamma \text{ is equivariant and } \gamma|_{\partial B_m^n} = \mathrm{id}\}, \\ c_m^n &= \sup_{\gamma \in \Gamma_m^n} \min_{u \in B_m^n} \Phi(\gamma(u)). \end{aligned}$$

Now, apply Lemma 3.6 to the functional $-\Phi$, defined on the space $Y^n \oplus Z^n$. It follows that $c_m^n \le d_m$ and there exists $u_n \in Y^n \oplus Z^n$ such that

$$c_m^n - 2/n \le \Phi(u_n) \le c_m^n + 2/n, \qquad \|\Phi'|_{Y^n \oplus Z^n}(u_n)\| \le 8/n.$$

Since the $(\mathrm{PS})_c^*$-condition holds at the appropriate levels, $\{c_m^n\}_{n \ge m}$ converges along a subsequence to a critical value $c_m \in [b_m, d_m]$ of Φ as $n \to \infty$. From (A3), it follows that $c_m \to 0$ as $m \to \infty$. □

3.2. Fountain theorem. A similar extraction of a subspace can extend the fountain theorem. Let X be a Banach space, and $X = X^1 \oplus X^2$ where $\dim(X^1) = \infty$. Let $\{e^1_j\}_{j\in\mathbb{N}}$ form a basis for X^1 and define $X^1(j) = \mathrm{sp}\{e^1_j\}$. Let $\Phi \in C^1(X)$ be a functional with an admissible group action G as defined in Definition 3.1.

Define the progressive decomposition of X^1 as follows

$$Y^m = \bigoplus_{0\le j\le m} X^1(j) \quad \text{and} \quad Y_m = \bigoplus_{m\le j<\infty} X^1(j).$$

We introduce the following conditions:

(B2) For some $\rho_m > 0$, $a_m = \max\{\Phi(u) : u \in Y^m,\ \|u\| = \rho_m\} \le 0$.

(B3) For some $r_m > 0$, $b_m = \inf\{\Phi(u) : u \in Y_m \oplus X^2,\ \|u\| = r_m\} \to \infty$ as $m \to \infty$.

(B4) Suppose that Φ satisfies $(\mathrm{PS})_c$-condition for each $c > 0$.

Theorem 3.7. *Assume that the functional $\Phi \in C^1(X)$ satisfies the symmetry condition* (A1). *Further, suppose that for each $m \in \mathbb{N}$ there exists $\rho_m > r_m > 0$ such that conditions* (B2)–(B4) *hold. Then there exists an unbounded sequence of positive critical values.*

The inclusion of a formal proof for this perturbation of a well known result seems unwarranted since the concept and the proof is so close to Bartsch's traditional fountain theorem in [7]. The only difference is that a subspace X^2 is extracted, and the symmetric mountain pass theorem applied to the remaining subspaces.

Remark 3.8. If compactness conditions (A5) and (B4) are omitted, then Theorems 3.5 and 3.7 yield (PS)- and $(\mathrm{PS})^*$-sequences at the associated sequence of energies.

4. Proofs of theorems

For applications of Theorems 3.5 and 3.7, we replace the generic space X with the concrete example $D^{1,2}(\mathbb{R}^N)$. To verify that this space is appropriate, the existence of a countable basis and the ability to generate an orthogonal complement must be assured.

By functional analysis, for example [28], given any closed linear subspace M of a Banach space X, there is a complementary subspace N if and only if there is a projector P of X onto M. Using the obvious projector derived from the inner product in $D^{1,2}(\mathbb{R}^N)$, an orthogonal complement to any closed subspace is guaranteed.

According to Theorem II.7 in [26], a Hilbert space is separable if and only if it possesses a countable orthonormal basis. To see that $D^{1,2}(\mathbb{R}^N)$ is separable, recall the imbedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$. Consequently, $D^{1,2}(\mathbb{R}^N)$ must be isometrically isomorphic to a closed linear subspace V of L^{2^*}, with the isomorphism established by the map $u \mapsto Du$. Since $L^{2^*}(\mathbb{R}^N)$ is separable, so $D^{1,2}(\mathbb{R}^N)$ is too.

4.1. Proofs of Theorems 1.1 and 1.2. Let $\Omega = \operatorname{int}\{x \in \mathbb{R}^N : k(x) > 0\}$. Define the subspace X^2 spanned by those u which are zero on the support of $k(x)$: $X^2 = \{u \in D^{1,2}(\mathbb{R}^N) : u(x) = 0 \text{ a.e. } x \in \Omega\}$. For any bounded sequence $\{u_n\} \subset D^{1,2}$, it follows from the Sobolev imbedding theorem that for a subsequence $u_n(x) \to u_0(x)$ almost everywhere, and hence X^2 is a closed subspace. Define X^1 by the relation $X \equiv D^{1,2}(\mathbb{R}^N) = X^1 \oplus X^2$, an orthogonal decomposition. Note that for all $u \in X^2$, $\int k(x)|u|^q = 0$, and X^2 will form the positive definite subspace mentioned in Remark 3.4. If $k(x) > 0$ a.e., then $\Omega = \mathbb{R}^N$ and $X^2 = \{0\}$.

Lemma 4.1. *Suppose* (G1) *holds with* $g^{\pm} \not\equiv 0$. *Assume* $1 < q < 2 < p \leq 2^*$ *and* (K1) *is satisfied. Then, for* $\lambda \in (-\lambda_{-1}, \lambda_1)$, *any* $(\mathrm{PS})_c^*$*-sequence for* I_λ *is bounded in* $D^{1,2}(\mathbb{R}^N)$.

Proof. Specifying (H1) ensures I_λ is differentiable. Suppose $u_n \in Y^n \oplus Z^n$ is a $(\mathrm{PS})_c^*$-sequence, with $\|\nabla u_n\| \to \infty$. Let $0 \leq \lambda < \lambda_1$ then, for n sufficiently large,

$$
\begin{aligned}
c + 1 + \|\nabla u_n\| &\geq I_\lambda(u_n) - \frac{1}{p}\langle I'(u_n), u_n\rangle \\
&= \left(\frac{1}{2} - \frac{1}{p}\right)\left(\int |\nabla u_n|^2 - \lambda \int g(x) u_n^2\right) - \left(\frac{1}{q} - \frac{1}{p}\right)\int k(x)|u_n|^q \\
&\geq \left(\frac{1}{2} - \frac{1}{p}\right)\left(1 - \frac{\lambda}{\lambda_1}\right)\|\nabla u_n\|^2 - \left(\frac{1}{q} - \frac{1}{p}\right) S^{-q/2}\|k(x)\|_{q_0}\|\nabla u_n\|^q,
\end{aligned}
$$

which provides a contradiction. A symmetric argument holds for $-\lambda_{-1} < \lambda < 0$. □

Although simple, Lemma 4.1 is applicable for both subcritical and critical exponents, and independently of the signs of $h(x)$ and $k(x)$. For nonnegative $h(x)$, we need only seek solutions at negative levels according to the lemma below.

Lemma 4.2. *Suppose* (G1), (K1) *and* (H1) *hold and let* $k(x)$ *and* $h(x)$ *be nonnegative. Then there exist no solutions to* (1) *at positive energy.*

Proof. Let u be a weak solution at level c. Then

$$
\begin{aligned}
\int |\nabla u|^2 - \lambda \int g(x) u^2 - \frac{2}{q}\int k(x)|u|^q + \frac{2}{p}\int h(x)|u|^p &= 2c, \\
\int |\nabla u|^2 - \lambda \int g(x) u^2 - \int k(x)|u|^q + \int h(x)|u|^p &= 0.
\end{aligned}
$$

Consequently,

$$0 \le \left(1 - \frac{2}{p}\right) \int h(x)|u|^p = -2c + \left(1 - \frac{2}{q}\right) \int k(x)|u|^q. \tag{6}$$

Since $h(x), k(x) \ge 0$ almost everywhere, it is impossible for c to assume a positive value. $\square$

Construction of the fountain theorems develops a sequence of linking sets, each satisfying a variational principle to expose a solution. For this particular problem, the size of these sets shrinks to zero as the technique progresses through the decomposition of $D^{1,2}$. As a consequence, the levels of the solutions rise towards zero. However, it is not possible to immediately deduce the size of the solution without further information regarding the functional. The result below is pertinent to the query posed in [12].

LEMMA 4.3. *Suppose* (G1), (K1) *and* (H1) *hold, let $k(x)$ and $h(x)$ be non-negative and $-\lambda_{-1} < \lambda < \lambda_1$. Let $\{u^m\}$ be a sequence of solutions to* (1) *with $I_\lambda(u^m) = -\varepsilon_m < 0$, $\varepsilon_m \to 0$ as $m \to \infty$. Then $\|\nabla u^m\| \to 0$ as $m \to \infty$.*

PROOF. Let u^m be a (weak) solution at level $-\varepsilon_m$. Clearly it cannot be the trivial solution, and following the working in Lemma 4.2,

$$-2\varepsilon_m \le \left(1 - \frac{2}{q}\right) \int k(x)|u^m|^q \le 0 \Rightarrow \int k(x)|u^m|^q \le \frac{2q}{2-q}\varepsilon_m.$$

Inserting this expression into (6) gives

$$\frac{p-2}{p} \int h(x)|u^m|^p = 2\varepsilon_m + \left(1 - \frac{2}{q}\right) \int k(x)|u^m|^q \le 2\varepsilon_m.$$

Implying that

$$\int h|u^m|^p \le \frac{2p}{p-2}\varepsilon_m.$$

Now, u^m is a solution, and assuming $0 < \lambda < \lambda_1$ (the case of $-\lambda_{-1} < \lambda < 0$ follows symmetrically)

$$\begin{aligned}\left(1 - \frac{\lambda}{\lambda_1}\right) \int |\nabla u^m|^2 &\le \int |\nabla u^m|^2 - \lambda \int g(x)(u^m)^2 \\ &= \int k(x)|u^m|^q - \int h(x)|u^m|^p \le \int k|u^m|^q \le \frac{2q}{2-q}\varepsilon_m\end{aligned}$$

establishing that $\|\nabla u^m\| \to 0$ as $\varepsilon_m \to 0$. $\square$

PROOF OF THEOREM 1.1. Verifying condition (A1), we firstly remark that functional $I_\lambda \in C^1(D^{1,2}, \mathbb{R}^N)$ is even. We now verify conditions (A2)–(A5). Define μ_m by

$$\mu_m = \sup_{u \in Y_m \oplus X^2 \setminus \{0\}} \left(\frac{(\int k(x)|u|^q)^{1/q}}{(\int |\nabla u|^2)^{1/2}} \right).$$

It is clear that $0 < \mu_{m+1} \le \mu_m$, so $\mu_m \to \mu_0 \ge 0$ as $m \to \infty$. We now show that $\mu_0 = 0$. For every $m \ge 1$, there exists $u_m \in Y_m \oplus X^2$ such that $\|\nabla u_m\| = 1$ and

$(\int k(x)|u_m|^q)^{1/q} > \mu_m/2$. By the definition of Y_m, $u_m \rightharpoonup u_0$ in $D^{1,2}(\mathbb{R}^N)$, where $u_0 \in X^2$. Compactness of the operator K_k^q gives that $u_m \to u_0$ in L_k^q, and so

$$\int k(x)|u_m|^q \to \int k(x)|u_0|^q = 0.$$

Now, for $u \in Y_m \oplus X^2$, with $\lambda > 0$

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{2}\bigg(1 - \frac{\lambda}{\lambda_1}\bigg)\|\nabla u\|^2 - \frac{\mu_m^q}{q}\|\nabla u\|^q + \frac{1}{p}\int h(x)|u|^p \\ &\geq \frac{1}{2}\bigg(1 - \frac{\lambda}{\lambda_1}\bigg)\|\nabla u\|^2 - \frac{\mu_m^q}{q}\|\nabla u\|^q \end{aligned}$$

and this guarantees $I_\lambda(u)$ remains positive for

$$\|\nabla u\| > \bigg(\frac{q(1-\lambda/\lambda_1)}{2\mu_m^q}\bigg)^{1/(q-2)} \equiv A_m. \tag{7}$$

For $\lambda < 0$, A_m is similar to expression (7), replacing λ_1 with $-\lambda_{-1}$

Setting $R_m = 2A_m$ ensures condition (A2) is satisfied. As $m \to \infty$, $\mu_m \to 0$ and so $A_m \to 0$, fulfilling requirement (A3).

Suppose $k(x) > 0$ almost everywhere. Then $\|\cdot\|_{q,k}$ forms a norm (not simply a seminorm). Since Y^m is finite dimensional all norms are equivalent, and

$$c_1\|\nabla u\| \leq \bigg(\int k(x)|u|^q\bigg)^{1/q} \leq c_2\|\nabla u\|.$$

We estimate I_λ from above for $u \in Y^m$,

$$I_\lambda(u) \leq \frac{1}{2}(1 + |\lambda| S^{-1}\|g\|_{N/2})\|\nabla u\|^2 - \frac{c_1}{q}\|\nabla u\|^q + \frac{1}{p}S^{-p/2}\|h\|_{p_0}\|\nabla u\|^p.$$

Since $q < 2 < p$, taking $r_m > 0$ sufficiently small, (A4) is satisfied.

The uniform estimate c_1 is stronger than actually required, so suppose $k(x) \geq 0$ and $X^2 \neq \{0\}$. Let $u \in Y^m$. By orthogonality of the decomposition, $u \notin X^2$, so $u(x) \not\equiv 0$ for $x \in \Omega$, and thus $\int k(x)|u|^q \neq 0$. Since Y^m is finite dimensional, there is a constant c_m such that

$$c_m\|\nabla u\| \leq \bigg(\int k(x)|u|^q\bigg)^{1/q}$$

for all $u \in Y^m$. In contrast to the case where $X^2 = \{0\}$, $c_m \to 0$ as $m \to \infty$.

For $u \in Y^m$,

$$I_\lambda(u) \leq \frac{1}{2}(1 + |\lambda| S^{-1}\|g\|_{N/2})\|\nabla u\|^2 - \frac{c_m}{q}\|\nabla u\|^q + \frac{1}{p}S^{-p/2}\|h\|_{p_0}\|\nabla u\|^p,$$

and $I_\lambda(u) \leq d_m < 0$ for $\|\nabla u\| = r_m$ sufficiently small. Since $c_m \to 0$ as $m \to \infty$, it follows that $r_m \to 0$ and $d_m \to 0$.

Verification of the Palais–Smale condition (A5) follows from Lemma 4.1 which shows that any (PS)*-sequence, $\{u_n\}$, is bounded in $D^{1,2}$. A subsequence, again denoted $\{u_n\}$, converges weakly to u_0 in $D^{1,2}$. Consequently $u_n \to u_0$ in $L^p_h, L^q_k, L^{N/2}_g$ by Lemma 2.2. Taking limits of $\langle I'(u_n), u_n\rangle$ and $\langle I'(u_n), u_0\rangle$

$$\begin{aligned}\langle I'_\lambda(u_n), u_0\rangle &= \int |\nabla u_0|^2 - \lambda \int g(x)u_0^2 - \int k(x)|u_0|^q \\ &\quad + \int h(x)|u_0| + o(1) = o(1), \\ \langle I'_\lambda(u_n), u_n\rangle &= \int |\nabla u_n|^2 - \lambda \int g(x)u_0^2 - \int k(x)|u_0|^q \\ &\quad + \int h(x)|u_0| + o(1) = o(1),\end{aligned}$$

implying $\int_{\mathbb{R}^N} |\nabla u_n|^2 \to \int_{\mathbb{R}^N} |\nabla u_0|^2$ and u_0 must be a critical point. Convergence of the solutions to zero in $D^{1,2}$ is proved by a trivial application of Lemma 4.3.□

For a sign-changing $h(x)$, it is possible to recover an infinite number of solutions using the dual fountain theorem. In verifying the conditions (A1)–(A5), we find that the technique will only elicit solutions at small energies.

Lemma 4.2 no longer applies, and it appears that Lemma 4.3 also fails. This implies that solutions may have small energies, but not necessarily small magnitudes.

Proof of Theorem 1.2. Following precisely the proof to Theorem 1.1, we decompose $X = X^1 \oplus X^2$, where $X^2 = \{u \in D^{1,2}(\mathbb{R}^N N) : u(x) = 0 \text{ a.e. } x \in \Omega\}$. However, since $h(x)$ changes sign, I_λ is no longer positive definite on X^2.

Using the technique from [30], there exists a sufficiently small number $R > 0$ such that for all $u \in D^{1,2}(\mathbb{R}^N)$ with $\|\nabla u\| < R$,

$$\frac{1}{p}\int h(x)|u|^p \le \frac{1}{p} S^{-p/2}\|h\|_{p_0}\|\nabla u\|^p \le \frac{1}{4}\left(1 - \frac{\lambda}{\lambda_1}\right)\|\nabla u\|^2.$$

Consequently, for $u \in Y_m \oplus X^2$,

$$I_\lambda(u) \ge \frac{1}{4}\left(1 - \frac{\lambda}{\lambda_1}\right)\|\nabla u\|^2 - \frac{\mu_m^q}{q}\|\nabla u\|^q.$$

It follows that $I_\lambda(u) > 0$ provided that

$$\|\nabla u\| > \left(\frac{q(1-\lambda/\lambda_1)}{4\mu_m^q}\right)^{1/(q-2)} \equiv A_m,$$

where $A_m \to 0$ as $m \to \infty$. Set $R_m = 2A_m$, then by choosing m_0 sufficiently large in condition (A2), R_{m_0} will fall within the bound R. Conditions (A2) and (A3) follow immediately and (A4) follows from the proof to Theorem 1.1. Lemma 4.1 implies (A5) and Theorem 3.5 provides the result. □

4.2. Proof of Theorem 1.3. When $h(x)$ is non-positive and $k(x)$ is indefinite in sign, the nature of the problem is inverted. Solutions exist at positive energies, and become sequentially unbounded.

For this problem, a slight twist on the fountain theorem is introduced to facilitate the possibility of $h(x)$ assuming the value zero on a subset of $\mathbb{R}^N$. If $h(x) < 0$ almost everywhere, then the usual fountain theorem would suffice.

PROOF OF THEOREM 1.3. Define $\Upsilon = \operatorname{int}\{x \in \mathbb{R}^N : h(x) < 0\}$. Define $X^2 = \{u \in D^{1,2} : u(x) = 0 \text{ for a.e. } x \in \Upsilon\}$. As before, X^2 forms a closed subspace which shall be extracted in applying the fountain theorem. Define X^1 as the complementary subspace to X^2. Define

$$\mu_m = \sup_{u \in Y_m \oplus X^2} \frac{(\int -h(x)|u|^p)^{1/p}}{\|\nabla u\|}.$$

We claim that $\lim_{m\to\infty} \mu_m = 0$. To see this, firstly note that $0 \le \mu_m$ is a decreasing sequence as Z_m shrinks. For each $m \ge 1$, there exists $u_m \in Y_m \oplus X^2$ such that $\|\nabla u_m\| = 1$ and

$$\left(\int -h(x)|u_m|^p\right)^{1/p} > \frac{\mu_m}{2}.$$

Clearly such a sequence contains a weakly convergent subsequence, $u_m \rightharpoonup u_0 \in X^2$. Since $p < 2^*$, use Lemma 2.2.

$$\int -h(x)|u_m|^p \to \int -h(x)|u_0|^p = 0. \tag{8}$$

We remark that this property is lost when $p = 2^*$.

We now confirm condition (B2). Since $Y^m \perp X^2$, each $u \in Y^m$ satisfies $u(x) \not\equiv 0$ on Υ, and consequently $\|\cdot\|_{p,(-h)}$ forms a norm on Y^m. Since Y^m is finite dimensional, all norms are equivalent and the following estimates hold for positive constants $C_{i,m}$, $i = 1,2,3$: (where $C_{2,m} \to 0$ as $m \to \infty$).

$$\begin{aligned} 0 &\le \int |k(x)||u|^q \le C_{1,m}\|\nabla u\|^q, \\ C_{2,m}\|\nabla u\|^p &\le \int -h(x)|u|^p \le C_{3,m}\|\nabla u\|^p. \end{aligned}$$

Now,

$$I_\lambda(u) \le \frac{1}{2}(1 + |\lambda|S^{-1}\|g\|_{N/2})\|\nabla u\|^2 + C_{1,m}\|\nabla u\|^q - C_{2,m}\|\nabla u\|^p.$$

For sufficiently large ρ_m, the final term dominates and $I_\lambda(u) < 0$ for all $u \in Y^m$, $\|\nabla u\| > \rho_m$.

Next we confirm (B3). Take $u \in Y_m \oplus X^2$. Then $0 \le \int -h|u|^p \le \mu_m^p \|\nabla u\|^p$. Consequently,

$$I_\lambda(u) \ge \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_1}\right)\|\nabla u\|^2 - \frac{1}{q}\int |k(x)||u|^q - \frac{\mu_m^p}{p}\|\nabla u\|^p.$$

If $\|\nabla u\|$ is sufficiently large, then

$$\frac{1}{q}\int k(x)|u|^q \le \frac{1}{q} S^{-q/2}\|k(x)\|_{q_0}\|\nabla u\|^q \le \frac{1}{4}\left(1 - \frac{\lambda}{\lambda_1}\right)\|\nabla u\|^2.$$

Hence, if $u_m \in Y_m \oplus X^2$ lies outside a sufficiently large radius,

$$I_\lambda(u_m) \ge \frac{1}{4}\left(1 - \frac{\lambda}{\lambda_1}\right)\|\nabla u_m\|^2 - \frac{\mu_m^p}{p}\|\nabla u_m\|^p.$$

Setting

$$r_m = \left(\frac{8\mu_m^p}{p(1-\lambda/\lambda_1)}\right)^{1/(2-p)}$$

it follows that for $u_m \in Y_m \oplus X^2$, $\|\nabla u_m\| = r_m$, $I_\lambda(u_m) \to \infty$ as $m \to \infty$.

The Palais–Smale condition (B4) follows by Lemma 4.1.

The existence of an infinite sequence of solutions u^m follows by the fountain Theorem 3.7. Since $c_m \ge \inf\{I_\lambda(u_m) : u_m \in Y_m \oplus X^2,\ \|\nabla u_m\| = r_m\} \to \infty$, it follows trivially that $\|\nabla u^m\|$ must also diverge. □

4.3. Proof of Theorem 1.4. We now consider the case of $0 \le h(x) \in L^\infty$ and $p = 2^*$. A study by Alves et al [1], [2] and by Miyagaki [24], has concluded the existence of nonnegative solutions to a similar problem on $\mathbb{R}^N$, but without the weighting function $h(x)$. There, $g(x)$ is restricted to positive functions and a different growth condition imposed.

When $h(x)$ is fixed to a nonnegative function, then a degree of indefiniteness is eliminated, the (PS)*-condition is quite easily achieved, and the geometry of I_λ is suitable for application of the dual fountain theorem.

Development of the dual form of the Palais–Smale condition requires the introduction of a projection operator. Define P^n to be the orthogonal projector from $D^{1,2}$ into the space $Y^n \oplus Z^n$.

The space $D^{1,2}(\mathbb{R}^N)$ is decomposed in the same way as for the proofs of Theorems 1.1 and 1.2. For $\Omega = \operatorname{int}\{x \in \mathbb{R}^N : k(x) > 0\}$, we define $X^2 = \{u \in D^{1,2}(\mathbb{R}^N) : u(x) = 0 \text{ a.e. } x \in \mathbb{R}^N\}$ and X^1 as the complementary subspace.

Lemma 4.4. *Suppose* (G1), (K1) *and* (H1) *are satisfied with* $p = 2^*$ *Any bounded* (PS)**-sequence* $\{u_n\} \subset D^{1,2}$ *converges weakly to a weak solution* (*perhaps trivial*) *of equation* (1).

Proof. Let $\{u_n\}$ be a (PS)*-sequence, and suppose $u_n \rightharpoonup u_0$. Take arbitrary $\phi \in C_0^\infty(\mathbb{R}^N)$. With P^n defined as the orthogonal projector, let $\phi^n = P^n\phi$.

Now, u_n is bounded implying that $I'(u_n)$ is bounded, and $\phi^n - \phi \to 0$ strongly in $D^{1,2}$, giving that

$$\langle I'_\lambda(u_n), \phi - \phi^n\rangle \le \|I'_\lambda(u_n)\|\|\nabla(\phi - \phi^n)\| \to 0. \tag{9}$$

Combining this with a simple decomposition of ϕ,

$$\langle I'_\lambda(u_n), \phi\rangle = \langle I'_\lambda(u_n), \phi^n\rangle + \langle I'_\lambda(u_n), \phi - \phi^n\rangle = \langle I'(u_n), \phi^n\rangle + o(1).$$

Since $\{u_n\}$ is a (PS)*-sequence, $I|'_{Y^n\oplus Z^n} \to 0$ and according to (9) this implies $\langle I'(u_n), \phi\rangle \to 0$. Using the strong convergence of projection, a simple corollary of Lemma 2.4 states that for $\phi \in D^{1,2}$, $\int h(x)|u_n|^{2^*-2}u_n\phi^n \to \int h(x)|u_0|^{2^*-2}u_0\phi$. Now,

$$\begin{aligned}&\int \nabla u_n\nabla\phi - \lambda\int g(x)u_n\phi - \int k(x)|u_n|^{q-2}u_n\phi + \int h(x)|u_n|^{2^*-2}u_n\phi \to 0\\ \Rightarrow &\int \nabla u_0\nabla\phi - \lambda\int g(x)u_0\phi - \int k(x)|u_0|^{q-2}u_0\phi + \int h(x)|u_0|^{2^*-2}u_0\phi = 0,\end{aligned}$$

revealing u_0 is a weak solution. □

The dual formulation of the Palais–Smale condition can be achieved at all levels, when $h(x)$ and $k(x)$ are restricted to nonnegative functions.

Lemma 4.5. *Assume* (G1), (K1), (H1), $p = 2^*$, $h(x), k(x) \ge 0$ *are not identically zero and* $-\lambda_{-1} < \lambda < \lambda_1$. *Then* I_λ *satisfies the* (PS)**-condition.*

Proof. Let $\{u_n\} \subset D^{1,2}(\mathbb{R}^N)$ be a (PS)*-sequence. Lemma 4.1 guarantees that u_n is bounded, and so a relabelled subsequence converges weakly, $u_n \rightharpoonup u_0$. Let P^n be the orthogonal projector from $D^{1,2}$ into $Y^n \oplus Z^n$. Now,

$$\begin{aligned}\langle I'(u_n), u_n\rangle &= \int |\nabla u_n|^2 - \lambda\int g(x)u_n^2 - \int k(x)|u_n|^q + \int h(x)|u_n|^{2^*}\\ &= \int |\nabla u_n|^2 - \lambda\int g(x)u_0^2\\ &\quad - \int k(x)|u_0|^q + \int h(x)|u_n|^{2^*} + o(1) = o(1).\end{aligned} \tag{10}$$

Making use of the strong convergence $P^nu_0 \to u_0$ in $D^{1,2}(\mathbb{R}^N)$ and Lemma 2.4,

$$\begin{aligned}\langle I'(u_n), P^nu_0\rangle &= \int \nabla u_n\nabla(P^nu_0) - \lambda\int g(x)u_nP^nu_0\\ &\quad - \int k(x)|u_n|^{q-2}u_nP^nu_0 + \int h(x)|u_n|^{2^*-2}u_nP^nu_0\\ &= \int |\nabla u_0|^2 - \lambda\int g(x)u_0^2\\ &\quad - \int k(x)|u_0|^q + \int h(x)|u_0|^{2^*} + o(1) = o(1).\end{aligned} \tag{11}$$

Subtracting (11) from (10),

$$\int (|\nabla u_n|^2 - |\nabla u_0|^2) + \int (h(x)|u_n|^{2^*} - h(x)|u_0|^{2^*}) \to 0.$$

Lower-semicontinuity of norms and seminorms dictates that

$$\liminf_{n\to\infty} \int |\nabla u_n|^2 \geq \int |\nabla u_0|^2, \quad \liminf_{n\to\infty} \int h|u_n|^{2^*} \geq \int h|u_0|^{2^*}$$

implying that $\int |\nabla u_n|^2 \to \int |\nabla u_0|^2$. Combined with almost everywhere convergence, this means $u_n \to u_0$ strongly in $D^{1,2}(\mathbb{R}^N)$. □

Proof of Theorem 1.4. Identically to Theorem 1.1 the geometry of the functional I_λ satisfies conditions (A1)–(A4), guaranteeing the generation of a $(\mathrm{PS})^*_c$-sequence with $c < 0$. Lemma 4.1 shows that this sequence must be bounded in $D^{1,2}(\mathbb{R}^N)$. Strong convergence of the sequence follows from Lemma 4.5. Sizes of the solutions are restricted according to Lemma 4.3. □

4.4. Proof of Theorem 1.5. The publication [19] sought solutions in $H^1_0(\Omega)$ for the following problem on a bounded domain Ω:

$$-\Delta u = |u|^{2^*-2}u + \mu|u|^{q-2}u \tag{12}$$

which corresponds to seeking critical points of the functional

$$\Phi_\mu(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{2^*}\int |u|^{2^*} - \frac{\mu}{2}\int |u|^q.$$

A dual Palais–Smale condition may be formulated:

Lemma 4.6. *There exists $\widetilde{k} > 0$ such that, for any $\mu > 0$ and*

$$c < \frac{1}{N}S^{N/2} - \widetilde{k}\mu^{2^*/(2^*-q)}$$

the functional Φ_μ satisfies the $(\mathrm{PS})^*_c$-condition.

The functional Φ_μ can be checked to satisfy the requirements of the dual fountain theorem, and an infinite sequence of solutions at negative energy results provided that positive μ is sufficiently small.

For the problem (1), this strategy may be replicated to an extent. However, the weighting functions and an unbounded domain imply that a $(\mathrm{PS})^*_c$-condition is not easily achieved, and instead we will utilise the dual fountain theorem without a Palais–Smale condition.

Similar to the subcritical case, we shall later verify that the geometry of I_λ is appropriate for use of the dual fountain theorem. However, if $\{u^m\}$ is a sequence of solutions at negative levels $-\varepsilon_m \to 0$, the indefiniteness of $h(x)$ implies that $\|\nabla u^m\|$ does not necessarily converge to zero. As a consequence of the critical exponent $p = 2^*$, concentration is possible and compactness may be lost.

Proof of Theorem 1.5. Firstly we shall verify the conditions (A1)–(A4) to generate a family of $(\mathrm{PS})_c^*$-sequences at negative levels. Later we ensure the convergence of these sequences.

The symmetry condition (A1) follows trivially by the evenness of I_λ. Conditions (A2) and (A3) follow in an identical manner to the proof to Theorem 1.2. Condition (A4) is verified in the same way as Theorem 1.1. This is sufficient to generate a family of $(\mathrm{PS})_{-\varepsilon_m}^*$-sequences at levels $0 > -\varepsilon_m \to 0$.

Let u_n be a $(\mathrm{PS})_c^*$-sequence at level $c \in \mathbb{R}$. Then

$$I_\lambda(u_n) = \frac{1}{2}\|\nabla u_n\|^2 - \frac{\lambda}{2}\int g(x)u^2 - \frac{1}{q}\int k(x)|u_n|^q + \frac{1}{2^*}\int h(x)|u_n|^{2^*} \to c, \tag{13}$$

$$\langle I'_\lambda(u_n), u_n\rangle = \|\nabla u_n\|^2 - \lambda\int g(x)u_n^2 - \int k(x)|u_n|^q + \int h(x)|u_n|^{2^*} \to 0, \tag{14}$$

and hence

$$I_\lambda(u_n) = \left(\frac{1}{2} - \frac{1}{q}\right)\int k(x)|u_n|^q - \frac{1}{N}\int h(x)|u_n|^{2^*} + o(1). \tag{15}$$

By Lemma 4.1, u_n is bounded and so $u_n \rightharpoonup u$ where u is a weak solution by Lemma 4.4. Let $u_n = u + v_n$ with $v_n \rightharpoonup 0$. Using the Brézis–Lieb Lemma [13]

$$\int h(x)|u_n|^{2^*} = \int h(x)|u|^{2^*} + \int h(x)|v_n|^{2^*} + o(1),$$

and so (13) becomes

$$I_\lambda(u_n) = I_\lambda(u) + \frac{1}{2}\|\nabla v_n\|^2 + \frac{1}{2^*}\int h(x)|v_n|^{2^*} + o(1) = c + o(1). \tag{16}$$

Equation (14) gives that

$$\langle I'_\lambda(u_n), u_n\rangle = \langle I'(u), u\rangle + \|\nabla v_n\|^2 + \int h(x)|v_n|^{2^*} + o(1) = o(1).$$

Since u is a solution, it follows that for some $b \in \mathbb{R}^N$, $\|\nabla v_n\|^2 \to b$ and $\int h(x)|v_n|^{2^*} \to -b$.

If $h(x) \ge 0$ almost everywhere, then clearly $b = 0$, implying that concentration is impossible, and $u_n \to u$ strongly in $D^{1,2}(\mathbb{R}^N)$.

Suppose that $h^- \not\equiv 0$, allowing the possibility that $b > 0$. Then

$$-\int h(x)|v_n|^{2^*} \le \|h^-(x)\|_\infty \|v_n\|_{2^*}^{2^*},$$

and consequently,

$$\|v_n\|_{2^*}^2 \geq \left(\frac{1}{\|h^-(x)\|_\infty}\int -h(x)|v_n|^{2^*}\right)^{2/2^*}.$$

By Sobolev's inequality, $\|\nabla v_n\|^2 \geq S\|v_n\|_{2^*}^2$ and so

$$\|\nabla v_n\|^2 \geq \frac{S}{\|h^-(x)\|_\infty^{2/2^*}}\left(-\int h(x)|v_n|^{2^*}\right)^{2/2^*}.$$

Thus

$$b \geq \left(\frac{S}{\|h^-(x)\|_\infty^{2/2^*}}\right)^{N/2}. \tag{17}$$

So either $b = 0$, implying strong convergence, or the estimate (17) holds. Entertaining the latter scenario and using (15) and (16),

$$\begin{aligned} 0 \geq c &= \left(\frac{1}{2}-\frac{1}{q}\right)\int k(x)|u|^q - \frac{1}{N}\int h(x)|u|^{2^*} + \frac{1}{N}b \\ &\geq \frac{1}{N}\left(\frac{S}{\|h^-(x)\|_\infty^{2/2^*}}\right)^{N/2} - \frac{\|h(x)\|_\infty}{N}S^{-2^*/2}\|\nabla u\|^{2^*} \\ &\quad + \left(\frac{1}{2}-\frac{1}{q}\right)\|k(x)\|_{q_0}S^{-q/2}\|\nabla u\|^q. \end{aligned} \tag{18}$$

Now we show that the Palais–Smale sequences at negative energies have an upper bounded dependent upon $\|k(x)\|_{q_0}$.

Let u_n be a $(\mathrm{PS})_c^*$-sequence for I_λ at energy $-\varepsilon_m < 0$. For sufficiently large $n \in \mathbb{N}$, it must hold that $|\langle I_\lambda'(u_n), u_n\rangle| < \varepsilon_m/2$. Hence equation (15) gives

$$\frac{1}{N}\left(\|\nabla u_n\|^2 - \lambda\int g(x)u_n^2\right) - \left(\frac{1}{q}-\frac{1}{2^*}\right)\int k(x)|u_n|^q \leq -\frac{\varepsilon_m}{2}$$

and, by Hölder's and Sobolev's inequalities,

$$-\frac{\varepsilon_m}{2} \geq \frac{1}{N}\left(1-\frac{\lambda}{\lambda_1}\right)\|\nabla u_n\|^2 - \left(\frac{1}{q}-\frac{1}{2^*}\right)\|k(x)\|_{q_0}S^{-q/2}\|\nabla u_n\|^q.$$

Letting ε_m assume any positive value, and using lower semicontinuity of norms, it follows that $u_n \rightharpoonup u^m$ with

$$\|\nabla u^m\| \leq \left(\frac{N(1/q-1/2^*)S^{-q/2}\|k(x)\|_{q_0}}{1-\lambda/\lambda_1}\right)^{1/(2-q)}. \tag{19}$$

Substituting the estimate (19) into expression (18), we have, for positive constants A and B (independent of $\|k\|_{q_0}$) that

$$\frac{1}{N}\left(\frac{S}{\|h^-(x)\|_\infty^{2/2^*}}\right)^{N/2} \leq A\|k(x)\|_{q_0}^{2^*/(2-q)} + B\|k(x)\|_{q_0}^{2/(2-q)} \tag{20}$$

Clearly, if $\|k(x)\|_{q_0}$ is sufficiently small, then (20) cannot possibly hold, and from this contradiction we infer that (17) is impossible, $b = 0$ and concentration cannot occur. From this, strong convergence to a solution is verified. The generated $(\mathrm{PS})^*_{-\varepsilon_m}$-sequences provide an infinite number of solutions as $\varepsilon_m \to 0$. □

Remark 4.7. We see that as $\|h^-\|_\infty$ becomes larger, the above theorem requires that $\|k(x)\|_{q_0}$ shrink in order to maintain the convergence of $(\mathrm{PS})^*_c$-sequences.

Remark 4.8. It would be interesting to determine if Theorem 1.3 can be extended to the critical case.

Acknowledgement. The author would like to express appreciation to his supervisor Jan Chabrowski for assistance in the preparation of this paper. Funding was provided by the Australian Research Council through the APA program.

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Manuscript received October 29, 1998

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