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SOME EXISTENCE RESULTS FOR DYNAMICAL SYSTEMS ON NON-COMPLETE RIEMANNIAN MANIFOLDS

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Abstract. Let $\mathcal{M}^*$ be a non-complete Riemannian manifold with bounded topological boundary and $V : \mathcal{M} \to \mathbb{R}$ a C^2 potential function subquadratic at infinity.

In this paper we look for curves $x : [0,T] \to \mathcal{M}$ having prescribed period T or joining two fixed points of $\mathcal{M}$, satisfying the system

$$D_t(\dot{x}(t)) = -\nabla_R V(x(t)),$$

where $D_t(\dot{x}(t))$ is the covariant derivative of $\dot{x}$ along the direction of $\dot{x}$ and $\nabla_R V$ the Riemannian gradient of V.

We assume that $V(x) \to -\infty$ if $d(x, \partial\mathcal{M}) \to 0$ and, in the periodic case, suitable hypotheses on the sectional curvature of $\mathcal{M}$ at infinity.

We use variational methods in addition with a penalization technique and Morse index estimates.

1. Introduction and main results

Let $(\mathcal{M}^*, \langle\,\cdot\,,\,\cdot\,\rangle_R)$ be a finite dimensional Riemannian manifold and consider $\mathcal{M} \subseteq \mathcal{M}^*$, an open unbounded connected subset such that $(\mathcal{M}, \langle\,\cdot\,,\,\cdot\,\rangle_R)$ is a Riemannian manifold with bounded topological boundary $\partial\mathcal{M}$, and $V: \mathcal{M} \to \mathbb{R}$ a C^2 potential function.

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In this paper we want to look for curves $x : [0, T] \to \mathcal{M}$ having prescribed period T or joining two fixed points of $\mathcal{M}$, satisfying the system

$$D_t(\dot{x}(t)) = -\nabla_R V(x(t)), \tag{1.1}$$

where $D_t(\dot{x}(t))$ is the covariant derivative of $\dot{x}$ along the direction of $\dot{x}$ and $\nabla_R V$ the Riemannian gradient of V.

Those problems have been studied when V is subquadratic at infinity and $\mathcal{M}$ is a complete manifold, assuming, if $\mathcal{M}$ is non-compact, the existence of a function convex at infinity on $\mathcal{M}$, (see [10]) or suitable hypothesis on the sectional curvature at infinity (see [8], [9]).

Moreover, if $\mathcal{M}$ is non-complete, existence results of problem (1.1) have been obtained assuming that $\mathcal{M}$ has a convex boudary and V is bounded (see [1], [2], [7], [11], [17]).

In this paper we consider a potential V subquadratic at infinity and a non-complete manifold; the convexity assumptions on the boundary are replaced by suitable behaviour assumptions of the potential V nearby $\partial\mathcal{M}$. Moreover in the study of the periodic orbits we will need suitable assumptions on the sectional curvature of $\mathcal{M}$ at infinity.

Difficulties arise from the non-completeness of $\mathcal{M}$, and we will overcome them using a penalization technique, in addition to Morse index estimates.

We introduce now some notations and state the main theorems of the paper.

If $\mathcal{M}$ is a Riemannian manifold, denote $\Lambda(\mathcal{M})$ the free loop space on $\mathcal{M}$ and $K(x)$ $(x \in \mathcal{M})$ the supremum of the sectional curvature i.e.

$$K(x) = \sup\{K_\pi \mid \pi \subset T_x\mathcal{M}\},$$

where $T_x\mathcal{M}$ is the tangent space of $\mathcal{M}$ at x and K_π its sectional curvature with respect to the plane $\pi \subset T_x\mathcal{M}$.

Moreover let $d(\,\cdot\,,\,\cdot\,)$ denote the distance induced by the Riemannian structure of $\mathcal{M}$ and $H_d(x)$ the Hessian of the function $d(\,\cdot\,, \partial\mathcal{M})$ at x. Analogously if $f \in C^2(\mathcal{M}, \mathbb{R})$, $H_f(x)$ will denote the Hessian of the function f at x (see [12]).

The main theorems we prove are the following:

Theorem 1.1. *Let $(\mathcal{M}, \langle\,\cdot\,,\,\cdot\,\rangle_R)$ be a C^∞ connected, unbounded, finite dimensional Riemannian manifold having smooth topological bounded boundary, x_0 a fixed point of $\mathcal{M}$. Suppose that:*

(V_1) (i) $\displaystyle\lim_{d(x,x_0)\to+\infty} V(x) = +\infty,$

(ii) $\displaystyle\lim_{d(x,\partial\mathcal{M})\to 0} V(x) = -\infty,$

(V_2) (i) $\displaystyle\liminf_{d(x,x_0)\to+\infty} \sup_{\substack{v\neq 0\\ v\in T_x\mathcal{M}}} \frac{H_V(x)[v,v]}{\langle v,v\rangle_R} > 0,$

(ii) $\limsup_{d(x,x_0)\to+\infty} \sup_{\substack{v\neq 0\\ v\in T_x\mathcal{M}}} \dfrac{H_V(x)[v,v]}{\langle v,v\rangle_R} < +\infty$,

(iii) $\limsup_{d(x,\partial\mathcal{M})\to 0} \sup_{\substack{v\neq 0\\ v\in T_x\mathcal{M}}} \dfrac{H_V(x)[v,v]}{\langle v,v\rangle_R} < 0$,

(M_1) $\limsup_{d(x,x_0)\to+\infty} K(x) \leq 0$,

(M_2) *infinitely many integers $q \in \mathbb{N}$ exist, such that*

$$H_q(\Lambda(\mathcal{M}), \mathbf{K}) \neq 0 \tag{1.2}$$

$H_q(\,\cdot\,, \mathbf{K})$ being the q-th group of singular homology with coefficients in a field $\mathbf{K}$.

Moreover, suppose that $\delta > 0$ exists such that, for any $x \in \mathcal{M}$, with $d(x, \partial\mathcal{M}) < \delta$, it results:

(D_1) $\langle \nabla d(x, \partial\mathcal{M}), \nabla V(x)\rangle > 0$,

(D_2) $H_d(x)[v, v] \leq 0$ *for any* $v \in T_x\mathcal{M}$.

Then $T^* > 0$ exists such that, for any prescribed $T \in\,]0, T^*[$, at least one T-periodic non-constant solution of problem (1.1) *exists in $\mathcal{M}$.*

Remark 1.2. Hypothesis (V_2) implies that V is subquadratic at infinity. Indeed, it is possible to show that, if (ii) of (V_1) and (ii) of (V_2) hold and

$$\nu = \limsup_{d(x,x_0)\to+\infty} \sup_{\substack{v\neq 0\\ v\in T_x\mathcal{M}}} \frac{H_V(x)[v,v]}{\langle v,v\rangle_R}$$

then there exist two real constants c_1 and c_2 such that, for any $x \in \mathcal{M}$,

$$V(x) \leq \frac{\nu}{2} d^2(x, x_0) + c_1 d(x, x_0) + c_2 \tag{1.3}$$

(see Lemma 2.2 of [5]).

Remark 1.3. From Theorem 1.1 it follows that problem (1.1) admits periodic solutions for any prescribed period $T > 0$. Indeed, if $T > 0$, $p \in \mathbb{N}$ exists such that $T/p \in\,]0, T^*[$ and so the existing solution of period T/p has also period T.

Theorem 1.4. *Let $(\mathcal{M}, \langle\cdot,\cdot\rangle_R)$ be a C^∞ connected, unbounded, finite dimensional Riemannian manifold having bounded boundary, x_0 and x_1 two fixed points of $\mathcal{M}$. Assume that* (V_1)(i) *and* (V_2)(ii), (D_1), (D_2) *and* (M_2) *hold. Then there exist $T > 0$ and at least one solution $x : [0, T] \to \mathcal{M}$ of* (1.1) *such that $x(0) = x_0$ and $x(T) = x_1$.*

2. Preliminaries and functional framework

Let us introduce now some preliminary notations which will be used in the following sections.

Let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^N$ and $\langle\cdot,\cdot\rangle$ its usual inner product. Moreover, denote $S^1 = \mathbb{R}/T\mathbb{Z}$ and $H^1 = H^1([0,T],\mathbb{R}^N)$ the following Sobolev space

$$H^1 = \left\{ x : [0,T] \to \mathbb{R}^N \;\middle|\; \text{absolutely continuous}, \int_0^T \langle \dot{x}, \dot{x}\rangle \, dt < +\infty \right\}$$

endowed with its usual norm.

Let $(\mathcal{M}, \langle\cdot,\cdot\rangle_R)$ be a finite dimensional Riemannian manifold; the Nash embedding theorem (see [13]) assures that N large enough exists such that $\mathcal{M}$ can be isometrically embeddable in $\mathbb{R}^N$.

Thus, from now on we will identify $\mathcal{M}$ with a submanifold of the Euclidean space $\mathbb{R}^N$ and the Riemannian product $\langle\cdot,\cdot\rangle_R$ will be simply denoted $\langle\cdot,\cdot\rangle$.

Moreover, denote

$$\Lambda^1 = \Lambda^1(\mathcal{M}) = \{x \in H^1(S^1,\mathbb{R}^N) \mid x(t) \in \mathcal{M},\ x(0) = x(T)\}$$

and

$$\Omega^1 = \Omega^1(\mathcal{M}, x_0, x_1) = \{x \in H^1 \mid x(t) \in \mathcal{M},\ x(0) = x_0,\ x(T) = x_1\}.$$

It is known that Λ^1 and Ω^1 are Hilbert manifolds (see [11], [15], [17]) and their tangent spaces are

$$T_x\Lambda^1 = \{\xi \in H^1(S^1,\mathbb{R}^N) \mid \xi(t) \in T_{x(t)}\mathcal{M} \text{ for any } t \in [0,T]\}, \quad \text{if } x \in \Lambda^1,$$

equipped with the Riemannian product:

$$\langle \xi, \eta\rangle_1 = \int_0^T \langle D_t\xi, D_t\eta\rangle \, dt + \langle \xi(0), \eta(0)\rangle \quad \text{for } x \in \Lambda^1,\ \xi,\eta \in T_x\Lambda^1 \tag{2.1}$$

and

$$T_x\Omega^1 = \{\xi \in H^1 \mid \xi(0) = \xi(T) \text{ and } \xi(t) \in T_{x(t)}\mathcal{M} \text{ for any } t \in [0,T]\}, \quad \text{if } x \in \Omega^1,$$

equipped with the Riemannian product

$$\langle \xi, \eta\rangle_1 = \int_0^T \langle D_t\xi, D_t\eta\rangle \, dt \quad x \in \Omega^1,\ \xi,\eta \in T_x\Omega^1. \tag{2.2}$$

Both $T_x\Lambda^1$ and $T_x\Omega^1$ have a Riemannian structure.

Moreover, let us recall the Palais–Smale condition for a functional on a manifold.

Definition 2.1. Let $\mathcal{N}$ be a Riemannian manifold and $f : \mathcal{N} \to \mathbb{R}$ a C^1 functional and $b \in \mathbb{R}$; the functional f is said to satisfy the Palais–Smale condition in $f^b = \{x \in \mathcal{N} \mid f(x) \le b\}$, briefly (PS), if and only if any sequence $\{x_n\}$ in $\mathcal{N}$ such that

$$f(x_n) \le b \quad \text{for any } n \in \mathbb{N}$$

and

$$f'(x_n) \to 0 \quad \text{as } n \to +\infty,$$

admits a convergent subsequence in $\mathcal{N}$.

Definition 2.2. Let $\mathcal{N}$ be a Riemannian manifold, $f \in C^2(\mathcal{N}, \mathbb{R})$ and let $x \in \mathcal{N}$ be a critical point of f. The strict Morse index of x (possibly $+\infty$) is the dimension of the maximal subspace of $T_x\mathcal{N}$ where $H_f(x)$ is negative definite and will be denoted $m(x)$.

The large Morse index of x (possibly $+\infty$) is the dimension of the maximal subspace of $T_x\mathcal{N}$ where $H_f(x)$ is negative semidefinite and will be denoted $m^*(x)$. If $m^*(x) = m(x)$, x is said to be a non-degenerate critical point.

We recall now an abstract theorem on the Morse index that is a variant of some known theorems (see [8]) and will be used in the proofs of our results.

Theorem 2.3. *Let $\mathcal{N}$ be a complete Riemannian manifold of class C^2 and $f \in C^2(\mathcal{N}, \mathbb{R})$. Suppose that:*

(i) *for any critical point x of f, if 0 is an eigenvalue of $H_f(x)$, both it is isolated and it has finite multiplicity,*
(ii) *f satisfies the* (PS) *condition on f^b, for any $b \in \mathbb{R}$,*
(iii) $\inf_{\mathcal{N}} f > -\infty$,
(iv) *$q \geq 0$ is an integer such that $H_q(\mathcal{N}, \mathbf{K}) \neq 0$.*

Denote $\Gamma_q = \{A \subseteq \mathcal{N} \mid i_*(H_q(A, \mathbf{K})) \neq 0\}$, where $i : A \to \mathcal{N}$ is the inclusion map. Then there exists a critical point x^* of f corresponding to the critical value

$$c = \inf_{A \in \Gamma_q} \sup_{x \in A} f(x) \tag{2.3}$$

and satisfying

$$m(x^*) \leq q \leq m^*(x^*). \tag{2.4}$$

It is well known that the search of periodic solutions of problem (1.1) with prescribed period T or joining two fixed points of $\mathcal{M}$ can be reduced to the search of the critical points of the action functional

$$f(x) = \frac{1}{2} \int_0^T \langle \dot{x}, \dot{x} \rangle \, ds - \int_0^T V(x) \, ds \tag{2.5}$$

defined in Λ^1 (respectively, in Ω^1).

Let x be a critical point of the functional f; then the Hessian of f at x is:

$$\begin{aligned} H_f(x)[v,v] = & \int_0^T \langle D_s v, D_s v \rangle \, ds - \int_0^T \langle R_{\dot{x}v} \dot{x}, v \rangle \, ds \\ & - \int_0^T H_V(x)[v,v] \, ds \quad \text{for any } v \in T_x\Lambda^1 \text{ (resp. } v \in T_x\Omega^1) \end{aligned}$$

where $R_{\dot{x}v}$ denotes the Riemannian curvature tensor of $\mathcal{M}$ at $(\dot{x}, v)$ whose properties are: if $\dot{x}(s)$ and $v(s)$ are not linearly independent then $R_{\dot{x}(s)v(s)}\dot{x}(s) = 0$ otherwise

$$\langle R_{\dot{x}v}\dot{x}, v\rangle = K_\pi(\langle \dot{x}, \dot{x}\rangle\langle v, v\rangle - \langle \dot{x}, v\rangle^2) \tag{2.6}$$

where π is the plane generated by $\dot{x}$ and v (see [14]).

3. Periodic case

For any $\varepsilon > 0$, let $\psi_\varepsilon \in C^2(\mathbb{R}^+, \mathbb{R}^+)$ be such that $\psi_\varepsilon' \geq 0$ and

$$\psi_\varepsilon(t) = \begin{cases} 0 & \text{if } t \leq 1/2\varepsilon, \\ e^{2t-1/\varepsilon} - 1 & \text{if } t > 1/\varepsilon. \end{cases}$$

Denote, for any $\varepsilon > 0$,

$$U_\varepsilon(x) = \psi_\varepsilon(|V(x)|) \quad \text{for any } x \in \mathcal{M} \tag{3.1}$$

and consider the following penalized functional $f_\varepsilon : \Lambda^1 \to \mathbb{R}$ defined

$$f_\varepsilon(x) = f(x) + \int_0^T U_\varepsilon(x(t))\, dt.$$

Lemma 3.1. *Let $\{x_n\} \subset \Lambda^1$ be such that*

$$\left\{ \int_0^T \langle \dot{x}_n, \dot{x}_n\rangle\, dt \right\} \quad \textit{is bounded} \tag{3.2}$$

and let $\{s_n\} \subset [0, T]$ satisfy

$$\lim_n d(x_n(s_n), \partial\mathcal{M}) = 0. \tag{3.3}$$

Then, up to a subsequence, for any $\varepsilon > 0$,

$$\lim_n \int_0^T \psi_\varepsilon(|V(x_n(t))|)\, dt = +\infty. \tag{3.4}$$

Proof. Fix $\varepsilon > 0$. As $\partial\mathcal{M}$ is bounded, from (3.2) and (3.3) it follows that

$$\{x_n(s) \mid n \in \mathbb{N},\ s \in [0, T]\}$$

is bounded. Without loss of generality, we can assume that, for any $n \in \mathbb{N}$,

$$d(x_n(s_n), \partial\mathcal{M}) = \inf_{t\in[0,T]} d(x_n(t), \partial\mathcal{M}).$$

Moreover, denote $\{t_n\} \subseteq [0, T]$ a sequence such that

$$d(x_n(t_n), \partial\mathcal{M}) = \sup_{t\in[0,T]} d(x_n(t), \partial\mathcal{M}).$$

If

$$\liminf_n d(x_n(t_n), \partial\mathcal{M}) = 0, \tag{3.5}$$

then (3.4) can be easily proved. Indeed, in this case, up to a subsequence,

$$\lim_n \sup_{t\in[0,T]} d(x_n(t),\partial\mathcal{M}) = 0$$

and then, from (ii) of (V_1), it follows that for any $\sigma > 1/\varepsilon > 0$, $n^* \in \mathbb{N}$ exists such that, for any $n \in \mathbb{N},\ n > n^*$,

$$|V(x_n(t))| > \sigma > 1/\varepsilon \quad \text{for any } t \in [0,T]$$

and thus

$$\psi_\varepsilon(|V(x_n(t))|) = e^{2|V(x_n(t))|-1/\varepsilon} - 1 \quad \text{for any } t \in [0,T].$$

It follows that,

$$\int_0^T \psi_\varepsilon(|V(x_n(t))|)\,dt = \int_0^T e^{2|V(x_n(t))|-1/\varepsilon}\,dt - T \geq T(e^\sigma - 1).$$

Then

$$\lim_n \int_0^T \psi_\varepsilon(|V(x_n(t))|)\,dt = +\infty.$$

Let us consider now the case when

$$\liminf_n d(x_n(t_n),\partial\mathcal{M}) > 0.$$

Up to a subsequence, we can suppose that

$$\lim_n d(x_n(t_n),\partial\mathcal{M}) > 0.$$

We can choose $\eta > 0$ such that

$$e^{-|V(x_n(t_n))|} > \eta \quad \text{for any } n \in \mathbb{N}. \tag{3.6}$$

From (3.3) and (ii) of (V_1), it follows that

$$e^{-|V(x_n(s_n))|} \to 0,$$

and then, if n is large enough

$$e^{-|V(x_n(s_n))|} < \eta/2. \tag{3.7}$$

From (3.6) and (3.7) it follows that

$$e^{-|V(x_n(t_n))|} - e^{-|V(x_n(s_n))|} > \eta - \eta/2 = \eta/2 > 0. \tag{3.8}$$

In order to state (3.4), we need further evaluations. Fix $s > s_n$ (similar arguments hold if $s < s_n$), then

$$
\begin{aligned}
(3.9)\qquad e^{-|V(x_n(s))|} - e^{-|V(x_n(s_n))|} &= \int_{s_n}^{s} \langle \nabla(e^{-|V(x_n(\tau))|}), \dot{x}_n(\tau)\rangle \, d\tau \\
&\le \int_{s_n}^{s} ||\nabla(e^{-|V(x_n(\tau))|})|| \, ||\dot{x}_n(\tau)|| \, d\tau \\
&\le c_1 (s-s_n)^{1/2} \bigg(\int_0^T ||\dot{x}_n(\tau)||^2 \, d\tau \bigg)^{1/2} \\
&\le c_2 \sqrt{s-s_n}.
\end{aligned}
$$

From (3.9), it follows that, for any $s > s_n$,

$$
e^{-|V(x_n(s))|} \le c_2 \sqrt{s-s_n} + e^{-|V(x_n(s_n))|}
$$

and, from the inequality $(a+b)^2 \le 2(a^2+b^2)$, it results that, if $t_n > s_n$,

$$
\begin{aligned}
(3.10)\qquad \int_{s_n}^{t_n} & \frac{ds}{(c_2\sqrt{s-s_n} + e^{-|V(x_n(s_n))|})^2} \\
&\ge \int_{s_n}^{t_n} \frac{ds}{2(c_2^2(s-s_n) + e^{-2|V(x_n(s_n))|})} \\
&= c_3 \ln(1 + c_2^2(t_n - s_n) e^{2|V(x_n(s_n))|}).
\end{aligned}
$$

From (3.8) and (3.9), it follows that $\lim_n (t_n - s_n) > 0$ and then, from (3.10),

$$
(3.11)\qquad \lim_n \ln(1 + c_2^2(t_n - s_n) e^{2|V(x_n(s_n))|}) = +\infty.
$$

If it happens that, for an infinite number of integer:

$$
|V(x_n(t))| > 1/\varepsilon \quad \text{for any } t \in [s_n, T]
$$

then, for any $t \in [s_n, T]$,

$$
\psi_\varepsilon(|V(x_n(t))|) = e^{2|V(x_n(t))| - 1/\varepsilon} - 1,
$$

and then, if $t_n > s_n$,

$$
\begin{aligned}
\int_{s_n}^{t_n} \frac{ds}{(c_2\sqrt{s-s_n} + e^{-|V(x_n(s_n))|})^2} &\le \int_{s_n}^{t_n} \frac{ds}{e^{-2|V(x_n(s))|}} \\
&= e^{1/\varepsilon} \int_{s_n}^{t_n} \psi_\varepsilon(|V(x_n(s))|) \, ds + e^{1/\varepsilon} T
\end{aligned}
$$

and thus, from (3.10) and (3.11),

$$
\lim_n \int_0^T \psi_\varepsilon(|V(x_n(t))|) \, dt = +\infty.
$$

If, for infinite integers n, $\tau_n \in [s_n, T]$ exists such that

$$
|V(x_n(\tau_n))| \le 1/\varepsilon,
$$

denote

$$t_n^* = \inf\{t \in]s_n, T] \mid |V(x_n(t))| = 1/\varepsilon\}$$

and

$$s_n^* = \sup\{t \in]s_n, t_n^*[\mid |V(x_n(t))| = 2/\varepsilon\}.$$

Remark that, if $t \in]s_n^*, t_n^*[$, $1/\varepsilon \leq |V(x_n(t))| \leq 2/\varepsilon$.

Up to subsequences, we can suppose that

$$s_n \to s_0, \quad s_n^* \to s_0^*, \quad t_n^* \to t_0^*,$$

where s_0, s_0^* and t_0^* are distinct.

Let $\rho^* > 0$ be such that $[s_0^* - \rho^*, s_0^* + \rho^*] \cap [t_0^* - \rho^*, t_0^* + \rho^*] = \emptyset$ and take $u_n \in [s_0^* + \rho^*, t_0^* - \rho^*] \cap]s_n^*, t_n^*[$. We can assume that

$$2/\varepsilon \geq \lim_{n\to\infty} |V(x_n(u_n))| \geq 1/\varepsilon.$$

That implies that a constant $c^* \in \mathbb{R}$ exists such that, up to a subsequence,

$$e^{-|V(x_n(u_n))|} \geq c^*, \quad \text{for any } n \in \mathbb{N}.$$

As

$$\lim_n e^{-|V(x_n(s_n))|} = 0$$

it follows that, if n is large enough,

$$e^{-|V(x_n(s_n))|} < c^*/2$$

and then

$$e^{-|V(x_n(u_n))|} - e^{-|V(x_n(s_n))|} > c^* - c^*/2 > 0.$$

Reasoning as in the previous case with u_n instead of t_n, we can obtain (3.9) and (3.11). Moreover, if $t \in [s_n, u_n]$, then

$$|V(x_n(t))| \geq 1/\varepsilon$$

and thus

$$\psi_\varepsilon(|V(x_n(t))|) = e^{2|V(x_n(t))|-1/\varepsilon} - 1.$$

Reasoning in the same way, the claim follows. □

Lemma 3.2. *For any $b \in \mathbb{R}$ and for any $\varepsilon \in \mathbb{R}^+$, f_ε satisfies the* (PS) *condition in f_ε^b.*

Proof. Let $b > 0$ and $\{x_n\} \subset \Lambda^1$be such that

$$f_\varepsilon(x_n) \leq b \quad \text{for any } n \in \mathbb{N}, \tag{3.12}$$

$$df_\varepsilon(x_n) \to 0 \quad \text{if } n \to +\infty, \tag{3.13}$$

and let us prove that a convergent subsequence exists.

Indeed, if $I = \{t \in [0,T] \mid V(x(t)) \geq 0\}$, reasoning as in Lemma 3.6 of [9], it is possible to show that

$$\left\{ -\int_I V(x_n)\,ds + \int_I \psi_\varepsilon(|V(x_n)|)\,ds \right\}$$

is bounded from below and then

$$\left\{ -\int_0^T V(x_n)\,ds + \int_0^T \psi_\varepsilon(|V(x_n)|)\,ds \right\} \tag{3.14}$$

is bounded from below in Λ^1. From (3.12) and (3.14) it follows that:

$$\left\{ \int_0^T \langle \dot{x}_n, \dot{x}_n \rangle\,ds \right\} \quad \text{is bounded,} \tag{3.15}$$

and then $\big\{ \sup_{s\in[0,T]} d(x_0, x_n(s)) \big\}$ is bounded too. In fact, if

$$\sup_{s\in[0,T]} d((x_0, x_n(s)) \to +\infty$$

from (3.15) it follows that

$$\inf_{s\in[0,T]} d((x_0, x_n(s)) \to +\infty$$

and that contradicts (3.12). Then a subsequence exists, such that

$$x_n \rightharpoonup x \text{ weakly in } H^1 \text{ and strongly in } L^\infty.$$

Moreover, $\overline{\delta} > 0$ exists, such that

$$\{x_n\} \subset \Lambda^1(A_{\overline{\delta}}) = \{x \in \Lambda^1 \mid d(x(t), \partial\mathcal{M}) \geq \overline{\delta} \text{ for any } t \in [0,T]\}.$$

Indeed, if $\{s_n\} \subset [0,T]$ exists such that $\lim_n d(x_n(s_n), \partial\mathcal{M}) = 0$, then, from (3.15) and from Lemma 3.1, it follows that

$$\int_0^T \psi_\varepsilon(|V(x_n(s))|)ds \to +\infty,$$

which contradicts (3.12).

As $\Lambda^1(A_{\overline{\delta}})$ is a complete space, arguing as in Lemma 3.2 of [6], it can be proved that $\{x_n\}$ strongly converges to $x \in \Lambda^1(A_{\overline{\delta}}) \subseteq \Lambda^1$ in H^1 (see also Lemma 3.2 of [7] and Theorem 1.1 of [8]). □

Theorem 3.3. *Let $q \in \mathbb{N}$ be such that* (1.2) *holds. Then, for any $\varepsilon > 0$, f_ε has a critical point x_ε in Λ^1, corresponding to the critical value*

$$c_\varepsilon = \inf_{A\subset\Gamma_q} \sup_{x\in A} f_\varepsilon(x) \tag{3.16}$$

and such that

$$m(x_\varepsilon) \leq q \leq m^*(x_\varepsilon). \tag{3.17}$$

Proof. Reasoning as in Lemma 3.1 of [9] it is possible to show that, for any $\varepsilon > 0$, f_ε satisfies condition (i) of Theorem 2.3. Moreover, from Lemma 3.2 and from (3.14), it follows that (ii) and (iii) of Theorem 2.3 hold. Furthermore, as the inclusion of Λ^1 in $\Lambda(\mathcal{M})$ is a homotopy equivalence, the nontrivial homology groups of Λ^1 and $\Lambda(\mathcal{M})$ with respect to a field are the same (see [15], [16]). Thus also (iv) holds. All the hypotheses of Theorem 2.3 are satisfied, thus (3.16) and (3.17) follow. □

Lemma 3.4. *An $\varepsilon_0 > 0$ and $Q \in \mathbb{N}$ exist, such that for any $\varepsilon \in]0, \varepsilon_0[$ and, for any x_ε critical point of f_ε satisfying* (3.16) *and* (3.17)*, the following relation hold*

$$x_\varepsilon \text{ is constant} \Rightarrow m^*(x_\varepsilon) \leq Q.$$

Proof. Let $\varepsilon > 0$ and x_ε a constant critical point of f_ε, then the tangent space $T_{x_\varepsilon}\Lambda^1$ is given by

$$T_{x_\varepsilon}\Lambda^1 = \left\{ \xi \in H^1(S^1, \mathbb{R}^N) \,\middle|\, \exists v = (v_1, \dots, v_n) \in H^1(S^1, \mathbb{R}^N) \text{ s.t. } \xi = \sum_{i=1}^n v_i e_i \right\},$$

$\{e_1, \dots, e_n\}$ being an orthonormal basis of $T_{x_\varepsilon}\mathcal{M}$ and $n = \dim \mathcal{M}$.

From the definition of covariant derivative along a curve, it follows that, for any $\xi \in T_{x_\varepsilon}\Lambda^1$

$$D_s\xi(s) = \sum_{1=i}^n \dot{v}_i(s) e_i.$$

It means that

$$T_{x_\varepsilon}\Lambda^1 \equiv H^1(S^1, T_{x_\varepsilon}\mathcal{M})$$

that is, the covariant derivative is equal to the usual derivative and $T_{x_\varepsilon}\Lambda^1$ is isometric to $H^1(S^1, \mathbb{R}^n)$. Then, the Hessian of f_ε at x_ε reduces to

$$H_{f_\varepsilon}(x_\varepsilon)[v, v] = \int_0^T \langle \dot{v}, \dot{v} \rangle \, ds - \int_0^T H_V(x_\varepsilon)[v, v] \, ds + \int_0^T H_{U_\varepsilon}(x_\varepsilon)[v, v] \, ds, \tag{3.18}$$

for any $v \in T_{x_\varepsilon}\Lambda^1$. Let us consider the following decomposition of $H^1(S^1, \mathbb{R}^n)$ with respect to the metric (2.1):

$$H^1(S^1, \mathbb{R}^n) = \mathbb{R}^n \oplus H_0^1(S^1, \mathbb{R}^n),$$

where $\mathbb{R}^n$ is identified with the constant loop space and

$$H_0^1(S^1, \mathbb{R}^n) = \{v \in H^1(S^1, \mathbb{R}^n) \mid v(0) = v(T) = 0\}.$$

It is well known that the self-adjoint realization in $L^2([0, T], \mathbb{R}^n)$

$$v \to -\ddot{v}$$

with T-periodic boundary conditions has a sequence $(\lambda_k)_{k \in \mathbb{N}}$ of eigenvalues, each one to be counted with its multiplicity.

Denote ζ_k the eigenvector relative to λ_k and let H_r be the space spanned by $\{\zeta_1, \dots, \zeta_r\}$, then

$$H_0^1(S^1, \mathbb{R}^n) = H_r \oplus H_r^\perp.$$

From (i) and (ii) of (V_2), it follows that $K > 0$ and $\nu > 0$ exist, such that

$$0 < \sup_{\substack{v \neq 0 \\ v \in T_x\mathcal{M}}} \frac{H_V(x)[v,v]}{\langle v, v\rangle} < \nu \quad \text{for any } x \in \mathcal{M},\ d(x, x_0) > K, \tag{3.19}$$

and, from (ii) of (V_1) and from (iii) of (V_2) it follows that $\delta' < \delta$ exists, such that, for any $x \in \mathcal{M}$, $d(x, \partial\mathcal{M}) < \delta'$,

$$V(x) < 0 \quad \text{and} \quad \sup_{\substack{v \neq 0 \\ v \in T_x\mathcal{M}}} \frac{H_V(x)[v,v]}{\langle v, v\rangle} < 0. \tag{3.20}$$

Denote

$$I_1 = \{s \in [0,T] \mid d(x_\varepsilon(s), x_0) > K\},$$
$$I_2 = \{s \in [0,T] \mid d(x_\varepsilon(s), \partial\mathcal{M}) < \delta'\},$$

and $I_3 = [0,T] - (I_1 \cup I_2)$, then

$$\int_{I_1} H_{U_\varepsilon}(x_\varepsilon)[v,v]\, ds \geq 0 \tag{3.21}$$

and, from (3.19),

$$-\int_{I_1} H_V(x_\varepsilon)[v,v]\, ds \geq -\nu \int_{I_1} \langle v, v\rangle\, ds \tag{3.22}$$

and, from (3.20),

$$\int_{I_2} (-H_V(x_\varepsilon)[v,v] + H_{U_\varepsilon}(x_\varepsilon)[v,v])\, ds \geq 0. \tag{3.23}$$

The potential V is bounded on the set

$$D = \{x \in \mathcal{M} \mid d(x, x_0) \leq K,\ d(x, \partial\mathcal{M}) \geq \delta'\}$$

therefore $\varepsilon_0 > 0$ exists, such that

$$|V(x)| < \frac{1}{2\varepsilon_0} \quad \text{for any } x \in D$$

and so, for any $\varepsilon \in \,]0, \varepsilon_0[$,

$$\int_{I_3} H_{U_\varepsilon}(x_\varepsilon)[v,v]\, ds = 0. \tag{3.24}$$

Denote

$$\sup_{x \in D} \sup_{\substack{v \neq 0 \\ v \in T_x\mathcal{M}}} \left| \frac{H_V(x)[v,v]}{\langle v, v\rangle} \right| = K_D,$$

then

$$\left| \frac{H_V(x_\varepsilon(s))[v(s), v(s)]}{\langle v(s), v(s)\rangle} \right| \leq K_D \quad \text{for any } s \in I_3, \text{ and any } \varepsilon \in]0, \varepsilon_0[.$$

If we choose $r \in \mathbb{N}$ such that $\lambda_r > \lambda = \max\{\nu, K_D\}$, from (3.18), (3.21), (3.22), (3.23) and (3.24), it results that

$$\begin{aligned} H_{f_\varepsilon}(x_\varepsilon)[v, v] &\geq \int_0^T \langle \dot v, \dot v\rangle\, ds - \nu \int_{I_1} \langle v, v\rangle\, ds - \int_{I_3} H_V(x_\varepsilon)[v, v]\, ds \\ &\geq \lambda_r \int_0^T \langle v, v\rangle\, ds - \lambda \int_{I_1 \cup I_3} \langle v, v\rangle\, ds \\ &\geq (\lambda_r - \lambda) \int_{I_1 \cup I_3} \langle v, v\rangle\, ds > 0 \quad \text{for any } v \in H_r^\perp. \end{aligned}$$

Then it follows that

$$m^*(x_\varepsilon) \leq \dim H_r + \dim \mathcal{M} = Q. \tag{3.25}$$

□

Proof of Theorem 1.1. Fix $T > 0$ and let $q > 2\dim \mathcal{M}$ be such that $q > Q$ and (1.2) hold. If x_ε is a critical point of f_ε satisfying (3.16) and (3.17) and $\varepsilon < \varepsilon_0$, ε_0 being the one defined in Lemma 3.4; then x_ε is a non-costant solution. In order to prove the theorem it is enough to show that $\varepsilon_1 > 0$, $\varepsilon_1 < \varepsilon_0$ and $M > 0$ exist, such that, for any $\varepsilon \in]0, \varepsilon_1[$ and for any $s \in [0, T]$,

$$d(x_\varepsilon(s), x_0) \leq M, \tag{3.26}$$

$$d(x_\varepsilon(s), \partial\mathcal{M}) \geq \delta', \tag{3.27}$$

where δ' is such that (3.20) holds. Indeed, if (3.26) and (3.27) hold, the potential V is bounded on the set

$$\{x_\varepsilon(s) \in \mathcal{M} |\ s \in [0, T],\ \varepsilon \in]0, \varepsilon_1[\},$$

then we can choose $\varepsilon^* \in]0, \varepsilon_1[$ small enough, such that

$$|V(x_{\varepsilon^*}(s))| \leq K < \frac{1}{2\varepsilon^*} \quad \text{for any } s \in [0, T].$$

Then

$$\psi_{\varepsilon^*}(|V(x_{\varepsilon^*}(s))|) = 0 \quad \text{for any } s \in [0, T]$$

which implies that x_{ε^*} is a critical point of f.

Let us prove (3.26). Argue by contradiction and suppose there exist $\varepsilon_n \to 0^+$ and a sequence $\{x_n\}$ of critical points of $f_n = f_{\varepsilon_n}$ such that (3.16) and (3.17) hold and, moreover,

$$\sup_{s \in [0,T]} d(x_n(s), x_0) \to +\infty \quad \text{as } n \to +\infty. \tag{3.28}$$

As the singular homology has compact support, β independent of n exists, such that

$$f_n(x_n) \le \beta \quad \text{for any } n \in \mathbb{N},$$

and then

$$f(x_n) \le \beta \quad \text{for any } n \in \mathbb{N}. \tag{3.29}$$

We want to show that

$$\inf_{s\in[0,T]} d(x_n(s), x_0) \to +\infty \quad \text{as } n \to +\infty.$$

Indeed, if

$$\left\{ \int_0^T \langle \dot{x}_n, \dot{x}_n \rangle \, ds \right\}$$

is bounded, the claim is obvious because of (3.28); if

$$\int_0^T \langle \dot{x}_n, \dot{x}_n \rangle \, ds \to +\infty$$

and

$$\left\{ \inf_{s\in[0,T]} d(x_n(s), x_0) \right\} \quad \text{is bounded}$$

then

$$\left(\int_0^T \langle \dot{x}_n, \dot{x}_n \rangle \, ds \right)^{1/2} \ge \int_0^T \langle \dot{x}_n, \dot{x}_n \rangle^{1/2} \, ds \ge \sup_{s\in[0,T]} d(x_n(s), x_0).$$

Then, from (1.3) and (3.29) it follows that

$$\begin{aligned} \int_0^T \langle \dot{x}_n, \dot{x}_n \rangle \, ds &\le \beta + \int_0^T V(x_n) \, ds \le \beta' + \frac{\nu}{2} \int_0^T d^2(x_n, x_0) \, ds \\ &\quad + c_1 \int_0^T d(x_n, x_0) \, ds \\ &\le \beta' + \frac{\nu}{2} T \sup_{s\in[0,T]} d^2(x_n(s), x_0) + c_1 T \sup_{s\in[0,T]} d(x_n(s), x_0) \\ &\le \frac{\nu}{2} T \int_0^T \langle \dot{x}_n, \dot{x}_n \rangle \, ds + c_1 T \left(\int_0^T \langle \dot{x}_n, \dot{x}_n \rangle \, ds \right)^{1/2} + \beta', \end{aligned} \tag{3.30}$$

if n is large enough, and then

$$\left(1 - \frac{\nu}{2} T\right) \int_0^T \langle \dot{x}_n, \dot{x}_n \rangle \, ds - c_1 T \left(\int_0^T \langle \dot{x}_n, \dot{x}_n \rangle \, ds \right)^{1/2} \le \beta'.$$

Suppose $T \le 2/\nu$, then

$$\left\{ \int_0^T \langle \dot{x}_n, \dot{x}_n \rangle \, ds \right\} \quad \text{is bounded}$$

and it is a contradiction, thus

$$\inf_{s\in[0,T]} d(x_n(s),x_0) \to +\infty \quad \text{as } n \to +\infty.$$

As x_n is a critical point of f_n, for any $n \in \mathbb{N}$, $E_n > 0$ exists, such that

$$\frac{1}{2}\langle \dot{x}_n(s), \dot{x}_n(s)\rangle + V(x_n(s)) = E_n.$$

By virtue of (M_1) and (2.6) we can choose a sequence $\delta_n \to 0^+$ such that $\delta_n E_n \to 0^+$ and

$$\frac{\langle R_{vw}v, w\rangle}{\langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle^2} < \delta_n.$$

Then, for any $v \in W^n = \{w \in T_{x_n}\Lambda^1 \mid w(0) = 0\}$,

$$\begin{aligned}
H_{f_n}(x_n)[v,v] &= \int_0^T \langle D_s v, D_s v\rangle\, ds - \int_0^T \langle R_{\dot{x}_n v}\dot{x}_n, v\rangle\, ds \\
&\quad - \int_0^T H_V(x_n)[v,v]\, ds + \int_0^T H_{U_n}(x_n)[v,v]\, ds \\
&\geq \int_0^T \langle D_s v, D_s v\rangle\, ds \\
&\quad - \delta_n \bigg\{ \int_0^T \langle \dot{x}_n, \dot{x}_n\rangle\langle v,v\rangle\, ds - \int_0^T \langle \dot{x}_n, v\rangle^2\, ds \bigg\} \\
&\quad - \int_0^T H_V(x_n)[v,v]\, ds + \int_0^T H_{U_n}(x_n)[v,v]\, ds \\
&\geq \int_0^T \langle D_s v, D_s v\rangle\, ds - \delta_n \int_0^T \langle \dot{x}_n, \dot{x}_n\rangle\langle v,v\rangle\, ds \\
&\quad - \int_0^T H_V(x_n)[v,v]\, ds.
\end{aligned}$$

Denote, for any $n \in \mathbb{N}$,

$$M(x_n) = \sup_{\substack{v\neq 0 \\ v\in T_{x_n}\mathcal{M}}} \frac{H_V(x_n)[v,v]}{\langle v,v\rangle},$$

then, from (ii) of (V_2) and from the inequality

$$\bigg(\int_0^T \langle v,v\rangle\, ds\bigg)^{1/2} \leq 2T\bigg(\int_0^T \langle D_s v, D_s v\rangle\, ds\bigg)^{1/2},$$

it follows that, if n is large enough and $v \in W^n - \{0\}$,

$$\begin{aligned} H_{f_n}(x_n)[v,v] &\geq \int_0^T \langle D_s v, D_s v\rangle \, ds - \delta_n \int_0^T [2E_n - 2V(x_n)]\langle v,v\rangle \, ds \\ &\quad - \int_0^T M(x_n)\langle v,v\rangle \, ds \\ &\geq \int_0^T \left[\frac{1}{4T^2} - 2\delta_n E_n - M(x_n)\right]\langle v,v\rangle \, ds. \end{aligned}$$

Take $T^* > 0$ such that $T^* \leq \min\{2/\nu, 1/2\sqrt{\nu}\}$, then $H_{f_n}(x_n)[v,v] > 0$ and

$$m^*(x_n) \leq \dim(W^n)^\perp \leq 2\dim \mathcal{M},$$

which contradicts $q > 2\dim\mathcal{M}$.

Let us prove (3.27). We argue by contradiction and suppose $\{\varepsilon_n\} \to 0^+$, $\{s_n\} \subset [0,T]$ and $\{x_n\} \subset \Lambda^1$ exist such that x_n is a critical point of $f_n = f_{\varepsilon_n}$, (3.16) and (3.17) hold and, moreover,

$$d(x_n(s_n), \partial\mathcal{M}) = \inf_{s\in[0,T]} d(x_n(s), \partial\mathcal{M}) < \delta'.$$

Denote $v_n(s) = d(x_n(s), \partial\mathcal{M})$. Then, it results that

$$v_n'(s_n) = 0 \quad \text{and} \quad v_n''(s_n) \geq 0. \tag{3.31}$$

Moreover, as x_n is a critical point,

$$D_t\dot{x}_n = -\nabla V(x_n) + \psi_n'(|V(x_n))|)\nabla V(x_n)\mathrm{sign}(V(x_n)) \tag{3.32}$$

and, from (3.31),

$$0 \leq H_d(x_n(s_n))[\dot{x}_n(s_n), \dot{x}_n(s_n)] + \langle \nabla d(x_n(s_n), \partial\mathcal{M}), D_t\dot{x}_n(s_n))\rangle.$$

Then, from (D_1), (D_2) and (3.32)

$$\begin{aligned} 0 &\leq H_d(x_n(s_n))[\dot{x}_n(s_n), \dot{x}_n(s_n)] + \langle \nabla d(x_n(s_n), \partial\mathcal{M}), -\nabla V(x_n(s_n))\rangle \\ &\quad + \langle \nabla d(x_n(s_n), \partial\mathcal{M}), \nabla V(x_n(s_n))\rangle \psi_n'(|V(x_n(s_n))|)\,\mathrm{sign}(V(x_n(s_n))) \\ &\leq (-1 + \psi_n'(|V(x_n(s_n))|)\,\mathrm{sign}(V(x_n(s_n)))\langle \nabla d(x_n(s_n)), \partial\mathcal{M}), \nabla V(x_n(s_n))\rangle \\ &< 0. \end{aligned}$$

That is a contradiction, thus the claim follows. □

4. Case of curves joining two points

As in Section 3, for any $\varepsilon > 0$, we consider the penalized functional $f_\varepsilon : \Omega^1 \to \mathbb{R}$ defined,

$$f_\varepsilon(x) = f(x) + \int_0^T U_\varepsilon(x(t))\, dt.$$

It is possible to show that Lemma 3.1 still holds. The proof is obtained reasoning as in the case of $\Lambda^1$and observing that the inequality

$$e^{-|V(x_n(T))|} = e^{-|V(x_1)|} > 0$$

holds also in a neighbourhood I of T and then $\eta > 0$ and $\{t_n\} \subset I$, $t_n > s_n$ exist, such that

$$e^{-|V(x_n(t_n))|} > \eta \quad \text{for any } n \in \mathbb{N},$$

thus obtaining (3.6).

The functional f_ε can be proved to satisfy (PS) condition and also all the hypotheses of Theorem 2.3 in Ω^1 so $x_\varepsilon \in \Omega^1$ exists, such that (3.16) and (3.17) hold.

Proof of Theorem 1.4. Let us choose $q > 2\dim \mathcal{M}$; as in the periodic case, in order to prove the theorem it is enough to show that $\varepsilon_0 > 0$ and $M > 0$ exist, such that, for any $s \in [0,T]$ and for any $\varepsilon \in]0, \varepsilon_0[$, (3.26) and (3.27) hold.

In order to get (3.26) we argue by contradiction and suppose $\varepsilon_n \to 0^+$ and a sequence $\{x_n\}$ of critical points of $f_n = f_{\varepsilon_n}$ exist, such that (3.16), (3.17) and (3.28) hold.

Then, as the singular homology has compact support, β independent of n exists such that

$$f(x_n) \leq \beta \quad \text{for any } n \in \mathbb{N}.$$

Thus, as

$$\Big\{ \inf_{s\in[0,T]} d(x_n(s), x_0) \Big\} \quad \text{is bounded,}$$

and (3.30) holds, arguing as in Theorem 1.1, it is possible to show that, if n is large enough,

$$\Big\{ \int_0^T \langle \dot{x}_n, \dot{x}_n \rangle \, ds \Big\} \quad \text{is bounded.} \tag{4.1}$$

From (4.1) and (3.28) we obtain a contradiction, so (3.26) follows. The inequality (3.27) is obtained as in the periodic case. □

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