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MORSE DECOMPOSITIONS IN THE ABSENCE OF UNIQUENESS

Maria C. Carbinatto — Krzysztof P. Rybakowski

Abstract. In this paper we define attractors and Morse decompositions in an abstract framework of curves in a metric space. We establish some basic properties of these concepts including their stability under perturbations. This extends results known for flows and semiflows on metric spaces to large classes of ordinary or partial differential equations with possibly nonunique solutions of the Cauchy problem. As an application, we first prove a Morse equation in the context of a Conley index theory which was recently defined in [10] for problems without uniqueness, and then apply this equation to give an elementary proof of two multiplicity results for strongly indefinite elliptic systems previously obtained in [1] using Morse–Floer homology.

1. Introduction

Morse decompositions (see e.g. [3], [14], [13], [5], [6]) are a useful tool in the analysis of flows or semiflows defined by ordinary, functional and evolutionary partial differential equations. Combined with an appropriate version of the Conley index and a corresponding Morse equation, they often allow us to obtain multiplicity results for solutions of variational problems. Through the use of some more refined topological tools like the Conley connection matrix, Morse decompositions can also be used to detect connections, i.e. heteroclinic orbits in dynamical systems.

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However, in many situations of interest, e.g. in various applications to boundary value problems in Hilbert spaces, the resulting differential equation does not generate a (semi)flow simply because the nonlinearity of the equation is not regular enough and, as a consequence, the uniqueness property of the Cauchy problem is violated. In such cases concepts like attractors and Morse decompositions, as defined in the above mentioned works, are not applicable. Motivated by such applications, we develop in this paper an abstract theory of attractors and Morse decompositions, which contains as a special case the theory known for flows or semiflows but which also applies to various classes of ordinary or evolution equations with nonunique solutions.

Let us describe the main ideas of our approach. To this end let X be a metric space and $\mathcal{C} = C(\mathbb{R} \to X)$ be the set of all continuous maps from $\mathbb{R}$ to X endowed with the topology of uniform convergence on compact subsets of $\mathbb{R}$. Let π be a semiflow on X. As usual, we write $x\pi t$ instead of $\pi(t,x)$. Recall that a *full solution of* π is a map $\sigma\colon \mathbb{R} \to X$ such that for all $t \in [0,\infty[$ and $s \in \mathbb{R}$ we have $\sigma(s)\pi t = \sigma(s+t)$. Recall also that a subset S of X is called *invariant relative to* π if for every $x \in S$ there is a full solution σ of π lying in S, i.e. $\sigma(\mathbb{R}) \subset S$, and such that $\sigma(0) = x$.

Now let N be an arbitrary subset of X and let $\mathcal{T}$ be the set of all full solutions of π lying in N. It then follows that for every $S \subset N$, the set S is invariant relative to π if and only if for every $x \in S$ there is a $\sigma \in \mathcal{T}$ such that $\sigma(\mathbb{R}) \subset S$ and $\sigma(0) = x$. In other words, S is invariant relative to π if and only if S is $\mathcal{T}$-invariant, by which we mean that $S = \mathrm{Inv}_{\mathcal{T}}(S)$, where

$$\mathrm{Inv}_{\mathcal{T}}(S) := \{y \in X \mid \exists \sigma \in \mathcal{T} \text{ with } \sigma(\mathbb{R}) \subset S \text{ and } y = \sigma(0)\}.$$

Note that $\mathcal{T}$ is a subset of $\mathcal{C}$. Moreover, note that once $\mathcal{T}$ is given we do not need the semiflow π any more in order to define invariance of $S \subset N$ relative to π.

Similarly, if $S \subset N$ is compact and invariant relative to π, then in order to define attractors in S (relative to π) we only need the given set $\mathcal{T}$ of solutions. In fact, rewording the usual definition (see e.g. [13]) we see that $A \subset S$ is an attractor in S relative to π if and only if A is a $\mathcal{T}$-attractor, by which we mean that there is a neighbourhood Y of A such that $A = \omega_{\mathcal{T}}(Y)$.

Here, $\omega_{\mathcal{T}}(Y)$ is the set of all $y \in X$ for which there exist sequences $(\sigma_n)_{n\in\mathbb{N}}$ in $\mathcal{T}$ and $(t_n)_{n\in\mathbb{N}}$ in $[0,\infty[$ such that $\sigma_n(0) \in Y$ for all $n \in \mathbb{N}$, $t_n \to \infty$ and $\sigma_n(t_n) \to y$ as $n \to \infty$.

We can now proceed abstractly and first take $\mathcal{T}$ to be an *arbitrary* subset of $\mathcal{C}$. We can then define $\mathcal{T}$-invariant sets and $\mathcal{T}$-attractors as above. Similarly as in the semiflow case we can also define the dual $\mathcal{T}$-repellers and $\mathcal{T}$-attractor-repeller pairs.

As we show in Section 2, all the basic properties of attractor-repeller pairs known for flows or semiflows hold in this abstract situation if we assume that $\mathcal{T}$ is translation invariant and compact as a subset of $\mathcal{C}$. Here, translation invariance means, of course, that whenever σ is in $\mathcal{T}$ then so is every translate $\sigma(\cdot + s)$, $s \in \mathbb{R}$. In particular, if $\mathcal{T}$ is defined as above in the semiflow case, then $\mathcal{T}$ is obviously translation invariant. Moreover, $\mathcal{T}$ is compact if N is π-admissible in the sense of [13].

In the semiflow case, one can give two definitions of (totally ordered) Morse decompositions of S (one in terms of attractor filtrations and the other in terms of connecting orbits) and prove that these two definitions are equivalent. This can also be done in the present abstract setting, leading to the concepts of $\mathcal{T}$-Morse decompositions of the first and second kind. In Section 3 it is proved that these two definitions are equivalent provided that $\mathcal{T} \subset \mathcal{C}$ is compact, translation invariant and, in addition, cut-and-glue invariant. By cut-and-glue invariance of $\mathcal{T}$ we mean that whenever σ_1 and $\sigma_2 \in \mathcal{T}$ with $\sigma_1(0) = \sigma_2(0)$, then $\sigma \in \mathcal{T}$, where the map $\sigma\colon \mathbb{R} \to X$ is defined by

$$\sigma(t) = \begin{cases} \sigma_1(t) & \text{if } t \leq 0, \\ \sigma_2(t) & \text{if } t \geq 0. \end{cases}$$

In Sections 2 and 3 we also define convergence of sequences of subsets of $\mathcal{C}$ and show that, in some sense, $\mathcal{T}$-attractor-repeller pairs and $\mathcal{T}$-Morse decompositions are stable with respect to perturbations of $\mathcal{T}$.

Now the concept of a (full) solution makes sense not only for flows or semiflows but also for large classes of ordinary differential equations or evolution equations on a phase space X with merely continuous nonlinearities, which, in general, do not define a semiflow. Given a subset N of X we can then define $\mathcal{T}$ to be the set of all full solutions of such an equation lying in N. Then, under very general hypotheses on the set N and the given equation, the solution set $\mathcal{T}$ is compact, translation invariant and cut-and-glue invariant.

A specific application of our abstract results is given in Section 4. Using the perturbation stability result for $\mathcal{T}$-Morse decompositions we establish a Morse equation for the Galerkin-type Conley index theory developed in the recent paper [10] for problems with nonunique solutions.

In Section 5 we finally apply our theory to the strongly indefinite elliptic system

$$\begin{aligned} -\Delta u &= \partial_v H(u, v, x) \quad \text{in } \Omega, \\ -\Delta v &= \partial_u H(u, v, x) \quad \text{in } \Omega, \\ u &= 0, \quad v = 0 \qquad \text{in } \partial\Omega. \end{aligned} \tag{1.1}$$

on a smooth bounded domain Ω in $\mathbb{R}^N$, considered in the recent important paper [1] by Angenent and van der Vorst.

Under the growth assumptions on H made in [1] the solutions of (1.1) turn out to be equilibria of an abstract ordinary differential equation

$$\dot{z} = f(z) \tag{1.2}$$

on a Hilbert space X with the nonlinearity $f: X \to X$ being merely continuous but, in general, not differentiable nor even locally Lipschitzian. Therefore, in general, Equation (1.2) does not generate a semiflow on X.

However, the index theory of [10] and our abstract Morse decomposition theory are applicable in this situation. In particular, using the Morse equation from Section 4 we give new, Conley index based proofs of two multiplicity results for this system established in [1] by the use of Morse–Floer homology.

More applications of the abstract theory presented here will be given in the forthcoming publications [2] and [11].

In this paper we mostly use standard notation. In particular, by $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_0$ we denote the set of all real, all integer, all positive integer and all nonnegative integer numbers, respectively. Given a topological space X and $Y \subset X$, we write $\mathrm{Int}_X(Y)$, $\mathrm{Cl}_X(Y)$ and $\partial_X(Y)$ to denote the interior, the closure and the boundary of Y in X, respectively. Given topological spaces X_1 and X_2 we denote by $C(X_1 \to X_2)$ the set of all continuous maps from X_1 to X_2. Finally, for a and $b \in \mathbb{Z}$, we write $[\![a,b]\!] := [a,b] \cap \mathbb{Z}$. This less common notation is used here to replace the somewhat imprecise three dot $\ldots$ symbol. In particular, we will write $[\![1,n]\!]$ instead of $\{1,\ldots,n\}$ and x_i, $i \in [\![1,n]\!]$, instead of $x_1, \ldots, x_n$.

2. $\mathcal{T}$-invariance and attractor-repeller pairs

Throughout this paper, unless otherwise specified, let (X,d) be a metric space.

Let $\mathcal{C} = C(\mathbb{R} \to X)$ be the set of all continuous maps from $\mathbb{R}$ to X. We endow $\mathcal{C}$ with the metric

$$\overline{d}(x,y) = \sum_{n\in\mathbb{N}} 2^{-n}\overline{d}_n(x,y)/(\overline{d}_n(x,y)+1),$$

where

$$\overline{d}_n(x,y) = \sup_{t\in[-n,n]} d(x(t),y(t)), \quad x,y \in \mathcal{C}.$$

Note that $\overline{d}$ is indeed a metric on $\mathcal{C}$ inducing the topology of uniform convergence on compact sets in $\mathbb{R}$.

Let $\mathcal{T}$ be an arbitrary subset of $\mathcal{C}$. To aid intuition, the reader may think of X as a Hilbert or Banach space and $\mathcal{T}$ as a specified set of (full) solutions of a given ordinary differential equation or an evolution equation defined on (an open subset of) X.

In this section we define the concepts of invariance, attractors and repellers relative to this set $\mathcal{T}$ of "solutions". We will study some properties of these concepts under the assumption that $\mathcal{T}$ is compact in $\mathcal{C}$ and translation invariant. In particular, we will establish extensions of some fundamental results on attractor-repeller pairs known for flows or semiflows to the present more general case (cf. Theorem 2.11). In addition, we define perturbations of the solution sets $\mathcal{T}$ and show that attractor-repeller pairs have some stability properties with respect to such perturbations (see Theorem 2.19).

We first need a number of preliminary definitions. Let $\sigma\colon\mathbb{R}\to X$ be an arbitrary function. The function $\sigma^-\colon\mathbb{R}\to X$, $s\mapsto\sigma(-s)$, is called the *time inverse of* σ. Moreover, for every $t\in\mathbb{R}$ the function $\mathrm{tsl}_t\sigma\colon\mathbb{R}\to X$, $s\mapsto\sigma(s+t)$, is called the *t-translate of* σ. Furthermore, let $\omega(\sigma)$ be the set of all $y\in X$ for which there exists a sequence $(t_n)_{n\in\mathbb{N}}$ in $[0,\infty[$ with $t_n\to\infty$ and $\sigma(t_n)\to y$ as $n\to\infty$. Set $\alpha(\sigma)=\omega(\sigma^-)$. Given $Y\subset X$, $P\subset\mathbb{R}$, $y\in X$ and $t\in\mathbb{R}$ we now define the following sets:

$$S_{\mathcal{T}}=\bigcup_{\sigma\in\mathcal{T}}\sigma(\mathbb{R}),\tag{2.1}$$

$$\mathcal{T}(Y,P)=\{y\in X\mid\ \exists\sigma\in\mathcal{T}\,\exists t\in P\text{ with }\sigma(0)\in Y\text{ and }y=\sigma(t)\},\tag{2.2}$$

$$\mathcal{T}(y,P)=\mathcal{T}(\{y\},P),\tag{2.3}$$

$$\mathcal{T}(Y,t)=\mathcal{T}(Y,\{t\}),\tag{2.4}$$

$$\mathcal{T}^-=\{\sigma^-\mid\sigma\in\mathcal{T}\},\tag{2.5}$$

$$\omega_{\mathcal{T}}(Y)=\bigcap_{t\in[0,\infty[}\mathrm{Cl}_X(\mathcal{T}(Y,[t,\infty[)),\tag{2.6}$$

$$Y^*_{\mathcal{T}}=\{y\in X\mid\exists\sigma\in\mathcal{T}\text{ with }\omega(\sigma)\subset X\setminus Y\text{ and }y=\sigma(0)\},\tag{2.7}$$

$$\mathrm{Inv}_{\mathcal{T}}(Y)=\{y\in X\mid\exists\sigma\in\mathcal{T}\text{ with }\sigma(\mathbb{R})\subset Y\text{ and }y=\sigma(0)\}.\tag{2.8}$$

A set $S\subset X$ is called *$\mathcal{T}$-invariant* if $S=\mathrm{Inv}_{\mathcal{T}}(S)$, i.e. if and only if for every $y\in S$ there is a $\sigma\in\mathcal{T}$ such that $\sigma(\mathbb{R})\subset S$ and $y=\sigma(0)$.

A point $x\in X$ is called a *$\mathcal{T}$-equilibrium* if there is a $\sigma\in\mathcal{T}$ such that $\sigma(t)=x$ for all $t\in\mathbb{R}$.

$\mathcal{T}$ is called *translation invariant* if $\mathrm{tsl}_t\sigma\in\mathcal{T}$ for all $\sigma\in\mathcal{T}$ and all $t\in\mathbb{R}$.

$\mathcal{T}$ is called *gradient-like with respect to* φ if $\varphi\colon S_{\mathcal{T}}\to\mathbb{R}$ is a continuous function such that for every $\sigma\in\mathcal{T}$ the function $\varphi\circ\sigma\colon\mathbb{R}\to\mathbb{R}$ is nonincreasing and if $\varphi\circ\sigma$ is constant, then $\sigma\colon\mathbb{R}\to X$ is constant.

$\mathcal{T}$ is called *gradient-like* if there exists a function φ such that $\mathcal{T}$ is gradient-like with respect to φ.

A set $A \subset X$ is called a *$\mathcal{T}$-attractor* if there is a set $Y \subset X$ such that $A \subset \mathrm{Int}_X(Y)$ and $A = \omega_{\mathcal{T}}(Y)$. A is called a *$\mathcal{T}$-repeller* if A is a $\mathcal{T}^-$-attractor.

In the next propositions we will establish a few elementary properties of the sets and concepts just introduced.

PROPOSITION 2.1. *For all $Y \subset X$ and $y \in X$ the following conditions are equivalent:*

(2.9) $y \in \omega_{\mathcal{T}}(Y)$.

(2.10) *There exist sequences $(\sigma_n)_{n\in\mathbb{N}}$ in $\mathcal{T}$ and $(t_n)_{n\in\mathbb{N}}$ in $[0,\infty[$ such that $\sigma_n(0) \in Y$ for all $n \in \mathbb{N}$, $t_n \to \infty$ and $\sigma_n(t_n) \to y$ as $n \to \infty$.*

PROOF. Suppose $y \in \omega_{\mathcal{T}}(Y)$. Then by (2.6) for every $n \in \mathbb{N}$ there is a $y_n \in \mathcal{T}(Y, [n,\infty[)$ such that $d(y, y_n) < 1/n$. Hence there is a $\sigma_n \in \mathcal{T}$ and a $t_n \geq n$ with $\sigma_n(0) \in Y$ and $y_n = \sigma_n(t_n)$. Thus (2.10) is satisfied.

Now assume (2.10) and let $(\sigma_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ be as in (2.10). Let $t \in [0,\infty[$ be arbitrary. Then $t_n \geq t$ for some $n_0 \in \mathbb{N}$ and all $n \geq n_0$. It follows that $\sigma_n(t_n) \in \mathcal{T}(Y, [t,\infty[)$ for all $n \geq n_0$ and so $y \in \mathrm{Cl}_X \mathcal{T}(Y, [t,\infty[)$. This proves (2.9). □

Let $\sigma \in \mathcal{C}$ be arbitrary. For $\mathcal{T} := \{\sigma\}$ and $Y := X$ we see that $\mathcal{T}(Y, [t,\infty[) = \sigma([t,\infty[)$ for all $t \in \mathbb{R}$ and so, using Proposition 2.1, we obtain

$$\omega(\sigma) = \bigcap_{t\in[0,\infty[} \mathrm{Cl}_X(\sigma([t,\infty[)). \tag{2.11}$$

PROPOSITION 2.2. *If $\mathcal{T}$ is compact and translation invariant, then $S_{\mathcal{T}}$ is compact and $\mathcal{T}$-invariant. Moreover, for every $\sigma \in \mathcal{T}$ the sets $\alpha(\sigma)$ and $\omega(\sigma)$ are nonempty, compact, connected and $\mathcal{T}$-invariant. In addition, if $\mathcal{T}$ is gradient-like, then $\alpha(\sigma)$ and $\omega(\sigma)$ consist only of $\mathcal{T}$-equilibria. Finally, if $\mathcal{T}$ is gradient-like with respect to a function φ and $\sigma \in \mathcal{T}$ is not a constant map, then for all $x \in \alpha(\sigma)$ and $y \in \omega(\sigma)$*

$$\varphi(x) = \sup_{t\in\mathbb{R}} \varphi(\sigma(t)) > \inf_{t\in\mathbb{R}} \varphi(\sigma(t)) = \varphi(y)$$

so, in particular, $\alpha(\sigma) \cap \omega(\sigma) = \emptyset$.

PROOF. Let $(x_n)_{n\in\mathbb{N}}$ be any sequence in $S_{\mathcal{T}}$. Then there are sequences $(\sigma_n)_{n\in\mathbb{N}}$ in $\mathcal{T}$ and $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that $\sigma_n(t_n) = x_n$ for every $n \in \mathbb{N}$. Let $\tau_n = \mathrm{tsl}_{t_n}\sigma_n$. Since $\mathcal{T}$ is translation invariant, it follows that $\tau_n \in \mathcal{T}$ for all $n \in \mathbb{N}$. Since $\mathcal{T}$ is compact, we may assume, taking a subsequence if necessary, that there is a $\tau \in \mathcal{T}$ such that $\tau_n \to \tau$ in $\mathcal{C}$ as $n \to \infty$. Setting $x = \tau(0)$ we see that $x_n = \sigma_n(t_n) = \tau_n(0) \to \tau(0) = x$ as $n \to \infty$. This proves compactness of $S_{\mathcal{T}}$. Now let $x \in S_{\mathcal{T}}$ be arbitrary. Then there is a $\sigma \in \mathcal{T}$ and a $t \in \mathbb{R}$ with $\sigma(t) = x$. Setting $\tau = \mathrm{tsl}_t\sigma$ we have that $\tau \in \mathcal{T}$ and $\tau(0) = x$. This proves that $S_{\mathcal{T}}$ is $\mathcal{T}$-invariant.

Now let $\sigma \in \mathcal{T}$ be arbitrary and set $\tau_n := \mathrm{tsl}_n\sigma$, $n \in \mathbb{N}$. Then by the compactness and translation invariance of $\mathcal{T}$ we have that $\tau_n \in \mathcal{T}$ for all $n \in \mathbb{N}$ and there is a subsequence $(\tau_{n_m})_{m\in\mathbb{N}}$ of $(\tau_n)_{n\in\mathbb{N}}$ converging in $\mathcal{C}$ to some $\tau \in \mathcal{T}$. In particular, $\sigma(n_m) = \tau_{n_m}(0) \to x := \tau(0)$ as $m \to \infty$, so $x \in \omega(\sigma)$ and thus $\omega(\sigma)$ is nonempty.

To prove that $\omega(\sigma)$ is compact and connected, note that, by (2.11), the set $\omega(\sigma)$ is the intersection of the family $\mathrm{Cl}_X(\sigma([t,\infty[))$, $t \in [0,\infty[$, of closed subsets of $S_{\mathcal{T}}$ which is directed by the relation $\supset$. Since $S_{\mathcal{T}}$ is compact and σ is continuous it follows that $\mathrm{Cl}_X(\sigma([t,\infty[))$ is compact and connected for all $t \in [0,\infty[$. Now general topological results (e.g. Theorem 6.1.18 in [4]) imply that $\omega(\sigma)$ is compact and connected.

Now let $y \in \omega(\sigma)$ be arbitrary. Then there is a sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n \to \infty$ and $\sigma(t_n) \to y$ as $n \to \infty$. Set $\tau_n = \mathrm{tsl}_{t_n}\sigma$, $n \in \mathbb{N}$. Taking a subsequence, if necessary, we may assume that $\tau_n \to \tau$ as $n \to \infty$, for some $\tau \in \mathcal{T}$. It follows that for every $t \in \mathbb{R}$ we have $t_n + t \to \infty$ and $\sigma(t_n + t) = \tau_n(t) \to \tau(t)$ as $n \in \mathbb{N}$, so $\tau(t) \in \omega(\sigma)$. Thus $\tau(\mathbb{R}) \subset \omega(\sigma)$ and $y = \tau(0)$, which shows that $\omega(\sigma)$ is $\mathcal{T}$-invariant. Now suppose, in addition, that $\varphi\colon S_{\mathcal{T}} \to \mathbb{R}$ and $\mathcal{T}$ is gradient-like with respect to φ. Since φ is continuous and $\varphi\circ\sigma$ is nonincreasing, we obtain that

$$\varphi(\tau(t)) \equiv \sup_{s\in\mathbb{R}}(\varphi(\sigma(s))), \quad t \in \mathbb{R}.$$

It follows that $\varphi\circ\tau$ is constant, so τ is constant, i.e. y is a $\mathcal{T}$-equilibrium.

The analogous statements concerning $\alpha(\sigma)$ follow from the fact that the map $\mathcal{C} \to \mathcal{C}$, $\sigma \mapsto \sigma^-$, is continuous so $\mathcal{T}^-$ is compact and translation invariant. Since $\sigma^- \in \mathcal{T}^-$ we thus obtain, from what we have proved so far, that $\alpha(\sigma) = \omega(\sigma^-)$ is nonempty, compact, connected and $\mathcal{T}$-invariant. Moreover, if $\mathcal{T}$ is gradient-like with respect to φ then $\mathcal{T}^-$ is gradient-like with respect to $-\varphi$ and so $\alpha(\sigma) = \omega(\sigma^-)$ consists only of $\mathcal{T}^-$-equilibria, i.e. only of $\mathcal{T}$-equilibria.

The last statement of the proposition is obvious. □

PROPOSITION 2.3. *If $\mathcal{T}$ is translation invariant and $Y \subset X$, then* $\mathrm{Inv}_{\mathcal{T}}(Y)$ *is $\mathcal{T}$-invariant and* $\mathrm{Inv}_{\mathcal{T}}(Y)$ *is the largest $\mathcal{T}$-invariant set included in Y.*

PROOF. Let $x \in \mathrm{Inv}_{\mathcal{T}}(Y)$ be arbitrary. Then, by (2.8), there is a $\sigma \in \mathcal{T}$ with $x = \sigma(0)$ and $\sigma(\mathbb{R}) \subset Y$. Let $s \in \mathbb{R}$ be arbitrary and $\tau = \mathrm{tsl}_s\sigma$. Then, by our hypothesis, $\tau \in \mathcal{T}$ and $\tau(\mathbb{R}) = \sigma(\mathbb{R}) \subset Y$. Thus $\sigma(s) = \tau(0) \in \mathrm{Inv}_{\mathcal{T}}(Y)$, so $\mathrm{Inv}_{\mathcal{T}}(Y)$ is $\mathcal{T}$-invariant.

Now let $S \subset Y$ be $\mathcal{T}$-invariant. Then, for every $x \in S$, there is a $\sigma \in \mathcal{T}$ such that $x = \sigma(0)$ and $\sigma(\mathbb{R}) \subset S \subset Y$. It follows that $x \in \mathrm{Inv}_{\mathcal{T}}(Y)$. This proves that $S \subset \mathrm{Inv}_{\mathcal{T}}(Y)$ so $\mathrm{Inv}_{\mathcal{T}}(Y)$ is the largest $\mathcal{T}$-invariant set included in Y. □

Proposition 2.4. *If $\mathcal{T}$ is translation invariant and $Y \subset X$, then* $\mathrm{Inv}_{\mathcal{T}}(Y) \subset \omega_{\mathcal{T}}(Y)$.

Proof. Let $y \in \mathrm{Inv}_{\mathcal{T}}(Y)$ be arbitrary. Then there is a $\sigma \in \mathcal{T}$ such that $\sigma(\mathbb{R}) \subset Y$ and $\sigma(0) = y$. For $n \in \mathbb{N}$ let $\tau_n = \mathrm{tsl}_{-n}\sigma$. Then $\tau_n \in \mathcal{T}$, $\tau_n(0) = \sigma(-n) \in Y$ and $\tau_n(n) = y$. Proposition 2.1 implies that $y \in \omega_{\mathcal{T}}(Y)$. □

Corollary 2.5. $\omega_{\mathcal{T}}(\emptyset) = \emptyset$. *Moreover, if $\mathcal{T}$ is compact and translation invariant, then* $\omega_{\mathcal{T}}(X) = S_{\mathcal{T}}$.

Proof. It is clear that $\omega_{\mathcal{T}}(\emptyset) = \emptyset$. Assume that $\mathcal{T}$ is compact and translation invariant. Then $S_{\mathcal{T}} \subset X$ and so by Propositions 2.2, 2.3 and 2.4 we obtain that

$$S_{\mathcal{T}} \subset \mathrm{Inv}_{\mathcal{T}}(X) \subset \omega_{\mathcal{T}}(X).$$

On the other hand, let $y \in \omega_{\mathcal{T}}(X)$ be arbitrary. Then there is a sequence $(\sigma_n)_{n\in\mathbb{N}}$ in $\mathcal{T}$ and a sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n \to \infty$ and $\sigma_n(t_n) \to y$. Let $\tau_n := \mathrm{tsl}_{t_n}\sigma_n$ for all $n \in \mathbb{N}$. Then $\tau_n \in \mathcal{T}$ for all $n \in \mathbb{N}$ and, taking a subsequence, if necessary, we may assume that $\tau_n \to \tau$ in $\mathcal{C}$ for some $\tau \in \mathcal{T}$. It follows that $\tau(\mathbb{R}) \subset S_{\mathcal{T}}$ so $y = \tau(0) \in S_{\mathcal{T}}$. Consequently, $\omega_{\mathcal{T}}(X) \subset S_{\mathcal{T}}$ and the corollary is proved. □

Proposition 2.6. *If $\mathcal{T}$ is compact and translation invariant, then for every $Y \subset X$ the set $\omega_{\mathcal{T}}(Y)$ is compact and $\mathcal{T}$-invariant.*

Proof. Let $Y \subset X$ and $y \in \omega_{\mathcal{T}}(Y)$ be arbitrary. Let $(\sigma_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ be as in (2.10). By the compactness of $\mathcal{T}$ we may assume that there is a $\tau \in \mathcal{T}$ such that $\tau_n := \mathrm{tsl}_{t_n}\sigma_n \to \tau$ in $\mathcal{C}$ as $n \to \infty$. It follows that for every $t \in \mathbb{R}$ we have $\sigma_n(t_n + t) = \tau_n(t) \to \tau(t)$ and $t_n + t \to \infty$ as $n \to \infty$. Moreover, $\sigma_n(0) \in Y$ for all $n \in \mathbb{N}$. By Proposition 2.1 we now obtain that $\tau(t) \in \omega_{\mathcal{T}}(Y)$ for all $t \in \mathbb{R}$. This proves that $\omega_{\mathcal{T}}(Y)$ is $\mathcal{T}$-invariant and so $\omega_{\mathcal{T}}(Y) \subset S_{\mathcal{T}}$. Since $S_{\mathcal{T}}$ is compact by Proposition 2.2 and $\omega_{\mathcal{T}}(Y)$ is closed by (2.6), it follows that $\omega_{\mathcal{T}}(Y)$ is compact. □

Proposition 2.7. *If $\mathcal{T}$ is compact and translation invariant and if $Y \subset Y' \subset X$ and $\omega_{\mathcal{T}}(Y') \subset Y$ then $\omega_{\mathcal{T}}(Y) = \omega_{\mathcal{T}}(Y')$.*

Proof. Since $Y \subset Y'$ we have $\omega_{\mathcal{T}}(Y) \subset \omega_{\mathcal{T}}(Y')$, by (2.6). By Proposition 2.6 $\omega_{\mathcal{T}}(Y') \subset \mathrm{Inv}_{\mathcal{T}}(Y)$ and by Proposition 2.4 $\mathrm{Inv}_{\mathcal{T}}(Y) \subset \omega_{\mathcal{T}}(Y)$. □

The following result gives a useful characterization of $\mathcal{T}$-attractors:

Theorem 2.8. *Let $\mathcal{T}$ be compact and translation invariant and $Y \subset X$ be closed. Then, for every $A \subset X$, the following conditions are equivalent:*

(2.12) $A = \omega_{\mathcal{T}}(Y) \subset \mathrm{Int}_X(Y)$, (2.13) $A = \mathrm{Inv}_{\mathcal{T}}(Y)$ *and there is a* $t \in \,]0,\infty[$ *such that* $\mathcal{T}(Y,t) \subset \mathrm{Int}_X(Y)$.

Proof. Assume (2.12) and suppose that there is no $t \in \,]0,\infty[$ such that $\mathcal{T}(Y,t) \subset \mathrm{Int}_X(Y)$. Then there are sequences $(t_n)_{n\in\mathbb{N}}$ in $]0,\infty[$ and $(y_n)_{n\in\mathbb{N}}$ such that $t_n \to \infty$ as $n \to \infty$ and $y_n \in \mathcal{T}(Y,t_n)\setminus\mathrm{Int}_X(Y)$ for all $n \in \mathbb{N}$. Thus, for every $n \in \mathbb{N}$, there is a $\sigma_n \in \mathcal{T}$ with $\sigma_n(0) \in Y$ and $y_n = \sigma_n(t_n)$. By the compactness of $\mathcal{T}$ we may assume that there is a $\tau \in \mathcal{T}$ such that $\tau_n := \mathrm{tsl}_{t_n}\sigma_n \to \tau$ in $\mathcal{C}$ as $n \to \infty$. It follows that $y_n = \sigma_n(t_n) \to \tau(0)$ as $n \to \infty$ so $\tau(0) \in \omega_{\mathcal{T}}(Y)\setminus\mathrm{Int}_X(Y)$, a contradiction.

By Proposition 2.6 the set $A = \omega_{\mathcal{T}}(Y)$ is $\mathcal{T}$-invariant, so by (2.12) and Proposition 2.3 we have that $A \subset \mathrm{Inv}_{\mathcal{T}}(Y)$. On the other hand, $\mathrm{Inv}_{\mathcal{T}}(Y) \subset \omega_{\mathcal{T}}(Y)$ by Proposition 2.4. This proves (2.13).

Now assume (2.13). First we claim that

(2.14) there is an $\varepsilon \in \,]0,\infty[$ such that $\mathcal{T}(Y,[t-\varepsilon,t+\varepsilon]) \subset \mathrm{Int}_X(Y)$.

In fact, if the claim is not true, then we obtain a sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ with $t_n \to t$ as $n \to \infty$ and a sequence $(\sigma_n)_{n\in\mathbb{N}}$ in $\mathcal{T}$ with $\sigma_n(0) \in Y$ and $\sigma_n(t_n) \notin \mathrm{Int}_X(Y)$ for every $n \in \mathbb{N}$. We may assume that $\sigma_n \to \sigma$ as $n \to \infty$ for some $\sigma \in \mathcal{T}$. Thus $\sigma_n(t_n) \to \sigma(t)$ as $n \to \infty$, so $\sigma(t) \notin \mathrm{Int}_X(Y)$ and $\sigma_n(0) \to \sigma(0)$ as $n \to \infty$, so $\sigma(0) \in Y$ as Y is closed. Thus $\sigma(t) \in \mathcal{T}(Y,t) \setminus \mathrm{Int}_X(Y)$ which contradicts our assumption. This proves (2.14).

We also claim that

(2.15) the set $\mathcal{T}(Y,[t-\varepsilon,t+\varepsilon])$ is compact.

In fact, if $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{T}(Y,[t-\varepsilon,t+\varepsilon])$ then for every $n \in \mathbb{N}$ there is a $\sigma_n \in \mathcal{T}$ and $t_n \in [t-\varepsilon,t+\varepsilon]$ such that $\sigma_n(t_n) = x_n$. We may assume, taking a subsequence if necessary, that $\sigma_n \to \sigma$ and $t_n \to t_0$ as $n \to \infty$, for some $\sigma \in \mathcal{T}$ and some $t_0 \in [t-\varepsilon,t+\varepsilon]$. It follows that $x_n \to x_0 := \sigma(t_0)$. This proves compactness of $\mathcal{T}(Y,[t-\varepsilon,t+\varepsilon])$.

By (2.14) and (2.15) we see that there is an open set U such that

$$\mathcal{T}(Y,[t-\varepsilon,t+\varepsilon]) \subset U \subset \mathrm{Cl}_X(U) \subset \mathrm{Int}_X(Y) \subset Y. \tag{2.16}$$

It easily follows that whenever $\sigma \in \mathcal{T}$ satisfies $\sigma(0) \in Y$ then $\sigma(nr) \in U$ for all $n \in \mathbb{N}$ and $r \in [t-\varepsilon,t+\varepsilon]$.

Now let $s \in [t^2/\varepsilon,\infty[$ be arbitrary. Proposition 2.9 below implies that there is an $n \in \mathbb{N}$ and an $r \in \,]t-\varepsilon,t[$ such that $s = nr$. Thus, whenever $\sigma \in \mathcal{T}$ satisfies $\sigma(0) \in Y$ we obtain $\sigma(s) = \sigma(nr) \in U$. It follows that $\mathcal{T}(Y,[t^2/\varepsilon,\infty[) \subset U$ which implies that

$$\omega_{\mathcal{T}}(Y) \subset \mathrm{Cl}_X(\mathcal{T}(Y,[t^2/\varepsilon,\infty[)) \subset \mathrm{Cl}_X(U) \subset \mathrm{Int}_X(Y). \tag{2.17}$$

In particular, by (2.17) and Propositions 2.4 and 2.6 we have

$$A = \mathrm{Inv}_{\mathcal{T}}(Y) \subset \omega_{\mathcal{T}}(Y) \subset \mathrm{Inv}_{\mathcal{T}}(Y)$$

so

$$A = \omega_{\mathcal{T}}(Y). \tag{2.18}$$

(2.17) and (2.18) imply that (2.12) holds and the proof is complete. □

Proposition 2.9. *For all ε and $t \in \,]0,\infty[$ and all $s \in [t^2/\varepsilon, \infty[$ there is an $n \in \mathbb{N}$ and an $r \in \,]t-\varepsilon, t[$ such that $s = nr$.*

Proof. Given ε, $t \in \,]0,\infty[$ and $s \in [t^2/\varepsilon,\infty[$ there is an $n \in \mathbb{N}$ such that

$$(n-1)t \leq s < nt.$$

It follows that $t^2/\varepsilon \leq s < nt$ so $t/\varepsilon < n$ i.e. $t < \varepsilon n$. Moreover, $r := s/n < t$. Since $(n-1)t \leq s$ we also have $nt \leq t + s < \varepsilon n + s$ so $n(t-\varepsilon) < s$, i.e. $r = s/n > t - \varepsilon$. □

Let $A \subset X$ be a $\mathcal{T}$-attractor and let $Y \subset X$ be such that $A \subset \operatorname{Int}_X(Y)$ and $A = \omega_{\mathcal{T}}(Y)$. If $\mathcal{T}$ is compact and translation invariant then A is compact by Proposition 2.6 and so we may choose an open set U such that

$$A \subset U \subset \operatorname{Cl}_X(U) \subset \operatorname{Int}_X(Y).$$

Proposition 2.7 implies that $A = \omega_{\mathcal{T}}(\operatorname{Cl}_X(U))$ and so we can always assume that Y is closed. We will use this remark implicitly in the sequel.

Proposition 2.10. *Suppose that $\mathcal{T}$ is compact and translation invariant. Let $Y \subset X$ be arbitrary with $A := \omega_{\mathcal{T}}(Y) \subset \operatorname{Int}_X(Y)$. Then, for every $\sigma \in \mathcal{T}$, the following statements are equivalent:*

$$\omega(\sigma) \cap A \neq \emptyset \Leftrightarrow \sigma(\mathbb{R}) \cap \operatorname{Int}_X Y \neq \emptyset \Leftrightarrow \sigma(\mathbb{R}) \cap Y \neq \emptyset \Leftrightarrow \omega(\sigma) \subset A.$$

Proof. Suppose $\omega(\sigma) \cap A \neq \emptyset$. Since $A \subset \operatorname{Int}_X(Y)$, the definition of $\omega(\sigma)$ implies that $\sigma(\mathbb{R}) \cap \operatorname{Int}_X Y \neq \emptyset$. Now assume that $\sigma(\mathbb{R}) \cap Y \neq \emptyset$. Then $\omega(\sigma) \subset A$ by the definition of $\omega(\sigma)$ and the translation invariance of $\mathcal{T}$. Since $\omega(\sigma) \neq \emptyset$ by the compactness and translation invariance of $\mathcal{T}$, we conclude that $\omega(\sigma) \subset A$ implies that $\omega(\sigma) \cap A \neq \emptyset$. □

The following result defines $\mathcal{T}$-attractor-repeller pairs and establishes their main properties:

Theorem 2.11. *Let $\mathcal{T}$ be compact and translation invariant and let A be a $\mathcal{T}$-attractor. Then the set $A^* := A^*_{\mathcal{T}}$ is a $\mathcal{T}$-repeller. The sets A and A^* are compact, disjoint and $\mathcal{T}$-invariant. Moreover, for every $\sigma \in \mathcal{T}$, the following alternatives hold:*

(2.19) *if $\sigma(\mathbb{R}) \not\subset A^*$ then $\omega(\sigma) \subset A$,*

(2.20) *if $\sigma(\mathbb{R}) \not\subset A$ then $\alpha(\sigma) \subset A^*$.*

In particular, either $\sigma(\mathbb{R}) \subset A$, *or* $\sigma(\mathbb{R}) \subset A^*$ *or else* $\alpha(\sigma) \subset A^*$ *and* $\omega(\sigma) \subset A$. *Finally,* $A = (A^*_{\mathcal{T}})^*_{\mathcal{T}^-}$.

We call the set $A^*_{\mathcal{T}}$ the *dual* $\mathcal{T}$*-repeller of* A and the pair (A, A^*) a $\mathcal{T}$*-attractor-repeller pair*.

Proof. Let $y \in A^*$ be arbitrary. It follows from the definition of A^* that there is a $\sigma \in \mathcal{T}$ such that $\sigma(0) = y$ and $\omega(\sigma) \cap A = \emptyset$. For every $t \in \mathbb{R}$ we have $\sigma_t := \mathrm{tsl}_t \sigma \in \mathcal{T}$ and $\omega(\sigma_t) = \omega(\sigma)$ so $\omega(\sigma_t) \cap A = \emptyset$. Hence $\sigma(t) = \sigma_t(0) \in A^*$, so $\sigma(\mathbb{R}) \subset A^*$. It follows that A^* is $\mathcal{T}$-invariant.

There is a closed set $Y \subset X$ such that $A = \omega_{\mathcal{T}}(Y) \subset \mathrm{Int}_X(Y)$. It follows from Proposition 2.10 that

$$\text{whenever } \sigma \in \mathcal{T} \text{ then } \omega(\sigma) \cap A = \emptyset \Leftrightarrow \sigma(\mathbb{R}) \subset Y^* := X \setminus \mathrm{Int}_X(Y). \tag{2.21}$$

(2.21) and the compactness of $\mathcal{T}$ easily imply that A^* is compact.

We now show that $A^* = \mathrm{Inv}_{\mathcal{T}^-}(Y^*)$. In fact, obviously $\mathrm{Inv}_{\mathcal{T}^-}(Y^*) = \mathrm{Inv}_{\mathcal{T}}(Y^*)$. Since A^* is $\mathcal{T}$-invariant and $A^* \subset Y^*$ by (2.21), we obtain $A^* \subset \mathrm{Inv}_{\mathcal{T}}(Y^*)$. If $y \in \mathrm{Inv}_{\mathcal{T}}(Y^*)$ then there is a $\sigma \in \mathcal{T}$ with $\sigma(0) = y$ and $\sigma(\mathbb{R}) \subset Y^*$. Thus from (2.21) we conclude that $y \in A^*$.

We finally claim that there is a $t \in [0, \infty[$ with $\mathcal{T}^-(Y^*, t) \subset X \setminus Y \subset \mathrm{Int}_X(Y^*)$. Suppose this claim is not true. Then, since $\mathcal{T}(Y^*, t) = \mathcal{T}^-(Y^*, -t)$, we obtain the existence of a sequence $(t_n)_{n\in\mathbb{N}}$ in $]0, \infty[$ with $t_n \to \infty$ as $n \to \infty$, and a sequence $(\sigma_n)_{n\in\mathbb{N}}$ in $\mathcal{T}$ such that $\sigma_n(0) \in Y^*$ and $\sigma_n(-t_n) \in Y$ for all $n \in \mathbb{N}$. Taking a subsequence if necessary we may assume that $\sigma_n \to \sigma$ in $\mathcal{C}$. Let $\tau_n = \mathrm{tsl}_{-t_n}\sigma_n$, $n \in \mathbb{N}$. Then $\tau_n(0) = \sigma_n(-t_n) \in Y$ for all $n \in \mathbb{N}$ and $\tau_n(t_n) = \sigma_n(0) \to \sigma(0)$. Since Y^* is closed, we see that

$$\sigma(0) \in Y^* \cap \omega_{\mathcal{T}}(Y) = Y^* \cap A = \emptyset,$$

a contradiction, which proves the claim. Altogether, we obtain from Theorem 2.8 that

$$A^* = \omega_{\mathcal{T}^-}(Y^*) \subset \mathrm{Int}_X(Y^*) \tag{2.22}$$

so A^* is a $\mathcal{T}^-$-attractor, i.e. a $\mathcal{T}$-repeller. It also follows that $A \cap A^* = \emptyset$.

Now let $\sigma \in \mathcal{T}$ be arbitrary. If $\sigma(\mathbb{R}) \not\subset A = \mathrm{Inv}_{\mathcal{T}}(Y)$ then there is a $t \in \mathbb{R}$ with $\sigma(t) \notin Y$, so $\sigma(t) \in Y^*$. This implies that $\alpha(\sigma) \subset \omega_{\mathcal{T}^-}(Y^*) = A^*$.

On the other hand, if $\sigma(\mathbb{R}) \not\subset A^* = \mathrm{Inv}_{\mathcal{T}}(Y^*)$ then there is a $t \in \mathbb{R}$ with $\sigma(t) \notin Y^*$, i.e. $\sigma(t) \in \mathrm{Int}_X(Y)$. This implies that $\omega(\sigma) \subset A$.

Finally, $y \in (A^*_{\mathcal{T}})^*_{\mathcal{T}^-}$ if and only if there is a $\tau \in \mathcal{T}^-$ with $\tau(0) = y$ and $\omega(\tau) \cap A^*_{\mathcal{T}} = \emptyset$ if and only if there is a $\sigma \in \mathcal{T}$ with $\sigma(0) = y$ and $\alpha(\sigma) \cap A^*_{\mathcal{T}} = \emptyset$. Here, $\sigma = \tau^-$. Now, by what we have proved so far, $\alpha(\sigma) \cap A^*_{\mathcal{T}} = \emptyset$ if and only if $\sigma(\mathbb{R}) \subset A$. This clearly implies that $(A^*_{\mathcal{T}})^*_{\mathcal{T}^-} = A$. The theorem is proved. □

Theorem 2.11 clearly implies the following corollary:

Corollary 2.12. *Let $\mathcal{T}$ be compact and translation invariant. A pair (A_1, A_2) of subsets of X is a $\mathcal{T}$-attractor-repeller pair if and only if the pair (A_2, A_1) is a $\mathcal{T}^-$-attractor-repeller pair.*

We will now discuss perturbations of attractor-repeller pairs with respect to the set $\mathcal{T} \subset \mathcal{C}$. To this end we need the following convergence concept on the set of all subsets of $\mathcal{C}$.

Definition 2.13. Let $(\mathcal{T}_\kappa)_{\kappa \in \mathbb{N}}$ be a sequence of subsets of $\mathcal{C}$ and $\mathcal{T} \subset \mathcal{C}$ be arbitrary. We say that $(\mathcal{T}_\kappa)_{\kappa \in \mathbb{N}}$ *converges to* $\mathcal{T}$, and we write $\mathcal{T}_\kappa \to \mathcal{T}$ (as $\kappa \to \infty$), if for every sequence $(\kappa_n)_{n \in \mathbb{N}}$ in $\mathbb{N}$ with $\kappa_n \to \infty$ as $n \to \infty$ and every sequence $(\sigma_n)_{n \in \mathbb{N}}$ such that $\sigma_n \in \mathcal{T}_{\kappa_n}$ for all $n \in \mathbb{N}$ there is a subsequence $(\sigma_{n_m})_{m \in \mathbb{N}}$ and a $\sigma \in \mathcal{T}$ such that $\sigma_{n_m} \to \sigma$ in $\mathcal{C}$ as $m \to \infty$.

The next propositions contain some elementary consequences of the above definition.

Proposition 2.14. *Suppose N is closed in X, $\mathcal{T}_\kappa \to \mathcal{T}$ and $\operatorname{Inv}_{\mathcal{T}}(N) \subset \operatorname{Int}_X(N)$. Assume also that each $\mathcal{T}_\kappa$ is translation invariant. Then there is a $\kappa_0 \in \mathbb{N}$ such that $\operatorname{Inv}_{\mathcal{T}_\kappa}(N) \subset \operatorname{Int}_X(N)$ for all $\kappa \geq \kappa_0$.*

Proof. If the proposition is not true then, by Definition 2.13 and the translation invariance of $\mathcal{T}_\kappa$, there is a sequence $(\kappa_n)_{n \in \mathbb{N}}$ with $\kappa_n \to \infty$ as $n \to \infty$, a sequence $(\sigma_n)_{n \in \mathbb{N}}$ such that $\sigma_n \in \mathcal{T}_{\kappa_n}$ for all $n \in \mathbb{N}$ and a $\sigma \in \mathcal{T}$ such that $\sigma_n \to \sigma$ as $n \to \infty$, $\sigma_n(\mathbb{R}) \subset N$ and $\sigma_n(0) \in \partial_X(N)$ for every $n \in \mathbb{N}$. Since N is closed it follows that $\sigma(\mathbb{R}) \subset N$ and $\sigma(0) \in \partial_X(N)$, a contradiction. $\square$

Proposition 2.15. *Suppose N is closed and U is open in X, $\mathcal{T}_\kappa \to \mathcal{T}$ and $\operatorname{Inv}_{\mathcal{T}}(N) \subset \operatorname{Inv}_{\mathcal{T}}(U)$. Assume also that each $\mathcal{T}_\kappa$ is translation invariant. Then there is a $\kappa_0 \in \mathbb{N}$ such that $\operatorname{Inv}_{\mathcal{T}_\kappa}(N) \subset \operatorname{Inv}_{\mathcal{T}_\kappa}(U)$ for all $\kappa \geq \kappa_0$.*

Proof. If the proposition is not true, then, by Definition 2.13 and the translation invariance of $\mathcal{T}_\kappa$, there is a sequence $(\kappa_n)_{n \in \mathbb{N}}$ with $\kappa_n \to \infty$ as $n \to \infty$, a sequence $(\sigma_n)_{n \in \mathbb{N}}$ such that $\sigma_n \in \mathcal{T}_{\kappa_n}$ for all $n \in \mathbb{N}$ and a $\sigma \in \mathcal{T}$ such that $\sigma_n \to \sigma$ as $n \to \infty$, $\sigma_n(\mathbb{R}) \subset N$ and $\sigma_n(0) \in X \setminus U$ for every $n \in \mathbb{N}$. We thus obtain that $\sigma(0) \in X \setminus U$. However, since N is closed it follows that $\sigma(\mathbb{R}) \subset N$ and so $\sigma(0) \in \operatorname{Inv}_{\mathcal{T}}(N) \subset \operatorname{Inv}_{\mathcal{T}}(U) \subset U$, a contradiction. $\square$

Proposition 2.16. *Suppose N is closed in X, $N' \subset X$ is arbitrary, $\mathcal{T}_\kappa \to \mathcal{T}$ and $\operatorname{Inv}_{\mathcal{T}}(N) \subset \operatorname{Inv}_{\mathcal{T}}(N') \subset \operatorname{Int}_X(N')$. Assume also that each $\mathcal{T}_\kappa$ is translation invariant. Then there is a $\kappa_0 \in \mathbb{N}$ such that $\operatorname{Inv}_{\mathcal{T}_\kappa}(N) \subset \operatorname{Inv}_{\mathcal{T}_\kappa}(N')$ for all $\kappa \geq \kappa_0$.*

Proof. Let $U := \operatorname{Int}_X(N')$. Since $\operatorname{Inv}_{\mathcal{T}}(N') \subset U$ we obtain that $\operatorname{Inv}_{\mathcal{T}}(U) = \operatorname{Inv}_{\mathcal{T}}(N')$ so the proposition follows from Proposition 2.15. $\square$

PROPOSITION 2.17. *Suppose N and N' are closed in X, $\mathcal{T}_\kappa \to \mathcal{T}$,*

$$\operatorname{Inv}_{\mathcal{T}}(N) \subset \operatorname{Int}_X(N),\ \operatorname{Inv}_{\mathcal{T}}(N') \subset \operatorname{Int}_X(N') \text{ and } \operatorname{Inv}_{\mathcal{T}}(N) = \operatorname{Inv}_{\mathcal{T}}(N').$$

Assume also that each $\mathcal{T}_\kappa$ is translation invariant.

Then there is a $\kappa_0 \in \mathbb{N}$ such that $\operatorname{Inv}_{\mathcal{T}_\kappa}(N) = \operatorname{Inv}_{\mathcal{T}_\kappa}(N')$ for all $\kappa \geq \kappa_0$.

PROOF. This is an immediate consequence of Proposition 2.16. □

PROPOSITION 2.18. *Suppose N is closed in X, $\mathcal{T}_\kappa \to \mathcal{T}$, $t \in]0,\infty[$ and $\mathcal{T}(N,t) \subset \operatorname{Int}_X(N)$. Then there is a $\kappa_0 \in \mathbb{N}$ such that $\mathcal{T}_\kappa(N,t) \subset \operatorname{Int}_X(N)$ for all $\kappa \geq \kappa_0$.*

PROOF. If the proposition is not true then, by Definition 2.13, there is a sequence $(\kappa_n)_{n\in\mathbb{N}}$ with $\kappa_n \to \infty$ as $n \to \infty$, a sequence $(\sigma_n)_{n\in\mathbb{N}}$ such that $\sigma_n \in \mathcal{T}_{\kappa_n}$ for all $n \in \mathbb{N}$ and a $\sigma \in \mathcal{T}$ such that $\sigma_n \to \sigma$ as $n \to \infty$, $\sigma_n(0) \in N$ and $\sigma_n(t) \notin \operatorname{Int}_X(N)$ for every $n \in \mathbb{N}$. Since N is closed it follows that $\sigma(0) \in N$ and $\sigma(t) \notin \operatorname{Int}_X(N)$, a contradiction. □

We can now state a basic perturbation stability result for attractor-repeller pairs:

THEOREM 2.19. *Let $(\mathcal{T}_\kappa)_{\kappa\in\mathbb{N}}$ be a sequence of compact and translation invariant subsets of $\mathcal{C}$ and $\mathcal{T} \subset \mathcal{C}$ be compact and translation invariant. Suppose $\mathcal{T}_\kappa \to \mathcal{T}$ and let (A, A^*) be a $\mathcal{T}$-attractor-repeller pair. Let V (resp. V^*) be closed in X and such that $A = \operatorname{Inv}_{\mathcal{T}}(V) \subset \operatorname{Int}_X(V)$ (resp. $A^* = \operatorname{Inv}_{\mathcal{T}^-}(V^*) \subset \operatorname{Int}_X(V^*)$). Then there is a $\kappa_0 \in \mathbb{N}$ such that $(\operatorname{Inv}_{\mathcal{T}_\kappa}(V), \operatorname{Inv}_{\mathcal{T}_\kappa}(V^*))$ is a $\mathcal{T}_\kappa$-attractor-repeller pair for all $\kappa \geq \kappa_0$.*

PROOF. Let N and N^* be closed and such that $A = \omega_{\mathcal{T}}(N) \subset \operatorname{Int}_X(N)$ and $A^* = \omega_{\mathcal{T}^-}(N^*) \subset \operatorname{Int}_X(N^*)$. Since A and A^* are disjoint and closed by Theorem 2.11 we may use Proposition 2.7 and choose N and N^* smaller, if necessary, to ensure that N and N^* are disjoint. For $\kappa \in \mathbb{N}$ set $A_\kappa = \operatorname{Inv}_{\mathcal{T}_\kappa}(N)$ and $\widetilde{A}_\kappa = \operatorname{Inv}_{\mathcal{T}_\kappa}(N^*)$. By Theorem 2.8 there is a $t_0 \in]0,\infty[$ such that $\mathcal{T}(N, t_0) \subset \operatorname{Int}_X(N)$. Consequently, by Proposition 2.18 there is a $\kappa_0 \in \mathbb{N}$ such that $\mathcal{T}_\kappa(N, t_0) \subset \operatorname{Int}_X(N)$ for all $\kappa \geq \kappa_0$. Thus Theorem 2.8 implies that

$$A_\kappa = \omega_{\mathcal{T}_\kappa}(N) \subset \operatorname{Int}_X(N), \quad \kappa \geq \kappa_0, \tag{2.23}$$

so A_κ is a $\mathcal{T}_\kappa$-attractor for all $\kappa \geq \kappa_0$. Set $A^*_\kappa = A^*_{\mathcal{T}_\kappa}$. If $\kappa \geq \kappa_0$ and $x \in \widetilde{A}_\kappa$ then there is a $\sigma \in \mathcal{T}_\kappa$ with $\sigma(0) = x$ and $\sigma(\mathbb{R}) \subset N^*$. Since N^* is closed, we conclude that $\omega(\sigma) \subset N^* \subset X \setminus N$, so $\omega(\sigma) \cap A_\kappa = \emptyset$. Hence $x \in A^*_\kappa$ which proves that $\widetilde{A}_\kappa \subset A^*_\kappa$. Now suppose that $A^*_\kappa \not\subset \widetilde{A}_\kappa$ for infinitely many $\kappa \in \mathbb{N}$. Then there are sequences $(\kappa_n)_{n\in\mathbb{N}}$ with $\kappa_n \to \infty$ as $n \to \infty$ and $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in A^*_{\kappa_n} \setminus \widetilde{A}_{\kappa_n}$ for all $n \in \mathbb{N}$. Thus there is a sequence $(\sigma_n)_{n\in\mathbb{N}}$ with $\sigma_n \in \mathcal{T}_{\kappa_n}$, $x_n = \sigma_n(0)$ and $\omega(\sigma_n) \cap A_{\kappa_n} = \emptyset$ for all $n \in \mathbb{N}$. Proposition 2.10 and (2.23) imply that $\sigma_n(\mathbb{R}) \cap N = \emptyset$ for all $n \in \mathbb{N}$ large enough. On the other hand, for every $n \in \mathbb{N}$ we have $\sigma_n(\mathbb{R}) \not\subset N^*$ since otherwise $x_n \in \mathrm{Inv}_{\mathcal{T}_{\kappa_n}}(N^*) = \widetilde{A}_{\kappa_n}$, a contradiction. It follows that for every $n \in \mathbb{N}$ there is a $t_n \in \mathbb{R}$ with $\sigma_n(t_n) \notin N^*$. Let $\tau_n = \mathrm{tsl}_{t_n}\sigma_n$, $n \in \mathbb{N}$. Taking subsequences if necessary we may assume that there is a $\tau \in \mathcal{T}$ such that $\tau_n \to \tau$ in $\mathcal{C}$. It follows that $\tau(0) \notin \mathrm{Int}_X(N^*)$. By Theorem 2.11 this implies that $\omega(\tau) \subset A$ so $\tau(t) \in \mathrm{Int}_X(N)$ for some $t \in \mathbb{R}$. But $\tau_n(\mathbb{R}) = \sigma_n(\mathbb{R}) \subset X \setminus N$ for all $n \in \mathbb{N}$ so $\tau(\mathbb{R}) \subset X \setminus \mathrm{Int}_X(N)$, a contradiction. This proves that $A^*_\kappa \subset \widetilde{A}_\kappa$ so $A^*_\kappa = \widetilde{A}_\kappa$ for all κ large enough. Thus, for all such κ, the pair $(A_\kappa, \widetilde{A}_\kappa)$ is a $\mathcal{T}_\kappa$-attractor-repeller pair. Now, since $A = \mathrm{Inv}_{\mathcal{T}}(V) \subset \mathrm{Int}_X(V)$ and $A = \mathrm{Inv}_{\mathcal{T}}(N) \subset \mathrm{Int}_X(N)$, Proposition 2.17 implies that $\mathrm{Inv}_{\mathcal{T}_\kappa}(V) = \mathrm{Inv}_{\mathcal{T}_\kappa}(N) = A_\kappa$ for all $\kappa \in \mathbb{N}$ large enough. Similarly, $\mathrm{Inv}_{\mathcal{T}^-_\kappa}(V^*) = \mathrm{Inv}_{\mathcal{T}^-_\kappa}(N^*) = \widetilde{A}_\kappa$ for all $\kappa \in \mathbb{N}$ large enough. This completes the proof. □

3. $\mathcal{T}$-Morse decompositions

In this section we again assume that we have a fixed subset $\mathcal{T}$ of $\mathcal{C}$. We will define attractor filtrations relative to $\mathcal{T}$ (Definitions 3.1) and we will present two definitions of a Morse decomposition relative to $\mathcal{T}$ (Definitions 3.2 and 3.3). If $\mathcal{T}$ is compact, translation invariant and satisfies a so-called cut-and-glue invariance property, then, as we will show in Theorems 3.8 and 3.10 these two definitions are equivalent. Finally, we establish perturbation stability properties for attractor filtrations and Morse decompositions (Theorems 3.14 and 3.15).

We begin with the following definitions.

Definition 3.1. A *$\mathcal{T}$-attractor filtration* (*of length m*) is a sequence $(A_r)_{r=0}^m$ of $\mathcal{T}$-attractors such that $A_0 = \emptyset$, $A_m = S_{\mathcal{T}}$ and $A_r \subset A_{r+1}$ for $r \in [\![0, m-1]\!]$.

If $(A_r)_{r=0}^m$ is a $\mathcal{T}$-attractor filtration then the sequence $((A_r)^*_{\mathcal{T}})_{r=0}^m$ is called the *dual $\mathcal{T}$-repeller filtration of* $(A_r)_{r=0}^m$.

Definition 3.2. A finite sequence $(M_r)_{r=1}^m$ is called a *$\mathcal{T}$-Morse decomposition of the first kind* if there is a $\mathcal{T}$-attractor filtration $(A_r)_{r=0}^m$ such that $M_r = A_r \cap (A_{r-1})^*_{\mathcal{T}}$ for $r \in [\![1, m]\!]$.

Definition 3.3. A finite sequence $(M_r)_{r=1}^m$ is called a *$\mathcal{T}$-Morse decomposition of the second kind* if the following properties hold:

(3.1) The sets M_r, $r \in [\![1, m]\!]$, are closed, $\mathcal{T}$-invariant and pairwise disjoint.

(3.2) For every $\sigma \in \mathcal{T}$ either $\sigma(\mathbb{R}) \subset M_k$ for some $k \in [\![1, m]\!]$ or else there are k, $l \in [\![1, m]\!]$ with $k < l$, $\alpha(\sigma) \subset M_l$ and $\omega(\sigma) \subset M_k$.

The following simple result is important for applications:

Proposition 3.4. *Let $\mathcal{T}$ be compact, translation invariant and gradient-like with respect to a function $\varphi\colon S_{\mathcal{T}} \to \mathbb{R}$. Suppose that the set $\mathcal{E}$ of $\mathcal{T}$-equilibria has*

m elements for some $m \in \mathbb{N}$. Put all the elements of $\mathcal{E}$ in a sequence $(x_r)_{r=1}^m$ with $\varphi(x_r) \leq \varphi(x_{r+1})$ for all $r \in [\![1, m-1]\!]$. Then $(\{x_r\})_{r=1}^m$ is a $\mathcal{T}$-Morse decomposition of the second kind.

Proof. Clearly the sets $\{x_r\}$, $r \in [\![1,m]\!]$, are closed and pairwise disjoint. Moreover, the definition of $\mathcal{T}$-equilibria implies that for every $r \in [\![1,m]\!]$ the set $\{x_r\}$ is $\mathcal{T}$-invariant.

Let $\sigma \in \mathcal{T}$ be arbitrary. Either σ is a constant map so $\sigma(t) \equiv x_i$ for some $i \in [\![1,m]\!]$ and all $t \in \mathbb{R}$, or else σ is not constant and so, by Proposition 2.2 the sets $\alpha(\sigma)$ and $\omega(\sigma)$ are connected and contain only $\mathcal{T}$-equilibria. Moreover, $\phi(x) > \phi(y)$ for $x \in \alpha(\sigma)$ and $y \in \omega(\sigma)$. It follows that $\alpha(\sigma) = \{x_l\}$ and $\omega(\sigma) = \{x_k\}$ for some k and $l \in [\![1,m]\!]$ with $k < l$. □

We have the following simple result.

Proposition 3.5. *Let $\mathcal{T}$ be compact and translation invariant and $(M_r)_{r=1}^m$ be a $\mathcal{T}$-Morse decomposition of the second kind. Moreover, let k_1, $k_2 \in [\![1,m]\!]$, $k_1 \leq k_2$ and $\sigma \in \mathcal{T}$ be arbitrary with $\alpha(\sigma) \subset M_{k_1}$ and $\omega(\sigma) \subset M_{k_2}$. Then $k_1 = k_2$ and $\sigma(\mathbb{R}) \subset M_{k_1} = M_{k_2}$.*

Proof. Since $\mathcal{T}$ is compact and translation invariant, it follows that both $\omega(\sigma)$ and $\alpha(\sigma)$ are nonempty. By Definition 3.3 two possible cases can occur:

Case 1. There is a $k \in [\![1,m]\!]$ with $\sigma(\mathbb{R}) \subset M_k$.

Since M_k is closed we obtain

$$\emptyset \neq \alpha(\sigma) \subset M_k \cap M_{k_1} \quad \text{and} \quad \emptyset \neq \omega(\sigma) \subset M_k \cap M_{k_2}.$$

Since the sets M_r, $r \in [\![1,m]\!]$, are pairwise disjoint, we obtain $k = k_1 = k_2$ and so the conclusion follows in this case.

Case 2. There are k, $l \in [\![1,m]\!]$ with $k < l$, $\alpha(\sigma) \subset M_l$ and $\omega(\sigma) \subset M_k$.

However, this implies

$$\emptyset \neq \alpha(\sigma) \subset M_l \cap M_{k_1} \quad \text{and} \quad \emptyset \neq \omega(\sigma) \subset M_k \cap M_{k_2},$$

so $k_1 = l > k = k_2 \geq k_1$, a contradiction. □

We now introduce the following basic concept:

Definition 3.6. Given σ_1 and $\sigma_2 \in \mathcal{C}$ with $\sigma_1(0) = \sigma_2(0)$ the map

$$\sigma_1 \triangleright \sigma_2 \colon \mathbb{R} \to X, \quad (\sigma_1 \triangleright \sigma_2)(t) := \begin{cases} \sigma_1(t) & \text{if } t \leq 0, \\ \sigma_2(t) & \text{if } t \geq 0, \end{cases}$$

is called the *cut-and-glue of* (σ_1, σ_2).

Intuitively, we cut σ_k, $k = 1$, 2, into the "left" and "right" parts and glue the left part of σ_1 to the right part of σ_2.

A subset $\mathcal{T}$ of $\mathcal{C}$ is called *cut-and-glue invariant* if for all σ_1, $\sigma_2 \in \mathcal{T}$ with $\sigma_1(0) = \sigma_2(0)$ it follows that $\sigma_1 \triangleright \sigma_2 \in \mathcal{T}$.

Proposition 3.7. *Suppose that $\mathcal{T}$ is translation and cut-and-glue invariant, $s \in \mathbb{R}$ is arbitrary and σ_1, $\sigma_2 \in \mathcal{T}$ are arbitrary such that $\sigma_1(s) = \sigma_2(s)$. Then $\sigma_1 \triangleright_s \sigma_2 \in \mathcal{T}$, where*

$$\sigma_1 \triangleright_s \sigma_2 \colon \mathbb{R} \to X, \quad (\sigma_1 \triangleright_s \sigma_2)(t) := \begin{cases} \sigma_1(t) & \text{if } t \le s, \\ \sigma_2(t) & \text{if } t \ge s. \end{cases}$$

Proof. Set $\tau_k = \mathrm{tsl}_s \sigma_k$, $k = 1, 2$. Then $\tau_k \in \mathcal{T}$ for $k = 1, 2$ and $\tau_1(0) = \tau_2(0)$, so $\tau := \tau_1 \triangleright \tau_2 \in \mathcal{T}$. Hence $\sigma := \mathrm{tsl}_{-s}\tau \in \mathcal{T}$. Now

$$\sigma(t) = \tau(-s+t) = \begin{cases} \tau_1(-s+t) = \sigma_1(t) & \text{if } -s+t \le 0, \\ \tau_2(-s+t) = \sigma_2(t) & \text{if } -s+t \ge 0. \end{cases}$$

Hence $\sigma = \sigma_1 \triangleright_s \sigma_2$. □

We can now state the following theorem.

Theorem 3.8. *Suppose $\mathcal{T}$ is compact, translation and cut-and-glue invariant. Moreover, let $(M_r)_{r=1}^m$ be a $\mathcal{T}$-Morse-decomposition of the first kind. Then $(M_r)_{r=1}^m$ is a $\mathcal{T}$-Morse-decomposition of the second kind.*

Proof. Let $(A_r)_{r=0}^m$ be a $\mathcal{T}$-attractor filtration with $M_r = A_r \cap (A_{r-1})^*_{\mathcal{T}}$ for $r \in [\![1, m]\!]$. Since A_r and $(A_{r-1})^*_{\mathcal{T}}$ are closed it follows that M_r is closed, for all $r \in [\![1, m]\!]$. Let $r \in [\![1, m]\!]$ and $x \in M_r$ be arbitrary. Since A_r is $\mathcal{T}$-invariant, there is a $\sigma_1 \in \mathcal{T}$ such that $\sigma_1(0) = x$ and $\sigma_1(\mathbb{R}) \subset A_r$. By the definition of $(A_{r-1})^*_{\mathcal{T}}$, there is a $\sigma_2 \in \mathcal{T}$ such that $\sigma_2(0) = x$ and $\omega(\sigma_2) \subset X \setminus A_{r-1}$. Let $\sigma := \sigma_1 \triangleright \sigma_2$. Then $\sigma \in \mathcal{T}$. Let $t \in \mathbb{R}$ be arbitrary and $\sigma_t := \mathrm{tsl}_t \sigma$. Then $\sigma_t \in \mathcal{T}$, $\alpha(\sigma_t) = \alpha(\sigma) = \alpha(\sigma_1) \subset A_r$ and $\omega(\sigma_t) = \omega(\sigma) = \omega(\sigma_2) \subset X \setminus A_{r-1}$ so $\sigma(t) = \sigma_t(0) \in A_r \cap (A_{r-1})^*_{\mathcal{T}}$. Since $t \in \mathbb{R}$ is arbitrary, we conclude that $\sigma(\mathbb{R}) \subset A_r \cap (A_{r-1})^*_{\mathcal{T}}$. Since $x \in M_r$ is arbitrary, this implies that M_r is $\mathcal{T}$-invariant.

Now let k and $l \in [\![1, m]\!]$ be arbitrary with $k \ne l$. We may assume that $k < l$. Hence $k \le l - 1$ so $(A_{l-1})^*_{\mathcal{T}} \subset (A_k)^*_{\mathcal{T}}$ so

$$M_k \cap M_l = A_k \cap (A_{k-1})^*_{\mathcal{T}} \cap A_l \cap (A_{l-1})^*_{\mathcal{T}} \subset A_k \cap (A_k)^*_{\mathcal{T}} = \emptyset.$$

This concludes the proof of property (3.1).

Now let $\sigma \in \mathcal{T}$ be arbitrary. Since $\omega(\sigma) \subset S_{\mathcal{T}} = A_m$ and $\alpha(\sigma) \subset S_{\mathcal{T}} = (A_0)^*_{\mathcal{T}}$ it follows that there is a smallest $k \in [\![0, m]\!]$ and a largest $l \in [\![0, m]\!]$ such that $\omega(\sigma) \subset A_k$ and $\alpha(\sigma) \subset (A_l)^*_{\mathcal{T}}$. Since $S_{\mathcal{T}}$ is compact by Proposition 2.2, it follows that $\omega(\sigma)$ and $\alpha(\sigma)$ are both nonempty so, in particular, $k \ne 0$ and $l \ne m$ (as

$A_0 = \emptyset = (A_m)^*_{\mathcal{T}}$). We thus have $\omega(\sigma) \not\subset A_{k-1}$ and $\alpha(\sigma) \not\subset (A_{l+1})^*_{\mathcal{T}}$ which, by Theorem 2.11, implies that $\sigma(\mathbb{R}) \subset (A_{k-1})^*_{\mathcal{T}}$ and $\sigma(\mathbb{R}) \subset A_{l+1}$. Thus

$$\sigma(\mathbb{R}) \subset A_{l+1} \cap (A_{k-1})^*_{\mathcal{T}}. \tag{3.3}$$

If $l+1 = k$ then (3.3) implies that

$$\sigma(\mathbb{R}) \subset M_k. \tag{3.4}$$

Suppose that $l+1 \neq k$. We claim that $k < l+1$. In fact, otherwise $l+1 < k$ so $l+1 \leq k-1$ so (3.3) shows that

$$\sigma(\mathbb{R}) \subset A_{l+1} \cap (A_{k-1})^*_{\mathcal{T}} \subset A_{k-1} \cap (A_{k-1})^*_{\mathcal{T}} = \emptyset,$$

a contradiction, which proves the claim. Using (3.3) and the definition of k and l we also have

$$\omega(\sigma) \subset A_k \cap (A_{k-1})^*_{\mathcal{T}} = M_k, \tag{3.5}$$

$$\alpha(\sigma) \subset A_{l+1} \cap (A_l)^*_{\mathcal{T}} = M_{l+1}. \tag{3.6}$$

The above claim together with (3.4), (3.5) and (3.6) implies property (3.2). □

The next result shows that a $\mathcal{T}$-Morse decomposition of the first kind uniquely determines its $\mathcal{T}$-attractor filtration.

Proposition 3.9. *Suppose $\mathcal{T}$ is compact, translation and cut-and-glue invariant. Let $(A_r)_{r=0}^m$ be a $\mathcal{T}$-attractor filtration and set $M_r := A_r \cap (A_{r-1})^*_{\mathcal{T}}$ for $r \in [\![1, m]\!]$.*

Then, for every $k \in [\![0, m]\!]$,

$$A_k = \left\{ x \;\middle|\; \exists \sigma \in \mathcal{T} \text{ with } \sigma(0) = x \text{ and } \alpha(\sigma) \subset \bigcup_{r=1}^{k} M_r \right\}. \tag{3.7}$$

Proof. Note that, if $k = 0$ then $\bigcup_{r=1}^0 M_r = \emptyset$ so the right hand side of (3.7) is the empty set. Since $A_0 = \emptyset$, Formula (3.7) holds in this case.

Let $k \in [\![1, m]\!]$ and $x \in A_k$ be arbitrary. Since A_k is $\mathcal{T}$-invariant by Theorem 2.11, there is a $\sigma \in \mathcal{T}$ with $\sigma(0) = x$ and $\sigma(\mathbb{R}) \subset A_k$. Thus there is a smallest $i \in [\![1, k]\!]$ with $\sigma(\mathbb{R}) \subset A_i$. Hence $\sigma(\mathbb{R}) \not\subset A_{i-1}$ so $\alpha(\sigma) \subset A_i \cap (A_{i-1})^*_{\mathcal{T}} = M_i$. It follows that $\alpha(\sigma) \subset \bigcup_{r=1}^k M_r$. Conversely, if x is such that there is a $\sigma \in \mathcal{T}$ with $\sigma(0) = x$ and $\alpha(\sigma) \subset \bigcup_{r=1}^k M_r$, then, for some $i \in [\![1, k]\!]$, $\alpha(\sigma) \subset M_i = A_i \cap (A_{i-1})^*_{\mathcal{T}}$, so $\alpha(\sigma) \subset A_i \subset A_k$ and so, by Theorem 2.11, $\sigma(\mathbb{R}) \subset A_k$. Hence $x \in A_k$. □

Proposition 3.9 suggests the following converse of Theorem 3.8:

Theorem 3.10. *Suppose $\mathcal{T}$ is compact, translation and cut-and-glue invariant. Moreover, let $(M_r)_{r=1}^m$ be a $\mathcal{T}$-Morse-decomposition of the second kind. For $k \in [\![0, m]\!]$ define the sets*

$$A_k = \left\{ x \;\middle|\; \exists \sigma \in \mathcal{T} \text{ with } \sigma(0) = x \text{ and } \alpha(\sigma) \subset \bigcup_{r=1}^{k} M_r \right\}. \tag{3.8}$$

Then $(A_k)_{k=0}^m$ is a $\mathcal{T}$-attractor filtration and $M_k = A_k \cap (A_{k-1})^*_{\mathcal{T}}$ for $k \in [\![1, m]\!]$. In particular, $(M_r)_{r=1}^m$ is a $\mathcal{T}$-Morse-decomposition of the first kind.

Proof. Note that, by (3.8) and the translation invariance of $\mathcal{T}$, the set A_k is $\mathcal{T}$-invariant for every $k \in [\![0, m]\!]$. We first claim that for k, $l \in [\![1, m]\!]$,

$$M_k \subset A_l \text{ if } k \le l \text{ and } M_k \subset X \setminus A_l \text{ if } k > l. \tag{3.9}$$

In fact, suppose first that $k \le l$ and let $x \in M_k$ be arbitrary. Since M_k is $\mathcal{T}$-invariant, there is a $\sigma \in \mathcal{T}$ with $\sigma(0) = x$ and $\sigma(\mathbb{R}) \subset M_k$. Since M_k is closed and $k \le l$ we see that $\alpha(\sigma) \subset M_k \subset \bigcup_{r=1}^{l} M_l$ and so $x \in A_l$, as claimed.

Now assume that $k > l$ and suppose that there is an $x \in M_k \cap A_l$. Using the definition of A_l and the $\mathcal{T}$-invariance of M_k we obtain the existence of σ_1, $\sigma_2 \in \mathcal{T}$ with $\sigma_1(0) = x = \sigma_2(0)$ such that $\alpha(\sigma_1) \subset \bigcup_{r=1}^{l} M_r$ and $\sigma_2(\mathbb{R}) \subset M_k$. Set $\sigma = \sigma_1 \triangleright \sigma_2$. Thus there is an $r \in [\![1, l]\!]$ with $\alpha(\sigma) = \alpha(\sigma_1) \subset M_r$ and, since M_k is closed, we also have that $\omega(\sigma) = \omega(\sigma_2) \subset M_k$. Now Proposition 3.5 and the fact that $r < k$ immediately lead to a contradiction, proving the claim.

By (3.2) and (3.8) we have that $A_0 = \emptyset$ and $A_m = S_{\mathcal{T}}$. Thus Corollary 2.5 implies that A_0 and A_m are $\mathcal{T}$-attractors. Let $l \in [\![1, m-1]\!]$ be arbitrary and assume that A_{l+1} is a $\mathcal{T}$-attractor. We will prove that A_l is a $\mathcal{T}$-attractor. This will imply that A_k is a $\mathcal{T}$-attractor for all $k \in [\![0, m]\!]$.

We require three lemmas.

Lemma 3.11. *Let V be an open set with $M_{l+1} \subset V$ and $M_k \subset X \setminus V$ for all $k \ne l+1$. Let $x \in X$, $(\tau_\nu)_{\nu \in \mathbb{N}}$ be a sequence in $\mathcal{T}$ with $\tau_\nu(0) \to x$ as $\nu \to \infty$ and $(s_\nu)_{\nu \in \mathbb{N}}$ be a sequence in $[0, \infty[$ such that $\tau_\nu(t) \in \mathrm{Cl}_X(V)$ for all $\nu \in \mathbb{N}$ and all $t \in [0, s_\nu]$. Assume that there is a $z \in M_{l+1}$ such that $\tau_\nu(s_\nu) \to z$ as $\nu \to \infty$. Then there is a $\tau \in \mathcal{T}$ with*

$$\tau(0) = x \quad \text{and} \quad \omega(\tau) \subset M_{l+1}. \tag{3.10}$$

Proof. Taking subsequences if necessary, we may assume that $\tau_\nu \to \tau'$ for some $\tau' \in \mathcal{T}$. We have two possible cases to consider:

Case 1. The sequence $(s_\nu)_{\nu \in \mathbb{N}}$ is unbounded. Then, taking subsequences, we may assume that $s_\nu \to \infty$ as $\nu \to \infty$. It then follows that for every $t \in [0, \infty[$ there is a $\nu(t) \in \mathbb{N}$ such that $t \in [0, s_\nu]$ (and so $\tau_\nu(t) \in \mathrm{Cl}_X(V)$) for $\nu \ge \nu(t)$. Hence $\tau'(t) \in \mathrm{Cl}_X(V)$ for all $t \in [0, \infty[$ and so our choice of V imply that $\tau := \tau'$ satisfies (3.10).

Case 2. The sequence $(s_\nu)_{\nu\in\mathbb{N}}$ is bounded. Then, taking subsequences, we may assume that $s_\nu \to s$ as $\nu \to \infty$, for some $s \in [0,\infty[$. Then $\tau_\nu(s_\nu) \to \tau'(s)$ so $\tau'(s) = z \in M_{l+1}$. Since $\mathcal{T}$ is translation invariant and M_{l+1} is $\mathcal{T}$-invariant, we see that there is a $\tau'' \in \mathcal{T}$ with $\tau''(s) = \tau'(s)$ and $\tau''(\mathbb{R}) \subset M_{l+1}$. Let $\tau := \tau' \triangleright_s \tau''$. Then $\tau \in \mathcal{T}$ and since $0 \le s$ we have $\tau(0) = \tau'(0) = x$ and so τ satisfies (3.10). □

Lemma 3.12. *A_l is closed.*

Proof. In fact, let $(y_n)_{n\in\mathbb{N}}$ be an arbitrary sequence in A_l such that $y_n \to y$ as $n \to \infty$, for some $y \in X$. We want to prove that $y \in A_l$. From the definition of A_l we obtain a sequence $(\sigma_n)_{n\in\mathbb{N}}$ in $\mathcal{T}$ such that $\sigma_n(0) = y_n$ and

$$\alpha(\sigma_n) \subset \bigcup_{r=1}^{l} M_r \quad \text{for all } n \in \mathbb{N}. \tag{3.11}$$

It clearly follows that

$$\sigma_n(\mathbb{R}) \subset A_l \subset A_{l+1}, \quad n \in \mathbb{N}. \tag{3.12}$$

By taking subsequences if necessary we may suppose that $\sigma_n \to \sigma$ in $\mathcal{C}$ for some $\sigma \in \mathcal{T}$. Since A_{l+1} is closed, (3.12) implies that $\sigma(\mathbb{R}) \subset A_{l+1}$ and $\alpha(\sigma) \subset A_{l+1}$. There is an $r \in [\![1,m]\!]$ with $\alpha(\sigma) \subset M_r$. Hence $\alpha(\sigma) \subset M_r \cap A_{l+1}$ which in view of (3.9) implies that $r \le l+1$. If $r \le l$ then the fact that $\sigma(0) = y$ implies that $y \in A_l$ and we are done. Therefore, suppose that $r = l+1$. We will show that this leads to a contradiction, proving the lemma.

Since the sets M_k, $k \in [\![1,m]\!]$, are closed and pairwise disjoint, there is an open set V with $M_{l+1} \subset V$ and $M_k \subset X \setminus V$ for all $k \ne l+1$. It follows that there is a sequence $(t_\nu)_{\nu\in\mathbb{N}}$ with $t_\nu \to \infty$ and $\sigma(-t_\nu) \to z$ as $\nu \to \infty$, for some $z \in M_{l+1}$. We can choose a strictly increasing sequence $(n_\nu)_{\nu\in\mathbb{N}}$ such that $d(\sigma_{n_\nu}(-t_\nu), \sigma(-t_\nu)) \to 0$ as $\nu \to \infty$. It follows that $\sigma_{n_\nu}(-t_\nu) \to z$ as $\nu \to \infty$. Since $z \in V$ and V is open we may also assume that $\sigma_{n_\nu}(-t_\nu) \in V$ for all $\nu \in \mathbb{N}$. Now (3.11) implies that for every $\nu \in \mathbb{N}$ there is a $t'_\nu \in \mathbb{R}$ with $-t'_\nu < -t_\nu$, such that $\sigma_{n_\nu}(-t'_\nu) \in \partial_X(V)$ and $\sigma_{n_\nu}(t) \in \mathrm{Cl}_X(V)$ for all $t \in [-t'_\nu, -t_\nu]$. Set $s_\nu := t'_\nu - t_\nu$ and $\tau_\nu := \mathrm{tsl}_{-t'_\nu}\sigma_{n_\nu}$, $\nu \in \mathbb{N}$. It follows that, for all $\nu \in \mathbb{N}$, $\tau_\nu \in \mathcal{T}$ and, moreover, that

$$\tau_\nu(0) \in \partial_X(V) \text{ and } \tau_\nu(t) \in \mathrm{Cl}_X(V) \text{ for all } \nu \in \mathbb{N} \text{ and all } t \in [0, s_\nu]. \tag{3.13}$$

Taking subsequences if necessary we may assume that $\tau_\nu \to \tau'$ as $\nu \to \infty$, for some $\tau' \in \mathcal{T}$. Using (3.12) we obtain $\tau_\nu(\mathbb{R}) = \sigma_{n_\nu}(\mathbb{R}) \subset A_{l+1}$ so $\tau'(\mathbb{R}) \subset A_{l+1}$ and thus $\alpha(\tau') \subset A_{l+1}$, i.e. $\alpha(\tau') \subset M_r$ for some $r \in [\![1, l+1]\!]$. Let $x = \tau'(0)$. Since $\tau_\nu(s_\nu) \to z \in M_{l+1}$ as $\nu \to \infty$, Lemma 3.11 implies that there is a $\tau \in \mathcal{T}$ with $\tau(0) = x$ and $\omega(\tau) \subset M_{l+1}$. Let $\tau'' = \tau' \triangleright \tau$. Then $\tau'' \in \mathcal{T}$,

$\alpha(\tau'') = \alpha(\tau') \subset A_{l+1}$ and $\omega(\tau'') = \omega(\tau) \subset M_{l+1}$. Now Proposition 3.5 implies that $\tau''(\mathbb{R}) \subset M_{l+1}$ so $x = \tau(0) \in \partial_X(V) \cap M_{l+1} = \emptyset$ by (3.13), a contradiction.□

Lemma 3.13. *A_l is a $\mathcal{T}$-attractor.*

Proof. Given $Y \subset X$ and $\delta \in]0, \infty[$ we denote by $\mathcal{V}_\delta(Y)$ the closed δ-neighbourhood of Y, i.e.

$$\mathcal{V}_\delta(Y) = \{x \in X \mid \inf_{y \in Y} d(x, y) \leq \delta\}.$$

Since A_{l+1} is a $\mathcal{T}$-attractor, there is a closed set N such that $A_{l+1} = \omega_\mathcal{T}(N) \subset \mathrm{Int}_X(N)$. Since A_{l+1} is $\mathcal{T}$-invariant we have $A_{l+1} \subset S_\mathcal{T}$ and since A_{l+1} is closed and $S_\mathcal{T}$ is compact we conclude that A_{l+1} is compact. Thus there is a $\overline{\delta} \in]0, \infty[$ such that

$$\mathcal{V}_\delta(A_{l+1}) \subset \mathrm{Int}_X(N), \quad \delta \in]0, \overline{\delta}].$$

Now the $\mathcal{T}$-invariance of A_l implies that

$$A_l \subset \mathrm{Inv}_\mathcal{T}(\mathcal{V}_\delta(A_l)) \subset \omega_\mathcal{T}(\mathcal{V}_\delta(A_l)), \quad \delta \in]0, \overline{\delta}]. \tag{3.14}$$

We claim that

$$\omega_\mathcal{T}(\mathcal{V}_\delta(A_l)) \subset A_l, \quad \text{for some } \delta \in]0, \overline{\delta}]. \tag{3.15}$$

This claim, together with (3.14) and the fact that $A_l \subset \mathrm{Int}_X(\mathcal{V}_\delta(A_l))$ implies the lemma.

Suppose (3.15) is not true and let $(\delta_\nu)_{\nu \in \mathbb{N}}$ be a sequence in $]0, \overline{\delta}]$ with $\delta_\nu \to 0$ as $\nu \to \infty$. Let $\nu \in \mathbb{N}$ be arbitrary. Then there is a $y_\nu \in \omega_\mathcal{T}(\mathcal{V}_{\delta_\nu}(A_l)) \setminus A_l$. Hence there is a sequence $(\sigma_\nu^n)_{n \in \mathbb{N}}$ in $\mathcal{T}$ and a sequence $(t_\nu^n)_{n \in \mathbb{N}}$ in $\mathbb{R}$ such that $\sigma_\nu^n(0) \in \mathcal{V}_{\delta_\nu}(A_l)$ for all $n \in \mathbb{N}$ while $t_\nu^n \to \infty$ and $\sigma_\nu^n(t_\nu^n) \to y_\nu$ as $n \to \infty$. Taking subsequences if necessary we may assume that $\mathrm{tsl}_{t_\nu^n} \sigma_\nu^n \to \sigma_\nu$ for some $\sigma_\nu \in \mathcal{T}$. Then, for every $t \in \mathbb{R}$, it follows that $t_\nu^n + t \to \infty$ and $\sigma_\nu^n(t_\nu^n + t) \to \sigma_\nu(t)$ as $n \to \infty$, so

$$\sigma_\nu(t) \in \omega_\mathcal{T}(\mathcal{V}_{\delta_\nu}(A_l)) \subset \omega_\mathcal{T}(\mathcal{V}_{\delta_\nu}(A_{l+1})) \subset \omega_\mathcal{T}(N) = A_{l+1}.$$

It follows that $\sigma_\nu(\mathbb{R}) \subset A_{l+1}$, so, as A_{l+1} is closed, we conclude that $\alpha(\sigma_\nu) \subset A_{l+1}$. Hence, by (3.9), there is an $r \in [\![1, l+1]\!]$ such that $\alpha(\sigma_\nu) \subset M_r$. If $r \leq l$ then it follows that $y_\nu = \sigma_\nu(0) \in A_l$, a contradiction. Therefore,

$$\alpha(\sigma_\nu) \subset M_{l+1}, \quad \nu \in \mathbb{N}. \tag{3.16}$$

Let V be as in the proof of Lemma 3.12. Since A_l is closed by Lemma 3.12 and disjoint from M_{l+1}, we may assume, by taking V and $\overline{\delta}$ smaller, if necessary, that

$$\mathrm{Cl}_X(V) \cap \mathcal{V}_\delta(A_l) = \emptyset, \quad \delta \in]0, \overline{\delta}]. \tag{3.17}$$

Now (3.16) implies that, for every $\nu \in \mathbb{N}$, there is a $z_\nu \in M_{l+1}$ and a sequence $(r_\nu^\mu)_{\mu\in\mathbb{N}}$ in $\mathbb{R}$ with $r_\nu^\mu \to \infty$ as $\mu \to \infty$ and $d(\sigma_\nu(-r_\nu^\mu), z_\nu) < (1/\mu)$ for all $\mu \in \mathbb{N}$. In particular, $d(\sigma_\nu(-r_\nu^\nu), z_\nu) < (1/\nu)$. Taking subsequences if necessary we may also assume that there is a $z \in M_{l+1}$ such that $d(z_\nu, z) < (1/\nu)$ for all $\nu \in \mathbb{N}$.

Thus for every $\nu \in \mathbb{N}$ there is an $n(\nu) \in \mathbb{N}$ with $n(\nu) \geq \nu$, $s_\nu := t_\nu^{n(\nu)} - r_\nu^\nu > 0$ and $d(\sigma_\nu^{n(\nu)}(s_\nu), \sigma_\nu(-r_\nu^\nu)) < (1/\nu)$. Putting things together, we thus obtain that

$$d(\sigma_\nu^{n(\nu)}(s_\nu), z) < (3/\nu) \quad \text{for all } \nu \in \mathbb{N}.$$

Since $z \in V$ we may thus assume that $\sigma_\nu^{n(\nu)}(s_\nu) \in V$ and $\sigma_\nu^{n(\nu)}(0) \notin V$ for all $\nu \in \mathbb{N}$. Therefore for every $\nu \in \mathbb{N}$ there is a $\widetilde{s}_\nu \in]0, s_\nu[$ such that $\sigma_\nu^{n(\nu)}(\widetilde{s}_\nu) \in \partial_X(V)$ and $\sigma_\nu^{n(\nu)}(t) \in \mathrm{Cl}_X(V)$ for all $t \in [\widetilde{s}_\nu, s_\nu]$. Let $\tau_\nu := \mathrm{tsl}_{\widetilde{s}_\nu}\sigma_\nu^{n(\nu)}$, $\nu \in \mathbb{N}$. Then, for all $\nu \in \mathbb{N}$, $\tau_\nu(0) \in \partial_X(V)$ and $\tau_\nu(t) \in \mathrm{Cl}_X(V)$ for all $t \in [0, s_\nu - \widetilde{s}_\nu]$. Taking subsequences, if necessary, we may assume that $\tau_\nu(0) \to x$ as $\nu \to \infty$, for some $x \in \partial_X(V)$. Since $\tau_\nu(s_\nu - \widetilde{s}_\nu) \to z \in M_{l+1}$ as $\nu \to \infty$, an application of Lemma 3.11 shows that there is a $\tau \in \mathcal{T}$ with $\tau(0) = x$ and $\omega(\tau) \subset M_{l+1}$.

Now we have two possible cases:

Case 1. The sequence $(\widetilde{s}_\nu)_{\nu\in\mathbb{N}}$ is unbounded. We may then assume that $\widetilde{s}_\nu \to \infty$ as $\nu \to \infty$. Since $\sigma_\nu^{n(\nu)}(0) \in N$ for all $\nu \in \mathbb{N}$, it follows that $x \in \omega_{\mathcal{T}}(N) = A_{l+1}$, so there is a $\tau' \in \mathcal{T}$ with $\tau'(0) = x$ and $\alpha(\tau') \subset M_r$ for some $r \in [\![1, l+1]\!]$. Defining $\tau'' := \tau' \triangleright \tau$ we see that $\tau'' \in \mathcal{T}$, $\alpha(\tau'') \subset M_r$ and $\omega(\tau'') \subset M_{l+1}$, which by Proposition 3.5 implies that $\tau''(\mathbb{R}) \subset M_{l+1}$, and this is a contradiction since $\tau''(0) = x \in \partial_X(V)$.

Case 2. The sequence $(\widetilde{s}_\nu)_{\nu\in\mathbb{N}}$ is bounded. We may then assume that $\widetilde{s}_\nu \to s$ for some $s \in [0, \infty[$, $\tau_\nu \to \tau'$ and $\sigma_\nu^{n(\nu)}(0) \to w$ as $\nu \to \infty$, for some $\tau' \in \mathcal{T}$ and $w \in A_l$. Thus $\tau_\nu(-\widetilde{s}_\nu) \to w$ so $\tau'(-s) = w$ and $\tau'(0) = x$. The definition of A_l and the translation invariance of $\mathcal{T}$ imply that there is a $\tau'' \in \mathcal{T}$ with $\tau''(-s) = w$ and $\alpha(\tau'') \subset M_r$ for some $r \in [\![1, l]\!]$. Set $\tau''' = \tau'' \triangleright_{-s} \tau'$. Then $\tau''' \in \mathcal{T}$, $\alpha(\tau''') \subset M_r$ and, since $-s \leq 0$, we also have that $\tau'''(0) = \tau'(0) = x$. Set $\tau'''' = \tau''' \triangleright \tau$. Then $\tau'''' \in \mathcal{T}$, $\alpha(\tau'''') \subset M_r$ and $\omega(\tau'''') \subset M_{l+1}$, which contradicts Proposition 3.5, as $r < l+1$. The lemma is proved. $\square$

Lemma 3.13 and obvious induction shows that A_k is a $\mathcal{T}$-attractor for all $k \in [\![0, m]\!]$.

Now let $k \in [\![1, m]\!]$ be arbitrary. Let $x \in M_k$ be arbitrary. Since M_k is $\mathcal{T}$-invariant, there is a $\sigma \in \mathcal{T}$ with $\sigma(0) = x$ and $\sigma(\mathbb{R}) \subset M_k$. Since M_k is closed we have $\alpha(\sigma) \subset M_k$ so by the definition of A_k we have $x \in A_k$. If $\sigma(\mathbb{R}) \not\subset (A_{k-1})^*_{\mathcal{T}}$ then (3.9) and Theorem 2.11 imply that $\emptyset \neq \omega(\sigma) \subset M_k \cap A_{k-1} = \emptyset$, a contradiction. Thus $\sigma(\mathbb{R}) \subset (A_{k-1})^*_{\mathcal{T}}$ so $x \in A_k \cap (A_{k-1})^*_{\mathcal{T}}$. This proves that $M_k \subset A_k \cap (A_{k-1})^*_{\mathcal{T}}$. Conversely, let $x \in A_k \cap (A_{k-1})^*_{\mathcal{T}}$ be arbitrary. Then, by the definition of A_k and Definition 3.3 there is a $\sigma_1 \in \mathcal{T}$ with $\sigma_1(0) = x$ and an $r \in [\![1, k]\!]$ such that $\alpha(\sigma_1) \subset M_r$. Moreover, there is a $\sigma_2 \in \mathcal{T}$ such that $\sigma_2(0) = x$ and $\omega(\sigma_2) \cap A_{k-1} = \emptyset$, which, in view of (3.9) implies that $\omega(\sigma_2) \cap M_l = \emptyset$ for all $l \in [\![1, k-1]\!]$. It follows that $\omega(\sigma_2) \subset M_n$ for some $n \in [\![k, m]\!]$. Setting $\sigma = \sigma_1 \triangleright \sigma_2 \in \mathcal{T}$ we see that $\alpha(\sigma) \subset M_r$, $\omega(\sigma) \subset M_n$ and $r \le n$. Proposition 3.5 implies $r = n = k$ and $\sigma(\mathbb{R}) \subset M_k$, so $x \in M_k$. This proves that $A_k \cap (A_{k-1})^*_{\mathcal{T}} \subset M_k$ and completes the proof of the theorem. □

In the sequel, if $\mathcal{T}$ is compact, translation and cut-and-glue invariant, then, in view of Theorems 3.8 and 3.10 we have a well-defined concept of a $\mathcal{T}$-Morse decomposition, meaning a $\mathcal{T}$-Morse decomposition of the first kind or, equivalently, of the second kind.

We will now state and prove two perturbation stability results for attractor filtrations and Morse decompositions.

Theorem 3.14. *Suppose that $\mathcal{T}_\kappa \to \mathcal{T}$, where $\mathcal{T}$ and $\mathcal{T}_\kappa$, $\kappa \in \mathbb{N}$, are compact, translation and cut-and-glue invariant subsets of $\mathcal{C}$. Let $(A_r)_{r=0}^m$ be a $\mathcal{T}$-attractor filtration. For every $r \in [\![0, m]\!]$ let V_r and V_r^* be closed sets with $A_r = \mathrm{Inv}_{\mathcal{T}}(V_r) \subset \mathrm{Int}_X(V_r)$ and $(A_r)^*_{\mathcal{T}} = \mathrm{Inv}_{\mathcal{T}^-}(V_r^*) \subset \mathrm{Int}_X(V_r^*)$.*

For $\kappa \in \mathbb{N}$ and $r \in [\![0, m]\!]$ set

$$A_r^\kappa = \mathrm{Inv}_{\mathcal{T}_\kappa}(V_r), \quad \widetilde{A}_r^\kappa = \mathrm{Inv}_{\mathcal{T}_\kappa^-}(V_r^*).$$

Then there is a $\kappa_0 \in \mathbb{N}$ such that, for all $\kappa \in \mathbb{N}$ with $\kappa \ge \kappa_0$, the sequence $(A_r^\kappa)_{r=0}^m$ is a $\mathcal{T}_\kappa$-attractor filtration and $(\widetilde{A}_r^\kappa)_{r=0}^m$ is its dual $\mathcal{T}_\kappa$-repeller filtration.

Proof. An application of Theorem 2.19 shows that $(A_r^\kappa, \widetilde{A}_r^\kappa)$ is a $\mathcal{T}_\kappa$-attractor-repeller pair for all $r \in [\![0, m]\!]$ and all $\kappa \in \mathbb{N}$ large enough. Furthermore, we conclude from Proposition 2.16 that $A_r^\kappa \subset A_{r+1}^\kappa$ for all $r \in [\![0, m-1]\!]$ and all $\kappa \in \mathbb{N}$ large enough. Thus we only have to show that $A_0^\kappa = \emptyset$ and $A_m^\kappa = S_{\mathcal{T}_\kappa}$ for all $\kappa \in \mathbb{N}$ large enough. If there is a sequence $(\kappa_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ with $\kappa_n \to \infty$ and $A_0^{\kappa_n} \ne \emptyset$, then there is a sequence $(\sigma_n)_{n\in\mathbb{N}}$ such that $\sigma_n \in \mathcal{T}_{\kappa_n}$ and $\sigma_n(\mathbb{R}) \subset V_0$ for all $n \in \mathbb{N}$. Then, taking a subsequence if necessary, we may assume that $\sigma_n \to \sigma$ for some $\sigma \in \mathcal{T}$. Hence $\sigma(\mathbb{R}) \subset V_0$ so $A_0 = \mathrm{Inv}_{\mathcal{T}}(V_0) \ne \emptyset$, a contradiction. Now clearly $A_\kappa \subset S_{\mathcal{T}_\kappa}$ for every $\kappa \in \mathbb{N}$. Consequently, if there is a sequence $(\kappa_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ with $\kappa_n \to \infty$ and $A_m^{\kappa_n} \ne S_{\mathcal{T}_\kappa}$, then there is a sequence $(\sigma_n)_{n\in\mathbb{N}}$ such that $\sigma_n \in \mathcal{T}_{\kappa_n}$ and $\sigma_n(0) \notin V_m$ for all $n \in \mathbb{N}$. Taking a subsequence if necessary, we may assume that $\sigma_n \to \sigma$ for some $\sigma \in \mathcal{T}$. Hence $\sigma(0) \notin \mathrm{Int}_X(V_m)$ so $\sigma(\mathbb{R}) \not\subset A_m$ and thus $A_m \ne S_{\mathcal{T}}$, a contradiction. □

Theorem 3.15. *Suppose that $\mathcal{T}_\kappa \to \mathcal{T}$, where $\mathcal{T}$ and $\mathcal{T}_\kappa$, $\kappa \in \mathbb{N}$, are compact, translation and cut-and-glue invariant subsets of $\mathcal{C}$. Let $(M_r)_{r=1}^m$ be a $\mathcal{T}$-Morse decomposition. Let $(W_r)_{r=1}^m$ be a finite sequence of closed sets such that*

$$M_r = \mathrm{Inv}_{\mathcal{T}}(W_r) \subset \mathrm{Int}_X(W_r), \quad r \in [\![1, m]\!].$$

For $\kappa \in \mathbb{N}$ and $r \in [\![1, m]\!]$ set

$$M_r^\kappa = \mathrm{Inv}_{\mathcal{T}_\kappa}(W_r). \tag{3.18}$$

Then there is a $\kappa_1 \in \mathbb{N}$ such that for all $\kappa \in \mathbb{N}$ with $\kappa \geq \kappa_1$ the sequence $(M_r^\kappa)_{r=1}^m$ is a $\mathcal{T}_\kappa$-Morse decomposition.

Proof. Choose a $\mathcal{T}$-attractor filtration $(A_r)_{r=0}^m$ such that

$$M_r = A_r \cap (A_{r-1})^*_{\mathcal{T}}, \quad r \in [\![1, m]\!].$$

For every $r \in [\![0, m]\!]$ let V_r and V_r^* be closed sets with $A_r = \mathrm{Inv}_{\mathcal{T}}(V_r) \subset \mathrm{Int}_X(V_r)$ and $(A_r)^*_{\mathcal{T}} = \mathrm{Inv}_{\mathcal{T}^-}(V_r^*) \subset \mathrm{Int}_X(V_r^*)$. Let $r \in [\![1, m]\!]$ be arbitrary. Since M_r is $\mathcal{T}$-invariant and $M_r \subset V_r \cap V_{r-1}^*$, we see that

$$\begin{aligned} M_r &\subset \mathrm{Inv}_{\mathcal{T}}(V_r \cap V_{r-1}^*) \subset \mathrm{Inv}_{\mathcal{T}}(V_r) \cap \mathrm{Inv}_{\mathcal{T}}(V_{r-1}^*) \\ &= \mathrm{Inv}_{\mathcal{T}}(V_r) \cap \mathrm{Inv}_{\mathcal{T}^-}(V_{r-1}^*) = A_r \cap (A_{r-1})^*_{\mathcal{T}} = M_r \end{aligned}$$

so

$$\begin{aligned} M_r &= \mathrm{Inv}_{\mathcal{T}}(V_r \cap V_{r-1}^*) = \mathrm{Inv}_{\mathcal{T}}(V_r) \cap \mathrm{Inv}_{\mathcal{T}^-}(V_{r-1}^*) \\ &\subset \mathrm{Int}_X(V_r) \cap \mathrm{Int}_X(V_{r-1}^*) \subset \mathrm{Int}_X(V_r \cap V_{r-1}^*). \end{aligned} \tag{3.19}$$

For $r \in [\![0, m]\!]$ and $\kappa \in \mathbb{N}$ define $A_r^\kappa = \mathrm{Inv}_{\mathcal{T}_\kappa}(V_r)$, $\widetilde{A}_r^\kappa = \mathrm{Inv}_{\mathcal{T}_\kappa^-}(V_r^*)$. By Theorem 3.14 there is a $\kappa_0 \in \mathbb{N}$ such that, for all $\kappa \in \mathbb{N}$ with $\kappa \geq \kappa_0$, the sequence $(A_r^\kappa)_{r=0}^m$ is a $\mathcal{T}_\kappa$-attractor filtration and $(\widetilde{A}_r^\kappa)_{r=0}^m$ is its dual $\mathcal{T}_\kappa$-repeller filtration. It follows that, for all $\kappa \in \mathbb{N}$ with $\kappa \geq \kappa_0$, the sequence $(\widetilde{M}_r^\kappa)_{r=1}^m$ is a $\mathcal{T}_\kappa$-Morse decomposition, where $\widetilde{M}_r^\kappa = A_r^\kappa \cap \widetilde{A}_{r-1}^\kappa$, $r \in [\![1, m]\!]$. Proceeding as in the proof of Formula (3.19) we see that

$$\begin{aligned} \widetilde{M}_r^\kappa &= \mathrm{Inv}_{\mathcal{T}_\kappa}(V_r \cap V_{r-1}^*) = \mathrm{Inv}_{\mathcal{T}_\kappa}(V_r) \cap \mathrm{Inv}_{\mathcal{T}_\kappa^-}(V_{r-1}^*) \\ &\subset \mathrm{Int}_X(V_r) \cap \mathrm{Int}_X(V_{r-1}^*) \subset \mathrm{Int}_X(V_r \cap V_{r-1}^*). \end{aligned}$$

Now (3.18), (3.19), and Proposition 2.17 imply that there is a $\kappa_1 \in \mathbb{N}$, $\kappa_1 \geq \kappa_0$, such that

$$\widetilde{M}_r^\kappa = \mathrm{Inv}_{\mathcal{T}_\kappa}(V_r \cap V_{r-1}^*) = \mathrm{Inv}_{\mathcal{T}_\kappa}(W_r) = M_r^\kappa, \quad r \in [\![1, m]\!],\ \kappa \geq \kappa_1. \qquad \square$$

4. Applications to a Galerkin-type Conley index

In this section we will apply the abstract results obtained before to certain classes of ordinary differential equations on Banach spaces, considered in the paper [10], which do not necessarily satisfy the uniqueness property of the Cauchy problem. In [10] a Galerkin-type Conley index is defined for such equations, generalizing an index previously defined in [7].

We will establish a Morse equation for this Conley index theory (Theorems 4.7 and 4.16). This Morse equation can be used to prove multiplicity results for strongly indefinite problems in Hilbert spaces. An example of such an application will be given in the next section.

We assume the reader's familiarity with the paper [10] and only review some basic notation and those results from that paper which we require to prove the results of this section.

In this section let $(E, \|\cdot\|)$ be a Banach space and we set $X = E$ and $d(x,y) = \|x-y\|$ for x, $y \in X$. Given $N \subset U \subset E$ and $f: U \to E$ an arbitrary function, we set

$$|f|_N = \sup_{x \in N} \|f(x)\| \in [0, \infty].$$

If $U \subset X$ is open and $f \in C(U \to X)$ then by a *solution of* f we mean a differentiable function $\sigma: \mathbb{R} \to E$ with $\sigma(\mathbb{R}) \subset U$ and such that

$$\sigma'(t) = f(\sigma(t)), \quad \text{for all } t \in \mathbb{R}.$$

Note that any translate of a solution of f is again a solution of f. Furthermore, if σ_1 and σ_2 are two solutions of f with $\sigma_1(0) = \sigma_2(0)$, then $\sigma := \sigma_1 \triangleright \sigma_2$ is easily seen to be a solution of f.

By $\mathrm{Sol}(f)$ we denote the set of all solutions of f. Moreover, given $Y \subset U$ we denote by $\mathrm{Sol}(f,Y)$ the set of all solutions σ of f such that $\sigma(\mathbb{R}) \subset Y$. It follows that $\mathrm{Sol}(f,Y)$ is translation and cut-and-glue invariant.

Define $\mathrm{Inv}(f,Y)$ to be the set of all $y \in E$ for which there is a $\sigma \in \mathrm{Sol}(f,Y)$ with $\sigma(0) = y$. Note that $\mathrm{Inv}(f,Y) = \mathrm{Inv}_{\mathcal{T}}(Y)$, where $\mathcal{T} = \mathrm{Sol}(f)$. A set $S \subset U$ is called *invariant relative to* f if $\mathrm{Inv}(f,S) = S$. Thus S is invariant relative to f if and only if S is $\mathcal{T}$-invariant, where $\mathcal{T} = \mathrm{Sol}(f)$.

Now let $S \subset U$ be invariant relative to f. Set $\mathcal{T}_{(f,S)} = \mathrm{Sol}(f,S)$. Since S is invariant relative to f it follows that $S = S_{\mathcal{T}_{(f,S)}}$.

We say that a set A is an *attractor* (resp. a *repeller*) in S if A is a $\mathcal{T}_{(f,S)}$-attractor (resp. a $\mathcal{T}_{(f,S)}$-repeller). Analogously, attractor filtrations in S, resp. Morse decompositions of the first (resp. the second) kind of S are simply $\mathcal{T}_{(f,S)}$-attractor filtrations, resp. $\mathcal{T}_{(f,S)}$-Morse decompositions of the first (resp. the second) kind. If $\mathcal{T}_{(f,S)}$ is compact then in view of Theorems 3.8 and 3.10 we may simply speak of Morse-decompositions of S.

A bounded set $N \subset U$ is called an *isolating neighbourhood relative to* f if N is closed in X and $\mathrm{Inv}(f,N) \subset \mathrm{Int}_X(N)$. The set $\mathrm{Inv}(f,N)$ is then called an isolated invariant set relative to f.

The following result is obvious:

Proposition 4.1. *If $U \subset E$ is open, $f: U \to E$ is continuous, and $Y \subset Y' \subset U$ then*

$$\begin{aligned} \mathrm{Sol}(f,Y) &= \mathrm{Sol}(f,S), \quad \textit{where } S = \mathrm{Inv}(f,Y), \\ \mathrm{Inv}(f,Y) &= \mathrm{Inv}_{\mathcal{T}}(Y), \quad \textit{where } \mathcal{T} = \mathrm{Sol}(f,Y'). \end{aligned}$$

We also have the following result:

Proposition 4.2. *Suppose E is finite dimensional, U is open in E, N is bounded and closed in E and $|f_\kappa - f|_N \to 0$ as $\kappa \to \infty$ where $(f_\kappa)_{\kappa\in\mathbb{N}}$ is a sequence in $C(U \to E)$ and $f \in C(U \to E)$ is arbitrary. Define $\mathcal{T} := \mathrm{Sol}(f, N)$ and $\mathcal{T}_\kappa := \mathrm{Sol}(f_\kappa, N)$, $\kappa \in \mathbb{N}$. Then the sets $\mathcal{T}$, $\mathcal{T}_\kappa$, $\kappa \in \mathbb{N}$, are compact and $\mathcal{T}_\kappa \to \mathcal{T}$.*

Proof. This follows from Proposition 3.10 in [10]. The proof is an application of Kamke's Theorem for finite dimensional ordinary differential equations.□

We now obtain the following

Proposition 4.3. *Suppose E is finite dimensional, U is open in E, N is bounded and closed in E and $|f_\kappa - f|_N \to 0$ as $\kappa \to \infty$ where $(f_\kappa)_{\kappa\in\mathbb{N}}$ is a sequence in $C(U \to E)$ and $f \in C(U \to E)$ is arbitrary. Suppose that N is an isolating neighbourhood relative to f. Then there is a $\kappa_0 \in \mathbb{N}$ such that for every $\kappa \in \mathbb{N}$ with $\kappa \geq \kappa_0$, N is an isolating neighbourhood relative to f_κ.*

Proof. Using Proposition 4.1 we obtain $\mathrm{Inv}(f, N) = \mathrm{Inv}_{\mathcal{T}}(N)$ and

$$\mathrm{Inv}(f_\kappa, N) = \mathrm{Inv}_{\mathcal{T}_\kappa}(N), \quad \kappa \in \mathbb{N},$$

where $\mathcal{T} := \mathrm{Sol}(f, N)$ and $\mathcal{T}_\kappa := \mathrm{Sol}(f_\kappa, N)$, $\kappa \in \mathbb{N}$. Our hypothesis is that $\mathrm{Inv}(f, N) \subset \mathrm{Int}_X(N)$. Thus $\mathrm{Inv}_{\mathcal{T}}(N) \subset \mathrm{Int}_X(N)$ and so, by Proposition 2.14, $\mathrm{Inv}_{\mathcal{T}_\kappa}(N) \subset \mathrm{Int}_X(N)$ for some $\kappa_0 \in \mathbb{N}$ and all $\kappa \geq \kappa_0$. Thus

$$\mathrm{Inv}(f_\kappa, N) \subset \mathrm{Int}_X(N), \quad \kappa \geq \kappa_0. \qquad \square$$

The last result obviously implies the following corollary.

Corollary 4.4. *Suppose E is finite dimensional, U is open in E, N is bounded and closed in E and $f \in C(U \to E)$ is arbitrary. If N is an isolating neighbourhood relative to f then there is an $\varepsilon > 0$ such that whenever $g \in C(U \to E)$ is such that $|g - f|_N < \varepsilon$ then N is an isolating neighbourhood relative to g. We define $\varepsilon(f, N) > 0$ to be the supremum of such numbers ε.*

In the situation of the above corollary, if f is locally Lipschitzian, then the classical Conley index of $S_f := \mathrm{Inv}(f, N)$ relative to the local flow π_f generated by the ordinary differential equation

$$\dot{x} = f(x)$$

is defined, and we write $h(f, S_f)$ to denote this index. Actually, since the set N uniquely determines the invariant set S_f we also write $h(f, N)$ instead of S_f and call $h(f, N)$ the Conley index of the isolating neighbourhood N relative to f. If f is merely continuous, then there is a locally Lipschitzian map g with

$$|g - f|_N < \varepsilon(f, N).$$

Following [10] we now define the *Conley index* $h(f,N)$ *of the isolating neighbourhood* N *relative to* f as

$$h(f,N) := h(g,N).$$

It is shown in [10] that the index just defined only depends on the isolated invariant set S_f and not on the particular choice of the isolating neighbourhood. Moreover, this index enjoys all the properties of the classical Conley index like nontriviality or homotopy invariance.

We can now specialize the perturbation stability result for Morse decompositions, Theorem 3.15, to the present finite dimensional situation:

Theorem 4.5. *Suppose E is finite dimensional, U is open in E, N is bounded and closed in E and $|f_\kappa - f|_N \to 0$ as $\kappa \to \infty$ where $(f_\kappa)_{\kappa\in\mathbb{N}}$ is a sequence in $C(U \to E)$ and $f \in C(U \to E)$ is arbitrary. Suppose that N is an isolating neighbourhood relative to f. Moreover, for every $r \in [\![1,m]\!]$ let $W_r \subset N$ be a closed set which is an isolating neighbourhood relative to f and suppose that $(\mathrm{Inv}(f,W_r))_{r=1}^m$ is a Morse decomposition of $\mathrm{Inv}(f,N)$ relative to f. Then there is a $\kappa_0 \in \mathbb{N}$ such that for all $\kappa \in \mathbb{N}$ with $\kappa \geq \kappa_0$, the set N is an isolating neighbourhood relative to f_κ, for every $r \in [\![1,m]\!]$ the set $W_r \subset N$ is an isolating neighbourhood relative to f_κ and $(\mathrm{Inv}(f_\kappa,W_r))_{r=1}^m$ is a Morse decomposition of $\mathrm{Inv}(f_\kappa,N)$ relative to f_κ.*

Proof. Let $\mathcal{T}$ and $\mathcal{T}_\kappa$, $\kappa \in \mathbb{N}$ be as in of Proposition 4.2. By Proposition 4.1 we have $\mathrm{Inv}(f,N) = \mathrm{Inv}_{\mathcal{T}}(N) \subset \mathrm{Int}_X(N)$, $\mathrm{Inv}(f,W_r) = \mathrm{Inv}_{\mathcal{T}}(W_r) \subset \mathrm{Int}_X(W_r)$, $\mathrm{Inv}(f_\kappa,N) = \mathrm{Inv}_{\mathcal{T}_\kappa}(N)$ and $\mathrm{Inv}(f_\kappa,W_r) = \mathrm{Inv}_{\mathcal{T}_\kappa}(W_r)$, $r \in [\![1,m]\!]$, $\kappa \in \mathbb{N}$.

Since $\mathcal{T}$ and $\mathcal{T}_\kappa$, $\kappa \in \mathbb{N}$ are compact, translation and cut-and-glue invariant, an application of Proposition 4.2, Proposition 2.14 and Theorem 3.15 shows that there is a $\kappa_0 \in \mathbb{N}$ such that for all $\kappa \geq \kappa_0$, $\mathrm{Inv}(f_\kappa,N) = \mathrm{Inv}_{\mathcal{T}_\kappa}(N) \subset \mathrm{Int}_X(N)$ and $\mathrm{Inv}(f_\kappa,W_r) = \mathrm{Inv}_{\mathcal{T}_\kappa}(W_r) \subset \mathrm{Int}_X(W_r)$, $r \in [\![1,m]\!]$, and $(\mathrm{Inv}(f_\kappa,W_r))_{r=1}^m$ is a $\mathcal{T}_\kappa$-Morse decomposition, i.e. $(\mathrm{Inv}(f_\kappa,W_r))_{r=1}^m$ is a Morse decomposition of $\mathrm{Inv}(f_\kappa,N)$ relative to f_κ. □

The last result clearly implies the following theorem.

Theorem 4.6. *Suppose that E is finite dimensional, $U \subset E$ is open, $N \subset E$ is bounded and closed and $N \subset U$, $f\colon U \to E$ is continuous. Suppose that N is an isolating neighbourhood relative to f. Moreover, for every $r \in [\![1,m]\!]$ let $W_r \subset N$ be a closed set which is an isolating neighbourhood relative to f and suppose that $(\mathrm{Inv}(f,W_r))_{r=1}^m$ is a Morse decomposition of $\mathrm{Inv}(f,N)$ relative to f. Then there is an $\varepsilon \in \,]0,\infty[$ such that whenever $g\colon U \to E$ is continuous and $|f-g|_N < \varepsilon$ then N is an isolating neighbourhood relative to g, W_r is an isolating neighbourhood relative to g, $r \in [\![1,m]\!]$, and $(\mathrm{Inv}(g,W_r))_{r=1}^m$ is a Morse decomposition of $\mathrm{Inv}(g,N)$ relative to g. By $\varepsilon(f,N,(W_r)_{r=1}^m)$ we denote the supremum of all such numbers ε.*

We will now state and prove the Morse equation for the version of the Conley index defined above. To this end, let $(H_q)_{q\in\mathbb{Z}}$ (resp. $(H^q)_{q\in\mathbb{Z}}$) be an arbitrary homology (resp. cohomology) theory with coefficients in an R-module M, where R is an integral domain. If (Y, y_0) is a pointed space then we define the Betti numbers

$$\beta_q(Y, y_0) := \operatorname{rank} H_q(Y, \{y_0\}) \in \mathbb{N}_0 \cup \{\infty\}, \quad q \in \mathbb{Z},$$

resp.

$$\beta^q(Y, y_0) := \operatorname{rank} H^q(Y, \{y_0\}) \in \mathbb{N}_0 \cup \{\infty\}, \quad q \in \mathbb{Z}.$$

We also define the formal Poincaré polynomial

$$p(t, (Y, y_0)) = \sum_{q=0}^{\infty} \beta_q(Y, y_0) t^q, \quad t \in \mathbb{R},$$

resp.

$$p(t, (Y, y_0)) = \sum_{q=0}^{\infty} \beta^q(Y, y_0) t^q, \quad t \in \mathbb{R}.$$

In particular, whenever defined, the Conley index $h(f, N)$ is an equivalence class of homotopy equivalent pointed spaces, so the polynomial $p(t, h(f, N))$ is defined.

We now obtain the following Morse equation:

Theorem 4.7. *Let U, f, N and $(W_r)_{r=1}^m$ be as in Theorem* 4.6. *Then*

$$\sum_{r=1}^{m} p(t, h(f, W_r)) = p(t, h(f, N)) + (1 + t)Q(t), \quad t \in \mathbb{R},$$

where $Q(t) = \sum_{k=0}^{\infty} a_k t^k$, $t \in \mathbb{R}$, is a formal power series with coefficients $a_k \in \mathbb{N}_0 \cup \{\infty\}$, $k \in \mathbb{N}_0$.

Proof. Let $g \in C(U \to E)$ be a locally Lipschitzian map such that

$$|g - f|_N < \varepsilon(f, N, (W_r)_{r=1}^m).$$

Then from Theorem III.3.5 in [13] we obtain the usual Morse equation

$$\sum_{r=1}^{m} p(t, h(g, W_r)) = p(t, h(g, N)) + (1 + t)Q(t), \quad t \in \mathbb{R} \tag{4.1}$$

where $Q(t) = \sum_{k=0}^{\infty} a_k t^k$, $t \in \mathbb{R}$, is a formal power series with coefficients $a_k \in \mathbb{N}_0 \cup \{\infty\}$, $k \in \mathbb{N}_0$.

Since, clearly, $\varepsilon(f, N, (W_r)_{r=1}^m) \le \varepsilon(f, N)$ and $\varepsilon(f, N, (W_r)_{r=1}^m) \le \varepsilon(f, W_r)$ for all $r \in [\![1, m]\!]$ we see that $h(f, N) = h(g, N)$ and $h(f, W_r) = h(g, W_r)$ for all $r \in [\![1, m]\!]$. This together with (4.1) implies the assertion of the theorem. $\square$

We will now treat certain classes of ordinary differential equations on infinite dimensional Banach space.

We begin with the following useful definition.

Definition 4.8. The quadruple (L, E_{-1}, E_0, E_1) is called a *trichotomy on the Banach space E* if the following properties are satisfied:

(1) $L: E \to E$ is a bounded linear operator.
(2) E_j, $j \in [\![-1, 1]\!]$, are closed L-invariant subspaces of E with $E = E_{-1} \oplus E_0 \oplus E_1$ and E_0 is finite dimensional. For $j \in [\![-1, 1]\!]$ we denote by $L_j: E_j \to E_j$ the restriction of L to E_j.
(3) There are constants $M \in [0, \infty[$ and $\alpha \in]0, \infty[$ such that

$$\begin{aligned} \|e^{L_{-1}t}\|_{\mathcal{L}(E_{-1}, E_{-1})} &\le Me^{-\alpha t}, \quad t \in [0, \infty[, \\ \|e^{L_1 t}\|_{\mathcal{L}(E_1, E_1)} &\le Me^{\alpha t}, \quad\ \ t \in]-\infty, 0]. \end{aligned}$$

The triple (L, E_{-1}, E_1) is called a *dichotomy on E* if $(L, E_{-1}, \{0\}, E_1)$ is a trichotomy on E.

For the rest of this section assume that $(E, \|\cdot\|)$ is an infinite dimensional Banach space.

Assume the following hypothesis:

Hypothesis 4.9.

(1) *(L, E_{-1}, E_0, E_1) is a given trichotomy on E.*
(2) *$(P^\ell)_{\ell \in \mathbb{N}}$ is a sequence of bounded linear operators on E such that, for all $x \in E$, $P^\ell(x) \to x$ as $\ell \to \infty$.*
(3) *For every $\ell \in \mathbb{N}$ the subspace $E^\ell := P^\ell(E)$ is finite dimensional (hence closed in E) and L-invariant. By $L^\ell: E^\ell \to E^\ell$ we denote the restriction of L to E^ℓ for $\ell \in \mathbb{N}$.*

Remark 4.10. In view of the Uniform Boundedness Principle, item (2) of the above hypothesis is equivalent to the requirement that $P^\ell \to \mathrm{Id}_E$ as $\ell \to \infty$, uniformly on compact subsets of E.

We have the following result:

Proposition 4.11. *Suppose U is open in E, N is bounded and closed in E with $N \subset U$, and $K \in C(U \to E)$ is such that $K(N)$ is relatively compact in E. Define*

$$f: U \to E, \quad x \mapsto Lx + K(x),$$

and

$$f^\ell: U \cap E^\ell \to E^\ell, \quad x \mapsto L^\ell x + P^\ell K(x), \quad \ell \in \mathbb{N}.$$

Let

$$\mathcal{T} := \mathrm{Sol}(f, N) \subset C(\mathbb{R} \to E)$$

and

$$\mathcal{T}_\ell := \mathrm{Sol}(f^\ell, N \cap E^\ell) \subset C(\mathbb{R} \to E^\ell) \subset C(\mathbb{R} \to E), \quad \ell \in \mathbb{N}.$$

Then $\mathcal{T}$ and $\mathcal{T}_\ell$, $\ell \in \mathbb{N}$ are compact in $\mathcal{C} := C(\mathbb{R} \to E)$ and $\mathcal{T}_\ell \to \mathcal{T}$ in $\mathcal{C}$, as $\ell \to \infty$.

Proof. The proof follows from [10, Proposition 4.3 and the proof of Proposition 4.7]. $\square$

Now we obtain the following

Corollary 4.12. *Assume the hypotheses of Proposition* 4.11*. In addition, suppose that N is an isolating neighbourhood relative to f. Then there is an $\ell_0 \in \mathbb{N}$ such that for all $\ell \in \mathbb{N}$ with $\ell \geq \ell_0$ the set $N \cap E^\ell$ is an isolating neighbourhood relative to f^ℓ. By $\ell_0(K,N)$ we denote the smallest of such numbers ℓ_0.*

Proof. Note that, by Proposition 4.1, $\mathrm{Inv}(f,N) = \mathrm{Inv}_{\mathcal{T}}(N)$ and $\mathrm{Inv}(f^\ell, N \cap E^\ell) = \mathrm{Inv}_{\mathcal{T}_\ell}(N)$, $\ell \in \mathbb{N}$. Now Proposition 4.11 together with Proposition 2.14 imply the existence of an $\ell_0 \in \mathbb{N}$ such that whenever $\ell \geq \ell_0$, then $\mathrm{Inv}_{\mathcal{T}_\ell}(N) \subset \mathrm{Int}_X(N)$. Since $X = E$ and $\mathrm{Inv}_{\mathcal{T}_\ell}(N) \subset E^\ell$ for all $\ell \in \mathbb{N}$ it follows that $\mathrm{Inv}_{\mathcal{T}_\ell}(N) \subset \mathrm{Int}_{E^\ell}(N \cap E^\ell)$ for all $\ell \in \mathbb{N}$ with $\ell \geq \ell_0$. $\square$

Now assume all hypotheses of Corollary 4.12. Following [10] we define the *$\mathcal{LS}$-Conley index $h(f,N)$ of the isolating neighbourhood N relative to f* as

$$h(f,N) := (h(f,N)_\ell)_{\ell \geq \ell_0(K,N)},$$

where $h(f,N)_\ell = h(f^\ell, N \cap E^\ell)$, $\ell \geq \ell_0(K,N)$. Here, of course, $h(f^\ell, N \cap E^\ell)$ is the finite dimensional Conley index defined earlier in this section.

It is proved in [10] that whenever N and N' are two isolating neighbourhoods of the same isolated invariant set S (relative to f) with $K(N)$ and $K(N')$ relatively compact in E then

$$h(f,N)_\ell = h(f,N')_\ell, \quad \text{for all } \ell \in \mathbb{N} \text{ large enough.}$$

Consequently, given an isolated invariant set relative to f we may write $h(f,S)$ instead of $h(f,N)$, where N is an arbitrary isolating neighbourhood of S relative to f with $K(N)$ relatively compact in E.

As it is shown in [10] this version of Conley index again satisfies all the properties of the classical Conley index.

Now consider the following additional hypothesis.

Hypothesis 4.13. *For every sufficiently large $\ell \in \mathbb{N}$ there are linear L-invariant subspaces F^ℓ, F^ℓ_{-1} and F^ℓ_1 of E such that $E^{\ell+1} = F^\ell \oplus E^\ell$ and the triple*

$$(L|_{F^\ell}, F^\ell_{-1}, F^\ell_1)$$

is a dichotomy on F^ℓ. By i_ℓ we denote the dimension of F^ℓ_1.

We now have the following

Proposition 4.14. *Assume Hypothesis* 4.13 *in addition to the hypotheses of Corollary* 4.12. *Then there is an* $\ell_1 \geq \ell_0(K, N)$ *such that*

$$h(f, N)_{\ell+1} = \Sigma^{i_\ell} \wedge h(f, N)_\ell, \quad \ell \geq \ell_1.$$

Here, of course, Σ^k *is the homotopy type of a pointed* k*-dimensional sphere,* $k \in \mathbb{N}_0$.

Proof. This is just Proposition 4.18 in [10]. □

We can now state the following perturbation stability result for Morse decompositions:

Theorem 4.15. *Assume the hypotheses of Corollary* 4.12. *In addition, for every* $r \in [\![1, m]\!]$ *let* $W_r \subset N$ *be closed with* $\mathrm{Inv}(f, W_r) \subset \mathrm{Int}_E(W_r)$ *and suppose that* $(\mathrm{Inv}(f, W_r))_{r=1}^m$ *is a Morse decomposition of* $\mathrm{Inv}(f, N)$ *relative to* f. *Let* f^ℓ *be as in Proposition* 4.11. *Then there is an* $\ell_0 \in \mathbb{N}_0$ *such that whenever* $\ell \geq \ell_0$ *then*

$$\begin{aligned} \mathrm{Inv}(f^\ell, N \cap E^\ell) &\subset \mathrm{Int}_{E^\ell}(N \cap E^\ell), \\ \mathrm{Inv}(f^\ell, W_r \cap E^\ell) &\subset \mathrm{Int}_{E^\ell}(W_r \cap E^\ell) \quad \textit{for all } r \in [\![1, m]\!], \end{aligned}$$

and $(\mathrm{Inv}(f^\ell, W_r \cap E^\ell))_{r=1}^m$ *is a Morse decomposition of* $\mathrm{Inv}(f^\ell, N \cap E^\ell)$ *relative to* f^ℓ.

By $\ell_0(f, N, (W_r)_{r=1}^m)$ *we denote the minimum of all such numbers* ℓ_0.

Proof. Let $\mathcal{T}$ and $\mathcal{T}_\ell$, $\ell \in \mathbb{N}$ be as in Proposition 4.11. By Proposition 4.1 we have $\mathrm{Inv}(f, N) = \mathrm{Inv}_{\mathcal{T}}(N) \subset \mathrm{Int}_X(N)$, $\mathrm{Inv}(f, W_r) = \mathrm{Inv}_{\mathcal{T}}(W_r) \subset \mathrm{Int}_X(W_r)$, $\mathrm{Inv}(f^\ell, N) = \mathrm{Inv}_{\mathcal{T}_\ell}(N)$ and $\mathrm{Inv}(f^\ell, W_r) = \mathrm{Inv}_{\mathcal{T}_\ell}(W_r)$, $r \in [\![1, m]\!]$, $\ell \in \mathbb{N}$.

Since $\mathcal{T}$ and $\mathcal{T}_\ell$, $\ell \in \mathbb{N}$ are compact, translation and cut-and-glue invariant, an application of Proposition 2.14 and Theorem 3.15 shows that there is an $\ell_0 \in \mathbb{N}$ such that for all $\ell \geq \ell_0$, $\mathrm{Inv}(f^\ell, N) = \mathrm{Inv}_{\mathcal{T}_\ell}(N) \subset \mathrm{Int}_X(N)$ and $\mathrm{Inv}(f^\ell, W_r) = \mathrm{Inv}_{\mathcal{T}_\ell}(W_r) \subset \mathrm{Int}_X(W_r)$, $r \in [\![1, m]\!]$, and $(\mathrm{Inv}(f^\ell, W_r))_{r=1}^m$ is a $\mathcal{T}_\ell$-Morse decomposition, i.e. $(\mathrm{Inv}(f^\ell, W_r))_{r=1}^m$ is a Morse decomposition of $\mathrm{Inv}(f^\ell, N)$ relative to f^ℓ. Since $\mathrm{Inv}_{\mathcal{T}_\ell}(N) \subset E^\ell$ and $\mathrm{Inv}_{\mathcal{T}_\ell}(W_r) \subset E^\ell$, $\ell \in \mathbb{N}$, $r \in [\![1, m]\!]$, it follows that $\mathrm{Inv}(f^\ell, N) = \mathrm{Inv}_{\mathcal{T}_\ell}(N) \subset \mathrm{Int}_{E^\ell}(N \cap E^\ell)$ and $\mathrm{Inv}(f^\ell, W_r) = \mathrm{Inv}_{\mathcal{T}_\ell}(W_r) \subset \mathrm{Int}_{E^\ell}(W_r \cap E^\ell)$, $r \in [\![1, m]\!]$, $\ell \geq \ell_0$. □

We now obtain the following Morse equation:

Theorem 4.16. *Assume all the hypotheses of Theorem* 4.15. *Then, for every* $\ell \geq \ell_0(f, N, (W_r)_{r=1}^m)$,

$$\sum_{r=1}^m p(t, h(f, W_r)_\ell) = p(t, h(f, N)_\ell) + (1 + t)Q_\ell(t), \quad t \in \mathbb{R},$$

where $Q_\ell(t) = \sum_{k=0}^{\infty} a_{\ell,k} t^k$, $t \in \mathbb{R}$, *is a formal power series with coefficients* $a_{\ell,k} \in \mathbb{N}_0 \cup \{\infty\}$, $k \in \mathbb{N}_0$.

Proof. This is an application of Theorems 4.7 and 4.15. $\square$

5. An indefinite elliptic system

We will now apply the results of the preceding section to give Conley index proofs of two multiplicity results for a strongly indefinite elliptic system previously proved in [1] using the Morse–Floer homology.

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary. Consider the following elliptic system

$$\begin{aligned} -\Delta u &= \partial_v H(u,v,x) \quad \text{in } \Omega, \\ -\Delta v &= \partial_u H(u,v,x) \quad \text{in } \Omega, \\ u &= 0, \quad v = 0 \qquad \text{in } \partial\Omega. \end{aligned} \tag{5.1}$$

Throughout this section we make the following assumptions:

(5.2) *p and $q \in \,]1,\infty[$ are such that*

$$\begin{aligned} (1/p) &> (1/2) - (2/N), \\ (1/q) &> (1/2) - (2/N), \\ (1/p) + (1/q) &> 1 - (2/N). \end{aligned}$$

(5.3) *The function* $H\colon \mathbb{R} \times \mathbb{R} \times \overline{\Omega} \to \mathbb{R}$, $(\xi,\eta,x) \mapsto H(\xi,\eta,x)$, *is of class* C^2.

(5.4) *There is a constant* $c_1 \in \,]0,\infty[$ *such that for all* $(\xi,\eta,x) \in \mathbb{R} \times \mathbb{R} \times \overline{\Omega}$

$$\begin{aligned} |\partial_\xi H(\xi,\eta,x)| &\le c_1(|\xi|^{p-1} + |\eta|^{(p-1)q/p} + 1), \\ |\partial_\eta H(\xi,\eta,x)| &\le c_1(|\eta|^{q-1} + |\xi|^{(q-1)p/q} + 1). \end{aligned}$$

(5.5) *There are constants* c_2 *and* $\delta \in \,]0,\infty[$ *such that for all* $(\xi,\eta,x) \in \mathbb{R}\times\mathbb{R}\times\overline{\Omega}$

$$\partial_\xi H(\xi,\eta,x)\xi - \partial_\eta H(\xi,\eta,x)\eta \ge -c_2 + \delta(|\xi|^p + |\eta|^q).$$

Following [11] we will now briefly describe how to use Conley index to obtain solutions of (5.1). For more details, the reader is referred to [11] and the references contained there.

First of all, it is well-known that the linear operator

$$B\colon W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \to L^2(\Omega), \quad u \mapsto -\Delta u,$$

is positive self-adjoint and, consequently, sectorial in $X = L^2(\Omega)$. Thus B generates a family X^α, $\alpha \in [0,\infty[$, of fractional power spaces (cf e.g. [8]). We write

$A := B^{1/2}$. Moreover, for $\alpha \in [0,\infty[$ let $E^\alpha := X^{\alpha/2}$ and $E^{-\alpha} := E^{\alpha*}$ be the dual of E^α. Note that for $\alpha \in [0,\infty[$ the formula

$$\langle u,v\rangle_\alpha := \langle A^\alpha u, A^\alpha v\rangle_{L^2}, \quad u,v \in E^\alpha$$

defines a Hilbert product in E^α and A^α is an isometry between the Hilbert spaces E^α and $L^2(\Omega)$. Endow $E^{-\alpha} := E^{\alpha*}$ with the dual product. We write

$$A^{-\alpha} := (A^\alpha)^{-1}\colon L^2(\Omega) \to E^\alpha.$$

Whenever $\lambda > 0$ and $B\phi = \lambda\phi$ then $A^\beta\phi = \lambda^{\beta/2}\phi$ for every $\beta \in \mathbb{R}$.

It is also well-known that for every $\beta \in \mathbb{R}$ the operator A^β can be uniquely extended to a map

$$A^\beta\colon \bigcup_{\alpha\in\mathbb{R}} E^\alpha \to \bigcup_{\alpha\in\mathbb{R}} E^\alpha$$

such that whenever $\alpha \in \mathbb{R}$ then $A^\beta(E^\alpha) = E^{\alpha-\beta}$ and $A^\beta_{|E^\alpha}\colon E^\alpha \to E^{\alpha-\beta}$ is an isometry.

Moreover, Hypothesis (5.2) is easily seen to be equivalent to the following condition:

(5.6) p and $q \in]1,\infty[$ and there are s, $t \in]0,\infty[$ such that $s+t = 2$ and

$$\begin{aligned}(1/p) &> (1/2) - (s/N),\\ (1/q) &> (1/2) - (t/N).\end{aligned}$$

From now on choose s and t as in (5.6).

Define the product Hilbert space $E := E^s \times E^t$ with the Hilbert product

$$\langle z,z'\rangle := \langle u,u'\rangle_s + \langle v,v'\rangle_t, \quad z = (u,v),\ z' = (u',v') \in E.$$

We write $|\cdot|_E$ to denote the Hilbert space norm on E. Moreover, given $z = (u,v)$ we write $\overline{z} := (u,-v)$. Now set

$$L(u,v) := (A^{-s}A^t v, A^{-t}A^s u), \quad (u,v) \in E.$$

This defines a bounded $(E,\langle\,\cdot\,,\,\cdot\,\rangle)$-symmetric linear operator $L\colon E \to E$ which has two eigenvalues, $\lambda = -1$ and $\lambda = 1$ with the corresponding eigenspaces denoted by E_{-1} and E_1, respectively. The spaces E_{-1} and E_1 are E-orthogonal complements to each other, and so, in particular, $E = E_{-1} \oplus E_1$. Thus the triple (L, E_{-1}, E_1) is a dichotomy on E. Let $(\lambda_k)_{k\in\mathbb{N}}$ be the repeated nondecreasing sequence of eigenvalues of B and $(\phi_k)_{k\in\mathbb{N}}$, be a corresponding L^2-orthogonal sequence of eigenvectors such that $|\phi_k|^2_{L^2} = 1/2$ for every $k \in \mathbb{N}$. For every $k \in \mathbb{N}$ let

$$\chi_k = (A^{-s}\phi_k, A^{-t}\phi_k).$$

Then $(\chi_k)_{k\in\mathbb{N}}$, is an E-orthonormal basis of E_1, while $(\overline{\chi}_k)_{k\in\mathbb{N}}$, is an E-orthonormal basis of E_{-1}.

For every $\ell \in \mathbb{N}$ let E^ℓ be the linear subspace spanned by $\bigcup_{k=1}^{\ell}\{\chi_k, \overline{\chi}_k\}$. Moreover, let F^ℓ_{-1}, resp. F^ℓ_1, be the one-dimensional linear subspace of E spanned by $\overline{\chi}_{\ell+1}$, resp. $\chi_{\ell+1}$ and set $F^\ell = F^\ell_{-1} \oplus F^\ell_1$. Let $P^\ell : E \to E$ be the E-orthogonal projector of E onto E^ℓ. It follows that $P^\ell x \to x$ as $\ell \to \infty$, for every $x \in E$.

Altogether, we see that Hypotheses 4.9 and 4.13 are satisfied with $i_\ell = 1$ for all $\ell \in \mathbb{N}$.

Now let us note that, in view of (5.4), for $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$ the function $\partial_\xi H(u(\,\cdot\,), v(\,\cdot\,), \,\cdot\,)$ lies in $L^{p/(p-1)}(\Omega)$ so we may regard $\partial_\xi H(u(\,\cdot\,), v(\,\cdot\,), \cdot\,)$ as an element of the dual space of $L^p(\Omega)$. Since our choice of s implies that $E^s \subset L^p(\Omega)$ with compact inclusion induced map, we can regard $\partial_\xi H(u(\,\cdot\,), v(\,\cdot\,), \cdot\,)$ as an element of E^{-s}. Hence $A^{-2s}\partial_\xi H(u(\,\cdot\,), v(\,\cdot\,), \,\cdot\,)$ is a well-defined element of E^s. Similarly, we may regard the function $\partial_\eta H(u(\,\cdot\,), v(\,\cdot\,), \,\cdot\,)$ as an element of E^{-t} so $A^{-2t}\partial_\eta H(u(\,\cdot\,), v(\,\cdot\,), \,\cdot\,)$ is a well-defined element of E^t. We thus obtain a well-defined map

$$K : E \to E, \quad (u, v) \mapsto (K_1(u, v), K_2(u, v)) \tag{5.7}$$

where

$$\begin{aligned} K_1(u, v) &= -A^{-2s}\partial_\xi H(u(\,\cdot\,), v(\,\cdot\,), \,\cdot\,), \\ K_2(u, v) &= -A^{-2t}\partial_\eta H(u(\,\cdot\,), v(\,\cdot\,), \,\cdot\,). \end{aligned} \tag{5.8}$$

Set $f = f_K = L + K$. The map $K : E \to E$ is continuous and whenever $N \subset E$ is bounded, then the set $K(N)$ is relatively compact in E. Moreover, $K = -\nabla\psi$ where

$$\psi : E \to \mathbb{R}, \quad (u, v) \mapsto \int_\Omega H(u(x), v(x), x)\,\mathrm{d}x.$$

Here, and in the sequel, the symbol ∇ denotes the gradient (of a given function on E) with respect to the inner product on E.

Since L is E-symmetric, we thus obtain

$$f = f_K = \nabla\Phi \tag{5.9}$$

where

$$\Phi : E \to \mathbb{R}, \quad z \mapsto (1/2)\langle Lz, z\rangle - \psi(z).$$

By using important bootstrapping arguments established in [1] it is proved that $z = (u, v)$ is a classical solution of (5.1) if and only if $z \in E$ and $f_K(z) = 0$. Thus the study of solutions of system (5.1) is reduced to the study of equilibria of the gradient-like ordinary differential equation

$$\dot{z} = f_K(z) \tag{5.10}$$

on E.

However, note that we do not impose any growth restrictions on the second partial derivatives of H with respect to the variables (ξ, η). Therefore, no matter how smooth the function H is, the map $f_K: E \to E$, in general, is not differentiable nor even locally Lipschitzian, and so the Cauchy problem for Equation (5.10) may have nonunique solutions. This is where the Conley index developed in [10] and the results on Morse decompositions presented in the first part of this paper come into play.

We first need the following useful

Definition 5.1. Suppose $z_0 = (u_0, v_0) \in E$ is an equilibrium of (5.10), i.e.

$$f_K(z_0) = 0.$$

Define the linear map $K_{\mathrm{lin},z_0}: E \to E$ by

$$K_{\mathrm{lin},z_0}(u,v) = (A^{-2s}(-a(\,\cdot\,)u + c(\,\cdot\,)v), A^{-2t}(c(\,\cdot\,)u - b(\,\cdot\,)v)).$$

Here, the continuous functions a, b and $c: \overline{\Omega} \to \mathbb{R}$ are defined, for $x \in \overline{\Omega}$, by

$$\begin{aligned} a(x) &= \partial_{\xi\xi} H(z_0(x), x), \\ b(x) &= \partial_{\eta\eta} H(z_0(x), x), \\ -c(x) &= \partial_{\xi\eta} H(z_0(x), x) = \partial_{\eta\xi} H(z_0(x), x). \end{aligned}$$

We call the equilibrium z_0 *hyperbolic* if the linear operator $L + K_{\mathrm{lin},z_0}$ is injective.

Remark. Note that the operator $L + K_{\mathrm{lin},z_0}$ is the 'formal' Fréchet derivative of f_K at z_0. In general, the true Fréchet derivative $Df_K(z_0)$ does not exist.

We now state the following fundamental Linearization Principle.

Theorem 5.2. *Let $z_0 = (u_0, v_0) \in E$ be a hyperbolic equilibrium of* (5.10). *Then $\{z_0\}$ is an isolated invariant set for f_K and there is an integer $\gamma = \gamma(z_0)$, called the renormalized Morse index of z_0, and there is an $\ell_1 \in \mathbb{N}$ such that*

$$h(f_K, \{z_0\})_\ell = h(L + K_{\mathrm{lin},z_0}, \{0\})_\ell = \Sigma^{\gamma+\ell}, \quad \ell \geq \ell_1. \tag{5.11}$$

Proof. This is Theorem 2.8 in [11] and its corollary. The proof that

$$h(f_K, \{z_0\})_\ell = h(L + K_{\mathrm{lin},z_0}, \{0\})_\ell, \quad \text{for all } \ell \in \mathbb{N} \text{ large enough}$$

is technically involved since, in general, the map $f_K = L + K: E \to E$ is merely continuous but not differentiable. To prove that

$$h(L + K_{\mathrm{lin},z_0}, \{0\}) = \Sigma^{\gamma+\ell}, \quad \text{for all } \ell \in \mathbb{N} \text{ large enough},$$

note that, for all $\ell \in \mathbb{N}$ large enough, 0 is a hyperbolic equilibrium of the linear finite dimensional ODE

$$\dot{x} = g_\ell(x), \quad x \in E^\ell$$

where $g_\ell\colon E^\ell \to E^\ell$ is the linear map $g_\ell = P^\ell \circ (L + K_{\mathrm{lin},z_0})_{|E^\ell}$. Thus, for every ℓ large enough there is a $k_\ell \in \mathbb{N}_0$ such that

$$h(g_\ell, \{0\}) = \Sigma^{k_\ell}. \tag{5.12}$$

Since in our case Hypothesis 4.13 holds with $i_\ell \equiv 1$, formula (5.12) together with Proposition 4.14 immediately implies the existence of $\gamma \in \mathbb{Z}$ such that

$$h(L + K_{\mathrm{lin},z_0}, \{0\})_\ell = \Sigma^{\gamma+\ell},$$

for all $\ell \in \mathbb{N}$ large enough. □

The following result was proved in [10] using an important a-priori estimate established in [1].

Theorem 5.3. *Define S to be the set of all points $z_0 \in E$ for which there is a bounded solution $z\colon \mathbb{R} \to E$ of f_K such that $z(0) = z_0$. Then S is compact in E and the Conley index $h(f_K, S)$ is defined and*

$$h(f_K, S)_\ell = \Sigma^\ell \quad \text{for all } \ell \text{ sufficiently large.}$$

We say that the function Φ is a *Morse function* if every equilibrium of (5.10) is hyperbolic. It is proved in Section 7 of [1] that the property of being a Morse function is generic in a certain sense.

We can now state the main result of this section.

Theorem 5.4. *Suppose that Φ is a Morse function. Moreover, assume that* 0 *is a hyperbolic equilibrium of* (5.10) *with $\gamma(0) \neq 0$. Then system* (5.1) *has at least two nontrivial solutions. Furthermore, if $\gamma(0) > 0$ and H is even, i.e.*

$$H(u, v, x) = H(-u, -v, x), \quad (u, v, x) \in \mathbb{R} \times \mathbb{R} \times \overline{\Omega},$$

then for every $\gamma \in [\![0, \gamma(0) - 1]\!]$ equation (5.10) *has at least two different equilibria with renormalized Morse index γ. In particular, system* (5.1) *has at least $2\gamma(0)$ nontrivial solutions.*

Remark 5.5. This result was proved in sections 9.2.1 and 9.2.2 of [1] using a version of Morse–Floer homology.

Proof. By Theorem 5.3 there is a bounded and closed set $N \subset E$ such that N is an isolating neighbourhood of S relative to f_K. Let $\mathcal{T} := \mathrm{Sol}(f_K, N)$. Then $\mathcal{T}$ is compact in $\mathcal{C} = C(\mathbb{R} \to E)$, translation and cut-and-glue invariant. Moreover, $z_0 \in E$ is a $\mathcal{T}$-equilibrium if and only if z_0 is an equilibrium of (5.10). Every equilibrium z_0 of (5.10) is hyperbolic and so we conclude, by Theorem 5.2, that $\{z_0\}$ is an isolated invariant set for f_K. It follows that the set $\mathcal{E}$ of equilibria of (5.10) is finite and (as $0 \in \mathcal{E}$) this set has m elements z_r, $r \in [\![1, m]\!]$, for some $m \in \mathbb{N}$. Since by (5.9) the set $\mathcal{T}$ is gradient-like with respect to the function $-\Phi$ it follows from Proposition 3.4 that, after a possible reordering, the family

$(\{z_r\})_{r=1}^m$ is a $\mathcal{T}$-Morse decomposition, i.e $(\{z_r\})_{r=1}^m$ is a Morse decomposition of $S = \mathrm{Inv}_{\mathcal{T}}(N)$, relative to f_K.

Now, for every $r \in [\![1,m]\!]$ let $W_r \subset N$ be a bounded isolating neighbourhood of $\{z_r\}$ (relative to f_K). Let $\gamma_r = \gamma(z_r)$, $r \in [\![1,m]\!]$. Then we obtain, using Theorems 5.2 and 5.3, that, for all $\ell \in \mathbb{N}$ large enough,

$$\begin{aligned} p(t, h(f_K, \{z_r\})_\ell) &= t^{\gamma_r+\ell}, \quad r \in [\![1,m]\!], \\ p(t, h(f_K, S))_\ell) &= t^\ell. \end{aligned}$$

In view of Theorem 4.16 this implies that there is an $\ell_1 \in \mathbb{N}$ such that, for every $\ell \in \mathbb{N}$ with $\ell \geq \ell_1$, there is a formal power series

$$Q_\ell(t) = \sum_{k=0}^{\infty} a_{\ell,k} t^k, \quad t \in \mathbb{R}$$

with coefficients $a_{\ell,k}$, $k \in \mathbb{N}_0$ lying in $\mathbb{N}_0 \cup \{\infty\}$ and such that

$$\sum_{r=1}^{m} p(t, h(f_K, \{z_r\})_\ell) = p(t, h(f_K, S))_\ell) + (1+t)Q_\ell(t), \quad \ell \geq \ell_1,\ t \in \mathbb{R}. \tag{5.13}$$

Setting $a_{\ell,-1} \equiv 0$ we see that

$$(1+t)Q_\ell(t) = \sum_{k=0}^{\infty} b_{\ell,k} t^k, \quad \ell \geq \ell_1,\ t \in \mathbb{R}$$

where

$$b_{\ell,k} := a_{\ell,k} + a_{\ell,k-1}, \quad k \in \mathbb{N}_0 \tag{5.14}$$

so

$$\sum_{r=1}^{m} t^{\gamma_r+\ell} = t^\ell + \sum_{k=0}^{\infty} b_{\ell,k} t^k, \quad \ell \geq \ell_1,\ t \in \mathbb{R}. \tag{5.15}$$

For $\gamma \in \mathbb{Z}$ let $c_\gamma \in \mathbb{N}_0$ be the number of $r \in [\![1,m]\!]$ such that $\gamma_r = \gamma$. Since $\gamma_r + \ell \geq 0$ for all $\ell \geq \ell_1$ and $r \in [\![1,m]\!]$ we see that

$$\sum_{r=1}^{m} t^{\gamma_r+\ell} = \sum_{\gamma\in\mathbb{Z}} c_\gamma t^{\gamma+\ell} = \sum_{k=0}^{\infty} c_{k-\ell} t^k, \quad \ell \geq \ell_1,\ t \in \mathbb{R}.$$

We can thus rewrite (5.13) in the form

$$\sum_{k=0}^{\infty} c_{k-\ell} t^k = t^\ell + \sum_{k=0}^{\infty} b_{\ell,k} t^k, \quad \ell \geq \ell_1,\ t \in \mathbb{R}. \tag{5.16}$$

Fix $\ell \geq \ell_1$ arbitrarily. If $m = 1$, then $\gamma_1 = \gamma(0) \neq 0$. However, (5.15) implies $\gamma_1 = 0$, a contradiction.

If $m = 2$, then (5.15) implies that $\gamma_1 = 0$ or $\gamma_2 = 0$ and so there is a $\gamma \in \mathbb{Z}$ such that

$$t^{\gamma+\ell} = \sum_{k=0}^{\infty} b_{\ell,k} t^k, \quad t \in \mathbb{R}. \tag{5.17}$$

This means that $b_{\ell,k_0} \neq 0$ for some $k_0 \in \mathbb{N}_0$. But then (5.14) implies that $b_{\ell,k_0-1} \neq 0$ or else $b_{\ell,k_0+1} \neq 0$. However, this contradicts (5.17) and proves that $m \geq 3$. This proves the first part of the theorem.

Now assume that $\gamma(0) > 0$ and that H is even. This implies, in particular that whenever $z_0 \neq 0$ is an equilibrium of (5.10) then $-z_0 \neq z_0$ but $K_{\mathrm{lin},-z_0} = K_{\mathrm{lin},z_0}$ and so by Theorem 5.3 we have that

$$h(f_K, \{-z_0\})_\ell = h(f_K, \{z_0\})_\ell, \quad \text{for all } \ell \in \mathbb{N} \text{ large enough.}$$

It follows that c_γ is odd for $\gamma = \gamma(0)$ and c_γ is even otherwise. Hence it follows from (5.16) that $b_{\ell,k}$ is odd if $k = \ell$ or $k = \gamma(0) + \ell$ and even otherwise. It now follows from (5.14) that $a_{\ell,k}$ is even if $-1 \leq k < \ell$ and so $a_{\ell,\ell}$ is odd. This implies by simple induction that $a_{\ell,k}$ is odd for all $k \in [\![\ell, \gamma(0) + \ell - 1]\!]$. By comparing coefficients in (5.16) we thus see that $c_{k-\ell} \geq 2$ for all $k \in [\![\ell, \gamma(0) + \ell - 1]\!]$. Hence, for every $\gamma \in [\![0, \gamma(0) - 1]\!]$ we have at least two equilibria of (5.10) with renormalized Morse index γ. This proves the second part of the theorem. □

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Maria C. Carbinatto
Departamento de Matemática
ICMC-USP
Caixa Postal 668
13.560-970 São Carlos SP, BRAZIL

E-mail address: mdccarbi@icmc.sc.usp.br

Krzysztof P. Rybakowski
Universität Rostock
Fachbereich Mathematik
Universitätsplatz 1
18055 Rostock, GERMANY

E-mail address: krzysztof.rybakowski@mathematik.uni-rostock.de